RELATIVISTIC GRAVITY AND THE ORIGIN OF INERTIA AND INERTIAL MASS

K Tsarouchas

Relativistic Gravity and the Origin of Inertia and Inertial Mass

K. I. Tsarouchas

School of Mechanical Engineering
National Technical University of Athens, Greece
E-mail-1: ko.tsarouchas@gmail.com - E-mail-2: ktsarouc@central.ntua.gr

Abstract

If equilibrium is to be a frame-independent condition, it is necessary the gravitational force to have precisely the same transformation law as that of the Lorentz-force. Therefore, gravity should be described by a gravitomagnetic theory with equations which have the same mathematical form as those of the electromagnetic theory, with the gravitational mass as a Lorentz invariant. Using this gravitomagnetic theory, in order to ensure the relativity of all kinds of translatory motion, we accept the principle of covariance and the equivalence principle and thus we prove that,

1. The external inertial force, perceived by an accelerating body, is real gravitational force due to induction effects from the entire Universe.
2. The internal inertial force, depends on the body's internal structure, but in a free fall it is canceled because of the equivalence principle and the body experiences only the external inertial force. That is why all bodies, fall with the same acceleration.
3. The inertial mass of a body depends on the distribution of matter in the Universe and this seems very important for the explanation of dark matter and dark energy.
4. The gravitational field affects the spacetime metric and all freely moving bodies follow geodesic of the metric.
5. We can obtain the Schwarzschild metric and thus, the new theory is in agreement with all past experiments and observations.

Keywords: Gravitomagnetism , Mach' principle , origin of inertia , dark matter , dark energy.

1 Introduction

The origin of inertial forces is a problem which has been of great concern to many thinkers since the time of Newton, but which so far has escaped a satisfactory solution. So, there is space for a new attempt. Inertial forces appear in a non-inertial reference frame. But what determines an inertial reference frame?

The first answer comes from Descartes and Newton, according to which, an inertial reference frame is a frame that moves at a constant velocity, with respect to the absolute space and the motion is absolute. The inertial forces, such as the centrifugal force, must arise from acceleration with respect to the absolute space. This idea implies that space is an absolute physical structure with properties of its own and the inertia is an intrinsic property of the matter.

The second answer comes from Leibniz, Berkeley, Mach and is known as Mach' principle, according to which, an inertial reference frame is a frame that moves with constant velocity, with respect to the rest of the matter in the Universe, and the motion is relative. The inertial forces, such as the centrifugal force, are more likely caused by acceleration, with respect to the mass of the celestial bodies. This idea implies that the properties of space arise from the mater contained therein and are meaningless in an empty space.

The distinction between Newton's and Mach's considerations, is not one of metaphysics but of physics, for if Mach were right then a large mass could produce small changes in the inertial forces observed in its vicinity, whereas if Newton were right then no such effect could occur [1]. This seems to be very important when we consider subjects such as dark matter and dark energy.

The idea that the only meaningful motion of a particle, is motion relative to other matter in the Universe, has never found its complete expression in a physical theory. The Special theory of relativity eliminated absolute rest from physics, but acceleration remains absolute in this theory. Alfred Einstein was inspired by Mach's principle. The General theory of relativity, attempted to continue this relativization and interpret inertia considering that it is the gravitational effect of the whole Universe, but as pointed out by Einstein, it failed to do so. Einstein showed that the gravitational field equations of General relativity imply that a body, in an empty Universe, has inertial properties [2].

The principle of equivalence is an essential part of General relativity. But although the principle of equivalence has been confirmed experimentally to high precision, the gravitational field equations of General relativity have not as yet been tested so decisively. Thus, it is not a theory fully confirmed experimentally and competing theories cannot be ruled out [3]. Almost all of the results that have been the subject of experimental investigation can be described by the linear approximation of the field equations [4]. The linear approximations of the field equations give us equations analogous to the equations of electromagnetism.

Finally, as pointed out by Henri Poincare, if equilibrium is to be a frame-independent condition, it is necessary for all forces of non-electromagnetic origin to have precisely the same transformation law as that of the Lorentz-force [5][6]. But this does not happen with gravity as it is described by General relativity.

2 Relativistic Gravity

In an inertial reference frame K, let's have a system of two non spinning bodies with gravitational masses and positive electric charges, in a region of free space where there are no external forces. We suppose that the two bodies are at rest in the inertial frame K, under equilibrium conditions, i.e. the force of gravitational attraction balances that of electrostatic repulsion.

But what is observed by another inertial frame of reference K', moving with constant velocity relative to the frame K? Let's imagine that if the bodies collide, they will explode. It is impossible for one observer to see an explosion and for another to not see it. **Therefore, the equilibrium must be a frame-independent condition.**

- **In order for this to happen, the gravitational force should be transformed in exactly the same way as the Lorentz-force is transformed in different inertial frames.**

Therefore, gravity should be described by a gravitomagnetic theory with equations which have the same mathematical form as those of the electromagnetic theory with the gravitational mass as a Lorentz invariant.

Moreover, as we will show later, **if we want to ensure the relativity of all kinds of translatory motion, the gravitational field should affect the spacetime metric and all the freely moving test bodies follow geodesic of the metric.**

According to Richard Feynman, we can reconstruct the complete electrodynamics using the Lorentz transformations (for force, potential, velocity and co-ordinate) and the following series of remarks [7][8]:

1. The Coulomb potential for a stationary charge in vacuum is, $\varphi_e = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$
2. An electric charge produces a scalar potential and a vector potential, which together form a four-vector, $A_e = \left(\frac{\varphi_e}{c}, \boldsymbol{A}_e\right)$
3. The potentials produced by a charge moving in any way, depend only upon the velocity and position at the retarded time.

Of course we need to know how to get the Coulomb's law from the scalar potential. So, if we want to obtain a gravitomagnetic theory, where its equations have the same mathematical form, as those of the electromagnetic theory, the same series of remarks must be met for gravity. We already have the first remark, that is, the gravitational potential for a stationary gravitational mass m in vacuum is $\varphi_g = -\frac{1}{4\pi g_0}\frac{m}{r}$ where $g_0 = \frac{1}{4\pi G}$ and G is the Newton's universal gravitational constant, but this is only the one remark. So, we need the other two, as well. We will obtain them with the following two principles:

Principle 1

> **A gravitational mass produces a scalar potential and a vector potential, which together form a four-vector,** $A_g = \left(\frac{\varphi_g}{c}, \boldsymbol{A}_g\right)$

From principle 1, follows that the gravitational mass is an invariant quantity. As we will show later, **the inertial mass and the gravitational mass are different entities but all the bodies released from the same point in a gravitational field, fall with the same acceleration.**

Principle 2

> **The potentials produced by a gravitational mass moving in any way, depend only upon the velocity and position at the retarded time.**

So, the potentials produced by a gravitational point-mass m moving with any velocity have the same mathematical form as the Lienard-Wiechert potentials for an electric point-charge moving with any velocity, but with a negative sign,

$$\varphi_g = -\frac{1}{4\pi g_0}\left[\frac{m}{r - \boldsymbol{r}\boldsymbol{v}/c}\right] \tag{2.1}$$

$$\boldsymbol{A}_g = -\frac{1}{c^2}\frac{1}{4\pi g_0}\left[\frac{m\boldsymbol{v}}{r - \boldsymbol{r}\boldsymbol{v}/c}\right] = \frac{1}{c^2}\left[\varphi_g \boldsymbol{v}\right] \tag{2.2}$$

where $\boldsymbol{r}$ is the vector from the gravitational point-mass to the point where the potential is evaluated, c is the speed of light in vacuum and all the quantities in the square bracket are to have their values at the retarded time. Starting from the potentials, in order to find the fields, we have the equations

$$\boldsymbol{E}_g = -\boldsymbol{\nabla}\varphi_g - \frac{\partial \boldsymbol{A}_g}{\partial t} \tag{2.3}$$

$$\boldsymbol{B}_g = \boldsymbol{\nabla} \times \boldsymbol{A}_g \tag{2.4}$$

The force, that a gravitational mass m experiences, when it moves with velocity $\boldsymbol{v}$ in the above fields is,

$$\boldsymbol{F}_g = m(\boldsymbol{E}_g + \boldsymbol{v} \times \boldsymbol{B}_g) \tag{2.5}$$

where $\boldsymbol{E}_g$ is the gravitational field and $\boldsymbol{B}_g$ the gravitomagnetic field.

