

# Quantum geometrodynamics: whence, whither?

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**Abstract** Quantum geometrodynamics is canonical quantum gravity with the three-metric as the configuration variable. Its central equation is the Wheeler–DeWitt equation. Here I give an overview of the status of this approach. The issues discussed include the problem of time, the relation to the covariant theory, the semiclassical approximation as well as applications to black holes and cosmology. I conclude that quantum geometrodynamics is still a viable approach and provides insights into both the conceptual and technical aspects of quantum gravity.

**Keywords** Quantum gravity · Quantum cosmology · Black holes

These considerations reveal that the concepts of spacetime and time itself are not primary but secondary ideas in the structure of physical theory. These concepts are valid in the classical approximation. However, they have neither meaning nor application under circumstances when quantum-geometrodynamical effects become important. ... There is no spacetime, there is no time, there is no before, there is no after. The question what happens “next” is without meaning [1].

## 1 Introduction

The quantization of the gravitational field is still among the most important open problems in theoretical physics. Despite many attempts, a final theory, which has to be both

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Dedicated to the memory of John Archibald Wheeler.

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mathematically consistent and experimentally tested, remains elusive. John Wheeler once wrote: “No question about quantum gravity is more difficult than the question, ‘What is the question?’” [2]. One of the questions is, of course, which approach to quantum gravity one is motivated to pursue.

My contribution here is devoted to one particular approach—quantum geometrodynamics. Being one of the oldest, it is still an active field of research. Quantum geometrodynamics is one version of canonical quantum gravity, to which also loop quantum gravity belongs. All canonical theories contain as their central equations *constraint equations*, that is, quantum versions of classical constraints between the generalized positions and momenta of the theory. In the case of gravity, these are the Hamiltonian and diffeomorphism constraints augmented, in the case of the loop approach, by the Gauss constraints. But the various canonical approaches are distinguished by their choice of canonical variables: three-metric and extrinsic curvature in geometrodynamics, holonomies and fluxes in the loop version. The non-trivial relationship between the various canonical variables leads to different, most probably inequivalent, quantum theories with different mathematical structures. Only the experiment can decide, at the end, which of them is the correct one, if any.

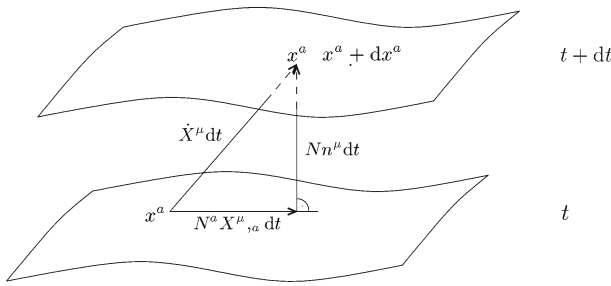
All the canonical theories are approaches which focus on the direct quantization of Einstein’s theory of general relativity. They thus do not necessarily entail a unification of gravity with the other interactions. Alternative approaches to a quantum theory of relativity are the covariant ones to which standard perturbation theory and path-integral quantization belong. Fundamentally different in spirit is string theory whose major aim *is* a unification of all interactions within one quantum framework. Quantum gravity as such emerges there only in an appropriate limit in which the various interactions becomes distinguishable. An introduction to all major approaches can be found in my monograph [3]. The reader can also find there a more complete list of references.

The purpose of this contribution is to provide a concise and critical review of the status of quantum geometrodynamics, its successes and shortcomings. I shall start in Sect. 2 with a brief introduction to the formalism of canonical gravity at both the classical and quantum level. I discuss in particular the problem of time and the relation of geometrodynamics to the covariant approaches. A brief historical overview is also included. Sect. 3 focuses on one of the successes: the relation of quantum geometrodynamics to quantum theory on a fixed background. This concerns in particular the recovery of the (functional) Schrödinger equation and its quantum gravitational corrections. Sects. 4 and 5 then give a brief overview of the main applications: quantum black holes and quantum cosmology. I shall end with some conclusions and an outlook.

