Gen Relativ Gravit (2009) 41:785–815
DOI 10.1007/s10714-009-0771-4

RESEARCH ARTICLE

The Superspace of geometrodynamics

Domenico Giulini

Received: 19 January 2009 / Accepted: 24 January 2009 / Published online: 14 February 2009

Abstract Wheeler's Superspace is the arena in which Geometrodynamics takes place. I review some aspects of its geometrical and topological structure that Wheeler urged us to take seriously in the context of canonical quantum gravity.

Keywords Quantum gravity · Superspace · Three-manifolds

> "The stage on which the space of the Universe moves is certainly not space itself. Nobody can be a stage for himself; he has to have a larger arena in which to move. The arena in which space does its changing is not even the space-time of Einstein, for space-time is the history of space changing with time. The arena must be a larger object: *Superspace*... It is not endowed with three or four dimensions—it's endowed with an *infinite* number of dimensions." (J.A. Wheeler: *Superspace*, Harper's Magazine, July 1974, p. 9)

1 Introduction

From somewhere in the 1950s on, John Wheeler repeatedly urged people who were interested in the quantum-gravity programme to understand the structure of a mathematical object that he called *Superspace* [79,80]. The intended meaning of 'Superspace' was that of a set, denoted by $\mathbb{S}(\Sigma)$, whose points faithfully correspond to all possible Riemannian geometries on a given three-manifold Σ. Hence, in fact,

I dedicate this contribution to the scientific legacy of John Wheeler.

D. Giulini (✉)
Max-Planck-Institute for Gravitational Physics (Albert-Einstein-Institute),
Am Mühlenberg 1, 14476 Potsdam OT Golm, Germany
e-mail: domenico.giulini@aei.mpg.de

there are infinitely many Superspaces, one for each three-manifold Σ. The physical significance of this concept is suggested by the dynamical picture of General Relativity (henceforth abbreviated by GR), according to which spacetime is the history (time evolution) of space. Accordingly, in Hamiltonian GR, Superspace plays the rôle of the configuration space the cotangent bundle of which gives the phase space of 3d-diffeomorphism reduced states. Moreover, in Canonical Quantum Gravity (henceforth abbreviated by CQG), Superspace plays the rôle of the domain for the wave function which is still subject to the infamous Wheeler–DeWitt equation. In fact, Bryce De Witt characterised the motivation for his seminal paper on CQG as follows:

> "The present paper is the direct outcome of conversations with Wheeler, during which one fundamental question in particular kept recurring: *What is the structure of the domain manifold for the quantum-mechanical state functional?"* ([16], p. 115)

More than 41 years after DeWitt's important contribution I simply wish to give a small overview over some of the answers given so far to the question: *What is the structure of Superspace?* Here I interpret 'structure' more concretely as 'metric structure' and 'topological structure'. But before answers can be attempted, we need to define the object at hand. This will be done in the next section; and before doing that, we wish to say a few more words on the overall motivation.

Minkowski space is the stage for relativistic particle physics. It comes equipped with some structure (topological, affine, causal, metric) that is *not* subject to dynamical changes. Likewise, as was emphasised by Wheeler, the arena for Geometrodynamics is Superspace, which also comes equipped with certain non-dynamical structures. The topological and geometric structures of Superspace are as much a background for GR as the Minkowski space is for relativistic particle physics. Now, Quantum Field Theory has much to do with the automorphism group of Minkowski space and, in particular, its representation theory. For example, all the linear relativistic wave equations (Klein–Gordan, Weyl, Dirac, Maxwell, Proca, Rarita–Schwinger, Dirac–Bargmann, etc.) can be understood in this group-theoretic fashion, namely as projection conditions onto irreducible subspaces in some auxiliary Hilbert space. (In the same spirit a characterisation of 'classical elementary system' has been given as one whose phase space supports a transitive symplectic action of the Poincaré group [4,5].) This is how we arrive at the classifying meaning of 'mass' and 'spin'. Could it be that Quantum Gravity has likewise much to do with the automorphism group of Superspace? Can we understand this group in any reasonable sense and what has it to do with four dimensional diffeomorphisms? If elementary particles *are* unitary irreducible representations of the Poincaré group, as Wigner once urged, what would the 'elementary systems' be that corresponded to irreducible representations of the automorphism group of Superspace?

I do not know any reasonably complete answer to any of these questions. But the analogies at least suggests the possibility of some progress *if* these structures and their automorphisms could be understood in any depth. This is a difficult task, as John Wheeler already foresaw forty years ago:

"Die Struktur des Superraumes enträtseln? Kaum in einem Sprung, und kaum heute!" ([79], p. 61)

Related in spirit is a recent approach in the larger context of 11-dimensional supergravity (see [14,15] and references therein), which is based on the observation that the supergravity dynamics in certain truncations corresponds to geodesic motion of a massless spinning particle on an E_{10} coset space. Here the Wheeler–DeWitt metric (9c) appears naturally with the right GR-value $\lambda = 1$, which in our context is the only value compatible with four-dimensional diffeomorphism invariance, as we will discuss. This may suggest an interesting relation between E_{10} and spacetime diffeomorphisms.

2 Defining Superspace

As already said, Superspace $\mathcal{S}(\Sigma)$ is the set of all Riemannian geometries on the three-manifold Σ. Here 'geometries' means 'metrics up to diffeomorphisms'. Hence $\mathcal{S}(\Sigma)$ is identified as set of equivalence classes in $\mathrm{Riem}(\Sigma)$, the set of all smooth (C^∞) Riemannian metrics in Σ under the equivalence relation of being related by a smooth diffeomorphism. In other words, the group of all (C^∞) diffeomorphisms, $\mathrm{Diff}(\Sigma)$, has a natural right action on $\mathrm{Riem}(\Sigma)$ via pullback and the orbit space is identified with $\mathcal{S}(\Sigma)$:

$$\mathcal{S}(\Sigma) := \mathrm{Riem}(\Sigma)/\mathrm{Diff}(\Sigma). \tag{1}$$

Let us now refine this definition. First, we shall restrict attention to those Σ which are connected and closed (compact without boundary). We note that Einstein's field equations by themselves do not exclude any such Σ. To see this, recall the form of the constraints for initial data (h, K), where $h \in \mathrm{Riem}(\Sigma)$ and K is a symmetric covariant 2nd rank tensor-field (to become the extrinsic curvature of Σ in spacetime, once the latter is constructed from the dynamical part of Einstein's equations)

$$\|K\|_h^2 - (\mathrm{Tr}_h(K))^2 - (R(h) - 2\Lambda) = -(2\kappa)\rho_m, \tag{2a}$$

$$\mathrm{div}_h\,(K - h\,\mathrm{Tr}_h(K)) = (c\kappa)\,j, \tag{2b}$$

where ρ_m and j_m are the densities of energy and momentum of matter respectively, $R(h)$ is the Ricci scalar for h, and $\kappa = 8\pi G/c^4$. Now, it is known that for any smooth function $f : \Sigma \to \mathbb{R}$ which is negative somewhere on Σ there exists an $h \in \mathrm{Riem}(\Sigma)$ so that $R(h) = f$ [47]. Given that strong result, we may easily solve (2) for $j = 0$ on any compact Σ as follows: First we make the Ansatz $K = \alpha h$ for some constant α and some $h \in \mathrm{Riem}(\Sigma)$. This solves (2b), whatever α, h will be. Geometrically this means that the initial Σ will be a totally umbillic hypersurface in spacetime. Next we solve (2a) by fixing α so that $\alpha^2 > (\Lambda + \kappa\ \sup_\Sigma(\rho_m))/3$ and then choosing h so that $R(h) = 2\Lambda + 2\kappa\rho_m - 6\alpha^2$, which is possible by the result just cited because the right-hand side is negative by construction. This argument can be generalised to non-compact manifolds with a finite number of ends and asymptotically flat data [84].

Next we refine the definition (1), in that we restrict the group of diffeomorphisms to the proper subgroup of those diffeomorphisms that fix a preferred point, called $\infty \in \Sigma$, and the tangent space at this point:

$$\mathrm{Diff_F}(\Sigma) := \left\{\phi \in \mathrm{Diff}(\Sigma) \mid \phi(\infty) = \infty,\ \phi_*(\infty) = \mathrm{id}|_{T_\infty \Sigma}\right\}. \tag{3}$$

The reason for this is twofold: First, if one is genuinely interested in closed Σ, the space $\mathcal{S}(\Sigma) := \mathrm{Riem}(\Sigma)/\mathrm{Diff}(\Sigma)$ is not a manifold if Σ allows for metrics with non-trivial isometry groups (not all Σ do; compare footnote 3). At those metrics $\mathrm{Diff}(\Sigma)$ clearly does not act freely, so that the quotient $\mathrm{Riem}(\Sigma)/\mathrm{Diff}(\Sigma)$ has the structure of a stratified manifold with nested sets of strata ordered according to the dimension of the isometry groups [22]. In that case there is a natural way to minimally resolve the singularities [23] which amounts to taking instead the quotient $\mathrm{Riem}(\Sigma) \times \mathrm{F}(\Sigma)/\mathrm{Diff}(\Sigma)$, where $\mathrm{F}(\Sigma)$ is the bundle of linear frames over Σ. The point here is that the action of $\mathrm{Diff}(\Sigma)$ is now free since there simply are no non-trivial isometries that fix a frame. Indeed, if ϕ is an isometry fixing some frame, we can use the exponential map and $\phi \circ \exp = \exp \circ \phi_*$ (valid for any isometry) to show that the subset of points in Σ fixed by ϕ is open. Since this set is also closed and Σ is connected, ϕ must be the identity.

Now, the quotient $\mathrm{Riem}(\Sigma) \times \mathrm{F}(\Sigma)/\mathrm{Diff}(\Sigma)$ is isomorphic[1] to

$$\mathcal{S}_\mathrm{F}(\Sigma) := \mathrm{Riem}(\Sigma)/\mathrm{Diff_F}(\Sigma)\,, \tag{4}$$

albeit not in a natural way, since one needs to choose a preferred point $\infty \in \Sigma$. This may seem somewhat artificial if really all points in Σ are considered to be equally real, but this is irrelevant for us as long as we are only interested in the isomorphicity class of Superspace. On the other hand, if we consider Σ as the one-point compactification of a manifold with one end[2], then (4) would be the right space to start with anyway since then diffeomorphisms have to respect the asymptotic geometry in that end, like, e.g., asymptotic flatness. *Therefore, from now on, we shall refer to* $\mathcal{S}_\mathrm{F}(\Sigma)$ *as defined in* (4) *as Superspace.* In view of the original definition (1) it is usually called 'extended Superspace' [22].

Clearly, the move from (1) to (4) would have been unnecessary in the closed case if one restricted attention to those manifolds Σ which do not allow for metrics with continuous symmetries, i.e. whose degree of symmetry[3] is zero. Even though these manifolds are not the 'obvious' ones one tends to think of first, they are, in a sense, 'most' three-manifolds. On the other hand, in order not to deprive ourselves form the possibility of physical idealisations in terms of prescribed exact symmetries, we prefer

[1] "Isomorphic as what?" one may ask. The answer is: as ILH (inverse-limit Hilbert) manifolds. In the ILH sense the action of $\mathrm{Diff}(\Sigma)$ on $\mathrm{Riem}(\Sigma) \times \mathrm{F}(\Sigma)$ is smooth, free, and proper; see [23] for more details and references.

[2] The condition of 'asymptotic flatness' of an end includes the topological condition that the one-point compactification is again a manifold. This is the case iff there exists a compact subset in the manifold the complement of which is homeomorphic to the complement of a closed solid ball in $\mathbb{R}^3$.

[3] Let $\mathcal{I}(\Sigma, h) := \{\phi \in \mathrm{Diff}(\Sigma) \mid \phi^* h = h\}$ be the isometry group of (Σ, h), then it is well known that $\dim \mathcal{I}(\Sigma, h) \leq \frac{1}{2} n(n+1)$, where $n = \dim \Sigma$. $\mathcal{I}(\Sigma, h)$ is compact if Σ is compact (see, e.g., Sect. 5 of [62]). Conversely, if Σ allows for an effective action of a compact group G then it clearly allows for a metric h on which G acts as isometries (just average any Riemannian metric over G.) The degree of symmetry of Σ, denoted by $\deg(\Sigma)$, is defined by $\deg(\Sigma) := \sup_{h \in \mathrm{Riem}(\Sigma)}\{\dim \mathcal{I}(\Sigma, h)\}$. For compact Σ the degree of symmetry is zero iff Σ cannot support an action of the circle group $SO(2)$. A list of three-manifolds with $\deg > 0$ can be found in [22] whereas [24] contains a characterisation of $\deg = 0$ manifolds.

to work with $\mathcal{S}_{\mathrm{F}}(\Sigma)$ defined in (4) (called 'extended Superspace' in [22], as already mentioned).

Let us add a few words on the point-set topology of $\mathrm{Riem}(\Sigma)$ and $\mathcal{S}_{\mathrm{F}}(\Sigma)$. First, $\mathrm{Riem}(\Sigma)$ is an open positive convex cone in the topological vector space of smooth (C^∞) symmetric covariant tensor fields over Σ. The latter space is a Fréchet space, that is, a locally convex topological vector space that admits a translation-invariant metric, $\overline{d}$, inducing its topology and with respect to which the space is complete. The metric can be chosen such that $\mathrm{Diff}(\Sigma)$ preserves distances. $\mathrm{Riem}(\Sigma)$ inherits this metric which makes it a metrisable topological space that is also second countable (recall also that metrisability implies paracompactness). $\mathcal{S}_{\mathrm{F}}(\Sigma)$ is given the quotient topology, i.e. the strongest topology in which the projection $\mathrm{Riem}(\Sigma) \to \mathcal{S}_{\mathrm{F}}(\Sigma)$ is continuous. This projection is also open since $\mathrm{Diff}_{\mathrm{F}}(\Sigma)$ acts continuously on $\mathrm{Riem}(\Sigma)$. A metric d on $\mathcal{S}_{\mathrm{F}}(\Sigma)$ is defined by

$$d([h_1],[h_2]) := \sup_{\phi_1,\phi_2\in\mathrm{Diff}_{\mathrm{F}}(\Sigma)} \overline{d}(\phi_1^* h_1, \phi_2^* h_2)\,, \tag{5}$$

which also turns $\mathcal{S}_{\mathrm{F}}(\Sigma)$ into a connected (being the continuous image of the connected $\mathrm{Riem}(\Sigma)$) metrisable and second countable topological space. Hence $\mathrm{Riem}(\Sigma)$ and $\mathcal{S}_{\mathrm{F}}(\Sigma)$ are perfectly decent connected topological spaces which satisfy the strongest separability and countability axioms. For more details we refer to [22,23,73].