So, we have now a gravitomagnetic theory, with equations that have the same mathematical form as those of the electromagnetic theory. Therefore, we expect that there are gravitomagnetic waves that propagate, in vacuum with the speed of light, and that they are described by equations which have the same mathematical form as the corresponding equations for electromagnetic waves, but with one important difference. An isolated electric source can radiate electric dipole radiation, with power proportional to the square of the second time derivative of the electric dipole moment. However, an isolated gravitational source cannot radiate gravitational dipole radiation, but quadrupole and radiation of higher polarity. The reason is simple. The electric dipole moment can move around with respect to the center of mass but the gravitational dipole moment is identical in location with the center off mass, and due to the law of conservation of momentum, cannot accelerate or radiate [9].

3 General relativity of translatory motion

We will follow now, the fundamental idea of relativity of all kinds of translatory motion. In accordance with this idea we can detect and measure the translatory motion of a given body, relative to other bodies, but cannot assign any meaning to its absolute motion.

In order for all kinds of translatory motion to be relative, the laws of physics should have the same mathematical form when referred to different reference frames, which are in relative translatory motion, since otherwise the difference in form could provide a criterion for judging the absolute motion. So, first we accept the next principle,

Principle 3 - The principle of general relativity - The principle of covariance

> **The laws of physics which are valid in an inertial reference frame, i.e. the laws of Special theory of relativity, have the same mathematical form in all reference frames which are in relative translatory motion.**

However, it does not end with the above principle. We need one more principle to ensure the relativity of all kinds of translatory motion. This arises from the fact that the expression of the equations of physics in a form which is independent of the reference frame does not in general prevent a change in their numerical content when we change from one reference frame to another and it is only by relating such changes in numerical content to conceivable changes in gravitational field that we are able to eliminate criteria for absolute motion and to preserve the idea of the relativity of all kinds of translatory motion [10].

We will consider now an accelerated reference frame by making a thought experiment, **the lab frame experiment**. Let's suppose that we have a space station, far from any massive body, that we use it as a laboratory. **We will call the local frame of the space station, lab frame. An observer L is at rest in the lab frame.** We assume that the distribution of matter in the Universe is such, that **the gravitational field in the lab frame is equal to zero.** This means that the gravitational scalar potential φ_g , from the entire Universe, has the same value everywhere in the lab frame, and so,

$$\nabla\varphi_g = 0 \tag{3.1}$$

We also suppose that the Universe expands symmetrically in all directions, with respect to the lab frame, so that the gravitational vector potential due to one part of the mass-current, is canceled out by the vector potential due to another part of the mass-current, owing to its symmetry. This means that the gravitational vector potential $\boldsymbol{A}_g$ from the entire Universe is equal to zero, everywhere in the lab frame,

$$\boldsymbol{A}_g = 0 \tag{3.2}$$

This would also happen if all the bodies of the Universe were at rest, relative to the laboratory. So, we can say that the lab frame is at rest relative to the Universe, or at rest relative to the fixed stars.

Let's suppose that in the lab frame, a small rocket starts to accelerate making translatory motion, with constant proper acceleration, i.e. feels a constant force in its instantaneous rest frame. **We will call the local reference frame of the rocket, rocket frame. An observer R is at rest in the rocket frame.** Let's apply the covariant laws of gravitomagnetism in the rocket frame, following the principle of general relativity. **Because we accept that acceleration is relative,** the fixed stars are accelerating relative to the rocket frame, in the same way for all parts of the rocket frame. Therefore the covariant laws of gravitomagnetism will give us an induced uniform gravitational field in the rocket frame, according to the Faraday's law for gravitomagnetism. We will consider this in detail, later.

So, the observer R is at rest in a local reference frame with a uniform gravitational field, while the observer L is at rest in a local reference frame without gravity, that is, an inertial reference frame. We assume that the gravitational forces between the rocket and the laboratory are negligible.

The observer L, who is at rest in a local inertia frame, using the Special theory of relativity is able to describe what physical effects are observed by the observer R who is at rest in the accelerating rocket frame. This is the well known study of a uniformly accelerated rigid reference frame, in Special relativity. So, from the viewpoint of the observer L, the well known physical effects, which are observed by the observer R, are [11]:

1. Redshift or blueshift of a light ray which moves parallel to the direction of the acceleration.
2. Varying coordinate speed of light; fixed local relative speed of light.
3. Spacetime is endowed with a metric.
4. Maximum proper time as the law of motion of freely moving bodies.
5. Horizons.

Let's consider now the description from the viewpoint of the observer R, who is at rest in the rocket frame. Because we accept that acceleration is relative, from the viewpoint of the observer R, it is the lab frame that makes accelerating motion, relative to the rocket frame. So, if the induced gravitational field did not exist, by symmetry, the rocket frame would be equivalent to the lab frame and the above physical effects would have to occur in the lab frame and not in the rocket frame. Nevertheless, the uniform gravitational field exists and the above physical effects occur in the rocket frame and not in the lab frame. **In order for this to happen, from the viewpoint of the observer R, the only way is to think that the induced uniform gravitational field, in the rocket frame, should be capable of causing all the above physical effects, with the same numerical values.** For this reason we accept the next principle,

Principle 4 - The principle of equivalence

> **Physics in an accelerating local reference frame with uniform acceleration $\boldsymbol{a} = -\boldsymbol{g}$, in a region without gravity, is equivalent to physics in a non accelerating local reference frame with a uniform gravitational field, where all the released bodies fall with acceleration $\boldsymbol{g}$.**

It must be said that the equivalence principle serves as a very useful guide indeed, in spite of it being only an approximation to truth. It is impossible to mimic a uniform gravitational field, over a finite distance, by using an accelerated frame [12].

From the viewpoint of the observer R, the lab frame makes a free fall in the gravitational field that he perceives but none of the above physical effects happen in the lab frame, although it is accelerating. **In order for this to happen, the only way is to think that the gravitational field must exactly cancel the acceleration of the lab frame so that, no sign of either acceleration or gravitation can be found by any physical means in the lab frame.** So, we have an alternative expression of the equivalence principle,

Principle 4 - The principle of equivalence

> **Physics in a local reference frame, freely falling in a gravitational field, is equivalent to physics in an inertial reference frame without gravity.**

From the viewpoint of the observer R, the fixed stars and the lab frame, fall in the gravitational field that he perceives. Thus, according to the equivalence principle, no sign of either acceleration or gravitation can be found by any physical means in the fixed stars and in the lab frame because they are freely falling in a gravitational field. **Therefore, the observer R doesn't observe any radiation field from them, because radiation is a sign of acceleration or gravitation. In order for this to happen, the only way is to think that the fields make a free fall just like the stars.**

So, for the the observer R, the instantaneous potentials of the free falling stars are the same with the potentials that they would have if they were moving with uniform velocity, equal to the instantaneous relative velocity.

Therefore, we can find the instantaneous potentials of the free falling stars in the rocket frame, from the potentials of them in the lab frame, using just the Lorentz transformation with the instantaneous relative velocity.

Thus, when the rocket accelerates relative to the fixed stars it is equivalent to say either that

1. the rocket accelerates and thus radiates because of the acceleration, while the fixed stars are stationary and thus they do not radiate, or that

2. the rocket is stationary in a universal gravitational field where all the fixed stars make free fall, and thus they do not radiate, whereas the rocket which is at rest in the gravitational field radiates because of the gravitational field.

So, with the principle of general relativity and the principle of equivalence we have ensured the relativity of all kinds of translatory motion.