## 2 What is quantum geometrodynamics?

### 2.1 The 3 + 1-decomposition

The usual starting point for developing the canonical formalism is the foliation of spacetime into three-dimensional spacelike hypersurfaces. A prerequisite for this is the



**Fig. 1** Two successive spacelike hypersurfaces in the 3+1-decomposition

global hyperbolicity of the spacetime. Figure 1 shows schematically two infinitesimally neighbored hypersurfaces. The vector  $\dot{X}^\mu dt$ , where

$$\dot{X}^\nu \equiv t^\nu = N n^\nu + N^a X^{\nu}_{,a}, \tag{1}$$

denotes the connection between points with the same spatial coordinates  $x^a$ . This connection can be decomposed into a normal and a tangential part. The amount of the normal separation is specified by the lapse function  $N$  (with  $n^\mu$  denoting a unit normal vector); the tangential separation is quantified by the components  $N^a$  of the shift vector. The four-dimensional line element between a point with coordinates  $x^a$  on the lower hypersurface to a point with coordinates  $x^a + dx^a$  on the upper hypersurface can then be decomposed as follows:

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ab} (dx^a + N^a dt)(dx^b + N^b dt) \\ &= (h_{ab} N^a N^b - N^2) dt^2 + 2h_{ab} N^a dx^b dt + h_{ab} dx^a dx^b, \end{aligned} \tag{2}$$

where  $h_{ab}$  denotes the components of the three-dimensional metric, in brief: the three-metric. In the canonical formalism, the three-metric will play the role of the configuration variable. To quote again John Wheeler: “The formalism of quantum gravity, in its best developed form, makes three-geometry a central concept” [2]. Instead of considering a three-metric on each hypersurface, we can imagine a given three-manifold  $\Sigma$  and a  $t$ -dependent three-metric on it. In fact, the canonical formalism depends on the chosen manifold  $\Sigma$ ; there is one canonical theory for each  $\Sigma$ .

This leads to a more fundamental viewpoint, cf. [4]. We can assume that in the beginning *only*  $\Sigma$  is given, not a spacetime. Only *after* solving the dynamical equations are we able to construct spacetime and *interpret* the time dependence of the metric  $h_{ab}$  on  $\Sigma$  as being brought about by ‘wafting’  $\Sigma$  through a four-manifold via a one-parameter family of embeddings.

The classical equations are six evolution equations for the  $h_{ab}$  and their momenta  $p^{ab}$  as well as four *constraints* for them. The momenta  $p^{ab}$  are linear combinations of the extrinsic curvature of the three-dimensional space. The six evolution equations and four constraints are the canonical version of the ten Einstein field equations. Only after the classical equations have been solved, can one interpret spacetime as a ‘trajectory of spaces’.

In the quantum theory, the trajectories will disappear as in ordinary quantum mechanics. There will thus be no spacetime at the most fundamental level; only the constraints for the three-dimensional space will remain. But before discussing them in the quantum theory, we shall have a brief look at their classical version.

## 2.2 Constraints

As mentioned in the last subsection, Einstein's equations can be written as a dynamical system of evolution equations together with constraints. The constraints are, at each space point, the *Hamiltonian constraint*  $H \approx 0$  and the three *momentum* or *diffeomorphism* constraints  $D^a \approx 0$ , where  $a = 1, 2, 3$ . The sign  $\approx$  denotes here the weak equality of Dirac, according to which the constraints can be used only after the evaluation of Poisson brackets. The explicit form of the constraints reads,

$$H[h_{ab}, p^{cd}] = 2\kappa G_{abcd} p^{ab} p^{cd} - (2\kappa)^{-1} \sqrt{h} ({}^3R - 2\Lambda) + \sqrt{h} \rho \approx 0, \quad (3)$$

$$D^a[h_{ab}, p^{cd}] = -2\nabla_b p^{ab} + \sqrt{h} j^a \approx 0, \quad (4)$$

where  $h$  is the determinant of the three-metric,  ${}^3R$  the three-dimensional Ricci scalar,  $\Lambda$  the cosmological constant,  $\rho$  ( $j^a$ ) denotes the energy density (current) of the non-gravitational fields and

$$\kappa = 8\pi G/c^4.$$

The coefficients  $G_{abcd}$  denote the "DeWitt metric" and are explicitly given by

$$G_{abcd} = \frac{1}{2\sqrt{h}} (h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd}). \quad (5)$$

The configuration space on which the constraints are defined is the space of all three-metrics and is called Riem  $\Sigma$ . The interpretation of the diffeomorphism constraints (4) is straightforward: they generate spatial coordinate transformations on  $\Sigma$ . What really counts is therefore the space of all three-geometries, which is obtained from Riem  $\Sigma$  after dividing out the diffeomorphisms. This space of all three-geometries has been baptized *superspace* by John Wheeler (it has nothing to do with supersymmetry) and is sometimes considered to be the real configuration space of canonical gravity. Its mathematical structure is highly non-trivial, see, for example, [4,3] and the references therein.