The basic geometric idea is now to regard $\mathrm{Riem}(\Sigma)$ as principal fibre bundle with structure group $\mathrm{Diff}_{\mathrm{F}}(\Sigma)$ and quotient $\mathcal{S}_{\mathrm{F}}(\Sigma)$:

$$\mathrm{Diff}_{\mathrm{F}}(\Sigma) \xrightarrow{\;i\;} \mathrm{Riem}(\Sigma) \xrightarrow{\;p\;} \mathcal{S}_{\mathrm{F}}(\Sigma) \tag{6}$$

where the maps i are the inclusion and projection maps respectively. This is made possible by the so-called 'slice theorems' (see [20,22]), and the fact that the group acts freely and properly. This bundle structure has two far-reaching consequences regarding the geometry and topology of $\mathcal{S}_{\mathrm{F}}(\Sigma)$. Let us discuss these in turn.

3 Geometry of Superspace

Elements of the Lie algebra $\mathfrak{diff}_{\mathrm{F}}(\Sigma)$ of $\mathrm{Diff}_{\mathrm{F}}(\Sigma)$ are vector fields on Σ. For any such vector field ξ on Σ there is a vector field V_ξ on $\mathrm{Riem}(\Sigma)$, called the vertical (or fundamental) vector field associated to ξ, whose value at $h \in \mathrm{Riem}(\Sigma)$ is just the infinitesimal change in h generated by ξ, that is,

$$V_\xi(h) = -L_\xi h, \tag{7}$$

where L_ξ denoted the Lie derivative with respect to ξ. Hence, for each $h \in \mathrm{Riem}(\Sigma)$, the map $V(h) : \xi \mapsto V_\xi(h)$ is an anti-Lie homomorphism (the 'anti' being due to the fact that we have a *right* action of $\mathrm{Diff}(\Sigma)$ on $\mathrm{Riem}(\Sigma)$), that is $[V_\xi, V_\eta] = -V_{[\xi,\eta]}$, if the Lie structure on $\mathfrak{diff}_{\mathrm{F}}(\Sigma)$ is that of ordinary commutators of vector fields. The kernel of the map $V(h) : \xi \mapsto V_\xi(h)$ consists of the finite-dimensional subspace of Killing fields on (Σ, h). The vertical vectors at $h \in \mathrm{Riem}(\Sigma)$ therefore form a linear subspace $\mathrm{Vert}_h \subset T_h\mathrm{Riem}(\Sigma)$, isomorphic to the vector fields on Σ modulo the Killing fields on (Σ, h). It is a closed subspace due to the fact that the operator $\xi \mapsto L_\xi h$ is overdetermined elliptic (cf. [11], Appendices G-I).

The family of ultralocal 'metrics' on $\mathrm{Riem}(\Sigma)$ is given by

$$\mathcal{G}_{(\alpha,\lambda)}(k,\ell) = \int_\Sigma d^3x\, \alpha \sqrt{\det(h)}\, \Big(h^{ab}h^{cd}\, k_{ac}\ell_{bd} - \lambda\, (h^{ab}k_{ab})(h^{cd}\ell_{cd})\Big)\,, \tag{8}$$

for each $k, \ell \in T_h\mathrm{Riem}(\Sigma)$. Here α is a positive real-valued function on Σ and λ a real number. An almost trivial but important observation is that $\mathrm{Diff}(\Sigma)$ is an isometry group with respect to all $\mathcal{G}_{(\alpha,\lambda)}$. The 'metric' picked by GR through the bilinear term in the constraint (2a) corresponds to $\lambda = 1$. The positive real-valued function α is not fixed and corresponds to the free choice of a lapse-function. In what follows we shall focus attention to $\alpha = 1$.

The pointwise bilinear form $(k,\ell) \mapsto (h\otimes h)(k,\ell) - \lambda \mathrm{Tr}_h(k)\mathrm{Tr}_h(\ell)$ in the integrand of (8) defines a symmetric bilinear form on the six-dimensional space of symmetric tensors which is positive definite for $\lambda < 1/3$, of signature $(1,5)$ for $\lambda > 1/3$, and degenerate of signature $(0,5)$ for $\lambda = 1/3$. It defines a metric on the homogeneous space $\mathrm{GL}(3)/\mathrm{O}(3)$, where the latter may be identified with the space of euclidean metrics on a three-dimensional vector space. Parametrising it by h_{ab}, we have

$$G_\lambda = G_\lambda^{ab\,cd} dh_{ab} \otimes dh_{cd} = -\epsilon d\tau \otimes d\tau + \frac{\tau^2}{c^2}\mathrm{Tr}(r^{-1}dr \otimes r^{-1}dr), \tag{9a}$$

where

$$r_{ab} := [\det(h)]^{-1/3} h_{ab},\ \ \tau := c\,[\det(h)]^{1/4},\ \ c^2 := 16|\lambda - 1/3|,\ \ \epsilon = \mathrm{sign}(\lambda - 1/3), \tag{9b}$$

and

$$G_\lambda^{ab\,cd} = \frac{1}{2}\sqrt{\det(h)}\,\Big(h^{ac}h^{bd} + h^{ad}h^{bc} - 2\lambda h^{ab}h^{cd}\Big)\,. \tag{9c}$$

This is a 1+5-dimensional warped-product geometry in the standard form of 'cosmological' models (Lorentzian for $\lambda > 1/3$), here corresponding to the 1+5 decomposition $\mathrm{GL}(3)/\mathrm{O}(3) \cong \mathbb{R} \times \mathrm{SL}(3)/\mathrm{SO}(3)$ with scale factor τ/c and homogeneous Riemannian metric on five-dimensional 'space' $\mathrm{SL}(3)/\mathrm{SO}(3)$, given by $\mathrm{Tr}(r^{-1}dr \otimes r^{-1}dr) = r^{ac}r^{bd}\, dr_{ab} \otimes dr_{cd}$. $\tau = 0$ is a genuine 'spacelike' ('cosmological') curvature singularity. An early discussion of this finite-dimensional geometry was given by DeWitt [16]. We stress that the Lorentzian nature of the Wheeler–DeWitt metric in GR (i.e. for $\lambda = 1$) has nothing to do with the Lorentzian nature of spacetime, as we will see below from the statement of Theorem 1 and formulae (33, 34); rather, it can be related to the attractivity of gravity [38].

As for the infinite-dimensional geometry of $\mathrm{Riem}(\Sigma)$, we remark that an element h of $\mathrm{Riem}(\Sigma)$ is a section in $T^*\Sigma \otimes T^*\Sigma$ and so is an element of $T_h\mathrm{Riem}(\Sigma)$. The latter has the fibre-metric (9). It is sometimes useful to use h (for index raising) in order to identify $T_h\mathrm{Riem}(\Sigma)$ with sections in $T\Sigma \otimes T^*\Sigma \cong \mathrm{End}(T\Sigma)$, also because the latter has a natural structure as associative- (and hence also Lie-) algebra. Then the inner product (8) for $\alpha = 1$ just reads (here and below $d\mu(h) = \sqrt{\det(h)}\, d^3x$)

$$\mathcal{G}_\lambda(k, \ell) = \int_\Sigma d\mu(h)\,(\mathrm{Tr}(k \cdot \ell) - \lambda\,\mathrm{Tr}(k)\mathrm{Tr}(\ell))\,. \tag{10}$$

For $\lambda = 0$ the infinite-dimensional geometry of $\mathrm{Riem}(\Sigma)$ has been studied in [27]. They showed that all curvature components involving one or more pure-trace directions vanish and that the curvature tensor for the trace-free directions is given by (now making use of the natural Lie-algebra structure of $T\Sigma \otimes T^*\Sigma$)

$$R(k, \ell)m = -\frac{1}{4}[[k, \ell], m]\,. \tag{11}$$

In particular, this implies that the sectional curvatures involving pure trace directions vanish and that the sectional curvatures for trace-free directions k, ℓ are non-positive:

$$\begin{aligned} K(k, \ell) &= -\frac{1}{4}\int_\Sigma d\mu(h)\,\mathrm{Tr}\,(k \cdot R(k, \ell)\ell) \\ &= -\frac{1}{4}\int_\Sigma d\mu(h)\,\mathrm{Tr}\,([k, \ell] \cdot [\ell, k]) \leq 0\,. \end{aligned} \tag{12}$$

Similar results hold for other values of λ, though some positivity statements cease to hold for $\lambda > 1/3$. We keep the generality in the value of λ for the moment in order to show that the value $\lambda = 1$ picked by GR is quite special. Since, as already stated, all elements of $\mathrm{Diff}_{\mathrm{F}}(\Sigma)$ are isometries of $\mathcal{G}_\lambda$, it is natural to try to define a bundle connection on $\mathrm{Riem}(\Sigma)$ by taking the *horizontal subspace* Hor^λ_h at each $T_h\mathrm{Riem}(\Sigma)$ to be the $\mathcal{G}_\lambda$–orthogonal complement to Vert_h, as suggested in [35]. From (8) one sees that $k \in T_h\mathrm{Riem}(\Sigma)$ is orthogonal to all $L_\xi h$ iff

$$(\mathcal{O}_\lambda k)^a := -\nabla^b(k_{ab} - \lambda h_{ab}h^{cd}k_{cd}) = 0\,. \tag{13}$$

But note that orthogonality does not imply transversality if the metric is indefinite, as for $\lambda = 1$. In that case the intersection $\mathrm{Vert}_h \cap \mathrm{Hor}^\lambda_h$ may well be non trivial, which implies that there is no well defined projection map

$$\mathrm{hor}^\lambda_h : T_h\mathrm{Riem}(\Sigma) \to \mathrm{Hor}^\lambda_h\,. \tag{14}$$

The definition of this map would be as follows: Let $k \in T_h\mathrm{Riem}(\Sigma)$, find a vector field ξ on Σ such that $k - V_\xi$ is horizontal. Then V_ξ is the (λ dependent) vertical component of k and the map $k \mapsto k - V_\xi$ is the (λ dependent) horizontal projection (14). When does that work? Well, according to (13), the condition for $k - V_\xi$ to be horizontal for given k is equivalent to the following differential equation for ξ:

$$D_\lambda \xi := (\delta d + 2(1-\lambda) d\delta - 2\mathrm{Ric})\, \xi = \mathcal{O}_h k. \tag{15}$$

Here we regarded ξ as one-form and d, δ denote the standard exterior differential and co-differential ($\delta\xi = -\nabla^a \xi_a$) respectively. Moreover, Ric is the endomorphism on one-forms induced by the Ricci tensor ($\xi_a \mapsto R^b_a \xi_b$). Note that the right-hand side of (15) is L^2-orthogonal for all k to precisely the Killing fields.

The singular nature of the GR value $\lambda = 1$ is now seen from writing down the principal symbol of the operator D_λ on the left-hand side of (15),

$$\sigma_\lambda(\zeta)^a_b = \|\zeta\| \left(\delta^a_b + (1-2\lambda) \frac{\zeta^a \zeta_b}{\|\zeta\|} \right), \tag{16}$$

whose determinant is $\|\xi\|^6 2(1-\lambda)$. Hence σ_λ is positive definite for $\lambda < 1$ and indefinite but still an isomorphism for $\lambda > 1$. This means that D_λ is elliptic for $\lambda \neq 1$ (strongly elliptic[4] for $\lambda < 1$) but fails to be elliptic for precisely the GR value $\lambda = 1$. This implies the possibility for the kernel of $D_{\lambda=1}$ to become infinite dimensional.

The last remark has a direct implication as regards the intersection of the horizontal and vertical subspaces. Recall that solutions ξ to $D_\lambda \xi = 0$ modulo Killing fields (which always solve this equation) correspond faithfully to vertical vectors at $h \in \mathrm{Riem}(\Sigma)$ (via $\xi \mapsto V_\xi(h)$) which are also horizontal. Since Killing vectors span at most a finite dimensional space, an infinite dimensional intersection $\mathrm{Vert}_h \cap \mathrm{Hor}^\lambda_h$ would be implied by an infinite dimensional kernel of D_1.

That this possibility for $\lambda = 1$ is actually realised for any Σ is easy to see: Take a metric h on Σ that is flat in an open region $U \subset \Sigma$, and consider $k \in T_h\mathrm{Riem}(\Sigma)$ of the form $k_{ab} = 2\nabla_a \nabla_b \phi$, where ϕ is a real-valued function on Σ whose support is contained in U. Then k is vertical since $k = L_\xi h$ for $\xi = \mathrm{grad}\,\phi$ (a non-zero gradient vector-field is never Killing on a compact Σ), and also horizontal since k satisfies (13). Such ξ clearly span an infinite-dimensional subspace in the kernel of D_1.

On the other hand, for $\lambda = 1$ there are also always open sets of $h \in \mathrm{Riem}(\Sigma)$ (and of $[h] \in \mathcal{S}_\mathrm{F}(\Sigma)$) for which the kernel of D_1 is trivial (the kernel clearly depends only on the diffeomorphism class $[h]$ of h). For example, consider metrics with negative definite Ricci tensor, which exist for any closed Σ [30]. (Note that Ricci-negative geometries never allow for non-trivial Killing fields.) Then it is clear from the definition of D_λ that it is a positive-definite operator for $\lambda \leq 1$. Hence the intersection $\mathrm{Vert}_h \cap \mathrm{Hor}^\lambda_h$ is trivial.