- **Acceleration, as well as velocity, is relative.**

Moreover, as we have seen, **the equivalence principle shows us that spacetime is endowed with a metric and the gravitational field affects the spacetime metric so that, the maximum proper time is the law of motion of a freely moving body in a gravitational field.** The two above physical effects are so important that we will elevate them to physical principles. So, we will accept the next two principles:

Principle 5 - The principle of spacetime metric

> **Spacetime is endowed with a metric. The spacetime interval between two events is:**
>
> $$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$
>
> **where $g_{\mu\nu}$ is the metric tensor.**

Principle 6 - The principle of geodesic motion or of maximum proper time

Freely moving test bodies, in a gravitational field, follow geodesic of the metric:

$$\delta \int ds = 0$$

The gravitational force affects the spacetime metric and so, we can say that gravity curves the spacetime. The spacetime of Special theory of relativity is the Minkowski spacetime which is a flat spacetime. While it is clear that flat and curved spaces are different entities, they are closely related. We are familiar from our experience with smoothly curved surfaces that any smoothly curved space can be approximated locally by a flat plane. This is the content of the local-flatness theorem. According to the local-flatness theorem, the metric in the immediate neighborhood of a point P is, to a close approximation, the Minkowski spacetime metric and the laws of Special relativity are valid there [13][14]. Therefore, **everywhere locally the laws of Special theory of relativity are valid.**

Let's follow now the principle of general relativity by applying the Newton's First Law, which is a law of Special relativity, **in the proper-frame of an accelerating body, i.e. the frame where the body is always at rest.** We assume that the the proper-frame is sufficiently small so that we can use the local-flatness theorem. Newton's First Law of motion states that a body, subject to no forces, remains at rest or continues to move in a straight line with constant speed. In its proper frame the body is at rest and so, according to the Newton's First Law the net force on it is zero. So, we come to the conclusion that: **In the proper-frame of any-body the total force on the body is equal to zero.**

So, in order for the equilibrium to be a frame-independent condition and for the acceleration to be relative, we have now a theory that describes gravity by equations which have the same mathematical form as those of the electromagnetic theory but also has all the principles of Einstein's General theory of relativity except the weak equivalence principle, i.e. the equality of the gravitational mass and the inertial mass. From all the above, without the field equations of Einstein's General relativity, we will obtain later the Schwarzschild metric and thus, the new theory will agree with all past experiments and observations. Now, having created all the tools we need, we can move on and consider what are the inertial forces.

4 Inertia

4.1 Gravitational inertial forces

Let's continue now, the lab frame experiment. We assume that the lab frame and the rocket frame have the three sets of axes parallel and common the x, x' axis. **The rocket, which is initially at rest** in the lab frame, begins to accelerate along the x axis. We assume that the rocket frame is sufficiently small, so that, according to the local-flatness theorem, we can consider that the spacetime is flat in the rocket frame, and so we can apply the laws of Special relativity. We have shown in the previous chapter, that **we can find the potentials of the fixed stars in the accelerated rocket frame, from their potentials in the lab frame, using the Lorentz transformations with the instantaneous relative velocity.**

The transformation laws which give the gravitational scalar potential φ'_g and the vector potential $\boldsymbol{A}'_g$ in a moving frame S', in terms of φ_g and $\boldsymbol{A}_g$ in a stationary frame S, as measured at the same point, at the same time by people in the two frames, are

$$\varphi'_g = \gamma(v)(\varphi_g - vA_{g-x}), \qquad A'_{g-y} = A_{g-y}$$
$$A'_{g-x} = \gamma(v)(A_{g-x} - \frac{v}{c^2}\varphi_g), \qquad A'_{g-z} = A_{g-z}, \qquad \gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{4.1}$$

This assumes that the primed reference frame is moving with speed v in the positive x-direction, as measured in the unprimed reference frame. We consider now that the rocket frame is the moving frame and the lab frame is the stationary frame. When the instantaneous speed of the rocket frame is v in the positive x-direction, as measured in the lab frame, it is straightforward to find the instantaneous potentials φ'_g and $\boldsymbol{A'}_g$ in the accelerated rocket frame, in terms of φ_g and $\boldsymbol{A}_g$ in the stationary lab frame. Substituting for A_{g-x} from equation (3.2) into equation (4.1) and using vector notation with $\boldsymbol{v}$ the velocity of the rocket in respect to the lab frame, the potentials in the rocket frame are [15][16],

$$\varphi'_g = \gamma(v)\varphi_g \tag{4.2}$$

$$\boldsymbol{A'}_g = -\frac{1}{c^2}\varphi'_g\boldsymbol{v} = -\frac{1}{c^2}\gamma(v)\varphi_g\boldsymbol{v} \tag{4.3}$$

The gravitational field in the rocket frame now, is given from the equation (2.3),

$$\boldsymbol{E'}_g = -\boldsymbol{\nabla'}\varphi'_g - \frac{\partial \boldsymbol{A}'_g}{\partial t'} \tag{4.4}$$

where $\partial t'$ is the time interval, in the rocket frame. The gravitomagnetic field, in the rocket frame, is zero because all the fixed stars make translatory motion in respect to the rocket frame and so, $\boldsymbol{\nabla'} \times \boldsymbol{A'}_g = 0$.

The $\gamma(v)$ factor is the same everywhere in the rocket frame. Hence, from the equation (4.2), the scalar potential φ'_g is always the same everywhere in the rocket frame and therefore,

$$\boldsymbol{\nabla'}\varphi'_g = 0 \tag{4.5}$$

The vector potential is also the same everywhere in the rocket frame. So, the gravitational field in the rocket frame becomes

$$\boldsymbol{E}'_g = -\frac{\partial \boldsymbol{A}'_g}{\partial t'} \tag{4.6}$$

Therefore, an induced uniform gravitational field appears in the accelerated rocket frame, whereas in the lab frame there is no gravitational field.

If a test-body K with gravitational mass m, is at rest in the rocket frame, will experience a gravitational force,

$$\boldsymbol{F'}_g = m\boldsymbol{E'}_g = -\frac{\partial\left(m\boldsymbol{A}'_g\right)}{\partial t'} \tag{4.7}$$

Substituting for $\boldsymbol{A}'_g$ from equation (4.3) into equation (4.7), we obtain,

$$\boldsymbol{F'}_g = \frac{m}{c^2}\frac{\partial(\varphi'_g\boldsymbol{v})}{\partial t'} = \frac{m}{c^2}\left(\frac{\partial\varphi'_g}{\partial t'}\boldsymbol{v} + \varphi'_g\frac{\partial\boldsymbol{v}}{\partial t'}\right) \tag{4.8}$$

We can have now, some very important results for non relativistic velocities before moving on and considering the subject in the relativistic domain. So, for $\gamma(v) = 1$, $\varphi'_g = \varphi_g$, $\partial t' = \partial t$ and $\frac{\partial\varphi'_g}{\partial t'} = \frac{\partial\varphi_g}{\partial t} = 0$. Thus, the equation (4.8) becomes

$$\boldsymbol{F}'_g = \frac{m\varphi_g}{c^2}\frac{\partial\boldsymbol{v}}{\partial t} = \frac{1}{c^2}m\varphi_g\,\boldsymbol{a} = \left(-\frac{1}{c^2}m\varphi_g\right)(-\boldsymbol{a}) \tag{4.9}$$

where $\boldsymbol{a}$ is the acceleration of the test-body K in respect to the lab frame. If we recall now that, the gravitational scalar potential is negative, it is obvious from the equation (4.9) that the induced gravitational force on the test-body K, is opposite in direction to the acceleration of the body and thus resists to any change of its speed. **It is an inertial force!**

We will call the force which is given from the equations (4.7), external gravitational inertial force because it is due to the acceleration in respect to the external bodies, i.e. in respect to the fixed stars. So, the external gravitational inertial force on the test-body K, for all speeds, is given by the equation

$$\boldsymbol{F}'_{ext-in-g} = \boldsymbol{F'}_g = -\frac{\partial\left(m\boldsymbol{A}'_g\right)}{\partial t'} \tag{4.10}$$

In addition to the external inertial force, there is also an internal inertial force. This is a well known effect [17]. The well known picture is something like this: We can think that a body consists of many particles. When the body is at rest or it's moving at uniform velocity, every particle exerts a force on every other, but the forces all balance in pairs, so that there is no net force. However, when the body is being accelerated, the internal forces will no longer be in balance, because of the fact that the influences take time to go from one particle to another. **With acceleration, if we look at the forces between the various particles of the body, action and reaction are not exactly equal, and the body exerts a force on itself that tries to hold back the acceleration.** Lorentz Hendrik originally calculated the electromagnetic self-force using a spherical charge distribution. **We will call this self-force, internal inertial force, because it depends on the internal structure of the body.**

