GEOMETRODYNAMICS IN MULTIDIMENSIONAL UNIFIED THEORY*)

L. Szabó

Institute for Theoretical Physics, Eötvös Lorand University,
Puskin u. 5—7, H-1088 Budapest VIII, Hungary

The unified theory of gravitation and a Yang-Mills field is formulated as a dynamical theory of $(r+3)$-geometries presumed to be principal bundles with Riemannian metric. Beyond the usual constraint equations the second fundamental form should satisfy a third constraint equation. It is shown that they have a wormhole type solution describing a pair of Yang-Mills charges.

1. GEOMETRODYNAMICS

About twenty years ago J. A. Wheeler, C. W. Misner and others tried to formulate general relativity as a well-defined dynamical theory. The final result of these investigations was that the general relativity is not a dynamical theory of the space-time geometry, but it is the dynamical theory of the space-like 3-geometries [2]. The description of space-time is nothing else but the description of the evolution of space-like hypersurfaces. Let σ be a space-like hypersurface in space-time. On this 3-dimensional manifold the metric ${}^{(\sigma)}g$ is positive definite. If we want to examine the evolution of this 3-geometry, the metric ${}^{(\sigma)}g$ and the speed of its change along the normal vector field, i.e. the Lie derivative $L_N\, {}^{(\sigma)}g = -{}^{(\sigma)}B$ must be given at $t = 0$, where ${}^{(\sigma)}B$ denotes the second fundamental form [4, 13].

Now, the following question arises: what conditions should be satisfied by ${}^{(\sigma)}g$ and ${}^{(\sigma)}B$, so that a space-time metric should exist which at $t = 0$ reduces to

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + {}^{(\sigma)}g$$

$$\frac{\partial}{\partial t}\, {}^{(\sigma)}g = -{}^{(\sigma)}B\,.$$

To answer this question we make use [13] of the Codazzi and Gauss equations, well known in differential geometry, and from the Einstein equations we obtain the constraint equations:

$${}^{(\sigma)}\nabla_Y\, {}^{(\sigma)}B(X_i, X_i) - {}^{(\sigma)}\nabla_{X_i}\, {}^{(\sigma)}B(X_i, Y) = 8\pi T(N, Y)\,, \quad \forall Y \in \Gamma(T(\sigma))\,,$$

$$\tfrac{1}{2}\left({}^{(\sigma)}R + 2\,\mathrm{Tr}\left({}^{(\sigma)}B \circ {}^{(\sigma)}B\right) - \left(\mathrm{Tr}\, {}^{(\sigma)}B\right)^2\right) = 8\pi T(N, N)\,,$$

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and the evolution equation

$$L_N {}^{(\sigma)}B(X, Y) = {}^{(\sigma)}R(X, Y) + 2\, {}^{(\sigma)}B \circ {}^{(\sigma)}B(X, Y) - (\mathrm{Tr}\, {}^{(\sigma)}B)\, {}^{(\sigma)}B(X, Y) - \\ - 8\pi T(X, Y) - 8\pi T(N, N)\, {}^{(\sigma)}g(X, Y), \quad \forall X, Y \in \Gamma(T(\sigma)).$$

Since σ is assumed to be initially at rest the constraint equations in vacuum reduce to

$${}^{(\sigma)}R = 0.$$

Misner has shown [15] that this constraint equation does have a solution in the $S^1 \times S^2$ wormhole topology. Around the mouths of the wormhole this solution is like a Schwarzschild one, therefore we can call it "mass without mass" [1]. It was also shown that the Einstein-Maxwell equations have a solution in wormhole topology. The electric lines of force are trapped in the topology of space, and it physically means "charge without charge".

In geometrodynamics [1] making use of the fact that the electromagnetic field leaves a very characteristic trace in spacetime geometry the description of the electromagnetic field can be reduced to the description of pure geometry. In the case of non-abelian gauge theories the implementation of this programme is impossible since there is no unambiguous relation between field-variables and the space-time geometry as in the case of electrodynamics. However, we show that the geometrodynamical concept can be extended to the non-abelian Yang-Mills fields within the framework of the multidimensional unified theory.