In the latter case it is interesting to observe that $\mathcal{G}_\lambda$ restricted to Vert_h (h Ricci negative) is positive definite, since

$$\mathcal{G}_\lambda \left(V_\xi(h), V_\xi(h) \right) = 2 \int_\Sigma d\mu(h)\, h(\xi, D_\lambda \xi). \tag{17}$$

[4] We follow the terminology of Appendix I in [11].

This means that $\mathcal{G}_\lambda$ restricted to the orthogonal complement, Hor_h^λ, contains infinitely many negative and infinitely may positive directions and the same (∞,∞) – signature is then directly inherited by $T_{[h]}\mathcal{S}_\mathrm{F}(\Sigma)$ for any Ricci-negative geometry $[h]$.

Far less generic but still interesting examples for trivial intersections $\mathrm{Vert}_h \cap \mathrm{Hor}_h^\lambda$ in case $\lambda = 1$ are given by Einstein metrics with positive Einstein constants. Since this condition implies constant positive sectional curvature, such metrics only exist on manifolds Σ with finite fundamental group, so that Σ must be a spherical space form S^3/G, where G is a finite subgroup of $SO(4)$ acting freely on S^3. Also note that the subspace in $\mathcal{S}_\mathrm{F}(\Sigma)$ of Einstein geometries is finite dimensional. Now, any solution ξ to $D_1\xi = 0$ must be divergenceless (take the co-differential δ of this equation) and hence Killing. The last statement follows without computation from the fact that $D_1\xi = 0$ is nothing but the condition that $L_\xi h$ is $\mathcal{G}_1$–orthogonal to all vertical vectors in $T_h\mathrm{Riem}(\Sigma)$, which for divergenceless ξ (traceless $L_\xi h$) is equivalent to $\mathcal{G}_0$–orthogonality, but then positive definiteness of $\mathcal{G}_0$ and $\mathcal{G}_0(L_\xi h, L_\xi h) = 0$ immediately imply $L_\xi h = 0$. This shows the triviality of $\mathrm{Vert}_h \cap \mathrm{Hor}_h^\lambda$.

The foregoing shows that $\mathcal{G}_1$ indeed defines a metric at $T_{[h]}\mathcal{S}_\mathrm{F}(\Sigma)$ for Ricci-positive Einstein geometries $[h]$. How does the signature of this metric compare to the signature (∞,∞) at Ricci-negative geometries? The answer is surprising: Take, e.g., for $[h]$ the round geometry on $\Sigma = S^3$. Then it can be shown that $\mathcal{G}_1$ defines a *Lorentz geometry* on $T_{[h]}\mathcal{S}_\mathrm{F}(\Sigma)$, that is with signature $(1,\infty)$, containing exactly one negative direction [35]. This means that the signature of the metric defined at various points in Superspace varies strongly, with intermediate transition regions where no metric can be defined at all due to signature change. Figure 1 is an attempt to picture this situation.

4 Intermezzo: GR as simplest representation of symmetry

It is well known that the field equations of GR have certain uniqueness properties and can accordingly be 'deduced' under suitable hypotheses involving a *symmetry principle* (diffeomorphism invariance), the equivalence principle, and some apparently mild technical hypotheses. More precisely, the equivalence principle suggests to only take the metric as dynamical variable [76] representing the gravitational field (to which matter then couples universally), whereas diffeomorphism invariance, derivability from an invariant Lagrangian (alternatively: local energy-momentum conservation in the sense of covariant divergencelessness), dependence of the equations on the metric up to at most second derivatives, and, finally, four-dimensionality lead uniquely to the left-hand side of Einstein's equation, including a possibly non-vanishing cosmological constant [54]. Here we will review how this 'deduction' works in the Hamiltonian setting on phase space $T^*\mathrm{Riem}(\Sigma)$, which goes back to [40,41,51,74].

4.1 3 + 1 decomposition

Since the 3 + 1 split of Einstein's equations has already been introduced in Claus Kiefer's contribution I can be brief on that point. The basic idea is to first imagine a spacetime (M, g) being given, where topologically M is a product $\mathbb{R}\times\Sigma$. Spacetime is then considered as the trajectory (history) of space in the following way: Let

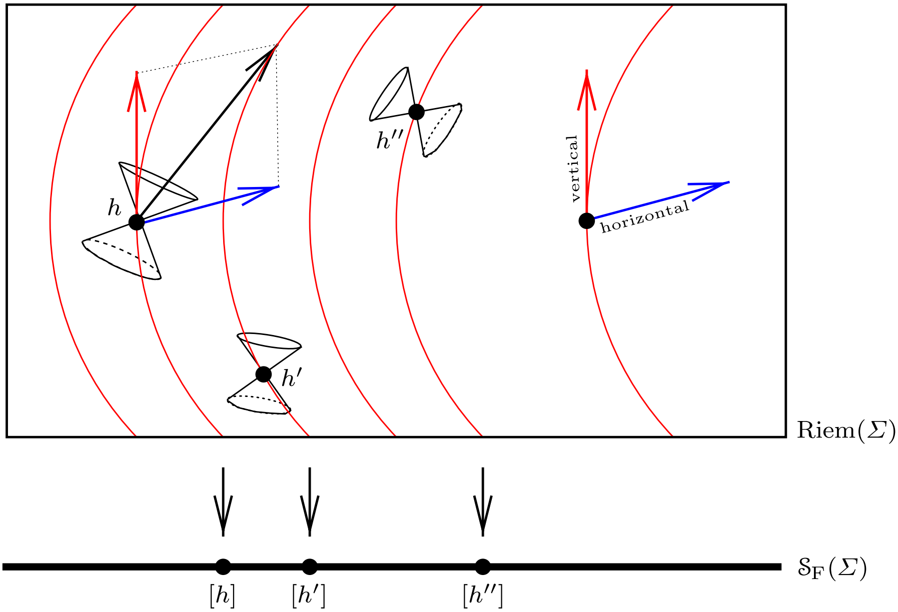

Fig. 1 The *rectangle* depicts the space $\mathrm{Riem}(\Sigma)$ which is fibred by the orbits of $\mathrm{Diff}_{\rm F}(\Sigma)$ (*curved vertical lines*). The metric $\mathcal{G}_1$ on $\mathrm{Riem}(\Sigma)$ is such that as we move along $\mathrm{Riem}(\Sigma)$ transversal to the fibres the "light-cones" tilt relative to the fibre directions. The process here shows a transition at $[h']$ where some fibre directions are lightlike and no metric can be defined in $T_{[h']}\mathcal{S}_{\rm F}(\Sigma)$, whereas they are timelike at $[h]$ and spacelike at $[h'']$. The *parallelogram* at h merely indicates the horizontal and vertical components of a vector in $T_h\mathrm{Riem}(\Sigma)$

$\mathrm{Emb}(\Sigma, M)$ denote the space of smooth spacelike embeddings $\Sigma \to M$. We consider a curve $\mathbb{R} \ni t \to \mathcal{E}_t \in \mathrm{Emb}(\Sigma, M)$ corresponding to a one-parameter family of smooth embeddings with spacelike images. We assume the images $\mathcal{E}_t(\Sigma) =: \mathcal{E}_t \subset M$ to be mutually disjoint and moreover that $\hat{\mathcal{E}} : \mathbb{R} \times \Sigma \to M$, $(t, p) \mapsto \mathcal{E}_t(p)$, is an embedding (it is sometimes found convenient to relax this condition, but this is of no importance here). The Lorentz manifold $(\mathbb{R} \times \Sigma, \mathcal{E}^* g)$ may now be taken as ($\mathcal{E}$–dependent) representative of M (or at least some open part of it) on which the leaves of the above foliation simply correspond to the $t = \text{const.}$ hypersurfaces. Let n denote a field of normalised timelike vectors normal to these leaves. n is unique up to orientation, so that the choice of n amounts to picking a 'future direction'.

The tangent vector $d\mathcal{E}_t/dt|_{t=0}$ at $\mathcal{E}_0 \in \mathrm{Emb}(\Sigma, M)$ corresponds to a vector field over $\mathcal{E}_0$ (i.e. section in $T(M)|_{\mathcal{E}_0}$), given by

$$\left.\frac{d\mathcal{E}_t(p)}{dt}\right|_{t=0} =: \left.\frac{\partial}{\partial t}\right|_{\mathcal{E}_0(p)} = \alpha n + \beta \tag{18}$$

with components $(\alpha, \beta) = $ (lapse, shift) normal and tangential to $\Sigma_0 \subset M$.

Conversely, each vector field V on M defines a vector field $X(V)$ on $\mathrm{Emb}(\Sigma, M)$, corresponding to the left action of $\mathrm{Diff}(M)$ on $\mathrm{Emb}(\Sigma, M)$ by composition. In local coordinates y^μ on M and x^k on Σ it can be written as

$$X(V) = \int_{\Sigma} d^3x \, V^\mu(y(x)) \frac{\delta}{\delta y^\mu(x)} \,. \tag{19}$$

One easily verifies that $X : V \mapsto X(V)$ is a Lie homomorphism:

$$[X(V), X(W)] = X([V, W]) \,. \tag{20}$$

In this sense, the Lie algebra of the four-dimensional diffeomorphism group is implemented on phase space of any generally covariant theory whose phase space includes the embedding variables [44] (so-called 'parametrised theories').

Alternatively, decomposing (19) into normal and tangential components with respect to the leaves of the embedding at which the tangent-vector field to $\mathrm{Emb}(\Sigma, M)$ is evaluated, yields an embedding-dependent parametrisation of $X(V)$ in terms of (α, β),

$$X(\alpha, \beta) = \int_{\Sigma} d^3x \left(\alpha(x) n^\mu[y](x) + \beta^m(x) \partial_m y^\mu(x)\right) \frac{\delta}{\delta y^\mu(x)}, \tag{21}$$

where y in square brackets indicates the functional dependence of n on the embedding. The functional derivatives of n with respect to y can be computed (see the Appendix of [74]) and the commutator of deformation generators then follows to be,

$$[X(\alpha_1, \beta_1) \,,\; X(\alpha_2, \beta_2)] = -X(\alpha', \beta'), \tag{22}$$

where

$$\alpha' = \beta_1(\alpha_2) - \beta_2(\alpha_1), \tag{23a}$$
$$\beta' = [\beta_1, \beta_2] + \sigma\alpha_1 \, \mathrm{grad}_h(\alpha_2) - \sigma\alpha_2 \mathrm{grad}_h\,(\alpha_1). \tag{23b}$$

Here we left open whether spacetime M is Lorentzian ($\sigma = 1$) or Euclidean ($\sigma = -1$), just in order to keep track how the signature of spacetime, $(-\sigma, +, +, +)$, enters. Note that the h-dependent gradient field for the scalar function α is given by $\mathrm{grad}_h(\alpha) = (h^{ab}\partial_b\alpha)\partial_a$. The geometric idea behind (23) is summarised in Fig. 2.

4.2 Hamiltonian geometrodynamics

The idea of Hamiltonian geometrodynamics is to realise these relations in terms of a Hamiltonian system on the phase space of physical fields. The most simple case is that where the latter merely include the spatial metric h on Σ, so that the phase space is the cotangent bundle $T^*\mathrm{Riem}(\Sigma)$ over $\mathrm{Riem}(\Sigma)$. One then seeks a correspondence

$$(\alpha, \beta) \mapsto \left(H(\alpha, \beta) \,:\, T^*\mathrm{Riem}(\Sigma) \to \mathbb{R}\right), \tag{24}$$

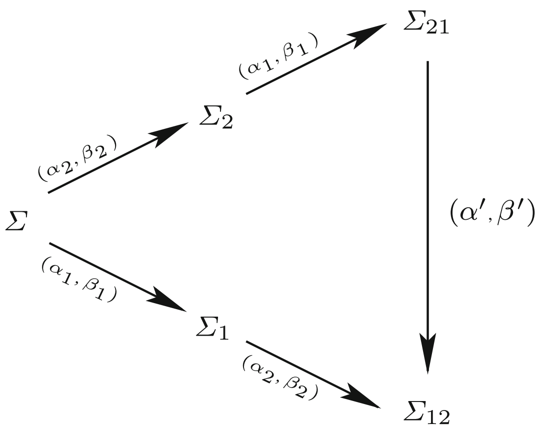

Fig. 2 An (infinitesimal) hypersurface deformation with parameters (α_1, β_1) that maps $\Sigma \mapsto \Sigma_1$, followed by one with parameters (α_2, β_2) that maps $\Sigma_1 \mapsto \Sigma_{12}$ differs by one with parameters (α', β') given by (23) from that in which the maps with the same parameters are composed in the opposite order

where

$$H(\alpha, \beta)[h, \pi] := \int_\Sigma d^3x \left(\alpha(x)\mathcal{H}[h, \pi](x) + h_{ab}(x)\beta^a(x)\mathcal{D}^b[h, \pi](x)\right), \quad (25)$$

with integrands $\mathcal{H}[h, \pi](x)$ and $\mathcal{D}^b[h, \pi](x)$ yet to be determined. H should be regarded as distribution (here the test functions are α and β^a) with values in real-valued functions on $T^*\mathrm{Riem}(\Sigma)$. Now, the essential requirement is that the Poisson brackets between the $H(\alpha, \beta)$ are, up to a minus sign,[5] as in (23):

$$\{H(\alpha_1, \beta_1)\,,\ H(\alpha_2, \beta_2)\} = H(\alpha', \beta'). \quad (26)$$

Once the distribution H satisfying (26) has been found, we can turn around the arguments given above and recover the action of the Lie algebra of four-dimensional diffeomorphism on the extended phase space including embedding variables [45]. That such an extension is indeed necessary has been shown in [64], where obstructions against the implementation of the action of the Lie algebra of four-dimensional diffeomorphisms have been identified in case the dynamical fields include non-scalar ones.