According to the equivalence principle, when a body makes free fall in a gravitational field, the gravitational field exactly cancels the acceleration so that, no sign of either acceleration or gravitation can be found by any physical means on the body. **Therefore, because the internal inertial force is a sign of acceleration, it should be canceled when the body makes free fall.** So, we come to the conclusion that,

- **When a body makes a free fall in a gravitational field, it experiences only the external inertial force.**

Let's imagine now, that the test-body K is a body without internal structure and thus, when it is accelerated by a force $\boldsymbol{F}$ in the lab frame, it does not experience any internal inertial force. The inertial force on the body is only the external gravitational inertial force. According to the Newton's First Law, as we restate it, in the proper-frame of the test-body K, the total force on the body, is zero. Therefore, for non relativistic velocities, from the equation (4.9), the force $\boldsymbol{F}$ that accelerates the test-body K is

$$\boldsymbol{F} = -\boldsymbol{F}'_{ext-in-g} = -\boldsymbol{F'}_g = \left(-\frac{1}{c^2}m\varphi_g\right)\boldsymbol{a} = m_{in}\boldsymbol{a} \tag{4.11}$$

The equation (4.11) is the Newton's Second Law for non relativistic velocities. Hence, the inertial mass m_{in} of a body without internal structure, with gravitational mass m, for non relativistic velocities, is

$$m_{in} = \left(-\frac{1}{c^2}m\varphi_g\right) \tag{4.12}$$

So, the inertial mass of a body, without internal structure, is not an intrinsic property of the body but is proportional to the gravitational scalar potential of the entire Universe.

It's noteworthy that if we consider that the density of matter is roughly uniform throughout space, then the most distant matter dominates the gravitational scalar potential. This is because, although the influence of matter decreases with the distance, the amount of matter goes up as the square of the distance. With this consideration, the distant matter is of predominant importance, while local matter has only a small effect on the gravitational scalar potential.

In the free fall of a body in a gravitational field, as we have shown, the internal structure of the body does not play any role and the body experiences only the external inertial

force. Let's suppose that a body A with gravitational mass m, makes free fall in the gravitational field of a body B with spherically symmetric gravitational mass M and $M \gg m$, in the lab frame. For non relativistic velocities, the Universal Newton's Law of gravitation and the Newton's Second Law gives for the magnitude of the acceleration of the body A,

$$G\frac{Mm}{r^2} = \left(-\frac{1}{c^2}m\varphi_g\right)a \tag{4.13}$$

where r the body's A distance from the centre of the body B. It is obvious that the gravitational mass m is canceled in equation (4.13).

Therefore, the acceleration of a body in a free fall, is independent of the gravitational mass of the body. So, for non relativistic velocities, all the bodies released from the same point in a gravitational field, fall with the same acceleration.

This is a fundamental experimental result that has been tested with enormous precision. In Einstein's General relativity, the above result is interpreted by accepting the equality of the gravitational mass and the inertial mass.

4.2 The momentum and the inertial mass of a body

Let's continue the lab frame experiment, to find out what is the inertial mass and the momentum of a body, for relativistic velocities now. We will start with the inertial mass and the momentum of a test-body K without internal structure, with gravitational mass m. We will find later the inertial mass and the momentum of a body without internal structure, with gravitational mass m and electric charge q and finally the total inertial mass and the total momentum of a body with internal structure, a composite body.

We shall call the momentum of the test-body K gravitational momentum $\boldsymbol{p}_g$ and its inertial mass, gravitational inertial mass m_{in-g} because they are due to external gravitational forces. We suppose that the test-body K, which is initially at rest in the lab frame, is accelerated by a force. In the instantaneous rest frame of the test-body K the force is measured $\boldsymbol{F'}$ and the same force in the lab frame is measured $\boldsymbol{F}$. The force is parallel to the relative motion of the two frames (longitudinal force) and so, according to the Lorentz transformations, it has the same value in the lab frame as it does in the rest frame of the test-body K. So,

$$\boldsymbol{F} = \boldsymbol{F'} \tag{4.14}$$

From the viewpoint of the lab frame, the test-body K is accelerating and its gravitational momentum changes. The Newton's Second Law for the motion of the test-body K in the lab frame, is

$$\boldsymbol{F} = \frac{\partial \boldsymbol{p}_g}{\partial t} \tag{4.15}$$

In equation (4.15), ∂t is the time interval in the lab frame. We use the partial derivative, for the Newton's Second Law, for a reason that will soon become apparent.

From the viewpoint of the test-body K, the lab frame and the fixed stars are accelerating and the gravitational vector potential from them changes and causes the external gravitational inertial force which is

$$\boldsymbol{F'}_{ext-in-g} = -\frac{\partial\left(m\boldsymbol{A}'_g\right)}{\partial t'} \tag{4.16}$$

In equation (4.16), $\partial t'$ is the time interval in the proper-frame of K.

We use the Lorentz transformation when we observe, for example, the same force from two different inertial reference frames as we have done for the force that accelerates the test-body K. In the example, we consider, which includes the lab frame and the proper-frame of K, each reference frame observes the motion of the other reference frame for the same time interval and so, we don't need to use the Lorentz transformations. Of course we assume that time is not affected by acceleration (Clock hypothesis). Therefore, because the proper time interval is an invariant quantity,

$$\partial t = \partial t' \tag{4.17}$$

The test-body K experiences the external gravitational inertial force $\boldsymbol{F}'_{ext-in-g}$ and the force $\boldsymbol{F}'$. According to the Newton's First Law, in the proper-frame of K the total force on it is zero. So,

$$\boldsymbol{F}' = -\boldsymbol{F}'_{ext-in-g} \tag{4.18}$$

Hence, from all the above equations we have

$$\frac{\partial \boldsymbol{p}_g}{\partial t} = \frac{\partial \left(m\boldsymbol{A}'_g\right)}{\partial t} \tag{4.19}$$

By integration we have for the gravitational momentum of the test-body K, in the lab frame

$$\boldsymbol{p}_g = m\boldsymbol{A'}_g \tag{4.20}$$

where the constant of integration is set equal to zero because we define the momentum to be zero when the velocity of the test-body K, in respect to the lab frame, is zero. When the instantaneous velocity of the test-body K, in respect to the lab frame is $\boldsymbol{v}$, substituting for $\boldsymbol{A'}_g$ from equation (4.3) into equation (4.20), we have the gravitational momentum of the test-body K, in the lab frame

$$\boldsymbol{p}_g = \gamma(v)\left(-\frac{1}{c^2}m\varphi_g\right)\boldsymbol{v} = m_{in-g}\boldsymbol{v} \tag{4.21}$$

where m_{in-g} the gravitational inertial mass of the test-body K, in the lab frame.

From the equation (4.21), it is obvious that the gravitational momentum of the test-body K is a function of the gravitational scalar potential which depends on position and of the velocity which depends on time. So, we should use partial derivative for the rate of change of the momentum in the Newton's Second Law. Because in the experiment that we consider, the gravitational scalar potential is the same everywhere in the lab frame and constant over time, we could also use the derivative of a function of a single variable, but this is not the general case.