2. MULTIDIMENSIONAL UNIFIED THEORY

The MUT is a generalization of the Kaluza-Klein model [11, 12]. The MUT is a unified geometrical description of gravitation and a Yang-Mills field in an $(r + 4)$-dimensional space, where r is the dimension of the gauge group G. This $(r + 4)$-dimensional space is assumed to be a principal bundle $H(M, G, \pi\{\psi\})$. The basis of this bundle denoted by M with Lorentz metric ${}^{(M)}g$ is the 4-dimensional space-time manifold. G is a compact semi-simple Lie group with an invariant metric ${}^{(G)}g_{ab} = f^c_{ad} f^d_{cb}$, where f^c_{ab} are the structure constants of the group. In this case the torsion-free connection coefficients are

$${}^{(G)}\Gamma^a_{bc} = \tfrac{1}{2} f^a_{bc},$$

and the group manifold is a Riemannian manifold with a constant positive curvature. $\pi: H \to M$ is the bundle projection, $\{\psi\}$ is the bundle atlas. Let a connection be given on the principal bundle H. The connection coefficients defined by the vertical part of the basis vector e_μ:

$$v e_\mu = A^a_\mu e_a,$$

will be identified with the Yang-Mills potentials. The indices $A, B, \ldots$ run from 1 to $r + 4$, $a, b, \ldots$ from 5 to $r + 4$, and $\mu, \nu, \ldots$ from 1 to 4. In this model the most important object is a pseudo-Riemannian metric defined on the total space H. The free particles move along the geodesics of this space. But this pseudo-Riemannian metric ${}^{(H)}g$ should be compatible with metrics ${}^{(M)}g$ and ${}^{(G)}g$ on the basis and the structure group, respectively:

$$ {}^{(H)}g|_{vTH} = \psi^*({}^{(G)}g)\,, \tag{1} $$

$$ {}^{(H)}g|_{hTH} = \pi^*({}^{(M)}g)\,, \tag{2} $$

$$ (vTH) \perp (hTH)\,. \tag{3} $$

If we choose the basis which is the natural one on the basis manifold and which is the left-invariant one on the fibre, the matrix of the metric ${}^{(H)}g$ is

$$ {}^{(H)}g_{AB} = \begin{pmatrix} {}^{(M)}g_{\mu\nu} + {}^{(G)}g_{ab}A^a_\mu A^b_\nu & {}^{(G)}g_{ab}A^a_\mu \\ {}^{(G)}g_{ab}A^a_\nu & {}^{(G)}g_{ab} \end{pmatrix}, $$

Further the horizontal lift basis seems practical [8]. In this basis the first four vectors $\tilde{e}_\mu$ are the horizontal lifts of e_μ. The commutation relations are

$$ [e_a, e_b] = f^c_{ab} e_c\,, $$
$$ [e_a, \tilde{e}_\mu] = [\tilde{e}_\mu \tilde{e}_\nu] = 0\,. $$

The torsion-free Christoffel coefficients in this basis are

$$ {}^{(H)}\Gamma^a_{bc} = \tfrac{1}{2} f^a_{bc}\,, \quad {}^{(H)}\Gamma^\mu_{bc} = 0\,, $$
$$ {}^{(H)}\Gamma^\nu_{\mu a} = {}^{(H)}\Gamma^\nu_{a\mu} = \tfrac{1}{2}\,{}^{(M)}g^{\nu\delta}\,{}^{(G)}g_{ab}\,F^b_{\mu\delta}\,, $$
$$ {}^{(H)}\Gamma^a_{\mu\nu} = -\tfrac{1}{2}F^a_{\mu\nu}\,, \quad {}^{(H)}\Gamma^\delta_{\mu\nu} = {}^{(M)}\Gamma^\delta_{\mu\nu}\,, $$

and the Ricci tensor is

$$ {}^{(H)}R_{ab} = {}^{(G)}R_{ab} + \tfrac{1}{4}\,{}^{(G)}g_{ac}\,{}^{(G)}g_{bd}\,{}^{(M)}g^{\alpha\beta}\,{}^{(M)}g^{\gamma\delta}\,F^c_{\alpha\gamma}F^d_{\beta\delta}\,, $$
$$ {}^{(H)}R_{\mu b} = {}^{(H)}R_{b\mu} = \tfrac{1}{2}\,{}^{(G)}g_{bc}\,{}^{(M)}g^{\alpha\beta}\,{}^{(M)}\nabla_\alpha F^c_{\mu\beta}\,, $$
$$ {}^{(H)}R_{\mu\nu} = {}^{(M)}R_{\mu\nu} - \tfrac{1}{2}\,{}^{(G)}g_{ab}\,{}^{(M)}g^{\alpha\beta}\,F^a_{\mu\alpha}F^b_{\nu\beta}\,, $$

and the scalar curvature is

$$ {}^{(H)}R = {}^{(M)}R + {}^{(G)}R - \tfrac{1}{4}\,{}^{(G)}g_{ab}\,{}^{(M)}g^{\mu\alpha}\,{}^{(M)}g^{\nu\beta}\,F^a_{\mu\nu}F^b_{\alpha\beta}\,, $$

where $F^a_{\mu\nu}$ denotes the curvature tensor of the connection describing the gauge field. The unified action integral is

$$ {}^{(H)}S = \int \sqrt{(-{}^{(H)}g)}\;{}^{(H)}R\,\mathrm{d}^4x\,\mathrm{d}^\gamma G\,. $$

From this action we can deduce the Einstein-Yang-Mills equations by variation of the metric ${}^{(M)}g_{\mu\nu}$ and the Yang-Mills potentials A^a_μ [7].