If the three-dimensional space  $\Sigma$  is compact without boundary, the full Hamiltonian is a sum of the above constraints. In the asymptotically flat case, it contains in addition boundary terms coming from the Poincaré charges at infinity, which include the ADM energy [5].

The Hamiltonian constraint can be mathematically interpreted as the generator of normal hypersurface deformations, that is, of deformations normal to the space-like hypersurfaces in the canonical formalism. Together with (4), it obeys the Poisson constraint algebra of all hypersurface deformations (normal and tangential) [3].

This symmetry is not equivalent to the four-dimensional symmetry of spacetime diffeomorphisms; however, the Hamiltonian formalism together with the hypersurface deformations is equivalent to the Lagrangian formalism with the spacetime diffeomorphisms. The constraint algebra closes, that is, the Poisson bracket between two constraints is proportional to a linear combination of the constraints. It is not a Lie algebra, though, because the Poisson bracket between two Hamiltonians (3) contains on the right-hand side explicit functions of the canonical variables.

There exists a subtle and intriguing connection between the constraints and the dynamical evolution [4,6]. Firstly, the constraints are preserved in time if and only if the energy–momentum tensor of matter has vanishing covariant divergence. This has an analogon in electrodynamics: the Gauss constraint is preserved in time if and only if the electric charge is conserved. Secondly, Einstein’s equations are the unique propagation law consistent with the constraint: if the constraints hold on every hypersurface, Einstein’s equations hold on spacetime; conversely, if the constraints are valid on a particular hypersurface and if Einstein’s equations hold on spacetime, the constraints hold on every hypersurface. This possesses, again, an analogon in electrodynamics: Maxwell’s equations are the unique propagation law consistent with the Gauss constraint. In a sense, the dynamical equations in general relativity follow entirely from the “laws of the instant”, that is, from the constraints [6].

### 2.3 Problem of time I

The fact that the laws of the instant suffice gives rise to the classical facet of the *problem of time*, cf. [7]. Let us restrict attention, for simplicity, to a compact three-space  $\Sigma$ . The total Hamiltonian is then a combination of the constraints only: the whole evolution is generated by the constraints. This shows again that the dynamical laws follow entirely from the constraints. No external time parameter exists, and all physical time variables, if needed, must be constructed from within the system, that is, as a functional of the canonical variables (Such physical time variables may come into play upon solving the constraints). A priori, there is no preferred choice of such an intrinsic time parameter. Still, in the classical theory a spacetime can be constructed after solving the field equations and can thus be described by a classical time function. This is no longer possible in the quantum theory where the spacetime itself (the “trajectory of spaces”) vanishes, giving rise to a more fundamental problem of time, see below.

The problem of time is connected with the problem of observables. The status of the latter is a subject of debate. The concept of observables was introduced by Peter Bergmann into the field of constrained dynamics to denote variables which have vanishing Poisson brackets with all of the constraints. Since constraints are believed to generate redundancy (“gauge”) transformations, these variables would be invariant under such transformations and would thus be candidates for physical variables. In fact, Bergmann coined the name observables in the hope that after quantization they would play the role of what is called observables in quantum theory.

This notion of observables may indeed be the appropriate one for gauge theories. It has, however, been disputed whether it is also the appropriate one for the situation

encountered here [6, 8]. A quantity having (weakly) vanishing<sup>1</sup> Poisson brackets with both the Hamiltonian and the diffeomorphism constraints (i.e. a quantity “commuting” with them) is a constant of motion because, at least in the spatially compact case, the full Hamiltonian is the sum of these constraints. This is another aspect of the problem of time—no time, no motion.

In order to avoid such a far-reaching conclusion, Kuchař has introduced the alternative concept of a *perennial* for a quantity that commutes with all constraints, that is, with both (3) and (4), and has instead reserved the notion *observable* for a quantity that commutes only with the diffeomorphism constraints (4) [6].