From the equation (4.21), we can see that the gravitational inertial mass of the test-body K , in the lab frame, is

$$m_{in-g} = \gamma(v)\left(-\frac{1}{c^2}m\varphi_g\right) \tag{4.22}$$

When the velocity of the test-body K, in respect to the lab frame, is zero, its gravitational inertial mass is equal to its gravitational inertial rest mass m_{in-g0}

$$m_{in-g0} = -\frac{1}{c^2}m\varphi_g \tag{4.23}$$

Let's consider now, what is the gravitational inertial rest mass of the test-body K. The instantaneous sum of the four-vector gravitational potentials of all the bodies in the Universe, at the position of the test body K, is also a four-vector. It is the total four-vector gravitational potential A_g

$$A_g = \left(\frac{\varphi_g}{c}, \boldsymbol{A}_g\right) \tag{4.24}$$

The four-velocity of the test-body K is,

$$U = \gamma(v)(c, \boldsymbol{v}) \tag{4.25}$$

The product mUA_g is Lorentz invariant, because the gravitational mass m is a Lorentz scalar, and the product of two four-vectors is also a scalar. This product give us the gravitational potential energy from the entire Universe relative to the test-body K. Evaluating the product mUA_g in the rest frame of the test-body K, when the test-body K is at rest in the lab frame, we obtain

$$mUA_g = m(c, o)\left(\frac{\varphi_g}{c}, \boldsymbol{A}_g\right) = m\varphi_g \tag{4.26}$$

This is the gravitational potential rest energy of the test-body K, in the lab frame and is Lorentz invariant. **So, from the equations (4.23), we can see that the gravitational inertial rest mass of the test-body K is Lorentz invariant.** Then, using the whell known equation from the Lorentz transformations, which relates the proper time interval in the rest frame of the test-body K with the time interval in the lab frame,

$$dt = \gamma(v)d\tau \tag{4.27}$$

we obtain, from the equations (4.21) and (4.22),

$$m_{in-g}c = \gamma(v)\left(-\frac{1}{c^2}m\varphi_g\right)c = \left(-\frac{1}{c^2}m\varphi_g\right)\frac{cdt}{d\tau} \tag{4.28}$$

$$\boldsymbol{p}_g = \gamma(v)\left(-\frac{1}{c^2}m\varphi_g\right)\frac{d\boldsymbol{r}}{dt} = \left(-\frac{1}{c^2}m\varphi_g\right)\frac{d\boldsymbol{r}}{d\tau} \tag{4.29}$$

We can now see that $m_{in-g}c$ and $\boldsymbol{p}_g$ are the components of a four vector, the gravitational four-momentum of the test-body K without internal structure

$$P_g = \left(-\frac{1}{c^2}m\varphi_g\right)\frac{d}{d\tau}(ct, \boldsymbol{r}) = m_{in-g0}U = (m_{in-g}c, \boldsymbol{p}_g) = \left(\frac{E_g}{c}, \boldsymbol{p}_g\right) \tag{4.30}$$

where

$$E_g = m_{in-g}c^2 = \gamma(v)\left(-\frac{1}{c^2}m\varphi_g\right)c^2 = -\gamma(v)m\varphi_g \tag{4.31}$$

So, we come to the conclusions that, the gravitational inertial rest mass of a body, without internal structure,

1. **is an invariant quantity.**
2. **is not an intrinsic property of the body but, in the lab frame, is the binding energy of the body with the entire Universe in respect to which it accelerates, when the body is at rest in the lab frame, divided by c^2.**

In another area in the Universe, we can have a second lab frame with different gravitational scalar potential φ'_g and gravitational vector potential also equal to zero. The gravitational inertial rest mass of the test-body K, in this second lab frame, will be

$$m_{in-g0} = -\frac{1}{c^2}m\varphi'_g \tag{4.32}$$

So, we come to the conclusions that, the gravitational inertial rest mass of a body without internal structure is Lorentz invariant but it does not have the same value everywhere.

From equation (4.31), we can see that when the test-body K is at rest in the lab frame, its gravitational rest energy is

$$E_{g0} = m_{in-g0}c^2 = \left(-\frac{1}{c^2}m\varphi_g\right)c^2 = -m\varphi_g \tag{4.33}$$

When the test-body K is at rest in the lab frame, its gravitational potential rest energy is

$$U_{g0} = m\varphi_g \tag{4.34}$$

Thus, the total gravitational rest energy of the test-body K, is

$$E_{g0} + U_{g0} = -m\varphi_g + m\varphi_g = 0 \tag{4.35}$$

Therefore, the total gravitational rest energy of the Universe is zero! It's noteworthy that Richard Feynman writes in the "Lectures on Gravitation" [18]:

> "Another spectacular coincidence relating the gravitational constant to the size of the universe comes in considering the total energy. The total gravitational energy of all the particles of the universe is something like GMM/R, where R=Tc, and T is the Hubble's time. [...] If now we compare this number to the total rest energy of the universe, Mc^2, lo and behold, we get the amazing result that $GM^2/R = Mc^2$, so that the total energy of the universe is zero. [...]Why this should be so is one of the great mysteries and therefore one of the important question of physics. After all, what would be the use of studying physics if the mysteries were not the most important things to investigate?"

Let's consider now, the gravitational inertial mass of a body when moves near to another body. We suppose that the test-body K, with gravitational mass m, is accelerated by a force, in respect to the lab frame, near to a stationary body B with gravitational mass M. From the viewpoint of the test-body K, the body B makes a free fall in the gravitational field that it perceives and thus, does not radiate. The instantaneous force from the body B on the test-body K when the test-body K moves at instantaneous velocity $\boldsymbol{v}$, is

$$\boldsymbol{F'}_{gB} = -m\boldsymbol{\nabla'}\varphi'_{gB} - \frac{\partial(m\boldsymbol{A'}_{gB})}{\partial t'} \tag{4.36}$$

where φ'_{gB} and $\boldsymbol{A'}_{gB}$ the potentials of the body B as perceived by the test body K. The term $-\frac{\partial(m\boldsymbol{A'}_{gB})}{\partial t'}$ from the equation (4.36), which is opposite in direction to the instantaneous velocity $\boldsymbol{v}$, contributes to the external gravitational inertial force. Following the previous procedure, we can find that the inertial mass of the test-body K near to the body B, which is at rest in the lab frame, is

$$m_{in-g} = \gamma(v)\left[-\frac{1}{c^2}m(\varphi_g + \varphi_{gB})\right] \tag{4.37}$$

Let's consider now, if the gravitational inertial mass of a body depends on whether its acceleration is in direction towards the center of our galaxy or in some other direction. This problem has the name "anisotropy of inertia" and was the subject of experimental investigation with negative results[19].

The gravitational scalar potential at a point P, in the field of a gravitational point-mass m, moving at a constant velocity $\boldsymbol{v}$, is given from an equation which have the same mathematical form as the well known corresponding equation of electromagnetism, but with a negative sign [20],

$$\varphi_g = -\frac{1}{4\pi g_0}\frac{m}{r\left(1-\beta^2 sin^2\theta\right)^{\frac{1}{2}}} \tag{4.38}$$

where $\boldsymbol{r}$ is a vector from the present position of the gravitational mass to the field point P, $\beta = \frac{v}{c}$ and θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{v}$. As we have shown, the gravitational rest mass of a body, is proportional to the gravitational scalar potential from the entire Universe at the position of the body. Hence, from the equation (4.38) we can see that any anisotropy of inertia comes from relativistic velocities.

From astronomical observations we know that, the very distant galaxies, move at relativistic velocities. However, their distribution in space is homogenous, as seen from afar. Because of this symmetrical distribution, the gravitational mass of a body, which is due to the distant galaxies, is independent of the direction of its acceleration relative to them. The nearest matter is certainly moving with non relativistic velocities. **Therefore, if a body moves also with non relativistic velocity, its gravitational inertial mass is independent of the direction of its acceleration.** The results are in agreement with past experiments.

Let's consider now the case where there are electric charges at rest in the lab frame. Because the equations of electromagnetism have the same mathematical form as the equations of gravitomagnetism and we have already done the lab frame experiment for gravitational inertial forces, it is not necessary to describe all the details again, but we will only state the important results.

Let's have a test-body B without internal structure, with gravitational mass m and electric charge q which is accelerated in the lab frame. **From the viewpoint of the test-body B all the other charges make a free fall in the gravitational field that it perceives and so, they do not radiate.** We shall call the momentum $\boldsymbol{p}_{ge}$ of the test-body B gravitoelectric momentum and its inertial mass m_{in-ge} gravitoelectric inertial mass because they are due to external gravitational and electric forces. We suppose that the electric scalar potential, from the electric charges, at the position of the test-body B is φ_e. In this case, the gravitoelectric momentum of the test-body B, is

$$\boldsymbol{p}_{ge} = \gamma(v)\left[-\frac{1}{c^2}(m\varphi_g + q\varphi_e)\right]\boldsymbol{v} \tag{4.39}$$

the gravitoelectric inertial mass m_{in-ge} of the test-body B is

$$m_{in-ge} = \gamma(v)\left[-\frac{1}{c^2}(m\varphi_g + q\varphi_e)\right] \tag{4.40}$$

and the gravitoelectric inertial rest mass m_{in-ge0} of the the test-body B is Lorentz invariant.

