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Why does a ball fall?: A new visualization for Einstein's model of gravity

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Many physics teachers seek a simple illustration of Einstein's model of gravity, suitable for the introductory physics classroom. In this article, we show that an ordinary wall map of the world can be used to contrast Newton's and Einstein's explanations for why a ball falls when released. Trajectories on the map are analogous to the trajectories of the ball through spacetime, because the geometry of the map is remarkably similar to the geometry of spacetime near Earth's surface. To aid in the pedagogy, we focus on the concept of *scale* rather than *curvature*. We show that, contrary to popular visualizations of Einstein's model, it is primarily the warping of time, not space, that causes a ball to fall, and we address the question of why we do not see the distortion of spacetime around us. Finally, we recover Newton's results for the falling ball from our geometrical treatment.
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I. INTRODUCTION

Einstein's model of gravity has been called "conceptually simple" and "the most sublime of all scientific creations,"[1] yet a century after its inception it is still largely absent from introductory physics courses. Pioneering efforts to introduce students to Einstein have tended to focus on the extraordinary predictions of Einstein's model that can capture students' imagination and kindle a lifelong interest in physics: the big bang,[2] the expanding universe, and black holes.[3]

This article describes a complementary approach, in which Einstein's model of gravity is applied to a bread-and-butter problem with which students are already familiar, and for which the Newtonian description seems perfectly satisfactory: "Why does a ball fall when you release it?" The falling ball problem turns out to offer an intriguing route to Einstein's model of gravity, suitable even for the high school physics classroom.

Our approach has three key features. First, we recast Newton's interpretation of the falling ball in a simple geometrical form (Sec. II), which enables us to show that Newton's and Einstein's models make *mutually incompatible* predictions about whether or not a force acts on the ball, and whether the ball's trajectory through spacetime is straight or curved. This dramatic tension puts the student in the role of detective, and prompts a deeper look at Einstein's model.

Second, we introduce the non-Euclidean geometry of Einstein's model by using an object familiar to students from grade school: an ordinary wall map of the world in Mercator projection. Remarkably, the geometry of the map closely mimics the geometry of spacetime surrounding a falling ball. As a consequence, the ball's trajectories through spacetime are analogous to simple routes on the map. We use this coincidence to further contrast Einstein's and Newton's models of gravity (Sec. III).

Third, we focus on the physical concept of scale, or the varying scale of distance and of time in our world, which allows us to sidestep the mathematical abstraction of curved spacetime. We are invited to do this, since the curvature of spacetime can be split into two parts: a *stretching* of distance and time (needed for the falling ball problem), and a *twist* (a more difficult concept, needed primarily for relativistic phenomena such as frame-dragging).[4]

We felt it useful to avoid curved spacetime at the introductory level, mainly because it requires more mathematics but also because students are often confused by literal illustrations of the concept. An example is the rubber-sheet illustration of curved spacetime, popularized by Carl Sagan in the television series *Cosmos*, and now ubiquitous in the popular media and introductory textbooks.[5] A bowling ball is placed on a rubber sheet, and the resulting depression deflects the path of a smaller ball (Fig. 1). Unfortunately, the illustration makes no sense. Students observe that space is not a rubber sheet, does not curve into an unseen dimension, and does not push objects into circular orbits. The rubber sheet does not even reflect the symmetry of the central mass—if you turn the illustration upside down the explanation fails.

In fact, the distortion of space is irrelevant to the falling ball problem. As we will show it is primarily warped time—not space—that causes a ball to fall and satellites to orbit.

By contrast, students are already familiar with the concept of scale from their experience with models and maps. The varying scale of time has been measured directly, providing experimental support for Einstein's model.[6] Also, the idea that the scale of distance and time might vary from place to place has an interesting history that provides a gentle introduction to Einstein. This idea is briefly sketched in Sec. IV, and developed quantitatively in Sec. V. There we address the

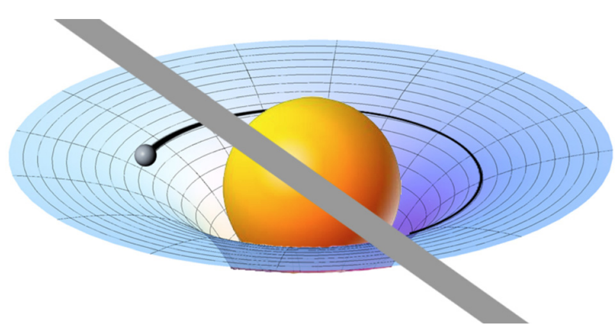

Fig. 1. The rubber-sheet illustration of curved spacetime is often confusing to students. Contrary to the illustration, the distortion of space has an insignificant effect on the motion of a ball.

question, "Why don't we see the distortion of space and time in the world around us?" Finally, in Sec. VI we use Einstein's geometric approach to recover Newton's results for the falling ball.