4.3 Why constraints

From this follows a remarkable uniqueness result. Before stating it with all its hypotheses, we show why the constraints $\mathcal{H}[h, \pi] = 0$ and $\mathcal{D}^b[h, \pi] = 0$ must be imposed.

Consider the set of smooth real-valued functions on phase space, $F : T^*\mathrm{Riem}(\Sigma) \to \mathbb{R}$. They are acted upon by all $H(\alpha, \beta)$ via Poisson bracketing:

[5] Due to the standard convention that the Hamiltonian action being defined as a *left* action, whereas the Lie bracket on a group is defined by the commutator of left-invariant vector fields which generate *right* translations.

$F \mapsto \{F, H(\alpha, \beta)\}$. This defines a map from (α, β) into the derivations of phase-space functions. We require this map to also respect the commutation relation (26), that is, we require

$$\{\{F, H(\alpha_1, \beta_1)\}, H(\alpha_2, \beta_2)\} - \{\{F, H(\alpha_2, \beta_2)\}, H(\alpha_1, \beta_1)\} = \{F, H\}(\alpha', \beta')\,. \tag{27}$$

The subtle point to be observed here is the following: Up to now the parameters (α_1, β_1) and (α_2, β_2) were considered as given functions of $x \in \Sigma$, independent of the fields $h(x)$ and $\pi(x)$, i.e. independent of the point of phase space. However, from (23b) we see that $\beta'(x)$ does depend on $h(x)$. This dependence may not give rise to extra terms $\propto \{F, \alpha'\}$ in the Poisson bracket, for, otherwise, the extra terms would prevent the map $(\alpha, \beta) \mapsto \{-, H(\alpha, \beta)\}$ from being a homomorphism from the algebraic structure of hypersurface deformations into the derivations of phase-space functions. This is necessary in order to interpret $\{-, H(\alpha, \beta)\}$ as a generator (on phase-space functions) of a *spacetime* evolution corresponding to a normal lapse α and tangential shift β. In other words, the evolution of observables from an initial hypersurface Σ_i to a final hypersurface Σ_f must be independent of the intermediate foliation ('integrability' or 'path independence' [40,41,74]). Therefore we placed the parameters (α', β') outside the Poisson bracket on the right-hand side of (27), to indicate that no differentiation with respect to h, π should act on them.

To see that this requirement implies the constraints, rewrite the left-hand side of (27) in the form

$$\begin{aligned}
&\{\{F, H(\alpha_1, \beta_1)\}, H(\alpha_2, \beta_2)\} - \{\{F, H(\alpha_2, \beta_2)\}, H(\alpha_1, \beta_1)\} \\
&\quad = \{F, \{H(\alpha_1, \beta_1), H(\alpha_2, \beta_2)\}\} \\
&\quad = \left\{F, H(\alpha', \beta')\right\} \\
&\quad = \{F, H\}(\alpha', \beta') + H\left(\{F, \alpha'\}, \{F, \beta'\}\right),
\end{aligned} \tag{28}$$

where the first equality follows from the Jacobi identity, the second from (26), and the third from the Leibniz rule. Hence the requirement (27) is equivalent to

$$H\left(\{F, \alpha'\}, \{F, \beta'\}\right) = 0 \tag{29}$$

for all phase-space functions F to be considered and all α', β' of the form (23). Since only β' depends on phase space, more precisely on h, this implies the vanishing of the phase-space functions $H\left(0, \{F, \beta'\}\right)$ for all F and all β' of the form (23b). This can be shown to imply $H(0, \beta) = 0$, i.e. $\mathcal{D}[h, \pi] = 0$. Now, in turn, for this to be preserved under all evolutions we need $\left\{H(\alpha, \tilde{\beta}), H(0, \beta)\right\} = 0$, and hence in particular $\{H(\alpha, 0), H(0, \beta)\} = 0$ for all α, β, which implies $H(\alpha, 0) = 0$, i.e. $\mathcal{H}[h, \pi] = 0$. So we see that the constraints indeed follow.

Sometimes the constraints $H(\alpha, \beta)=0$ are split into the *Hamiltonian* (*or scalar*) *constraints*, $H(\alpha, 0)=0$, and the *diffeomorphisms* (*or vector*) *constraints*, $H(0, \beta)=0$. The relations (26) with (23) then show that the vector constraints form a Lie-subalgebra which, because of $\{H(0, \beta), H(\alpha, 0)\} = H(\beta(\alpha), 0) \neq H(0, \beta')$, is not an ideal.

This means that the Hamiltonian vector fields for the scalar constraints are not tangent to the surface of vanishing vector constraints, except where it intersects the surface of vanishing scalar constraints. This implies that the scalar constraints do not act on the solution space for the vector constraints, so that one simply cannot first reduce the vector constraints and then, on the solutions of that, search for solutions to the scalar constraints. Also, it is sometimes argued that the scalar constraints should *not* be regarded as generators of gauge transformations but rather as generators of physically meaningful motions whose effect is to change the physical state in a fashion that is, in principle, observable. See [52] and also [7] and Sect. 2.3 of Claus Kiefer's contribution for a recent revival of that discussion. However, it seems inconsistent to me to simultaneously assume 1) physical states to always satisfy the scalar constraints and 2) physical observables to exist which do not Poisson commute with the scalar constraints: The Hamiltonian vector field corresponding to such an 'observable' will not be tangent to the surface of vanishing scalar constraints and hence will transform physical to unphysical states upon being actually measured.

4.4 Uniqueness of Einstein's geometrodynamics

It is sometimes stated that the relations (26) together with (23) determine the function $H(\alpha,\beta) : T^*\mathrm{Riem}(\Sigma) \to \mathbb{R}$, i.e. the integrands $\mathcal{H}[h,\pi]$ and $\mathcal{D}[h,\pi]$, uniquely up to two free parameters, which may be identified with the gravitational and the cosmological constants. This is a mathematical overstatement if read literally, since the result can only be shown if certain additional assumptions are made concerning the action of $H(\alpha,\beta)$ on the basic variables h and π.

The first such assumption concerns the intended ('semantic' or 'physical') meaning of $H(0,\beta)$, namely that the action of $H(0,\beta)\}$ on h or π is that of an infinitesimal spatial diffeomorphism of Σ. Hence it should be the spatial Lie derivative, L_β, applied to h or π. It then follows from the general Hamiltonian theory that $H(0,\beta)$ is given by the *momentum map* that maps the vector field β (viewed as element of the Lie algebra of the group of spatial diffeomorphisms) into the function on phase space given by the contraction of the momentum with the β-induced vector field $h \to L_\beta h$ on $\mathrm{Riem}(\Sigma)$:

$$H(0,\beta) = \int_\Sigma d^3x\, \pi^{ab}(L_\beta h)_{ab} = -2\int_\Sigma d^3x (\nabla_a\pi^{ab})h_{bc}\beta^c\,. \tag{30}$$

Comparison with (25) yields

$$\mathcal{D}^b[h,\pi] = -2\nabla_a\pi^{ab}. \tag{31}$$

The second assumption concerns the intended ('semantic' or 'physical') meaning of $H(\alpha,0)$, namely that $\{-,H(\alpha,0)\}$ acting on h or π is that of an infinitesimal 'timelike' diffeomorphism of M normal to the leaves $\mathcal{E}_t(\Sigma)$. If M were given, it is easy to prove that we would have $L_{\alpha n}h = 2\alpha K$, where n is the timelike field of normals to the leaves $\mathcal{E}_t(\Sigma)$ and K is their extrinsic curvature. Hence one requires

$$\{h, H(\alpha, 0)\} = 2\alpha\, K. \tag{32}$$

Note that both sides are symmetric covariant tensor fields over Σ. The important fact to be observed here is that α appears without differentiation. This means that $H(\alpha, 0)$ is an ultralocal functional of π, which is further assumed to be a polynomial. (Note that we do not assume any relation between π and K at this point).

Quite generally, we wish to stress the importance of such 'semantic' assumptions concerning the intended meanings of symmetry operations when it comes to 'derivations' of physical laws from 'symmetry principles'. Such derivations often suffer from the same sort of overstatement that tends to give the impression that the mere requirement that some group G acts as symmetries alone distinguishes some dynamical laws from others. Often, however, additional assumptions are made that severely restrict the form in which G is allowed to act. For example, in field theory, the requirement of locality often enters decisively, like in the statement that Maxwell's vacuum equations are Poincaré- but not Galilei invariant. In fact, without locality the Galilei group, too, is a symmetry group of vacuum electrodynamics [29]. Coming back to the case at hand, I do not know of a uniqueness result that does not make the assumptions concerning the spacetime interpretation of the generators $H(\alpha, \beta)$. Compare also the related discussion in [66,67].

The uniqueness result for Einstein's equation, which in its space-time form is spelled out in Lovelock's theorem [54] already mentioned above, now takes the following form in Geometrodynamics [51]:

Theorem 1 *In four spacetime dimensions* (*Lorentzian for* $\sigma = 1$, *Euclidean for* $\sigma = -1$), *the most general functional* (25) *satisfying* (26) *with* (23), *subject to the conditions discussed above, is given by* (31) *and the two-parameter* (κ, Λ) *family*

$$\mathcal{H}[h, \pi] = \sigma\,(2\kappa)\, G_{ab\,cd}\pi^{ab}\pi^{cd} - (2\kappa)^{-1}\sqrt{\det(h)}\,(R(h) - \Lambda)\,, \tag{33}$$

where

$$G_{ab\,cd} = \frac{1}{2\sqrt{\det(h)}}\left(h_{ac}h_{bd} + h_{ad}h_{bc} - \frac{1}{2}h_{ab}h_{cd}\right), \tag{34}$$

and $R(h)$ *is the Ricci scalar of* (h, Σ). *Note that* (34) *is just the "contravariant version" of the metric* (9c) *for* $\lambda = 1$, *i.e.*, $G_{ab\,nm}G^{nm\,cd} = \frac{1}{2}(\delta^c_a\delta^d_b + \delta^d_a\delta^c_b)$.

The Hamiltonian evolution so obtained is precisely that of General Relativity (without matter) with gravitational constant $\kappa = 8\pi G/c^4$ and cosmological constant Λ. The proof of the theorem is given in [51], which improves on earlier versions [41,74] in that the latter assumes in addition that $\mathcal{H}[h, \pi]$ be an even function of π, corresponding to the requirement of time reversibility of the generated evolution. This was overcome in [51] by the clever move to write the condition set by $\{H(\alpha_1, 0), H(\alpha_2, 0)\} = H(0, \beta')$ (the right-hand side being already known) on $H(\alpha, 0)$ in terms of the corresponding Lagrangian functional L, which is then immediately seen to turn into a condition which is *linear* in L, so that terms with even powers in velocity decouple form those with odd powers. However, a small topological subtlety remains that is neglected in all these references and which potentially introduces a little more ambiguity that encoded in the two parameters κ and Λ, though its significance is more in the quantum theory. To see this recall that we can always perform a canonical transformation of the form

$$\pi \mapsto \pi' := \pi + \Theta \tag{35}$$

where Θ is a closed one-form on $\mathrm{Riem}(\Sigma)$. The latter condition ensures that all Poisson brackets remain the same if π is replaced with π'. Since $\mathrm{Riem}(\Sigma)$ is an open positive convex cone in a vector space and hence contractible, it is immediate that $\Theta = d\theta$ for some function $\theta : \mathrm{Riem}(\Sigma) \to \mathbb{R}$. However, π and π' must satisfy the diffeomorphism constraint, which is equivalent to saying that the kernel of π (considered as one-form on $\mathrm{Riem}(\Sigma)$) contains the vertical vector fields, which implies that Θ, too, must annihilate all V_ξ so that θ is constant on each connected component of the $\mathrm{Diff}_{\mathrm{F}}(\Sigma)$ orbit in $\mathrm{Riem}(\Sigma)$. But unless the $\mathrm{Diff}_{\mathrm{F}}(\Sigma)$ orbits in $\mathrm{Riem}(\Sigma)$ are connected, this does not mean that θ is the pull back of a function on Superspace, as assumed in [51]. We can only conclude that Θ is the pull back of a closed but not necessarily exact one-form on Superspace. Hence there is an analogue of the Bohm-Aharonov-like ambiguity that one always encounters if the configuration space is not simply connected. Whether this is the case depends in a determinate fashion on the topology of Σ: One has, due to the contractibility of $\mathrm{Riem}(\Sigma)$,

$$\pi_n\left(\mathrm{Riem}(\Sigma)/\mathrm{Diff}_{\mathrm{F}}(\Sigma)\right) \cong \pi_{n-1}\left(\mathrm{Diff}_{\mathrm{F}}(\Sigma)\right) \qquad (n \geq 1)\,. \tag{36}$$

For $n = 1$ the right hand side is

$$\pi_0\left(\mathrm{Diff}_{\mathrm{F}}(\Sigma)\right) := \mathrm{Diff}_{\mathrm{F}}(\Sigma)/\mathrm{Diff}^0_{\mathrm{F}}(\Sigma) =: \mathrm{MCG}_{\mathrm{F}}(\Sigma) \tag{37}$$

where $\mathrm{Diff}^0_{\mathrm{F}}(\Sigma)$ is the identity component of $\mathrm{Diff}_{\mathrm{F}}(\Sigma)$ and where we introduced the name $\mathrm{MCG}_{\mathrm{F}}(\Sigma)$ (Mapping-Class Group for Frame fixing diffeomorphisms) for the quotient group of components.

In view of the uniqueness result above, one might wonder what goes wrong when using (the contravariant version of) the metric G_λ for $\lambda \neq 1$ in (34). The answer is that it would spoil (26). More precisely, it would contradict $\{H(\alpha_1, 0)\,,\; H(\alpha_2, 0)\} = H(0, \alpha_1\nabla\alpha_2 - \alpha_2\nabla\alpha_1)$ due to an extra term $\propto (h_{ab}\pi^{ab})^2$ in $\mathcal{H}[h, \pi]$, unless the additional constraint $h_{ab}\pi^{ab} = 0$ were imposed, which is equivalent to $\mathrm{Tr}_h(K) = 0$ and hence to the condition that only maximal slices are allowed [35]. But this is clearly unacceptable (cf. Sect.6 of [8]).