We assume now, that the Universe consists of n discrete gravitational masses, and m discrete electric charges, each at a different distance r_i from the the test-body B, as measured in the lab frame. For non relativistic velocities, according to the equations (4.40), the gravitoelectric inertial mass of the the test-body B, in the lab frame, is

$$m_{in-ge} = \frac{1}{c^2}\left(\frac{1}{4\pi g_0}\sum_{i=1}^{n}\frac{mm_i}{r_i} - \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{m}\frac{qq_i}{r_i}\right) \tag{4.41}$$

The above equation shows us that, the gravitoelectric inertial mass of a body, can take any value, positive or negative. It depends on the position of the body in relation to the other gravitational masses and electric charges. This seems to be very important for the explanation of dark matter, dark energy and certainly for phenomena of nuclear physics, as we will show below.

Having finished with the gravitoelectric inertial mass of a body without internal structure, let's consider now the total inertial rest mass of a composite body M, a body with internal structure. We shall call it just inertial mass. The Special theory of relativity accepts the principle of conservation of four-momentum, i.e. the sum of the four-momentum of all the particles going into a collision, is the same as the sum of the four- momentum of all those coming out. If we apply the conservation of four-momentum in a inelastic collision where n free moving point particles without internal structure, collide and create a composite body M which is at rest in a reference frame, the inertial rest mass m_{in-0} of the composite body M is

$$m_{in-0} = \sum_{i=1}^{i=n} m_{in-ge0i} + \sum T/c^2 + E_{field}/c^2 \tag{4.42}$$

where $m_{in-ge0i}$ is the gravitoelectric inertial rest mass of each particle, without internal structure, that makes up the body M, T is the kinetic energy of the relative motion of the particles that make up the body M, and E_{field} is the potential energy of the interaction of the particles that make up the body M [21]. **The inertial rest mass m_{in-0} of the composite body M, is also Lorentz invariant as is well known from the Special theory of relativity** [22].

Therefore, if we have the gravitoelectric inertial rest mass of each point particle, without internal structure, using the principle of conservation of four-momentum, we end up with the relation that gives us the inertial rest mass of the body M with internal structure, which is Lorentz invariant.

4.3 Dark matter

From equation (4.41), which is valid for non relativistic velocities, it follows that the inertial mass of a star depends on the gravitational scalar potential of the entire Universe, i.e. the inertial mass of a star depends on the distribution of matter in the Universe. From astronomical observation, we know that the distribution of matter in the Universe is highly inhomogeneous; there are planets, the sun, stars, galaxies, clusters of galaxies and so on. So, it seems that the position where a star is located, affects significantly the inertial mass of the star. In places with higher density of matter the inertial mass of a star will be greater than the inertial mass of the same star, in a place with lower density of matter. This phenomenon has been observed, but the inability to explain it has led to the theory that in the Universe most of the matter is dark matter. It is very likely that the equation (4.41) provides a solution to this problem.

4.4 Zero inertial mass

let us now calculate the order of magnitude of the distance between two charged particles, in order for the effect of the electric scalar potential on their inertial mass to be significant.

For non relativistic velocities, the gravitoelectric inertial mass of a particle O without internal structure, with gravitational mass m and electric charge q, near to an electric charge Q, according to the equation (4.41), is

$$m_{in-ge} = m_{in-g} + m_{in-e} = \frac{1}{c^2}\left(\frac{1}{4\pi g_0}\sum_{i=1}^{N}\frac{mm_i}{r_i} - \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r}\right) = m_{in-g} - \frac{1}{c^2}\frac{1}{4\pi\varepsilon_0}\frac{qQ}{r} \tag{4.43}$$

For like charges there is a critical distance, between them, where the inertial mass of the particle O becomes zero. Let's calculate approximately this critical distance. When the inertial mass of the particle O is zero, equation (4.43) becomes

$$m_{in-g} = \frac{1}{c^2}\frac{1}{4\pi\varepsilon_0}\frac{qQ}{r_{critical}} \Rightarrow r_{critical} = \frac{1}{c^2}\frac{1}{4\pi\varepsilon_0}\frac{qQ}{m_{in-g}} \tag{4.44}$$

For two protons $r_{critical} \approx 1,53 \times 10^{-18} m$ and for two electrons $r_{critical} \approx 2,81 \times 10^{-15} m$. We can see that the effect of electric potential becomes significant in the subatomic world. **What if two protons approach at a shorter distance? What if the inertial rest mass becomes negative? Do we really need the nuclear forces to explain phenomena of nuclear physics?** A door is opening and many questions arise.

5 Spacetime metric

5.1 Spacetime metric outside of a stationary and spherically symmetric gravitational mass

The equivalence principle shows us that, spacetime is endowed with a metric and the gravitational forces affects the spacetime metric. So, we will find now the spacetime metric outside of a stationary body B with spherically symmetric distribution of gravitational mass M. We will consider the freely motion of a test-body A with gravitational mass m in the radial direction of the gravitational field of the body B. We suppose that $M \gg m$. We will follow a new method based on a paper of F. Tangherlini [23][24][25].

From the principle of spacetime metric we have the spacetime interval between two events

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{5.1}$$

It is well documented that we can bring the spacetime interval outside of a stationary body with spherically symmetric distribution of gravitational mass, into the standard Schwarzschild form [26]

$$ds^2 = g_{00} c^2 dt^2 + g_{11} dr^2 - r^2 (d\theta^2 + sin^2\theta d\varphi^2) \tag{5.2}$$

We assume now that,

1. The scalar functions g_{00} and g_{11}, of the metric tensor, are functions only of the distance from the centre of the body B.

2. The gravitational inertial rest mass m_{in-g0}, of the test body A, is constant during the radial motion (We use the gravitational inertial rest mass because the test body A makes a free fall and thus experiences only the external gravitational inertial force).

3. The metric (5.2) should give us to infinity the Minkowski metric in spherical coordinates

$$ds^2 = c^2 dt^2 - dr^2 - r^2 (d\theta^2 + sin^2\theta d\varphi^2) \tag{5.3}$$

   So we must have the boundary conditions

$$\lim_{r\to\infty} g_{00}(r) \to 1 \qquad \lim_{r\to\infty} g_{11}(r) \to -1 \qquad \lim_{r\to\infty} g_{00}(r) g_{11}(r) \to -1 \tag{5.4}$$

For radial motion of the test-body A, the spacetime interval of equation (5.2), becomes

$$ds^2 = g_{00}(r) c^2 dt^2 + g_{11}(r) dr^2 \tag{5.5}$$

From the principle of geodesic motion we have

$$\delta \int ds = 0 \Rightarrow \delta \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau = 0 \Rightarrow \delta \int \sqrt{L} d\tau = 0 \tag{5.6}$$

The L may be termed a 'lagrangian'. Using the relations

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau} \qquad \dot{x}^\nu = \frac{dx^\nu}{d\tau} \tag{5.7}$$

the lagrangian L becomes

$$L = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \tag{5.8}$$

For radial motion the lagrangian becomes

$$L = g_{00}c^2\left(\frac{dt}{d\tau}\right)^2 + g_{11}\left(\frac{dr}{d\tau}\right)^2 \tag{5.9}$$

From the lagrangian, using the calculus of variation, we obtain the Euler-Lagrange system of equations

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) - \frac{\partial L}{\partial t} = 0 \qquad and \qquad \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{5.10}$$

where

$$\dot{t} = \frac{dt}{d\tau} \qquad and \qquad \dot{r} = \frac{dr}{d\tau}$$

Because the Lagrangian does not depend on time,

$$\frac{\partial L}{\partial t} = 0 \tag{5.11}$$

Therefore, from equations (5.10) we have

$$\frac{\partial L}{\partial \dot{t}} = const. \tag{5.12}$$

So, the term $\frac{\partial L}{\partial \dot{t}}$ is a conserved quantity. Performing the differentiation in the equation (5.9), using the fact that the metric function $g_{00}(r)$ doesn't depend on $\dot{t}$ and the relation $ds^2 = c^2 d\tau^2$, we obtain

$$\frac{\partial L}{\partial \dot{t}} = 2g_{00}c^2\frac{dt}{d\tau} = const. \Rightarrow g_{00}\frac{dt}{ds} = k_0 \tag{5.13}$$