II. NEWTON, EINSTEIN, AND THE FALLING BALL

A. Newton: "Gravity *must* be a force"

Release a ball from waist height. Why does it fall to the ground?

According to Newton's model of gravity, a *force* pulls the ball to the ground. Two lines of evidence support the claim that gravity is a force. First, we can feel a force pulling downward, when we hold the ball. Second, we can see the force's effect when we release the ball: it sets the ball in motion. (According to Newton's first law of motion, an object at rest will remain at rest unless a force acts on it. Since the ball does not remain at rest, a force must be present.) We will assess later whether these two lines of evidence hold up.

In order to compare Newton's model of gravity with Einstein's, it helps to recast Newton's first law of motion in a geometrical form. First, chart the ball's position as time passes, as shown in the *spacetime* diagram in Fig. 2. According to Newton, the resulting line will be straight if and only if no net force acts on the ball. Since the ball follows the graceful parabola B through spacetime, we conclude that a force must be acting on the ball.

Newton's model of gravity goes on to provide a mathematical description of this force, based on the masses of Earth and the ball and the distance between them.

Although the model seems reasonable at first, on closer inspection it is troubling. What *is* the force of gravity? How can Earth and the ball exert a force on each other when there is nothing between them to mediate that force? How does the ball even know of Earth's existence? Newton himself called the notion of action-at-a-distance an "absurdity."

Furthermore, we see that Newton's conception of force—and therefore his model of gravity—in part rests on the ability to distinguish a straight line from a curved line. How do we know, for example, that path A in Fig. 2 is straight and path B is curved? It certainly looks that way from the diagram. *But is it true?* As we will see, looks are deceiving; intuition is not a reliable guide to nature.

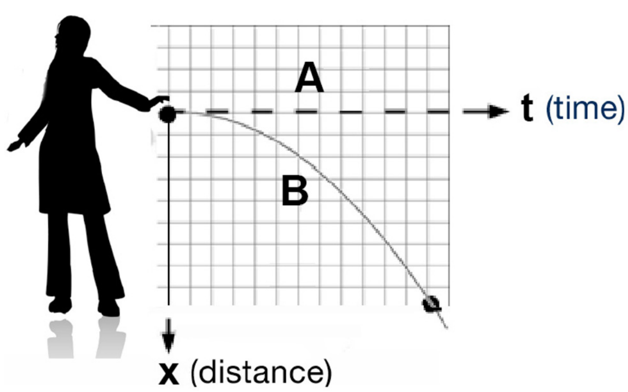

Fig. 2. The spacetime diagram for a ball released at waist height. The ball falls to ground (path B) rather than hovers in mid-air (path A). Which path is straight: A or B? *Are you sure?*.

B. Einstein: "Gravity *cannot* be a force"

Einstein's analysis of the falling ball starts with the same premise as Newton's: The ball's trajectory through spacetime will be straight if and only if no force is present. On that much they agree.

But Einstein makes a truly astonishing claim. Einstein claims that Newton's interpretation of Fig. 2 is backward: In reality, the ball's trajectory (path B) *is* the straight line, while path A is curved! And since the falling ball's trajectory through spacetime is a straight line, then there *cannot be a force acting on the ball.* Thus, Einstein's explanation of the falling ball is that there is nothing to explain! There is no force of gravity; the ball merely traverses the straightest line it can through spacetime. What could be simpler than that?

In fact, Einstein tells us that the trajectory through spacetime of *every* freely moving object is a straight line. Our eyes tell us otherwise, but Einstein is insistent.

Clearly, Einstein and Newton cannot both be right. Either path B is straight or it is not. Either a force acts on the ball or it does not. Which is it?

C. Objections to Einstein's claim

Students will object to Einstein's audacious claim.

Objection 1: "When I hold the ball, I feel its weight. What is that force if not gravity pulling down on the ball?"