As a final comment about uniqueness of representations of (26) we mention the apparently larger ambiguity—labelled by an additional $\mathbb{C}$-valued parameter, the *Barbero-Immirzi parameter*—that one gets if one uses connection variables rather than metric variables (cf. [6,42], Sect. 4.2.2 of [75], and Sect. 4.3.1 of [48]). However, in this case one does not represent (26) but a semi-direct product of it with the Lie algebra of $SU(2)$ gauge transformations, so that after taking the quotient with respect to the latter (which form an ideal) our original (26) is represented non locally. Also, unless the Barbero-Immirzi parameter takes the very special values $\pm i$ (for Lorentzian signature; ± 1 for Euclidean signature) the connection variable does not admit an interpretation as a space-time gauge field restricted to spacelike hypersurfaces (cf. [42,67]). For example, the holonomy of a spacelike curve γ varies with the choice of the spacelike hypersurface containing γ, which would be impossible if the spatial connection were the restriction of a space-time connection [67]. Accordingly, the dynamics generated by the constraints does then not admit the interpretation of being induced by appropriately moving a hypersurface through a spacetime with fixed geometric structures on it. Consequently, the argument provided here for why one should require (26) in the first place does, strictly speaking, not seem to apply in case of connection variables. It is therefore presently unclear to me on what set of assumptions a uniqueness result could be based in this case.

5 Topology of configuration space

Much of the global topology of $\mathcal{S}_{\mathrm{F}}(\Sigma)$ is encoded in its homotopy groups, which, in turn are given by those of $\mathrm{Diff}_{\mathrm{F}}(\Sigma)$ according to (36). Their structures were investigated in [33,36,83]. Early references as to their possible relevance in quantum gravity are [28,43,69,71].

We start by remarking that topological invariants of $\mathcal{S}_{\mathrm{F}}(\Sigma)$ are also topological invariants of Σ, which need not be homotopy invariant of Σ even if they are homotopy invariants of $\mathcal{S}_{\mathrm{F}}(\Sigma)$. This is, e.g., the case for the mapping-class group of homeomorphisms [55] and hence (in 3 dimensions) also for the mapping-class group $\mathrm{MCG}_{\mathrm{F}}(\Sigma)$. Remarkably, this means that we may distinguish homotopy equivalent but non homeomorphic three-manifolds by looking at homotopy invariants of their associated

Superspaces. Examples for this are given by certain types of lens spaces. First recall the definition of lens spaces $L(p, q)$ in 3 dimensions: $L(p, q) = S^3/\sim$, where (p, q) is a pair of positive coprime integers with $p > 1$, $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$, and $(z_1, z_2) \sim (z_1', z_2') \Leftrightarrow z_1' = \exp(2\pi i/p)z_1$, and $z_2' = \exp(2\pi i\, q/p)z_2$. One way to picture them is to take a solid ball in $\mathbb{R}^3$ and identify each point on the upper hemisphere with a points on the lower hemisphere after a rotation by $2\pi q/p$ about the vertical symmetry axis. (Usually one depicts the ball in a way in which it is slightly squashed along the vertical axis so that the equator develops a sharp edge and the whole body looks like a lens; see e.g. Fig. in [68].) In this way each set of p equidistant points on the equator is identified to a single point. The fundamental group of $L(p, q)$ is $\mathbb{Z}_p$, independent of q, and the higher homotopy groups are those of its universal cover, S^3. Moreover, for connected closed orientable three-manifolds the homology and cohomology groups are also determined by the fundamental group in an easy fashion: If A denotes the operation of abelianisation of a group, F the operation of taking the free part of a finitely generated abelian group, then the first four (zeroth to third, the only non-trivial ones) homology and cohomology groups are respectively given by $H_* = (\mathbb{Z}, A\pi_1, FA\pi_1, \mathbb{Z})$ and $H^* = (\mathbb{Z}, FA\pi_1, A\pi_1, \mathbb{Z})$ respectively. Hence, if taken of $L(p, q)$, all these standard invariants are sensitive only to p. However, it is known that $L(p, q)$ and $L(p, q')$ are

1. homotopy equivalent iff $qq' = \pm n^2 \pmod p$ for some integer n,
2. homeomorphic iff (all four possibilities) $q' = \pm q^{\pm 1} \pmod p$, and
3. orientation-preserving homeomorphic iff $q' = q^{\pm 1} \pmod p$.

The first statement is Theorem 10 in [82] and the second and third statement follow, e.g., from the like combinatorial classification of lens spaces [65] together with the validity of the 'Hauptvermutung' (the equivalence of the combinatorial and topological classifications) in 3 dimensions [60]. So, for example, $L(15,1)$ is homotopy equivalent but not homeomorphic to $L(15,4)$. On the other hand, it is known that the mapping-class group $\mathrm{MCG_F}(\Sigma)$ for $L(p,q)$ is $\mathbb{Z}\times\mathbb{Z}$ if $q^2 = 1 \pmod p$ with $q \neq \pm 1 \pmod p$, which applies to $p = 15$ and $q = 4$, and that in the remaining cases for $p > 2$ it is just $\mathbb{Z}$ (see Table IV on p. 591 of [83]). Hence $\mathrm{MCG_F}(\Sigma) \cong \mathbb{Z}\times\mathbb{Z}$ for $\Sigma = L(15,4)$ and $\mathrm{MCG_F}(\Sigma) \cong \mathbb{Z}$ for $\Sigma = L(15,1)$, even though $L(15,1)$ and $L(15,4)$ are homotopy equivalent!

Quite generally it turns out that Superspace stores much information about the topology of the underlying three-manifold Σ. This can be seen from the table in Fig. 3, which we reproduced form [32], and where properties of certain prime manifolds (see below for an explanation of 'prime') are listed. There is one interesting observation from that list which we shall mention right away: From gauge theories it is known that there is a relation between topological invariants of the classical configuration space and certain features of the corresponding quantum-field theory [46], in particular the emergence of certain anomalies which represent non-trivial topological invariants [1]. By analogy one could conjecture similar relations to hold quantum gravity. An interesting question is then whether there are preferred manifolds Σ for which all these invariants are trivial. From those represented on the table there is indeed a unique pair of manifolds for which this is the case, namely the three-sphere and the three-dimensional real projective space. To understand more of the information collected in the table we have to say more about general three-manifolds.

Of particular interest is the fundamental group of Superspace. Experience with ordinary quantum mechanics (cf. [34] and references therein) already suggests that its classes of inequivalent irreducible unitary representations correspond to a superselection structure which here might serve as fingerprint of the topology of Σ in the quantum theory. The sectors might, e.g., correspond to various statistics (in the presence of diffeomorphic primes) that preserve or violate a naively expected spin-statistics correlation [3,2,17,18] (see also below).

5.1 General three-manifolds and specific examples

The way to understand general three-manifolds is by cutting them along certain embedded two manifolds so that the remaining pieces are simpler in an appropriate sense. Here we shall only consider those simplifications that are achieved by cutting along embedded two-spheres. (Further decompositions by cutting along two-tori provide further simplifications, but these are not directly relevant here.) The two-spheres should be 'essential' and 'splitting'. An essential two-sphere is one which does not bound a three-ball and a splitting two-sphere is one whose complement has two (rather than just one) connected components. Figure 4 is intended to visualise the analogues

Prime Π	HC	S	C	N	$H_1(\Pi)$	$\pi_0(D_F(\Pi))$	$\pi_1(D_F(\Pi))$	$\pi_k(D_F(\Pi))$
S^3/D_8^*	+	+	+	−	$Z_2 \times Z_2$	O^*	0	$\pi_k(S^3)$
S^3/D_{8n}^*	+	+	+	−	$Z_2 \times Z_2$	D_{16n}^*	0	$\pi_k(S^3)$
$S^3/D_{4(2n+1)}^*$	+	+	+	+	Z_4	$D_{8(2n+1)}^*$	0	$\pi_k(S^3)$
S^3/T^*	?	+	+	−	Z_3	O^*	0	$\pi_k(S^3)$
S^3/O^*	w	+	+	+	Z_2	O^*	0	$\pi_k(S^3)$
S^3/I^*	?	+	+	−	0	I^*	0	$\pi_k(S^3)$
$S^3/D_8^* \times Z_p$	+	+	+	−	$Z_2 \times Z_{2p}$	$Z_2 \times O^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/D_{8n}^* \times Z_p$	+	+	+	−	$Z_2 \times Z_{2p}$	$Z_2 \times D_{16n}^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/D_{4(2n+1)}^* \times Z_p$	+	+	+	+	Z_{4p}	$Z_2 \times D_{8(2n+1)}^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/T^* \times Z_p$	?	+	+	−	Z_{3p}	$Z_2 \times O^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/O^* \times Z_p$	w	+	+	+	Z_{2p}	$Z_2 \times O^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/I^* \times Z_p$	?	+	+	−	Z_p	$Z_2 \times I^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/D'_{2^k(2n+1)} \times Z_p$	+	+	+	+	$Z_p \times Z_{2^k}$	$Z_2 \times D_{8(2n+1)}^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/T'_{8\cdot 3^m} \times Z_p$	?	+	+	−	$Z_p \times Z_{3^m}$	O^*	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$L(p,q_1)$	w	−	+	$(-)^p$	Z_p	Z_2	Z	$\pi_k(S^3)$
$L(p,q_2)$	$w+$	−	+	$(-)^p$	Z_p	$Z_2 \times Z_2$	$Z \times Z$	$\pi_k(S^3) \times \pi_k(S^3)$
$L(p,q_3)$	w	−	−	$(-)^p$	Z_p	Z_2	$Z \times Z$	$\pi_k(S^3) \times \pi_k(S^3)$
$L(p,q_4)$	w	−	+	$(-)^p$	Z_p	Z_2	$Z \times Z$	$\pi_k(S^3) \times \pi_k(S^3)$
RP^3	+	−	−	+	Z_2	1	0	0
S^3	+	−	−	−	1	1	0	0
$S^2 \times S^1$	/	−	−	+	Z	$Z_2 \times Z_2$	Z	$\pi_k(S^3) \times \pi_k(S^2)$
R^3/G_1	/	+	−	+	$Z \times Z \times Z$	$\mathrm{St}(3,Z)$	0	$\pi_k(S^3)$
R^3/G_2	/	+	−	+	$Z \times Z_2 \times Z_2$	$\mathrm{Aut}_+^{Z_2}(G_2)$	0	$\pi_k(S^3)$
R^3/G_3	/	+	+	+	$Z \times Z_3$	$\mathrm{Aut}_+^{Z_2}(G_3)$	0	$\pi_k(S^3)$
R^3/G_4	/	+	+	−	$Z \times Z_2$	$\mathrm{Aut}_+^{Z_2}(G_4)$	0	$\pi_k(S^3)$
R^3/G_5	/	+	+	+	Z	$\mathrm{Aut}_+^{Z_2}(G_5)$	0	$\pi_k(S^3)$
R^3/G_6	/	+	+	−	$Z_4 \times Z_4$	$\mathrm{Aut}_+^{Z_2}(G_5)$	0	$\pi_k(S^3)$
$S^1 \times R_g$	/	+	−	−	$Z \times Z_{2g}$	$\mathrm{Aut}_+^{Z_2}(Z \times F_g)$	0	$\pi_k(S^3)$
$K(\pi,1)_{\mathrm{sl}}$	/	+	*	*	$A\pi$	$\mathrm{Aut}_+^{Z_2}(\pi)$	0	$\pi_k(S^3)$

Fig. 3 This table, taken from [32], lists various properties of certain prime three-manifolds. The manifolds are grouped into those of finite fundamental group, which are of the form S^3/G, the exceptional one, $S^1 \times S^2$, which is prime but not irreducible ($\pi_2(S^1 \times S^2) = \mathbb{Z}$), those six which can carry a flat metric and which are of the form $\mathbb{R}^3/G$, and so-called Haken manifolds (sufficiently large $K(\pi, 1)$ primes). For the lens spaces q_1 stands for $q = \pm 1$, q_2 for $q \neq \pm 1$ and $q^2 = 1$, q_3 for $q^2 = -1$, and q_4 for the remaining cases, where all equalities are taken mod p. The *third* and *fourth column* list spinoriality and chirality, the *last three columns* the homotopy groups of their corresponding Superspace. We refer to [32] for the meanings of the other columns

of these notions in two dimensions. Given a closed three-manifold Σ, consider the following process: Cut it along an essential splitting two-sphere and cap off the two-sphere boundary of each remaining component by a three-disk. Now repeat the process for each of the remaining closed three-manifolds. This process stops after a finite number of steps [50] where the resulting components are uniquely determined up to

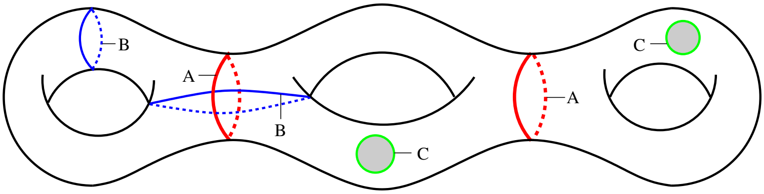

Fig. 4 A Riemann surface of genus 3 with three pairs of embedded 1-spheres (*circles*) of type A, B, and C. Type A is essential and splitting, type B is essential but not splitting, and type C is splitting but not essential. Any third essential and splitting 1-sphere can be continuously deformed via embeddings into one of the two drawn here

diffeomorphisms (orientation preserving if oriented manifolds are considered) and permutation [58]; see [39] for a lucid discussion. The process stops at that stage at which none of the remaining components, $\Pi_1, \ldots, \Pi_n$, allows for essential splitting two-spheres, i.e. at which each Π_i is a prime manifold. A three-manifold is called *prime* if each embedded two-sphere either bounds a three-disc or does not split; it is called *irreducible* if each embedded two-sphere bounds a three-disc. In the latter case the second homotopy group, π_2, must be trivial, since, if it were not, the so-called sphere theorem (see, e.g., [39]) ensured the existence of a non-trivial element of π_2 which could be represented by an *embedded* two-sphere. Conversely, it follows from the validity of the Poincaré conjecture that a trivial π_2 implies irreducibility. Hence irreducibility is equivalent to a trivial π_2. There is precisely one non-irreducible prime three-manifold, and that is the handle $S^1 \times S^2$. Hence a three-manifold is prime iff it is either a handle or if its π_2 is trivial.