3. GENERALIZATION OF GEOMETRODYNAMICS TO MUT

Instead of generalization of space-time we consider an $(r + 3)$-dimensional generalized space-like hypersurface Σ. Let Σ be a principal bundle: $\Sigma(\sigma, G, \pi)$, where σ is a 3-dimensional manifold and G is a compact semi-simple Lie group. On the bundle the vertical distribution $vT(\Sigma)$ is automatically given. Let us consider a metric ${}^{(\Sigma)}g$ on Σ, which satisfies the following conditions:

i. ${}^{(\Sigma)}g$ is positive definite

ii. The restriction of ${}^{(\Sigma)}g$ to the vertical distribution is G-invariant:

$$L_Z({}^{(\Sigma)}g/vT(\Sigma)) = 0\,, \tag{4}$$

where Z is an arbitrary fundamental field.

Proposition: In this case

$$hT(\Sigma) = (vT(\Sigma))^{\perp}\,,$$

the orthogonal complement to the vertical distribution defines a connection on $\Sigma(\sigma, G, \pi)$.

Proof: We show that $hT(\Sigma)$ is a G-invariant horizontal distribution. It is trivial that $T(\Sigma) = hT(\Sigma) \oplus vT(\Sigma)$. For any fundamental vector field Z and $X \in \Gamma(hT(\Sigma))$,

$$[Z, X] \in \Gamma(hT(\Sigma))\,.$$

Indeed, for any $V \in \Gamma(vT(\Sigma))$ we have ${}^{(\Sigma)}g(V, X) \equiv 0$. Thus

$$0 = Z\,{}^{(\Sigma)}g(X, V) = {}^{(\Sigma)}g([Z, X], V) + {}^{(\Sigma)}g(X, [Z, V])\,.$$

The second term is zero since the commutator of vertical fields is also a vertical vector field, thus we have

$$[Z, X] \in \Gamma((vT(\Sigma))^{\perp}) = \Gamma(hT(\Sigma))\,. \quad \text{Q.E.D.}$$

Furthermore, the metric ${}^{(\Sigma)}g$ induces a metric ${}^{(\sigma)}g$ on the basis $\sigma \subset M$ as follows: ${}^{(\sigma)}g(X, Y) := {}^{(\Sigma)}g(\tilde{X}, \tilde{Y})$, where $\tilde{X}$ and $\tilde{Y}$ denote the horizontal lifts of the vector fields X and Y. ${}^{(\sigma)}g$ is also positive definite. Let a second fundamental form ${}^{(\Sigma)}B$ be given on Σ. There exists a metric ${}^{(H)}g$ on the principal bundle $H(M, G, \pi)$ which at $t = 0$ reduces to

$${}^{(H)}g = -\mathrm{d}t^2 + {}^{(\Sigma)}g$$

$$\frac{\partial}{\partial t}\,{}^{(\Sigma)}g - -{}^{(\Sigma)}B\,,$$

and which satisfies the Einstein equations on H, if and only if the usual constraint equations are satisfied by ${}^{(\Sigma)}g$ and ${}^{(\Sigma)}B$:

$$(5)\qquad {}^{(\Sigma)}\nabla_Y\, {}^{(\Sigma)}B(X_i, X_i) - {}^{(\Sigma)}\nabla_{X_i}\, {}^{(\Sigma)}B(X_i, Y) = 0\,, \quad \forall Y \in \Gamma(T(\Sigma))\,,$$

$$(6)\qquad {}^{(\Sigma)}R + 2\,\mathrm{Tr}\left({}^{(\Sigma)}B \circ {}^{(\Sigma)}B\right) - \left(\mathrm{Tr}\, {}^{(\Sigma)}B\right)^2 = 0\,.$$

What further condition must be satisfied by ${}^{(\Sigma)}B$ so that equation (4) should be valid? This question is related to the problem in MUT that the $\delta\, {}^{(H)}g^{AB}$ cannot be arbitrary in the variation equation

$$\left({}^{(H)}R_{AB} - \tfrac{1}{2}\, {}^{(H)}g_{AB}\, {}^{(H)}R\right) \delta\, {}^{(H)}g^{AB} = 0\,,$$

but they should satisfy the compatibility conditions [7].