Simple, says Einstein: The force comes from *you*, pushing up on the ball. It is the force you apply in order to deflect the ball from its natural, straight-line motion (path B) and force it to follow path A. This is no different, in principle, from a tugboat pushing on an ocean liner to deflect its path.

Note that you only feel this force when you *prevent* an object from moving freely through spacetime. Astronauts, skydivers, and other freely moving explorers report feeling weightless.[7]

Objection 2: "The ball was initially at rest but didn't remain at rest. Are you saying that Newton's first law is wrong?"

A moot point, says Einstein. The proper arena for analyzing motion is spacetime,[8] and in spacetime the ball is never at rest, because it is always moving through time. So the only question is whether the falling ball's spacetime trajectory is a straight line.

We have now called into question both of Newton's lines of evidence that gravity is a force.

Objection 3: "Fine. But you can't expect me to believe that path A is straight and path B is curved. And surely tennis balls and baseballs follow curved paths through space, let alone through spacetime. How can their trajectories be considered straight lines? This defies the evidence of our own senses."

Remarkably, path B is indeed a straight line, as we will see. The reason, according to Einstein, is that every mass distorts the scale of distance and time around it. Newton's assumption that distance and time are measured the same everywhere is simply not true. A yardstick at waist height is *not* the same length as a yardstick on the floor. Neither is a minute at waist height the same length of time as a minute at floor level. The scales of length and time vary from place to place in our world; as we will see, this causes straight lines in spacetime to appear curved, and vice versa.

The price we pay for the simplicity of Einstein's description of gravity is the realization that the geometry of our world is no longer simple. In Einstein's model, the absurdity of Newton's "action at a distance" is traded for the absurdity that mass distorts the geometry of the space and time around it.

A final student comment: "Non-Euclidean geometry makes my head spin! If you want me to put common sense aside, you will have to answer three questions:

(1) What example can you show me that will convince me that a line that looks curved is in fact straight, and vice versa?
(2) If the world is really non-Euclidean, then why doesn't it look distorted?
(3) And if the distortion of space and time turns out to be a very small effect, then how could it cause something as dramatic as the fall of a ball?"

The remainder of this article addresses these questions.

III. A WORLD MAP AS MODEL FOR THE FALLING BALL PROBLEM

Remarkably, an ordinary wall map of the world provides a close analog to the falling ball problem, and is very useful in helping students to make sense of Einstein's audacious claim, for several reasons:

(1) It provides a simple and familiar example of straight lines that look curved to the eye.
(2) The scale of distance on the map varies with location on the map, analogous to the varying scale of distance and time in the world around us. In fact, the map's geometry turns out to closely mimic the geometry of the spacetime diagram for the falling ball, as we shall see.
(3) The map enables us to contrast Newton's and Einstein's models of gravity in a particularly visual and intuitive way.

To see how this works, consider the following parable.

A. A parable of two aviators

Alice and Bob prepare to fly from Boston to Madrid. They will chart their progress on a map of the world in equirectangular projection, as shown in Fig. 3. However, neither Alice nor Bob is aware that Earth's surface is curved, and they have no experience with a world map. They think that *the scale of distance* on their world map is uniform, as it is on their local maps of Boston and Madrid—and as appears to be the case in the world around them.

Bob, the navigator, plans their route. They will take off due east and fly as straight as possible. He reasons that they will reach Madrid along path A, as shown in Fig. 3. Alice, the pilot, confirms the flight plan. "To ensure that we fly as straight as possible," she says, "I'll keep the wings perfectly level." At the moment of takeoff, they are indeed heading directly for Spain along path A—just as our falling ball, when released, initially moves "due east" through spacetime in Fig. 2.

But Bob notices that their flight path starts curving to the south (Fig. 3, path B), heading for Africa and the equator. "What are you doing?!" Bob cries out. "Why aren't you flying straight?"

"I am!" Alice protests. "I am not banking the plane to the right or left, so this path *must* be straight!" But as they chart their progress on the map, they continue to veer southward.

Bob (whose last name is Newton) has a theory: "There must be a force pulling us towards the equator. Otherwise, our path would have been straight."