This makes sense. As Barbour and Foster have convincingly argued, it is misleading to think of the Hamiltonian constraint (3) as a generator of pure gauge transformations [8]. To support this claim they have focused on a particle model with a Hamiltonian constraint, where this constraint only generates reparametrizations of the curve parameter. They show that the presence of this constraint has to do with the fact that the initial condition for a geodesic in configuration space is a point and a direction at that point, not the absolute value of a velocity, and that the Hamiltonian does generate physical change. Extrapolating this insight to the gravitational situation, one would conclude that physical quantities are only required to commute with the diffeomorphism constraints (4), that is, that they do not need to be perennials. The Hamiltonian constraint yields a transformation from one configuration to a different one.

## 2.4 Quantization

Within the canonical formalism one can employ two approaches towards quantization. In the first one, one tries to solve the constraints (3) and (4) at the classical level in order to arrive at a formulation with unconstrained, “physical”, variables only. This is called reduced quantization. In practice, this approach is hardly feasible; it is even in quantum electrodynamics impossible to work with a reduced formulation—only in the non-interacting case can one identify the free transversal fields as the unconstrained variables.

One thus usually follows the second path, which is Dirac quantization [9]. In general, one would not expect this approach to be equivalent with reduced quantization, cf. [10]. However, using path-integral methods (cf. Sect. 2.6) one can show that at least in the one-loop (linear in  $\hbar$ -) approximation, reduced and Dirac quantization are equivalent if a particular factor ordering for the operators is chosen [11–13].

Let us focus on Dirac quantization. Poisson brackets of the canonical variables are translated into commutators, and the classical constraints are translated into restrictions on physically allowed wave functions. In Dirac’s words [9, p. 145]:

Weak equations between the classical variables correspond to linear conditions on the vectors  $\psi$ , according to the formula

$$X(q, p) = 0 \quad \text{corresponds to} \quad X\psi = 0.$$

<sup>1</sup> For the notion of a weakly vanishing quantity, see below.

(The weak equality sign  $\approx$  for the constraints was introduced later [14]). In our case it is the three-metric and its canonical momentum which in the quantum theory obey the canonical commutation relation. In the Schrödinger representation, the components of the momentum are substituted by  $\hbar/i$  times the functional derivative with respect to the metric,

$$\hat{p}^{ab} \longrightarrow \frac{\hbar}{i} \frac{\delta}{\delta h_{ab}}. \tag{6}$$

This is a formal heuristic prescription only, since one cannot expect the momentum to be represented by a self-adjoint operator, the reason being its non-commutation with the constraints.

In fact, the rule (6) does not implement one important property of the three-metric: the positivity property that demands  $\det h_{ab} > 0$ . It has thus been suggested to replace (6) by a modified prescription, leading to a variant of the canonical approach known as affine quantization [15]. The question as to which prescription is correct has to do with the problem of factor ordering.

With these formal rules, the classical Hamiltonian constraint (3) becomes the quantum Hamiltonian constraint, also known as the *Wheeler–DeWitt equation* [1, 16],

$$\hat{H}\Psi \equiv \left( -2\kappa\hbar^2 G_{abcd} \frac{\delta^2}{\delta h_{ab} \delta h_{cd}} - (2\kappa)^{-1} \sqrt{\hbar} ({}^{(3)}R - 2\Lambda) + \sqrt{\hbar} \hat{\rho} \right) \Psi = 0. \tag{7}$$

Similarly, the diffeomorphism constraints (4) are translated into their quantum version,

$$\hat{D}^a \Psi \equiv -2\nabla_b \frac{\hbar}{i} \frac{\delta \Psi}{\delta h_{ab}} + \sqrt{\hbar} \hat{j}^a \Psi = 0. \tag{8}$$

In these equations, a “naive” factor ordering has been chosen in the sense that all momenta are written to the right of the metric-dependent terms. The argument of the quantum geometrodynamical wave functional  $\Psi$  is the three-metric  $h_{ab}$  together with the non-gravitational degrees of freedom defined on  $\Sigma$  (in the simplest situations taken to be a scalar field). It is easy to see that (8) guarantees that the wave functional is independent under a spatial coordinate transformation which is connected with the identity (It can acquire a phase under a so-called large diffeomorphism). A similar feature is the quantized Gauss constraint in electrodynamics and Yang–Mills theories, which guarantees the invariance of the wave functional under infinitesimal gauge transformations.