Equation (5.13) is a first integral for the equation of motion of the test-body A, in the radial direction and states that the body's energy k_0 (per unit mass) is a constant of the motion. We can have a second first integral for the equation of motion in the radial direction by dividing the equation (5.5) by the spacetime interval ds^2

$$1 = g_{00}c^2\left(\frac{dt}{ds}\right)^2 + g_{11}\left(\frac{dr}{ds}\right)^2 \tag{5.14}$$

Equation (5.14) is the second first integral for the equation of motion of the test-body A, in the radial direction and states the invariant relation between energy and momentum (per unit mass). Eliminating $\frac{dt}{ds}$ from the equation (5.14), using the first integral from equation (5.13), we have

$$1 = \frac{c^2k_0^2}{g_{00}} + g_{11}\left(\frac{dr}{ds}\right)^2 \tag{5.15}$$

Define a matrix $g^{\nu\sigma}$ as the inverse of $g_{\nu\sigma}$, that is, $g^{\nu\sigma}g_{\kappa\nu} = \delta^{\sigma}_{\kappa}$. So,

$$g^{00} = \frac{1}{g_{00}} \qquad and \qquad g^{11} = \frac{1}{g_{11}} \tag{5.16}$$

Dividing the equation (5.15) by g_{11} and using the equations (5.16) we have

$$\frac{1}{g_{11}} = \frac{c^2k_0^2}{g_{00}g_{11}} + \left(\frac{dr}{ds}\right)^2 \Rightarrow g^{11} = c^2k_0^2(g^{00}g^{11}) + \left(\frac{dr}{ds}\right)^2 \tag{5.17}$$

Because we assume that g_{00} and g_{11} are functions only of r

$$\frac{dg^{00}}{ds} = \frac{\partial g^{00}}{\partial r}\frac{dr}{ds} = g^{00}{}_{,r}\frac{dr}{ds} \qquad and \qquad \frac{dg^{11}}{ds} = \frac{\partial g^{11}}{\partial r}\frac{dr}{ds} = g^{11}{}_{,r}\frac{dr}{ds} \tag{5.18}$$

where a comma denotes ordinary differentiation. Using the relations (5.18) we differentiate the equation (5.17) in respect to s and we obtain the following equation

$$g^{11}{}_{,r}\frac{dr}{ds} = c^2k_0^2\left(g^{00}g^{11}\right)_{,r}\frac{dr}{ds} + 2\frac{dr}{ds}\frac{d^2r}{ds^2} \Rightarrow \frac{d^2r}{ds^2} = -\frac{c^2k_0^2}{2}\left(g^{00}g^{11}\right)_{,r} + \frac{1}{2}g^{11}{}_{,r} \tag{5.19}$$

Using the relation $ds^2 = c^2d\tau^2$, we obtain the radial geodesic equation (5.19) in the form,

$$\frac{d^2r}{d\tau^2} = -\frac{c^4k_0^2}{2}\left(g^{00}g^{11}\right)_{,r} + \frac{c^2}{2}g^{11}{}_{,r} \tag{5.20}$$

In the equation (5.20) we can see that because of the k_0 term, the radial acceleration depends on the energy which the test-body A had initially, i.e. the radial velocity with which the test-body A was launched. The radial acceleration (acceleration of gravity) is equal to the acceleration of a freely falling particle instantaneously at rest, as observed by a static observer in the coordinate system [27]. So, the radial acceleration is the proper acceleration.

However, it is well known that the electric force on a charge is strictly independent of the charge's velocity [28]. Therefore, the proper acceleration of a charge due to an electric force, for a motion of the charge along the line of the force, is strictly independent of the charge's velocity [29]. The same happens with the gravitational force, because we accept that both forces are described by equations with the same mathematical form. **Therefore, the radial acceleration of the test-body A is independent of the radial velocity with which it was launched.**

Thus, in order for the radial acceleration to be indepndent of the energy which the test-body A had initially, this must be valid,

$$\left(g^{00}g^{11}\right)_{,r} = 0 \Rightarrow g^{00}g^{11} = const. \tag{5.21}$$

Taking into consideration the boundary conditions (5.4) and the equation (5.16), we obtain

$$g^{00}g^{11} = -1 \Rightarrow g^{11} = -\frac{1}{g^{00}} \Rightarrow g_{11} = -\frac{1}{g_{00}} \tag{5.22}$$

This shows that the line element, wich is given from the equation (5.2), can be written as

$$ds^2 = g_{00}(r)c^2dt^2 - g_{00}(r)^{-1}dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{5.23}$$

From the equation (5.22), using the equation (5.16), we have

$$g^{00}g^{11} = -1 \Rightarrow g^{11} = -\frac{1}{g^{00}} \Rightarrow g^{11} = -g_{00} \tag{5.24}$$

Finally, using the relations (5.21) and (5.24), the radial geodesic equation of motion (5.20), becomes

$$\frac{d^2r}{d\tau^2} = -\frac{c^2}{2}g_{00,r} \tag{5.25}$$

where r is the radial coordinate and τ the proper time.

Let's consider now the radial motion of the test-body A, in the gravitational field $\boldsymbol{E}_{gB}$ of the stationary body B, using the gravitomagnetic theory.

In flat spacetime, the flux of the gravitational field of the body B through a spherical surface of radius ρ with center the center of the spherical body B and outside of the body B, is equal to $4\pi\rho^2 E_{gB}$. According to the Gauss's law, for the gravitomagnetism, this is proportional to the gravitational mass M, inside the spherical surface. So, the magnitude E_{gB} of the gravitational field is given by the equation

$$E_{gB} = \frac{1}{4\pi g_0}\frac{M}{\rho^2} \tag{5.26}$$

In curved spacetime, the above law does not hold good because the space is non-Euclidean and the area of the spherical surface is not equal to $4\pi\rho^2$. However, if we make a transformation of the coordinate $\rho \to r$, where r the radial coordinate, so as to make the spherical surface always equal to $4\pi r^2$, which is valid for the metric given by the equation (5.22), then, the magnitude E_{gB} of the gravitational field is correctly given by the equation [30],

$$E_{gB} = \frac{1}{4\pi g_0}\frac{M}{r^2} \tag{5.27}$$

So, the well-known equation of motion of an electric charge in the electric field of another stationary electric charge, which apply now also for the motion of the test-body A in the gravitational field of the stationary body B, for purely radial motion, using a system of spherical coordinates originating at the centre of the body B and the radial coordinate r, becomes

$$m_{in-g0}\frac{d^2\boldsymbol{r}}{d\tau^2} = \gamma(v)m\boldsymbol{E}_{gB} \to m_{in-g0}\frac{d^2 r}{d\tau^2} = -\gamma(v)m\frac{\partial\varphi_{gB}}{\partial r} \tag{5.28}$$

For proper acceleration $\gamma(v) = 1$. So, the equation (5.28) for proper acceleration, becomes

$$m_{in-g0}\frac{d^2 r}{d\tau^2} = -m\frac{\partial\varphi_{gB}}{\partial r} \Rightarrow \frac{d^2 r}{d\tau^2} = -\frac{m}{m_{in-g0}}\frac{\partial\varphi_{gB}}{\partial r} \tag{5.29}$$

Comparing the equations (5.25) and (5.29), we have

$$\frac{c^2}{2}\frac{\partial g_{00}}{\partial r} = \frac{m}{m_{in-g0}}\frac{\partial\varphi_{gB}}{\partial r} \Rightarrow \frac{\partial g_{00}}{\partial r} = \frac{2m}{c^2 m_{in-g0}}\frac{\partial\varphi_{gB}}{\partial r} \tag{5.30}$$

Because we assume that the gravitational inertial rest mass m_{in-g0} of the test body A, is constant during the radial motion, using the boundary conditions (5.4) we obtain for g_{00}

$$g_{00} = 1 + \frac{2m}{c^2 m_{in-g0}}\varphi_{gB} \tag{5.31}$$

and for g_{11}, because of the equation (5.22),

$$g_{11} = -\frac{1}{1 + \frac{2m}{c^2 m_{in-g0}}\varphi_{gB}} \tag{5.32}$$

So, the spacetime interval outside of the body B, is

$$ds^2 = \left(1 - \frac{m}{m_{in-g0}}\frac{2GM}{c^2 r}\right)c^2dt^2 - \frac{dr^2}{\left(1 - \frac{m}{m_{in-g0}}\frac{2GM}{c^2 r}\right)} - r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{5.33}$$

In the International system of units the ratio of the gravitational mass to the gravitational inertial rest mass of the test-body A, is equal to unity

$$\frac{m}{m_{in-g0}} = 1 \tag{5.34}$$

So, in the International system of units, equation (5.33) becomes the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \frac{dr^2}{1 - \dfrac{2GM}{c^2 r}} - r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{5.35}$$

Therefore this theory is in agreement with all the past experiments and observations.