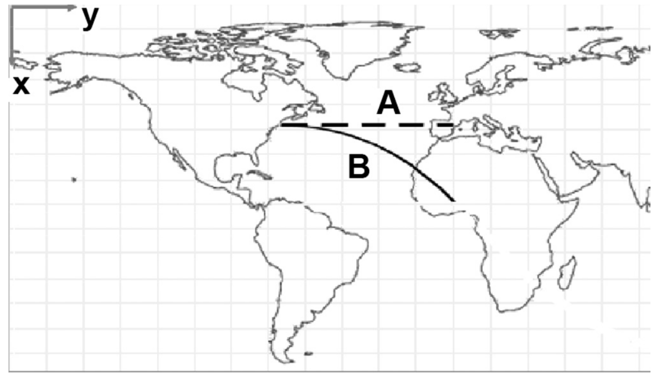

Fig. 3. Starting off due east and flying as straight as possible, a pilot expects to traverse path A, yet finds herself veering southward on path B instead. Why? Which line is straight: A or B?.

But Alice (whose last name is Einstein) feels no force, and she is convinced her route has been as straight as possible, even though it looks curved on the map. She has a competing theory: There is something strange about the scale of her map, but she is not sure what.

Puzzled, they return to Boston and try a simple experiment to test their competing theories. They head for Spain again, but this time they take off to the northeast, again flying as straight as possible (Fig. 4, path B). Bob predicts that the mysterious force will pull the plane southwards, bringing them to Madrid. Their route veers just as predicted, bringing them to their intended destination.

"So you see," says Bob triumphantly, "there must be a force pulling the plane towards the equator—just as the force of gravity pulls on a ball." To make the comparison, he tosses a ball upwards and watches it return to his hand, noting the similarity of the ball's path through spacetime (Fig. 5, path B) with the plane's flight path on the map.

For the second part of their experiment, they fly back to Boston, hewing to path A in Fig. 4, the line on their map that Bob claims is straight. But to accomplish this Alice must bank the plane continuously to the right.

Bob has brought along a sensitive force detector and he triumphantly notes that it registers a force, directed southwards. What further evidence for his theory could one want?

Alas, his excitement is short-lived, for Alice has gathered evidence supporting her own theory. She points out that the force detector has registered a southward force *only along path A*, but not along path B, and she can account for that

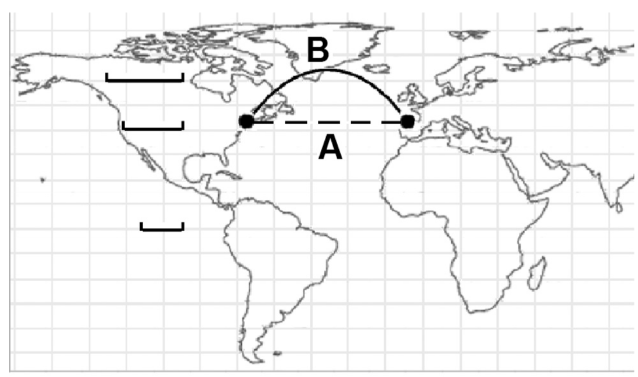

Fig. 4. A pilot's route from Boston to Spain along path B is the straightest, shortest route between its endpoints. The scale markers show that the scale of the map varies with latitude.

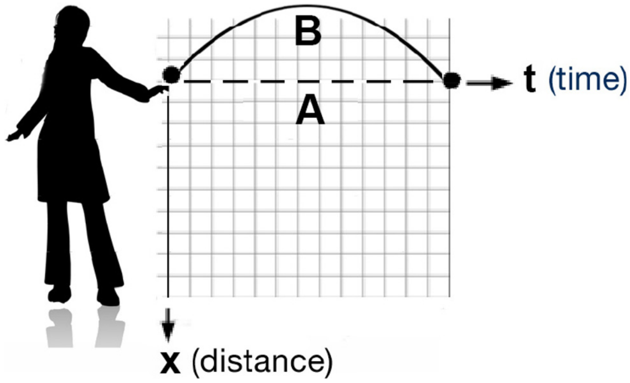

Fig. 5. A ball tossed upwards returns to the hand a moment later. The ball's route through spacetime (path B) is the shortest, straightest route between its endpoints, even though it looks longer than path A.

force: The "mysterious" southward force was merely the centrifugal force generated by banking the plane continuously to the right, to force it to follow path A. (This is analogous to the upward force you apply to a ball to hold it your hand, deflecting the ball from its natural path B to the curved path A in Fig. 2).

Furthermore, when Alice checks her flight log, she finds to her amazement that the number of miles flown along path B is much less than along path A, even though path B appears much longer! This is further evidence that path B is indeed the straight line, not path A.