Given a general three-manifold Σ as connected sum of primes $\Pi_1, \ldots, \Pi_n$, there is a general method to establish $\mathrm{MCG_F}(\Sigma)$ in terms of the individual mapping-class groups of the primes. The strategy is to look at the effect of elements in $\mathrm{MCG_F}(\Sigma)$ on the fundamental group of Σ. As Σ is the connected sum of primes, and as connected sums in d dimensions are taken along $d-1$ spheres which are simply-connected for $d \geq 3$, the fundamental group of a connected sum is the free product of the fundamental groups of the primes for $d \geq 3$. The group $\mathrm{MCG_F}(\Sigma)$ now naturally acts as automorphisms of $\pi(\Sigma)$ by simply taking the image of a based loop that represents an element in $\pi(\Sigma)$ by a based (same basepoint) diffeomorphism that represents the class in $\mathrm{MCG_F}(\Sigma)$. Hence there is a natural map

$$d_F : \mathrm{MCG_F}(\Sigma) \to \mathrm{Aut}\,(\pi_1(\Sigma))\,. \tag{38}$$

The known presentations[6] of automorphism groups of free products in terms of presentations of the automorphisms of the individual factors and additional generators (basically exchanging isomorphic factors and conjugating whole factors by individual elements of others) can now be used to establish (finite) presentations of $\mathrm{MCG_F}(\Sigma)$,

[6] A (finite) presentation of a group is its characterisation in terms of (finitely many) generators and (finitely many) relations.

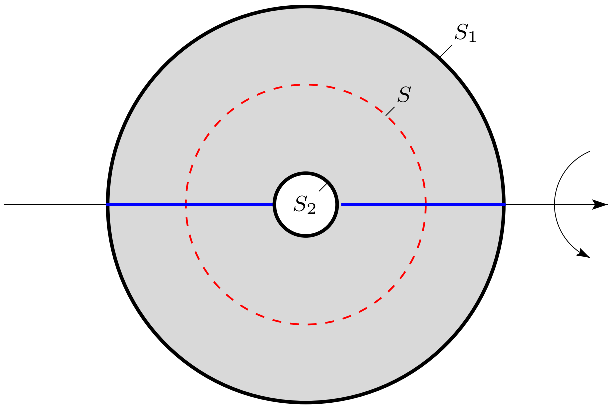

Fig. 5 The connected sum of two real projective spaces may be visualised by the shaded region obtained from the spherical shell that is obtained by rotating the shaded annulus about the vertical symmetry axis as indicated. Antipodal points on the outer two-sphere boundary S_1, as well as on the inner two-sphere boundary S_2, are pairwise identified. This results in the connected sum of two $\mathbb{R}P^3$ along the connecting sphere S. Due to the antipodal identifications, the two *thick horizontal segments* in the *shaded region* become a single loop, showing that the entire space is fibred by circles over $\mathbb{R}P^2$. Remarkably, the handle $S^1 \times S^2$, which is prime, is a double cover of this reducible manifold

provided (finite) presentations for all prime factors are known.[7] Here I wish to stress that this situation would be more complicated if $\text{Diff}(\Sigma)$ rather than $\text{Diff}_{\text{F}}(\Sigma)$ (or at least the diffeomorphisms fixing a preferred point) had been considered; that is, had we not made the transition from (1) to (4). Only for $\text{Diff}_{\text{F}}(\Sigma)$ (or the slightly larger group of diffeomorphisms fixing the point) is it generally true that the mapping-class group of a prime factor injects into the mapping-class group of the connected sum in which it appears. For more on this, compare the discussion on p. 182-3 in [37]. Clearly, one also needs to know which elements are in the kernel of the map (38). This will be commented on below in connection with Fig 7.

5.2 The connected sum of two real-projective spaces

In some (in fact many) cases the map d_F is an isomorphism. For example, this is the case if Σ is the connected sum of two $\mathbb{R}P^3$ (Fig. 5), so that $\pi_1(\Sigma)$ is the free product $\mathbb{Z}_2 * \mathbb{Z}_2$, a presentation of which is $\langle a, b \mid a^2 = b^2 = 1\rangle$. For the automorphisms we have $\text{Aut}(\mathbb{Z}_2 * \mathbb{Z}_2) \cong \mathbb{Z}_2 * \mathbb{Z}_2 = \langle E, S \mid E^2 = S^2 = 1\rangle$, where $E : (a, b) \to (b, a)$ and $S : (a, b) \to (a, aba^{-1})$. In this sense the infinite discrete group $\mathbb{Z}_2 * \mathbb{Z}_2$ is a quotient of the automorphism group of Superspace $\mathcal{S}_{\text{F}}(\Sigma)$ for Σ being the connected sum of two real projective spaces. It is therefore of interest to study its unitary irreducible representations. This can be done directly in a rather elementary fashion, or more systematically by a simple application of the method of induced representations (Mackey theory) using the isomorphicity $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z}_2 \ltimes \mathbb{Z}$

[7] This presentation of the automorphism group of free products is originally due to Fouxe-Rabinovitch [25,26]. Modern forms with corrections are given in [57] and [31].

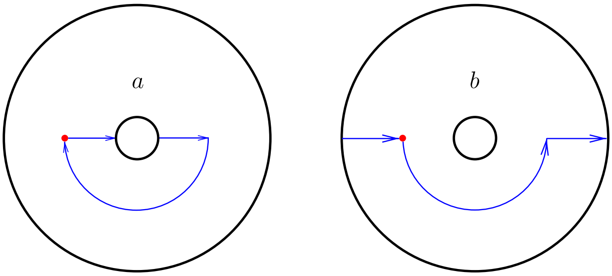

Fig. 6 The shown loops represent the generators a and b of the fundamental group $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = 1\rangle$. The product loop $c = ab$ is then seen to be one of the fibres mentioned in the caption of the previous Figure. In terms of a and c we have the presentation $\langle a, c \mid a^2 = 1,\ aca^{-1} = c\rangle$, showing the isomorphicity $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z}_2 \ltimes \mathbb{Z}$

(cf. caption to Fig. 6). The result is that, apart form the obvious four one-dimensional ones, given by $(E, S) \mapsto (\pm\mathrm{id}, \pm\mathrm{id})$, there is a continuous set of mutually inequivalent two-dimensional ones, given by

$$E \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad S \mapsto \begin{pmatrix} \cos\tau & \sin\tau \\ \sin\tau & -\cos\tau \end{pmatrix}, \quad \tau \in (0, \pi). \tag{39}$$

Already in this most simple example of a non-trivial connected sum we have an interesting structure in which the two 'statistics sectors' corresponding to the irreducible representations of the permutation subgroup (here just given by the $\mathbb{Z}_2$ subgroup generated by E) get mixed by S, where the 'mixing angle' τ uniquely characterises the representation. This behaviour can also be studied in more complicated examples [32,72]. For a more geometric understanding of the maps representing E and S, see [37].

5.3 Spinoriality

I also wish to mention one very surprising observation that was made by Rafael Sorkin and John Friedman in 1980 [28] and which has to do with the physical interpretation of the elements in the kernel of the map (38), leading to the conclusion that pure (i.e. without matter) quantum gravity should already contain states with half-integer angular momenta. The reason being a purely topological one, depending entirely on the topology of Σ. In fact, given the right topology of Σ, its one-point decompactification used in the context of asymptotically flat initial data will describe an isolated system whose asymptotic (at spacelike infinity) symmetry group is not the ordinary Poincaré group [10] but rather its double (= universal) cover. This gives an intriguing answer to Wheeler's quest to find a natural place for spin 1/2 in Einstein's standard geometrodynamics (cf. [59] Box 44.3).

I briefly recall that after introducing the concept of a 'Geon' ('gravitational-electromagnetic entity') in 1955 [77], and inspired by the observation that electric charge (in the sense of non-vanishing flux integrals of $\star F$ over closed two-dimensional

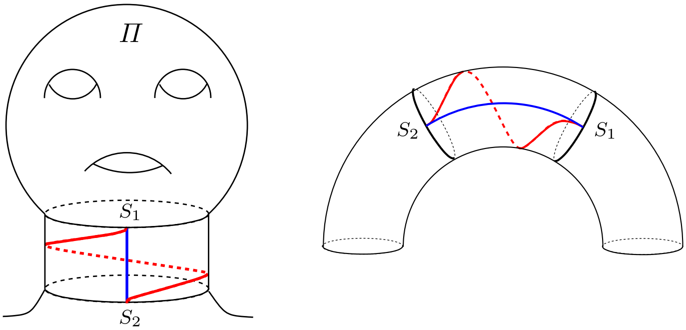

Fig. 7 Both pictures show rotations parallel to spheres S_1 and S_2: On the left, a rotation of a prime manifold in a connected sum parallel to the connecting sphere, on the right a rotation parallel to two meridian spheres in a 'handle' $S^1 \times S^2$. The support of the diffeomorphism is on the cylinder bound by S_1 and S_2. In either case its effect is depicted by the two curves connecting the two spheres. The two-dimensional representation given here is deceptive insofar, as in two dimensions the original and the mapped curves are not homotopic (keeping their endpoints fixed), due to the one-sphere not being simply connected, whereas they are in three (and higher) higher dimensions they are due to the higher-dimensional spheres being simply connected

surfaces) could be realised in Einstein-Maxwell theory without sources ('charge without charge'), Wheeler and collaborators turned to the Einstein–Weyl theory [13] and tried to find a 'neutrino analog of electric charge' [49]. Though this last attempt failed, the programme of 'matter as geometry' in the context of geometrodynamics, as outlined in the contributions to the anthology [78], survived in Wheeler's thinking well into the 1980s [81].

Back to the 'spin without spin' topologies, the elements of the kernel of (38) can be pictured as rotation parallel to certain spheres, as depicted in Fig. 7. (In many—and possibly all—cases the group generated by such maps actually exhaust the kernel; compare Theorem. 1.5 in [56] and footnote 21 in [37]). The point we wish to focus on here is that for *some* prime manifolds the diffeomorphism depicted on the left in Fig. 7 is indeed not in the identity component of all diffeomorphisms that fix a frame exterior to the outer (S_2) two-sphere. Such manifolds are called *spinorial*. For each prime it is known whether it is spinorial or not, and the easy-to-state but hard-to-prove result is, that the only non-spinorial manifolds[8] are the lens spaces $L(p, q)$, the handle $S^1 \times S^2$, and connected sums amongst them. That these manifolds are not spinorial is, in fact, very easy to visualise. Hence, given the proof of the 'only' part and of the fact that a connected sum is spinorial iff it contains at least one spinorial prime, one may summarise the situation by saying that the only non-spinorial manifolds are the 'obvious' ones.

Even though being a generic property in the sense just stated, spinoriality is generally hard to prove in dimensions three or greater. This is in marked contrast to two dimensions, where the corresponding transformation shown in the left picture of Fig.7 acts non trivially on the fundamental group. Indeed, consider a base point outside (below) S_2 in the left picture in Fig. 7, then the rotation acts by conjugating each of the $2g$ generators $(a_1, \ldots, a_g, b_1, \ldots, b_2)$ that Π adds to $\pi_1(\Sigma)$ by the element

[8] We remind the reader that 'manifold' here stands for 'three-dimensional closed orientable manifold'.

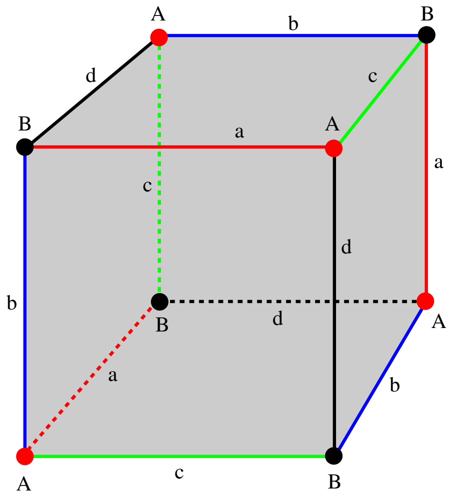

Fig. 8 Fundamental domain for the space S^3/D_8^*. Opposite faces are identified after a right-handed screw motion with 90° rotation, as indicated by the *coinciding labels* for the edges and vertices. The corresponding space with a left-handed identification is not orientation-preserving diffeomorphic to this one

$\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}$, which is non-trivial in $\pi_1(\Sigma)$ if other primes exist or otherwise if the point with the fixed frame is removed (as one may do, due to the restriction to diffeomorphisms fixing that point).