As they are written down, the Eqs. (7) and (8) are of a formal nature only, that is, they require a precise mathematical formulation. Such a formulation is not yet available, except within a one-loop approximation scheme, cf. Sect. 2.6. Using a different set of canonical variables, one arrives at alternatives, most likely inequivalent, versions of canonical quantum gravity. One of them is loop quantum gravity, which has its own advantages and shortcomings, see [17] as well as other contributions to this volume.

The mathematical problems of quantum geometrodynamics have to do with factor ordering, regularization, and Dirac consistency, which are themselves intertwined

problems. The latter refers to the quantum version of the classical constraint algebra, for which it is not clear that it closes on the constraints in the way the classical algebra does; the algebra may contain additional ‘anomalous’ terms. If it does not close, the Eqs. (7) and (8) will not be consistent because the anomaly would yield a non-vanishing term. This is what happens, in fact, for the quantum Virasoro algebra in string theory, where it is of the utmost importance. It is not at all obvious that such an anomaly is absent for geometrodynamics. This question can, of course, only be consistently addressed after the constraints have been regularized. It must be emphasized that many of these problems are not peculiar to quantum geometrodynamics, but occur in other approaches as well, in which general relativity is directly being quantized.

The main purpose of the Eqs. (7) and (8) is then twofold: on the one hand, it can give intuitive insight by formal manipulations of the equations. On the other hand, they may be truncated into well-defined equations in the context of particular models, notably in quantum cosmology. We shall encounter applications of both kinds below. In most of the formal applications as well as in the concrete models, the subtle features connected with the choice of factor ordering and possible anomalies is less relevant. Situations where they *are* definitely of relevance include discussions of the singularity avoidance in quantum gravity.

## 2.5 Problem of time II

In the quantum theory, the problem of time becomes more pressing. Not only the external time, but also spacetime as such has disappeared! This conclusion is unavoidable as long as one sticks to the usual quantum formalism (as we do here). In quantum mechanics, particle trajectories are absent. In quantum gravity, spacetime is the entity that is analogous to a particle trajectory; consequently, it is absent at the most fundamental level. In the canonical formalism discussed so far, space (in the form of the three-dimensional manifold) still exists. This is also the case in loop quantum gravity, although there geometric operators such as the area or volume operator can assume under certain conditions discrete spectra. It is, however, imaginable that space as such may vanish in the final theory of quantum gravity.

In spite of the absence of spacetime, the structure of the Wheeler–DeWitt equation (7) suggests the introduction of a novel concept: *intrinsic time*, which can be defined by the local hyperbolic structure of this equation. In contrast to the Schrödinger equation, its kinetic term has the same form as in a wave equation. The kinetic term thus distinguishes a time-like variable by the presence of different signs. One can show that the time-like sign occurs for the local size (as given by the square root of the determinant of the three-metric,  $\sqrt{h}$ ); in cosmological examples, it is usually the volume of the universe that plays the role of intrinsic time, see below.

This formal structure of the Wheeler–DeWitt equation with its concept of an intrinsic time has important consequences for the imposition of boundary data [18]. For a wave equation one usually specifies the function and its derivative at hypersurfaces of constant time (here: intrinsic time). We shall encounter some important consequences of this fact when discussing quantum cosmology below.



A problem related to the problem of time is the “Hilbert-space problem” [3]. The standard (“Schrödinger”) inner product in quantum mechanics is conserved *in* time  $t$ , reflecting the conservation of probability. But do we need such a product in the absence of an external time? After all, the concepts of probability and measurement are not obvious ones in a timeless world. Motivated by the wave-equation-like structure of the Wheeler–DeWitt equation, one might instead consider a “Klein–Gordon inner product” because such an inner product is conserved with respect to (intrinsic) time. However, it possesses the usual problem of such an inner product, which is the occurrence of negative probabilities. This would then perhaps lead to the need of a “third quantization” in which the wave functional itself would become an operator, similar to the necessary transition from relativistic quantum mechanics to quantum field theory. This would open a Pandora’s box of possibilities which with the current limited status of understanding should be avoided. It must be emphasized, however, that at least at a formal level (not discussing potential anomalies) and in the one-loop approximation, the various inner products lead to an equivalent formalism if a certain factor ordering is chosen [11, 12].