Who is right: Bob or Alice? Is a force or geometry at work?

Alice is correct, of course, and here's why: On the world map, *the horizontal scale of distance varies from place to place*. As shown in Fig. 4, the scale marker for distance is larger near the poles than near the equator. Therefore, if we select any two points in the northern hemisphere and seek the shortest path between them, the result will always be a line that goes slightly out of its way northward in order to minimize its length. Equivalently, *every straight line on the map is concave towards the equator*.[9] Thus, an airplane that moves "straight ahead" will always veer towards the equator. Geometry, rather than a force, is at work.

The map enables us to compare two very different explanations of the airplane's flight path. Ultimately, the force-free geometric interpretation wins out, and does so only because the scale of distance varies from place to place in the "world" of the map.

The same reasoning applies to the falling ball. When released, the ball follows the straightest path that it can, through a spacetime in which the scale of distance and the scale of time vary with location. Einstein's force-free model of gravity wins out over Newton's model, provided that the world really has the non-Euclidean geometry predicted by Einstein.

IV. A PERSPECTIVE ON THE SCALE OF DISTANCE AND TIME

A. Where did Einstein's idea come from?

Students know that the varying scale of distance on a map is merely an artifact, as it arises from trying to represent the curved surface of Earth on a flat sheet of paper. So why should we believe Einstein's claim that the scale of distance and the scale of time really *do* vary from place to place in our world. Why should that be? Where did the idea come from?

While the origin of Einstein's model of gravity is beyond the scope of this article and most introductory courses, the idea that space and time are malleable is so strange an idea that it is worth mentioning some of its less-familiar roots. Even revolutionary ideas may have a long and innocent gestation!

As early as 1623, Shakespeare considered whether time passes swiftly or lazily. As Rosalind observed in *As You Like It*, "time travels in divers paces for divers persons;" for a bride-to-be time seems to pass slowly, while for a condemned prisoner time passes all too swiftly.

Shakespeare was referring merely to the psychological perception of time. Yet by 1715, Isaac Newton and Gottfried Leibniz were heatedly debating whether the scale of distance and time really are the same everywhere. The debate was kindled by Caroline, Princess of Wales, an extraordinary woman who had been one of Leibniz's students.[10] She realized that Leibniz's view of space and time was fundamentally different from Newton's, so she encouraged the two to exchange letters in order to resolve the issue.[11] Newton thought it obvious that the scale of space and time are everywhere the same, forming a fixed and immutable backdrop to the universe. But Leibniz realized that the size of an object only has meaning in comparison to other objects. Without a comparison, how do we know that a yardstick on the moon has the same length as a yardstick on Earth, for example?

As Leibniz put it, "these gentlemen [Newton and his proxy, Clarke] maintain that space is a real, absolute being… but I held space to be something purely relative, as time is." Newton's view held sway for the next two centuries; rulers and clocks certainly appear to function the same everywhere. But Leibniz would turn out to be right.

By the 1860s, Bernhard Riemann had developed the mathematics of spaces in which the scale of distance varies from place to place. He noted that the geometry of the world around us is not to be decided by philosophy and debate, but instead is a matter for observation and experiment.[12]

These early pioneers showed that, in principle, the scale of distance and time *might* not be the same everywhere. But in the twentieth century, two revolutionary developments led to the conclusion that the scales of distance and time *cannot* be the same everywhere. The world must be non-Euclidean.

The first was the Michelson-Morley experiment and the inferences that flowed from it. That experiment, repeated with greater precision over the years, showed that the speed of light appears the same to all observers, regardless of their motion relative to the source of light. This paradoxical finding led Lorentz and Einstein to realize that two observers who are in relative motion do not share the same scale of distance or time.

Second, Einstein realized that in order to make special relativity compatible with the conservation of energy, *it must be true* that every mass distorts the surrounding geometry of distance and time. Einstein's decade-long struggle with this idea culminated in his model of gravity, one of the great milestones of scientific discovery.

B. What does "the scale of time" mean?

Many students have difficulty understanding what the *scale of time* refers to. The misleading statement that "time slows down near a massive object" especially confuses them.

No matter where you happen to be, time appears to flow normally. For example, to an observer atop Mt. Everest and an observer at sea level, time appears to flow normally—but it does not flow the *same* at those two places. How can we visualize that?