An example of a spinorial manifold is the spherical space form S^3/D_8^*, where S^3 is thought of as the sphere of unit quaternions and D_8^* is the subgroup in the group of unit quaternions given by the eight elements $\{\pm 1, \pm i, \pm j, \pm k\}$. The coset space S^3/D_8^* may be visualised as solid cube whose opposite faces are identified after a 90-degree rotation by either a right- or a left-handed screw motion; see Fig. 8. Drawings of fundamental domains in form of (partially truncated) solid polyhedra with suitable boundary identifications for spaces S^3/G are given in [19]. Let us take the basepoint $\infty \in S^3/D_8^*$ to be the centre of the cube in Fig. 8. The two generators, a and b, of the fundamental group

$$\pi_1(S^3/D_8^*) \cong D_8^* = \langle a, b \mid a^2 = b^2 = (ab)^2 \rangle \tag{40}$$

are then represented by two of the three oriented straight segments connecting the midpoints of opposite faces. The third corresponds to the product ab. A rotation of the cube about its centre by any element of its crystallographic symmetry group defines a diffeomorphism of S^3/D_8^* fixing ∞ since it is compatible with the boundary identification. It is not in the identity component of ∞-fixing diffeomorphisms since it obviously acts non-trivially on the generators of the fundamental group. Clearly, each such rigid rotation may be modified in an arbitrarily small neighbourhood of ∞ so as to also fix the tangent space at this point. That S^3/D_8^* is spinorial means that in going from the point-fixing to the frame-fixing diffeomorphisms one acquires more diffeomorphisms not connected to the identity. More precisely, the mapping-class group of frame-fixing diffeomorphisms is a $\mathbb{Z}_2$-extension of the mapping-class group of merely point-fixing diffeomorphisms. The generator of this extending $\mathbb{Z}_2$ is a full 360° rotation parallel to two small concentric spheres centred at ∞. In this way, the spinoriality of S^3/D_8^* extends the crystallographic symmetry group $O \subset SO(3)$ of the cube to its double cover $O^* \subset SU(2)$ and one finally gets

$$\mathrm{MCG_F}(\Pi) \cong O^* \quad \text{for} \quad \Pi = S^3/D_8^* \,. \tag{41}$$

This is precisely what one finds at the intersection of the 2nd row and 7th column of the table in Fig. 3, with corresponding results for the other spherical space forms.

Coming back to the previous example of the connected sum of two (or more [32]) real projective spaces, we can make the following observation: First of all, real projective three-space is a non-spinorial prime. This is obvious once one visualises it as a solid three-ball whose two-sphere boundary points are pairwise identified in an antipodal fashion, since this identification is compatible with a rigid rotation. A full rotation about the, say, centre-point of the ball may therefore be continuously undone by a rigid rotation outside a small ball about the centre, suitably 'bumped off' towards the centre. Second, as we have seen above, the irreducible representations of the mapping-class group of the connected sum contains both statistics sectors independently for one-dimensional representations and in a mixed form for the continuum of two-dimensional irreducible representations. This already shows [2] that there is no general kinematical spin-statistics relation as in other non-linear theories [21,70]. Such a relation may at best be re-introduced for some manifolds by restricting the way in which states are constructed, e.g., via the sum-over-histories approach [17].

5.4 Chirality

There is one last aspect about diffeomorphisms that can be explained in terms of Fig. 8. As stated in the caption of this figure, there are two versions of this space: one where the identification of opposite faces is done via a 90° right-handed screw motion and one where one uses a left-handed screw motion. These spaces are not related by an orientation preserving diffeomorphism. This is equivalent to saying that, say, the first of these spaces has no orientation-reversing self-diffeomorphism. Manifolds for which this is the case are called *chiral*. There are no examples in two dimensions. To see this, just consider the usual picture of a Riemannian genus g surface embedded into $\mathbb{R}^3$ and map it onto itself by a reflection at any of its planes of symmetry. So chiral manifolds start to exist in 3 dimensions and continue to do so in all higher dimensions, as was just recently shown [61]. If one tries to reflect the cube in Fig. 8 at one of its symmetry planes one finds that this is incompatible with the boundary identifications, that is, pairs of identified points are not mapped to pairs of identified points. Hence this reflection simply does not define a map of the quotient space. This clearly does not prove the nonexistence of orientation reversing maps, since there could be others than these obvious candidates.

In fact, following an idea of proof in [83], the chirality of S^G/D_8^* (and others of the form S^3/G) can be reduced to that of $L(4,1)$. That the latter is chiral follows from the following argument: Above we have already stated that $L(p,q)$ and $L(p,q')$ are orientation preserving diffeomorphic iff $q' = q^{\pm 1}$ (mod p). Since taking the mirror image in $\mathbb{R}^3$ of the lens representing $L(p,q)$ gives the lens representing

$L(p,-q)$, $L(p,q)$ admits an orientation reversing diffeomorphism iff $L(p,q)$ and $L(p,-q)$ are orientation preserving diffeomorphic. But, as just stated, this is the case iff $-q = q^{\pm 1} \pmod p$, i.e. if either $p = 2$, $q = 1$ (recall that p and q must be coprime) or $q^2 = -1 \pmod p$.[9] Hence, in particular, all $L(p,1)$ are chiral. Now, $G = D_8^*$ has three subgroups isomorphic to $\mathbb{Z}_4$, the fundamental group of $L(4,1)$, namely the ones generated by i, j, and k. They are normal so that we have a regular covering $L(4,1) \xrightarrow{p} S^3/G$. Now suppose $f : S^3/G \to S^3/G$ were orientation reversing, i.e. a diffeomorphism with $\deg(f) = -1$. Consider the diagram

$$\begin{tikzcd} L(4,1) \arrow[r, dotted, "\tilde{f}"] \arrow[d, "p"'] \arrow[dr, "f\circ p"] & L(4,1) \arrow[d, "p"] \\ S^3/G \arrow[r, "f"'] & S^3/G \end{tikzcd} \tag{42}$$

If the lift $\tilde{f}$ existed we would immediately get a contradiction since from commutativity of (42) we would get $\deg(\tilde{f}) \cdot \deg(p) = \deg(p) \cdot \deg(f)$ and hence $\deg(\tilde{f}) = -1$, which contradicts chirality of $L(4,1)$. Now, according to the theory of covering spaces the lift $\tilde{f}$ of $f \circ p$ exists iff the image of $\pi_1(L(4,1))$ under $(f \circ p)_*$, which is a subgroup $\mathbb{Z}_4 \subset D_8^*$, is conjugate to the image of $\pi_1(L(4,1))$ under p_*. This need not be the case, however, as different subgroups $\mathbb{Z}_4$ in D_8^* are normal and hence never conjugate (here we deviate from the argument in [83] which seems incorrect). However, by composing a given orientation reversing f with an orientation preserving diffeomorphism that undoes the $\mathbb{Z}_4$ subgroup permutation introduced by f, we can always create a new orientation reversing diffeomorphism that does not permute the $\mathbb{Z}_4$ subgroups. That new orientation reversing diffeomorphism—call it again f—now indeed has a lift $\tilde{f}$, so that finally we arrive at the contradiction envisaged above.

As prime manifolds, the two versions of a chiral prime corresponding to the two different orientations count as different. This means the following: Two connected sums which differ only insofar as a particular chiral prime enters with different orientations are not orientation-preserving diffeomorphic; they are not diffeomorphic at all if the complement of the selected chiral prime also chiral (i.e. iff another chiral prime exists). For example, the connected sum of two oriented S^3/D_8^* is not diffeomorphic to the connected sum of S^3/D_8^* with $\overline{S^3/D_8^*}$, where the overbar indicates the opposite orientation. Note that the latter case also leads to two non-homeomorphic three-manifolds whose classic invariants (homotopy, homology, cohomology) coincide. This provides an example of a topological feature of Σ that is not encoded into the structure of $\mathcal{S}_F(\Sigma)$.

[9] Remarkably, this result was already stated in footnote 1 of p. 256 of [50]. An early published proof is that in Sect. 77 of [68].

6 Summary and outlook

Superspace is for geometrodynamics what gauge-orbit space is for non-abelian gauge theories, though Superspace has generally a much richer topological and metric structure. Its topological structure encodes much of the topology of the underlying three-manifold and one may conjecture that some of its topological invariants bear the same relation to anomalies and sectorial structure as in the case of non-abelian gauge theories. Recent progress in three-manifold theory now allows to make more complete statements, in particular concerning the fundamental groups of Superspaces associated to more complicated three-manifolds. Its metric structure is piecewise nice but also suffers from singularities, corresponding to signature changes, whose physical significance is unclear. Even for simple three-manifolds, like the three-sphere, there are regions in Superspace where the metric is strictly Lorentzian (just one negative signature and infinitely many pluses), like at the round three-sphere used in the FLRW cosmological models, so that the Wheeler–DeWitt equation becomes strictly hyperbolic, but there are also regions with infinitely many negative signs in the signature.

Note that the cotangent bundle over Superspace is not the fully reduced phase space for matter-free General Relativity. It only takes account of the vector constraints and leaves the scalar constraint unreduced. However, under certain conditions, the scalar constraints can be solved by the ‘conformal method’ which leaves only the conformal equivalence class of three-dimensional geometries as physical configurations. In those cases the fully reduced phase space is the cotangent bundle over conformal Superspace, whose analog to (1) is given by replacing $\mathrm{Diff}(\Sigma)$ by the semi-direct product $\mathrm{C}(\Sigma) \rtimes \mathrm{Diff}(\Sigma)$, where $\mathrm{C}(\Sigma)$ is the abelian group of conformal rescalings that acts on $\mathrm{Riem}(\Sigma)$ via $(f, h) \mapsto f\,h$ (pointwise multiplication), where $f : \Sigma \to \mathbb{R}_+$. The right action of $(f, \phi) \in \mathrm{C}(\Sigma) \rtimes \mathrm{Diff}(\Sigma)$ on $h \in \mathrm{Riem}(\Sigma)$ is then given by $R_{(f,\phi)}(h) = f\phi^*h$, so that, using $R_{(f_2,\phi_2)}R_{(f_1,\phi_1)} = R_{(f_1,\phi_1)(f_2,\phi_2)}$, the semi-direct product structure is seen to be $(f_1, \phi_1)(f_2, \phi_2) = (f_2(f_1 \circ \phi_2), \phi_1 \circ \phi_2)$. Note that because of $(f_1 f_2) \circ \phi = (f_1 \circ \phi)(f_2 \circ \phi)$ $\mathrm{Diff}(\Sigma)$ indeed acts as automorphisms of $\mathrm{C}(\Sigma)$. Conformal Superspace and extended conformal superspace would then, in analogy to (1) and (4), be defined as $\mathcal{CS}(\Sigma) := \mathrm{Riem}(\Sigma)/\mathrm{C}(\Sigma) \rtimes \mathrm{Diff}(\Sigma)$ and $\mathcal{CS}_\mathrm{F}(\Sigma) := \mathrm{Riem}(\Sigma)/\mathrm{C}(\Sigma) \rtimes \mathrm{Diff}_\mathrm{F}(\Sigma)$ respectively. The first definition was used in [24] as applied to manifolds with 0° of symmetry (cf. footnote 3). In any case, since $\mathrm{C}(\Sigma)$ is contractible, the topologies of $\mathrm{C}(\Sigma) \rtimes \mathrm{Diff}(\Sigma)$ and $\mathrm{C}(\Sigma) \rtimes \mathrm{Diff}_\mathrm{F}(\Sigma)$ are those of $\mathrm{Diff}(\Sigma)$ and $\mathrm{Diff}_\mathrm{F}(\Sigma)$ which also transcend to the quotient spaces analogously to (36) whenever the groups act freely. In the first case this is essentially achieved by restricting to manifolds of vanishing degree of symmetry, whereas in the second case this follows almost as before, with the sole exception being (S^3, h) with h conformal to the round metric. [10] Hence the topological results obtained before also apply to this case. In contrast, the geometry for conformal Superspace differs

[10] Let $\mathcal{CI}(\Sigma, h) := \{\phi \in \mathrm{Diff}(\Sigma) \mid \phi^*h = fh,\ f : \Sigma \to \mathbb{R}_+\}$ be the group of conformal isometries. For compact Σ it is known to be compact except iff $\Sigma = S^3$ and h conformal to the round metric [53]. Hence, for $\Sigma \neq S^3$, we can average h over the compact group $\mathcal{CI}(\Sigma, h)$ and obtain a new Riemannian metric h' in the conformal equivalence class of h for which $\mathcal{CI}(\Sigma, h)$ acts as proper isometries. Therefore, by the argument presented in Sect. 2, it cannot contain non-trivial elements fixing a frame.

insofar from that discussed above as the conformal modes that formed the negative directions of the Wheeler–DeWitt metric (cf. (9a) are now absent. The horizontal subspaces [orthogonal to the orbits of $\mathrm{C}(\Sigma) \rtimes \mathrm{Diff_F}(\Sigma)$] are now given by the transverse and traceless [rather than just obeying (13)] symmetric two-tensors. In that sense the geometry of conformal Superspace, if defined as before by some ultralocal bilinear form on $\mathrm{Riem}(\Sigma)$, is manifestly positive (due to the absence of trace terms) and hence less pathological than the Superspace metric discussed above. It might seem that its physical significance is less clear, as there is now no constraint left that may be said to induce this particular geometry; see however [9].

Whether it is a realistic hope to understand Superspace and conformal Superspace (its cotangent bundle being the space of solutions to Einstein's equations) well enough to actually gain a sufficiently complete understanding of its automorphism group is hard to say. An interesting strategy lies in the attempt to understand the solution space directly in a group- (or Lie algebra-) theoretic fashion in terms of a quotient G_∞/H_∞, where G_∞ is an infinite dimensional group (Lie algebra) that (locally) acts transitively on the space of solutions and H_∞ is a suitable subgroup (algebra), usually the fixed-point set of an involutive automorphism of G. The basis for the hope that this might work in general is the fact that it works for the subset of stationary and axially symmetric solutions, where G^∞ is the Geroch Group; cf. [12]. The idea for generalisation, even to $d = 11$ supergravity, is expressed in [63] and further developed in [14].

Acknowledgments I thank Hermann Nicolai and Stefan Theisen for inviting me to this stimulating 405th WE-Heraeus-Seminar on Quantum Gravity and for giving me the opportunity to contribute this paper. I also thank Hermann Nicolai for pointing out [63] and Ulrich Pinkall for pointing out [53].