Using the equation (4.23), the equation (5.34) becomes

$$\frac{m}{m_{in-g0}} = -\frac{m}{m\varphi_g} c^2 = -\frac{c^2}{\varphi_g} \tag{5.36}$$

where φ_g the gravitational scalar potential of the entire Universe in the area where the body B is. So, in another area in our Galaxy or in the Universe, the ratio of gravitational to inertial mass is not equal to unity. There, the phenomena will be the same qualitatively but not quantitatively. So, finally, the equation (5.33) becomes

$$ds^2 = \left(1 + \frac{2\varphi_{gB}}{\varphi_g}\right) c^2 dt^2 - \frac{dr^2}{\left(1 + \dfrac{2\varphi_{gB}}{\varphi_g}\right)} - r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{5.37}$$

So, the spacetime interval, outside of a stationary body B with spherically symmetric gravitational mass, depends on the gravitational scalar potential φ_{gB} of the body B and on the gravitational scalar potential φ_g of the entire Universe at the body's B position.

5.2 Dark Energy

Let's consider now the light emitted by an atom, with gravitational mass m and gravitational inertial rest mass m_{in-g0}, at a distance r_{em} from the centre of a star, which is stationary with spherically symmetric gravitational mass M. From equation (5.33) arises the equation relating the proper time $d\tau_{em}$ at the point of emission, with the proper time $d\tau_\infty$ at infinity where is the point of observation [31]

$$d\tau_\infty = \frac{d\tau_{em}}{\sqrt{1 - \dfrac{2GMm}{c^2 r_{em} m_{in-g0}}}} \tag{5.38}$$

So, the frequency of light observed by an observer, in the point of observation, will be

$$f_\infty = f_{em}\sqrt{1 - \frac{2GMm}{c^2 r_{em} m_{in-g0}}} \tag{5.39}$$

The equation (5.39) describes the red shift of spectral lines, which is emitted by an atom in a gravitational field and is received by a body, which is out of the gravitational field. This phenomenon is known as gravitational red shift.

As the Universe expands, according to the equation (4.41), the gravitational inertial rest mass of a star decreases. Thus, the gravitational inertial rest mass of an atom, which

emits light, decreases over time. As it emerges from equation (5.39) the light emitted by two identical supernovas Ia which move at the same speed in respect to us, at different moments in the history of the Universe, will have different red shift. Because the atoms in a younger supernova have smaller gravitational inertial rest mass than the gravitational inertial rest mass of the atoms in an older supernova, the light emitted by a younger supernova Ia has greater red shift than the light emitted by an older supernova Ia. This phenomenon has been observed, but the inability to explain why the red shift of spectral lines is greater, has led to the theory that the Universe expands in an accelerating way, because of dark energy. It is very likely that the equations (4.41) and (5.39) provide a solution to this problem.

Conclusions

Finally, the physics that comes from the attempt to explore the origin of inertia shows us that the motion of a tiny body is affected by the entire Universe. As Dennis Sciama ended an article on inertia in Scientific American [32]: "If atomic properties are in fact so determined, we shall again be faced with the dual situation: Distant matter influencing local phenomena and local phenomena giving us information about distant matter. The scientist would then be able to claim that his imagination had out-stripped the poet's. For he would see the world not in a "grain of sand" but in an atom". The Universe never stops surprising us!!!

References

[1] Weinberg, S.: Gravitation and Cosmology, pp.86. Wiley, New York (1972)

[2] Einstein, A.: Kosmologische Betrachtungen zur allgemeinen relativitats theorie, pp.142. Sitzb.Preuss. Akad. Wiss. (1917)

[3] Sard, R.D.: Relativistic Mechanics, pp. 309, W.A. Benjamin New York (1970)

[4] Ohanian, H.C., Ruffini, R.: Gravitation and Spacetime, pp.95 and pp.115 Cambridge University Press (2013)

[5] Poincare, H. : Rend. Circ. Mat. Palermo, 21, (1906)

[6] Pauli, W.: Theory of Relativity, pp. 116, Pergamon Press (1958)

[7] Feynman, R., Leighton, R., Sands, M.: The Feynman Lectures on Physics, Vol.2. 26-1 Addison-Wesley, Publishing Company, Inc. (1964)

[8] Rosser, W. G. V.: Classical electromagnetism via relativity, pp. 29-77, Plenum, New York (1968)

[9] Ciufolini, I., Wheeler, J.A.: Gravitation and Inertia, pp.391. Princeton University Press (1995)

[10] Tolman, R.C.: Relativity thermodynamics and cosmology, pp. 165-166 Clarendon Press, Oxford (1934)

[11] Steane, A.: Relativity Made Relatively Easy, pp.238, Oxford University Press (2012)

[12] Morin, D.: Introduction to Classical Mechanics, ch.13.3.2 Cambridge University Press (2013)

[13] Misner, C., Thorne, K., Wheeler, J.A.: Gravitation, pp.19, W.H. Freeman and Company (1973)

[14] Schutz, B.F.: A First Course in General Relativity, pp.149 Cambridge University Press, (2009)

[15] Feynman, R., Leighton, R., Sands, M.: The Feynman Lectures on Physics, Vol.2. 25-5 Addison-Wesley, Publishing Company, Inc. (1964)

[16] Rosser,W. G. V.: An introduction to special relativity, Butterworth and Co. (1964)

[17] Feynman, R., Leighton, R., Sands, M.: The Feynman Lectures on Physics, Vol.2. 28-4 Addison-Wesley, Publishing Company, Inc. (1964)

[18] Feynman, R., Morinigo, F.B., Wagner, W.G.: Feynman Lectures on Gravitation, pp.10 Addison-Wesley Publishing Company, (1995)

[19] Cocconi, G., Salpeter, E.E.: Upper limit for the Anisotropy of Inertia from the Mossbauer Effect, Phys. Rev. Let., 4, 176, (1960)

[20] Lorrain, P., Corson, D.R.: Electromagnetic field and waves, 3rd ed. pp.317, Freeman, San Francisco (1988)

[21] Shepherd, P.J.: A course in Theoretical Physics, pp.320, John Wiley and Sons, (2013)

[22] Moller, C.: Theory of relativity, pp.77-80, Clarendon Press, Oxford (1952)

[23] Tangherlini, F.R.: Postulational approach to Schwarzschild's exterior solution with application to a class of interior solutions, Nuovo Cimento, 25, 1081, (1962)

[24] Rindler, W.: Counterexample to the Tangerlini argument, American Journal of Physics, Vol. 37, issue 1, pp. 72-73, (1969)

[25] Sacks, M., Ball, A.: Simple Derivation of the Schwarzschild Metric, American Journal of Physics, 36, 240, (1968)

[26] Cheng, Ta-Pei. : A college course on relativity and cosmology, 1rd ed. pp.115, Oxford University Press, (2015)

[27] Gron, O., Naess, A.: Einstein's Theory. A Rigorous Introduction for the Mathematically Untrained, pp.242, Springer, (2011)

[28] Purcell,E.M., Morin, D.J.: Electricity and magnetism, 3rd ed. pp.255, Cambridge university press, (2013)

[29] Steane, A.: Relativity Made Relatively Easy, pp.66-67, Oxford University Press (2012)

[30] Wild, J.P.: A revival of Newton's theory of gravitation, Monthly Notices of the Royal Astronomical Society, 282, (1996)

[31] Lambourne, R.J.A.: Relativity, gravitation and cosmology, pp.158, Cambridge University Press, (2010)

[32] Sciama, D.: Inertia, Scientific American, February 1957