Imagine two identical movies of a chef boiling a three-minute egg, using an ordinary kitchen timer. Play one movie at double-speed, the other at half-speed. In both movies, the egg comes out perfectly; in both, exactly three minutes elapse, as indicated by the timer. Yet the double-time movie finishes first; its scale of time is shorter than the other movie's. The scale of time has meaning only in comparison between two or more "movies." Thus, the movie of the universe doesn't play at a single speed; it varies depending on where you are.

V. METRICS FOR THE FALLING BALL AND WALL MAP

Students will have a second objection to the wall map analogy. On a world map, the varying scale of distance is obvious. For example, Brazil appears smaller than Greenland, though in reality it is four times larger. In contrast, we don't notice any distortion of space or time in our surroundings. A yardstick or clock at waist-level, certainly appear the same as they do at floor level. If it exists at all, this distortion must be extremely small. But in that case, how could it so dramatically influence the path of the ball? To answer this question, we need a quantitative measure of the geometry of both the map and the ball's spacetime diagram.

A. The metric for the wall map

The map's geometry is encapsulated in a mathematical expression known as the *metric*, which gives the distance between any two points on the map.

For the map used in Fig. 4, we take the x-coordinate to run from 0° (north pole) to 180° (south pole), while the y-coordinate runs from 0° to 360°. For this particular map projection, the vertical scale of distance is constant, and the horizontal scale of distance varies as $\sin x$. As a result, the distance ds between two nearby points on the map is not the familiar result from Euclidean geometry

$$ds^2 = dx^2 + dy^2, \tag{1}$$

but rather

$$ds^2 = dx^2 + (\sin^2 x)\ dy^2 \tag{2}$$

in the limit of nearby points.

Equation (2) is the metric for the world map. It can be used to determine any straight line on the map, given the line's starting point and initial direction. These straight lines are called *geodesics,* a useful term because it avoids confusion with lines that merely look straight but that are not the shortest distance between any two of their points. Examples of geodesics are path B in Figs. 3–5. A geodesic is the generalization of a straight line to a world governed by non-Euclidean geometry.

The procedure for starting with the metric and constructing a geodesic has been known for several hundred years. However, as Lagrange lamented in 1760, actually solving geodesic equations is "not as easy as one might hope," intractable in fact, except for special cases. Fortunately, it is straightforward to solve geodesic equations numerically. The reader can construct the geodesic starting due east at Boston, using computer programs available online.[2] The result of such an exercise is the sinusoidal curve shown as path B in Fig. 3—Alice's path really is a geodesic. We now turn now to the metric for the falling ball.

B. Metric for the falling ball

1. First turn gravity off

In the absence of gravity, the separation between two nearby points (events) in spacetime is given by

$$ds^2 = dx^2 - c^2 dt^2, \tag{3}$$

where dx is the vertical distance between the points, dt is the time interval between them, and c is the speed of light.[8] This is the result from Einstein's special theory of relativity.[13]

In this gravity-free world, freely moving objects follow geodesics that look like straight lines, just as they do in a Newtonian treatment. An example is path A in Fig. 2.

2. Turn on gravity

In Einstein's model of gravity, the mass of Earth distorts the surrounding spacetime. By how much? The answer was first worked out by the German physicist Karl Schwarzschild in 1916. Remarkably, Schwarzschild was able to master Einstein's theory of gravity while serving on the Russian front in World War I. He used the theory to calculate the metric for spacetime surrounding a non-spinning, spherical mass, and he was able to report his results before, sadly, he perished a few months later.[14]

In the Appendix, we apply Schwarzschild's metric to the falling ball problem. The result is the approximation

$$ds^2 \approx \left(1 + \frac{2MG}{R^2c^2}x\right)dx^2 - \left(1 - \frac{2MG}{R^2c^2}x\right)c^2dt^2, \tag{4}$$

where x is the distance the ball falls below waist height, M is the mass of Earth, R is the distance from the center of Earth to waist height, G is the gravitational constant, and c is the speed of light. (We ignore the spin of Earth, which would introduce additional terms in the metric that are negligible in this problem.)

Notice from the coefficients in the metric that the distortion of both distance and time depends on location x. As expected, the distortion grows larger with increasing mass and smaller with increasing distance from Earth.

The metric embodies the following physics: as you move from waist-level to ground level, the vertical scale of distance *shrinks*; and as you move from waist-level to ground level, the scale of time *expands* (i.e., the "video of the world" plays more slowly).