References

1. Alvarez-Gaumé, L., Ginsparg, P.: The structure of gauge and gravitational anomalies. Ann. Phys. **161**, 423–490 (1985)
2. Aneziris, C., et al.: Aspects of spin and statistics in general covariant theories. Int. J. Mod. Phys. A **14**(20), 5459–5510 (1989)
3. Aneziris, C., et al.: Statistics and general relativity. Mod. Phys. Lett. A **4**(4), 331–338 (1989)
4. Arens, R.: Classical Lorentz invariant particles. J. Math. Phys. **12**(12), 2415–2422 (1971)
5. Bacry, H.: Space-time and degrees of freedom of the elementary particle. Commun. Math. Phys. **5**(2), 97–105 (1967)
6. Barbero, F.: Real Ashtekar variables for Lorentzian signature space-times. Phys. Rev. D **51**(10), 5507–5510 (1995)
7. Barbour, J., Foster, B.Z.: Constraints and gauge transformations: Dirac's theorem is not always valid. arXiv:0808.1223v1, (2008)
8. Barbour, J., Foster, B.Z., Murchadha, N.Ó.: Relativity without relativity. Classical Quantum Gravity **19**(12), 3217–3248 (2002)
9. Barbour, J., Murchadha, N.Ó.: Classical and quantum gravity on conformal superspace. arXiv: gr-qc/9911071v1 (1999)
10. Beig, R., Murchadha, N.Ó.: The Poincaré group as symmetry group of canonical general relativity. Ann. Phys. **174**, 463–498 (1987)
11. Besse, A.L.: Einstein Manifolds. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge Band 10. Springer, Berlin (1987)

12. Breitenlohner, P., Maison, D.: On the Geroch group. Ann. Inst. Henri Poincaré A **46**(2), 215–246 (1987)
13. Brill, D.R., Wheeler, J.A.: Interaction of neutrinos and gravitational fields. Rev. Mod. Phys. **29**(3), 465–479 (1957)
14. Damour, T., Kleinschmidt, A., Nicolai, H.: Constraints and the E_{10} coset model. Classical Quantum Gravity **24**(23), 6097–6120 (2007)
15. Damour, T., Nicolai, H.: Symmetries, singularities and the de-emergence of space. arXiv:0705.2643v1 (2007)
16. DeWitt, B.S.: Quantum theory of gravity. I. The canonical theory. Physical Review, **160**(5), 1113–1148, (1967). Erratum, ibid. **171**(5), 1834 (1968)
17. Dowker, F., Sorkin, R.: A spin-statistics theorem for certain topological geons. Classical Quantum Gravity **15**, 1153–1167 (1998)
18. Dowker, F., Sorkin, R.: Spin and statistics in quantum gravity. In: Hilborn, R.C., Tino, G.M. (eds.) Spin-Statistics Connections and Commutation Relations: Experimental Tests and Theoretical Implications, pp. 205–218. American Institute of Physics, New York (2000)
19. Du Val, P.: Homographies, Quaternions, and Rotations. Clarendon Press, Oxford (1964)
20. Ebin, D.G.: On the space of Riemannian metrics. Bull. Am. Math. Soc. **74**(5), 1001–1003 (1968)
21. Finkelstein, D., Rubinstein, J.: Connection between spin, statistics, and kinks. J. Math. Phys. **9**(11), 1762–1779 (1968)
22. Fischer, A.E.: The theory of superspace. In: Carmeli, M., Fickler, S.I., Witten, L. (eds.) Relativity, proceedings of the Relativity Conference in the Midwest, held June 2–6, 1969, at Cincinnati Ohio, pp. 303–357. Plenum Press, New York (1970)
23. Fischer, A.E.: Resolving the singularities in the space of Riemannian geometries. J. Math. Phys. **27**, 718–738 (1986)
24. Fischer, A.E., Moncrief, V.E.: The structure of quantum conformal superspace. In: Cotsakis , S., Gibbons, G.W. (eds.) Global Structure and Evolution in General Relativity, Lecture Notes in Physics, vol. 460, pp. 111–173. Springer, Berlin (1996)
25. Fouxe-Rabinovitch, D.I.: Über die Automorphismengruppen der freien Produkte I. Matematicheskii Sb. **8**(50), 265–276 (1940)
26. Fouxe-Rabinovitch, D.I.: Über die Automorphismengruppen der freien Produkte II. Matematicheskii Sb. **9**(51), 297–318 (1941)
27. Freed, D.S., Groisser, D.: The basic geometry of the manifold of Riemannian metrics and of its quotient by the diffeomorphism group. Mich. Math. J. **36**(3), 323–344 (1989)
28. Friedman, J., Sorkin, R.: Spin 1/2 from gravity. Phys. Rev. Lett. **44**, 1100–1103 (1980)
29. Fushchich, W., Shtelen, V.: Are Maxwell's equations invariant under the Galilei transformations? Doklady Akademii Nauk Ukrainy A **3**, 22–26 (1991) (In Russian)
30. Gao, L.Z., Yau, S.-T.: The existence of negatively Ricci curved metrics on three-manifolds. Invent. Math. **85**(3), 637–652 (1986)
31. Gilbert, N.D.: Presentations of the automorphims group of a free product. Proc. Lond. Math. Soc. **54**, 115–140 (1987)
32. Giulini, D.: 3-manifolds for relativists. Int. J. Theor. Phys. **33**, 913–930 (1994)
33. Giulini, D.: On the configuration-space topology in general relativity. Helv. Phys. Acta **68**, 86–111 (1995)
34. Giulini, D.: Quantum mechanics on spaces with finite fundamental group. Helv. Phys. Acta **68**, 439–469 (1995)
35. Giulini, D.: What is the geometry of superspace? Phys. Rev. D **51**(10), 5630–5635 (1995)
36. Giulini, D.: The Group Of Large Diffeomorphisms In General Relativity Banach Center Publications **39**, 303–315 (1997)
37. Giulini D. (2007) Mapping-class groups of 3-manifolds in canonical quantum gravity. In: Fauser, B., Tolksdorf, J., Zeidler, E. (eds.) Quantum Gravity: Mathematical Models and Experimental Bounds. Birkhäuser Verlag, Basel, (2007). Online available at ⟨arxiv.org/pdf/gr-qc/0606066⟩.
38. Giulini, D., Kiefer, C.: Wheeler–DeWitt metric and the attractivity of gravity. Phys. Lett. A **193**(1), 21–24 (1994)
39. Hatcher, A.E.: Notes on basic 3-manifold topology. Online available at www.math.cornell.edu/~hatcher/3M/3Mdownloads.html
40. Hojman, S.A., Kuchař, K., Teitelboim, C.: New approach to general relativity. Nat. Phys. Sci. **245**, 97–98 (1973)
41. Hojman, S.A., Kuchař, K., Teitelboim, C.: Geometrodynamics regained. Ann. Phys. **96**, 88–135 (1976)

42. Immirzi, G.: Real and complex connections for canonical gravity. Classical Quantum Gravity **14**(10), L117–L181 (1997)
43. Isham, C.J.: Theta–states induced by the diffeomorphism group in canonically quantized gravity. In: Duff, J.J., Isham, C.J. (eds.) Quantum Structure of Space and Time. Proceedings of the Nuffield Workshop, August 3–21 1981, Imperial College London, pp. 37–52. Cambridge University Press, London (1982)
44. Isham, C.J., Kuchař, K.V.: Representations of spacetime diffeomorphisms. I. Canonical parametrized field theories. Ann. Phys. **164**, 288–315 (1985)
45. Isham, C.J, Kuchař, K.V.: Representations of spacetime diffeomorphisms. II. Canonical geometrodynamics. Ann. Phys. **164**, 316–333 (1985)
46. Jackiw, R.: Topological investigations of quantized gauge theories. In: DeWitt, B., Stora, R. (eds.) Relativity, Groups, and Topology II, Les Houches, Session XL, pp. 37–52. North-Holland, Amsterdam (1984)
47. Kazdan, J.L., Warner, F.W.: Scalar curvature and conformal deformation of Riemannian structure. J. Differ. Geom. **10**(1), 113–134 (1975)
48. Kiefer, C.: Quantum Gravity, International Series of Monographs on Physics, vol. 124, 2nd edn. Clarendon Press, Oxford (2007)
49. Klauder, J., Wheeler, J.A.: On the question of a neutrino analog to electric charge. Rev. Mod. Phys. **29**(3), 516–517 (1957)
50. Kneser, H.: Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten. Jahresberichte der Deutschen Mathematiker Vereinigung **38**, 248–260 (1929)
51. Kuchař, K.: Geometrodynamics regained: a Lagrangian approach. J. Math. Phys. **15**(6), 708–715 (1973)
52. Kuchař, K. (1993) Canonical quantum gravity. In: Gleiser, R.J., Kosameh, C.N., Moreschi, O.M. (eds.) General Relativity and Gravitation, pp. 119–150. IOP Publishing, Bristol (1993)
53. Lelong-Ferrand, J.: Transformation conformes et quasiconformes des variétés riemanniennes compacts (démonstration de la conjecture de A. Lichnerowicz). Mémoires de la Classe Des Sciences de l'Académie Royale Des Sciences, Des Lettres Et Des Beaux-Arts de Belgique **39**(5), 3–44 (1971)
54. Lovelock, D.: The four-dimensionality of space and the Einstein tensor. J. Math. Phys. **13**(6), 874–876 (1972)
55. McCarty, G.S.: Homeotopy groups. Trans. Am. Math. Soc. **106**, 293–303 (1963)
56. McCullough, D.: Topological and algebraic automorphisms of 3-manifolds. In: Piccinini, R. (ed.) Groups of Homotopy Equivalences and Related Topics. Springer Lecture Notes in Mathematics, vol. 1425, pp. 102–113. Springer, Berlin (1990)
57. McCullough, D., Miller, A.: Homeomorphisms of 3-manifolds with compressible boundary. Mem. Am. Math. Soc. **61**(344) (1986)
58. Milnor, J.W.: A unique decomposition theorem for 3-manifolds. Am. J. Math. **84**(1), 1–7 (1962)
59. Misner, C.W., Thorne, K.S., Wheeler, J.A.: Gravitation. W.H. Freeman and Company, New York (1973)
60. Moise, E.E.: Affine structures in 3-manifolds V The triangulation theorem and Hauptvermutung. Ann. Math **56**(1), 96–114 (1952)
61. Müllner, D. (2008) Orientation Reversal of Manifolds. PhD thesis, Friedrich-Wilhelms-Universität Bonn, October (2008)
62. Myers, S.B., Steenrod, N.E.: The group of isometries of a Riemannian manifold. Ann. Math. **40**(2), 400–416 (1939)
63. Nicolai, H.: On M-theory. J. Astrophys. Astronomy **20**(3–4), 149–164 (1999)
64. Pons, J.M.: Generally covariant theories: The Noether obstruction for realizing certain space-time diffeomorphisms inphase space. Classical Quantum Gravity **20**(15), 3279–3294 (2003)
65. Reidemeister, K.: Homotopieringe und Linsenräume. Abhandlungen Aus Dem Mathematischen Seminar der Universität Hamburg **11**(1), 102–109 (1935)
66. Samuel, J.: Canonical gravity, diffeomorphisms and objective histories. Classical Quantum Gravity **17**(22), 4645–4654 (2000)
67. Samuel, J.: Is Barbero's Hamiltonian formulation a gauge theory of Lorentzian gravity? Classical Quantum Gravity **17**(20), L141–L148 (2000)
68. Seifert, H., Threlfall, W.: A Textbook of Topology. Academic Press, Orlando, Florida, 1980. Translation of the 1934 german edition (1980)

69. Sorkin, R.: Introduction to topological geons. In: Bergmann, P.G., De Sabbata, V. (eds.), Topological Properties and Global Structure of Space–Time, NATO Advanced Study Institutes Series, vol. B138, p. 249. D. Reidel Publishing Company, Dordrecht-Holland (1986)
70. Sorkin, R.: A general relation between kink-exchange and kink-rotation. Commun. Math. Phys. **115**, 421–434 (1988)
71. Sorkin, R.: Classical topology and quantum phases: Quantum geons. In: De Filippo, S., Marinaro, M., Marmo, G., Vilasi, G. (eds.) Geometrical and Algebraic Aspects of Nonlinear Field Theory, pp. 201–218. Elsevier, Amsterdam (1989)
72. Sorkin, R., Surya, S.: An analysis of the representations of the mapping class group of a multi-geon three-manifold. Int. J. Mod. Phys. A **13**(21), 3749–3790 (1998)
73. Stern, M.D.: Investigations of the Topology of Superspace. PhD thesis, Department of Physics, Princeton University, 28 April (1967)
74. Teitelboim, C.: How commutators of constraints reflect the spacetime structure. Ann. Phys. **79**(2), 542–557 (1973)
75. Thiemann, T.: Modern Canonical Quantum General Relativity. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge (2007)
76. Thorne, K.S., Lee, D.L., Lightman, A.P.: Foundations for a theory of gravitation theories. Phys. Rev. D **7**(12), 563–3578 (1973)
77. Wheeler, J.A.: Phys. Rev. **97**(2), 511–536 (1955)
78. Wheeler, J.A.: Geometrodynamics. Academic Press, New York (1962)
79. Wheeler, J.A.: Einsteins Vision. Springer, Berlin (1968)
80. Wheeler, J.A.: Superspace and the nature of quantum geometrodynamics. In: DeWitt, C.M., Wheeler, J.A. (eds.) Battelle Rencontres, 1967 Lectures in Mathematics and Physics. pp. 242–307. W.A. Benjamin, New York (1968)
81. Wheeler, J.A.: Particles and geometry. In: Breitenlohner, P., Dürr, H.P. (eds.) Unified Theories of Elementary Particles, Lecture Notes in Physics, vol. 160, pp. 189–217. Springer, Berlin (1982)
82. Whitehead, J. H. C.: On incidence matrices, nuclei and homotopy types. Ann. Math. **42**(5), 1197–1239 (1941)
83. Witt, D.: Symmetry groups of state vectors in canonical quantum gravity. J. Math. Phys. **27**(2), 573–592 (1986)
84. Witt, D.: Vacuum space-times that admit no maximal slices. Phys. Rev. Lett. **57**(12), 1386–1389 (1986)