Most of the work in quantum geometrodynamics thus leaves the question of the inner product open and focuses on topics which are thought to be independent of it. This is different, for example, in loop quantum gravity where a consistent (Schrödinger-type) inner product exists at least at the kinematical level, that is, before the constraints are imposed. A necessary requirement is, of course, the recovery of standard quantum field theory with its standard Hilbert-space structure in an approximate limit. This is met successfully, see Sect. 3.

## 2.6 Relation to covariant quantum gravity

Quantum geometrodynamics aims to arrive at a quantum theory of gravity by a direct quantization of Einstein’s theory of general relativity. There are, however, alternative methods to achieve this goal. The oldest is perturbation theory around a fixed (usually flat) background. Another approach, which is intrinsically non-perturbative, is path-integral quantization. Such approaches are called *covariant* because they employ a notion of spacetime covariance as an important ingredient in the formalism (even if at the end there is no spacetime). In the case of the perturbation theory, “covariant” refers only to the isometry group of the chosen background, for example, Minkowski space. On the other hand, path-integral quantization is fully covariant in the sense of the diffeomorphism group. The same holds, despite its name, for DeWitt’s “background-field method” [3].

The question then arises whether there is any connection between the canonical and covariant approaches [3]. This question also occurs in standard quantum field theory, but becomes more pressing in quantum gravity because of the absence of spacetime in the canonical theory. The connection between both approaches is therefore best understood in the light of the path integral in which one integrates over the spacetime metric, in analogy to the integration over the formal particle paths in quantum mechanics.

The quantum gravitational path integral is formally given by the following expression,

$$Z = \int \mathcal{D}g \mathcal{D}\phi e^{iS[g,\phi]/\hbar}, \quad (9)$$

where the integration over  $\mathcal{D}g$  includes an integration over the three-metric as well as lapse function  $N$  and shift vector  $N^a$ , and where a matter field denoted by  $\phi$  has been taken into account. The non-trivial (and not yet fully solved) issue is, of course, the precise definition of the measure. Other contributions to this volume deal with this question.

At the formal level, one can find from the demand that  $Z$  be independent of  $N$  and  $N^a$  at the three-dimensional boundaries the result that the path integral must satisfy the constraints (7) and (8) [19]. In this sense one can disclose a connection between the covariant (path integral) and the canonical approaches. Of course, to put these formal derivations on a rigorous footing is far from trivial. Most of the work at the rigorous level has thus focused on the one-loop approximation of the path integral. The corresponding results have been derived by Andrei Barvinsky in a series of papers, see [13, 20, 21] and the references therein. They describe the state of our knowledge about the connection between the path-integral and the canonical approach.

## 2.7 A brief history of quantum geometrodynamics

The term *quantum geometrodynamics* was already used by John Wheeler to denote quite generally a quantum version of Einstein's theory, cf. [22]. Here, we shall use this term exclusively for the canonical version of quantum gravity based on the three-metric and its canonical momentum. The concept should also not be confused with the name "quantum geometry" which is used synonymously for loop quantum gravity [23].

The first traces of the canonical formalism can be found in an early paper by Klein [24], where he discovered that the first four Einstein equations are "Hamiltonian" and "momentum density" equations. A general concept for constraints was put forward by Rosenfeld [25]. He found that the first four Einstein equations are constraints in this general sense. He also discussed the issue of the consistency conditions in the quantum theory, that is, that the commutator between the constraints must close on a constraint. Following the corresponding discussion by Dirac in [9], this requirement is known as Dirac consistency.

A general formalism for constrained systems was developed by Dirac in his papers [9] and [14]. In [26] he applied it to the gravitational field and essentially derived, in fact, the Eqs. (3) and (4). He also discussed the reduced-quantization approach. Important contributions to canonical gravity came in addition from Peter Bergmann's group (see e.g. his short review in [27]) and from Arnowitt et al. (summarized in [5]). The latter gave, in particular, a rigorous definition of gravitational energy and a representation of the local gravitational degrees of freedom by canonical methods. As has been mentioned above, the notion of an observable in this context is due to Bergmann. Moreover, in 1966 he noted that the wave functional in canonical quantum gravity (in fact, in general constrained systems of this kind) is timeless [28]. To quote him: "To this extent the Heisenberg and Schrödinger pictures are indistinguishable in any

theory whose Hamiltonian is a constraint.” He did not, however, discuss the explicit form of the quantum constraints (7) and (8).