Although the scale of distance and time behave in opposite directions, they also have opposite signs in the metric, so they both have the same effect on the metric. Thus for the falling ball diagram in Fig. 2, the distance between grid lines increases as you move closer to the floor. This is analogous to the world map, where distances increase nearer the equator. We will return to the analogy in a moment.

C. Why doesn't the world look distorted?

Our senses are oblivious to the distortion of distance and time because the magnitude of the distortion is extremely small. For example, for the falling ball in Fig. 2, the scale of distance (scale_d) is given by the coefficient of dx in the metric, Eq. (4), as

$$\text{scale}_d = \left(1 + \frac{2MG}{R^2c^2}x\right)^{1/2}. \tag{5}$$

Evaluating this expression at waist height ($x = 0$ meters) and ground level ($x = 1$ meter), we find that the scale of vertical distance changes by a factor of about $1 + 10^{-16}$. Thus, a cat sprawled on the floor is shorter than her twin at waist height by less than the width of a proton! No wonder the world looks perfectly Euclidean to us. The distortion of space predicted by Einstein's model of gravity is far too small to see.

But this also means that the distortion of space is so very small that it has an insignificant effect on the ball's path. For example, we could never hope to represent such a small distortion graphically. Contrary to the rubber-sheet illustration of gravity, the distortion of space is not responsible for the fall of a ball.

Since the distortion of space is so small, we can safely ignore it, further simplifying the metric to

$$ds^2 \approx dx^2 - \left(1 - \frac{2MG}{R^2c^2}x\right)c^2dt^2. \tag{6}$$

D. Warped time causes the ball to fall

At first glance, the distortion of time seems just as insignificant as the distortion of distance. The scale of time (scale_t) varies by the factor

$$\text{scale}_t = \left(1 - \frac{2MG}{R^2c^2}x\right)^{1/2} \tag{7}$$

between waist-height and floor level, again a change of only 1 part in 10^{-16}. This tiny variation with height is much too small for our senses to detect, although it has been measured using atomic clocks,[6] providing direct proof of Einstein's claim that the scale of time varies with location.

Why does this minuscule warping of time influence the ball's trajectory so dramatically? The reason is that even a small interval of time corresponds to a large distance, when time is measured in meters. Imagine redrawing Fig. 2 with the time axis in meters rather than seconds. The half-second it takes the ball to fall corresponds to about 150×10^6 m, so if both axes were drawn at the same scale, the spacetime diagram would stretch halfway to the Moon! Then we would see that the ball's trajectory in spacetime is very nearly a straight line. The distortion of time is indeed tiny. Yes, the ball falls 1 m to the floor, but it has to travel through 150×10^6 m of time to get there!

We conclude that the distortion of time—not space—causes a ball to fall. In essence, an object is deflected towards the region in which time flows more slowly. In this way, the invisible dimension of time makes its presence palpable in our three-dimensional world.

E. Comparing the metrics for the map and the ball

To see why the world map is such a good analog for the falling ball problem, compare the metrics for the map and the ball. Rearrange the spacetime metric in Eq. (6) to the form

$$ds^2 \approx dx^2 - c^2dt^2 + 2ax\,dt^2, \tag{8}$$

where the constant $a = MG/R^2$. The first two terms are the metric for empty spacetime, while the third term shows how the scale of time varies with the ball's height above the ground. This third term plays the same role as the $\sin^2 x$ term in the metric for the world map, Eq. (2). In both cases, the scale changes as we move downward, and as we have seen in Sec. III, this change of scale curves geodesics southward.

F. Geodesic for the falling ball

We can now verify Einstein's claim that the falling ball follows the straightest path it can through spacetime. To find the geodesic, we start with the metric for the falling ball from Eq. (8), along with the initial conditions that the ball starts at waist height ($x = 0$) and heads "due east," since the ball is initially at rest ($dx/dt = 0$). The intrepid reader can set up the geodesic equations using instructions found online,[2] and solve the equations numerically using *Mathematica* or a similar program. The output of the calculation is the geodesic plotted in Fig. 6. On inspection, we see that the curve is just the parabola $x = \frac{1}{2}at^2$, where the constant $a = MG/R^2$.