Proposition: Condition (4) yields a third constraint equation:

$$(7)\qquad L_Z\left({}^{(\Sigma)}B\big|_{vT(\Sigma)}\right) = 0\,,$$

or any fundamental field Z.

Proof: Making use of the relation

$$L_Z L_Y = L_{[Z,Y]} + L_Y L_Z\,,$$

for any fundamental field Z we have

$$L_Z L_N\left({}^{(\Sigma)}g\big|_{vT(\Sigma)}\right) = L_{[Z,N]}\left({}^{(\Sigma)}g\big|_{vT(\Sigma)}\right) + L_N L_Z\left({}^{(\Sigma)}g\big|_{vT(\Sigma)}\right).$$

(Here and in the next formulas the fields Z and N are regarded as extended fields to an open neighbourhood of $\Sigma \subset H$). Since $L_Z\left({}^{(\Sigma)}g\big|_{vT(\Sigma)}\right) = 0$,

$$L_Z\, {}^{(\Sigma)}B = L_{[Z,N]}\left({}^{(\Sigma)}g\big|_{vT(\Sigma)}\right).$$

But $[Z, N] = 0$. Indeed, ${}^{(H)}g(N, .) \equiv 0$ since for any $X \in \Gamma(T(\Sigma))$

$${}^{(H)}g(N, .)\,(X) = {}^{(H)}g(N, X) = 0\,.$$

From here we have

$$L_Z\, {}^{(H)}g(N, X) = Z\left({}^{(H)}g(N, X)\right) -$$
$$- {}^{(H)}g([Z, N], X) - {}^{(H)}g(N, [Z, X]) = 0\,.$$

But $Z({}^{(H)}g(N, X)) = 0$ and ${}^{(H)}g(N, [Z, X]) = 0$ since N is normal to the hypersurface Σ. Thus we have found that $[Z, N]$ is parallel with N. Further we have

$${}^{(H)}g([Z, N], N) = {}^{(H)}g({}^{(H)}\nabla_Z N - {}^{(H)}\nabla_N Z, N) =$$
$$= {}^{(H)}g({}^{(H)}\nabla_Z N, N) - {}^{(H)}g({}^{(H)}\nabla_N Z, N)\,.$$

Taking into account that ${}^{(H)}g(N, N) \equiv -1$ and ${}^{(H)}g(Z, N) \equiv 0$ we find $[Z, N] = 0$. Q.E.D.

Finally the evolution equation is

$$L_N\, {}^{(\Sigma)}B(X, Y) = {}^{(\Sigma)}R(X, Y) + 2\, {}^{(\Sigma)}B \circ {}^{(\Sigma)}B(X, Y) - (\mathrm{Tr}\, {}^{(\Sigma)}B)\, {}^{(\Sigma)}B(X, Y)\,.$$

Let us now consider a very interesting special case when the topology of Σ is $\Sigma(S^1 \times S_2, G, \pi)$ which is the MUT-generalization of the simplest wormhole topology [14]. Since Σ is assumed to be initially at rest, the constraint equations reduce to

$${}^{(\Sigma)}R = 0\,.$$

${}^{(G)}R > 0$, and F^2 is the energy of the Yang-Mills field. Let F be assumed to be such a field configuration that

$$ {}^{(G)}R - \tfrac{1}{4}F^2 = 0\,. \tag{8}$$

For the metric of σ we have

$${}^{(\sigma)}R = 0\,,$$

which does have a solution in wormhole topology. This solution physically means that the Yang-Mills field is trapped by a wormhole. Examining the motion of test particles along the geodesics the mouth of the wormhole seems to be a Yang-Mills charged particle, that is we found "Yang-Mills charge without Yang-Mills charge".

At first sight it may happen that the metric of the basis $\sigma = S^1 \times S^2$ is non-compatible with the connection determined by equation (8). However, the metric ${}^{(\sigma)}g$ prescribes only the scalar product of horizontal vectors. This has nothing to do with the connection giving the horizontal part of a tangent vector.

The evolution of much more complicated hypersurfaces may be examined by approximations [4]. It is also a very important question, with respect to the quantized theory, how to change the topology in the course of the evolution of an $(r + 3)$-hypersurface [2, 14, 16, 17].

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