G. Recovering Newtonian physics

The geodesic in Fig. 6 was arrived at purely geometrically, and *any* two-dimensional region whose geometry is summarized by the metric in Eq. (8) will have geodesics that are parabolas. In finding the geodesic, we considered x and t to be dimensionless variables. But if we identify x with distance and t with time, then the coefficient $a = MG/R^2$ has units of acceleration and its value is $9.8\,\text{m/s}^2$, which is the free-fall value of acceleration at the surface of Earth.

Thus the geometric approach yields the same spacetime trajectory for the falling ball as is predicted by Newtonian mechanics. But here, the path of the ball is determined, not by the supposed force of gravity, but purely by the geometry of the non-Euclidean world we inhabit. The geometry is encoded in the metric. Thus, in order to calculate the trajectory of an object subject only to gravity, we do not need to know *anything* about force laws. All we need to know is how to draw a straight line in our non-Euclidean world.

VI. CONCLUSION

In short, the unremarkable fall of a ball reflects some truly remarkable features of nature: that nature's drama unfolds in the unity of spacetime rather than in space alone; that objects moving freely through space and time manage to follow the straightest possible path, even if that path looks curved to the eye; that the scale of distance and the scale of time vary from place to place; and that geometry, not force, rules the world of gravity.

It is also remarkable that Newton's and Einstein's radically different theories can both describe the same phenomenon. Einstein's theory stands alone, however, as a bridge to the future of science. The theory already has predicted three phenomena that would have been inconceivable a century ago: the big bang, black holes, and the dark energy thought

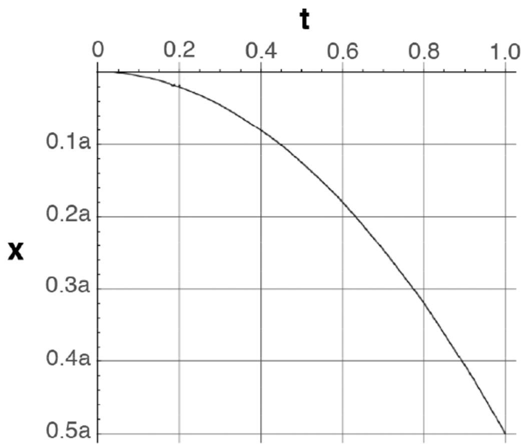

Fig. 6. The geodesic for a falling ball in a world whose metric is given by Eq. (8). It is also the plot of the distance x the ball falls in time t, as predicted by Newtonian physics.

to pervade our universe. Each of these mysteries has firm experimental support, yet none can be understood fully on the basis of current physics. In this way, Einstein's theory of gravity contains the seeds of its own demise and its rebirth in another, more complete theory.

We also saw that the motions of a ball through spacetime can be modeled by geodesics on a wall map of the world, providing a useful teaching tool for comparing Einstein's and Newton's models of gravity.

Our approach avoided mention of the curvature of spacetime, focusing instead on the physical concept of the scale of distance and of time, a route to Einstein that is more in line with students' prior experiences with scale models. But this approach is suitable only for simple examples like the falling ball, as there are important aspects of gravity that cannot be treated using the idea of scale. One example is gravitational waves.[15] Another is the spacetime around a spinning sphere, described not by the Schwarzschild metric but by the more complicated Kerr metric, which contains "cross-terms" that do not correspond to the scale of space or time. (Such cross-terms correspond to the physical phenomenon of "frame-dragging," in which orbiting objects are carried along with the spin of the sphere.[4])

Despite these limitations, the concept of scale allows for a simple entrée to Einstein's model of gravity. It is comforting that a theory as profound as Einstein's can be introduced, at least in part, by a prop as simple and familiar to us as the wall map that was our constant companion in grade school.

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APPENDIX: SPACETIME METRIC FOR THE FALLING BALL PROBLEM

The metric for space and time surrounding a non-rotating, spherical mass M is[14]

$$ds^2 = \left(1 - \frac{2MG}{rc^2}\right)^{-1} dr^2 - \left(1 - \frac{2MG}{rc^2}\right) c^2 dt^2, \qquad \text{(A1)}$$

where we have included only the radial direction of space r, and where the constants are as described in the text. Take x to be the distance that the ball falls below the tabletop, and R to be the distance from the tabletop to Earth's center. Replace the variable r by $R - x$, and simplify the resulting metric by noting that $2MG/Rc^2 \ll 1$ and using the approximation $(1-a)^{-1} \approx 1 + a$, for $a \ll 1$. The result is the metric in Eq. (4).

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