This was then achieved in the already mentioned papers by Wheeler and DeWitt [1, 16]. While the general formalism was discussed extensively in [16], conceptual issues form the main part of [1]. In fact, the Wheeler–DeWitt equation (as it was, of course, only later called) can be found in [1] only in an appendix and in a shorthand notation. However, in his pioneering paper [16] DeWitt acknowledges John Wheeler’s important influence: “The present paper is the direct outcome of conversations with Wheeler, during which one fundamental question in particular kept recurring: *What is the structure of the domain manifold for the quantum-gravitational state functional?*” (see [16, p. 1115]). In fact, much space in [16] is devoted to the configuration space, the inner product (for which he suggested to use the Klein–Gordon inner product), but also to the semiclassical limit and, for the first time, to quantum cosmology. He suggests a first criterion of singularity avoidance in demanding that the wave function vanish in the region of a classical singularity. DeWitt also addresses the problem of the interpretation of quantum theory in the light of cosmology, which motivates him to adopt the Everett interpretation.

This concludes the early history of quantum geometrodynamics. From 1968 on, the work in this field concentrates on the general issues and models which are the topic of my contribution. It is somewhat surprising that Dirac, who contributed so much to the early development of the field, seems to have lost interest. In a contribution to a conference which took place in Trieste in 1968 he gave a talk entitled “The quantization of the gravitational field” [29]. In it he mentions only his own work and a paper by Schwinger and focuses attention to the open problem of the constraint algebra, concluding that “the problem of the quantization of the gravitational field is thus left in a rather uncertain state” ([29, p. 543]). This is perhaps due to his instrumentalist attitude towards physics (in addition to his emphasis on mathematical beauty) which forbade him to continue with a physical investigation before these consistency conditions were solved. Even in such a small field as geometrodynamics, the tastes of the contributors are highly diverse. It should, however, be remarked that at least at a formal level (without addressing the question of regularization), the factor ordering can be fixed by the requirement that different quantization approaches be equivalent [11, 12].

### 3 The bridge to quantum theory on a fixed background

#### 3.1 Hamilton–Jacobi equation

The fundamental quantum equations (7) and (8) are usually derived from a three-plus-one-decomposition of the classical spacetime and the imposition of heuristic quantization rules. One may, however, arrive at those equations from a different conceptual direction, which is analogous to Schrödinger’s original derivation of his famous wave equation. Let us quote Schrödinger himself:

*...we know today, in fact, that our classical mechanics fails for very small dimensions of the path and for very great curvatures. Perhaps this failure is*

in strict analogy with the failure of geometrical optics ...that becomes evident as soon as the obstacles or apertures are no longer great compared with the real, finite, wavelength. ...Then it becomes a question of searching for an ‘undulatory mechanics’—and the most obvious way is by an elaboration of the Hamiltonian analogy on the lines of undulatory optics.<sup>2</sup>

The essential idea here is to “guess” a wave equation that yields the Hamilton–Jacobi equation of classical mechanics in an appropriate limit. We can try the same for general relativity: “guess” a wave equation that gives in the classical limit Einstein’s equations in their Hamilton–Jacobi version. But what is the Hamilton–Jacobi version of these equations? Asher Peres derived it in 1962 [31]: instead of the ten Einstein field equations, which are partial differential equations, one gets the following four functional differential equations, which are nothing but the four constraint Eqs. (3) and (4) in the Hamilton–Jacobi form,

$$16\pi G G_{abcd} \frac{\delta S}{\delta h_{ab}} \frac{\delta S}{\delta h_{cd}} - \frac{\sqrt{h}}{16\pi G} ({}^{(3)}R - 2\Lambda) = 0, \quad (10)$$

$$D_a \frac{\delta S}{\delta h_{ab}} = 0.$$

(Restriction has here been made to the vacuum case). The eikonal  $S$  is a functional of the three-metric,  $S[h_{ab}(\mathbf{x})]$ . Using the principle of constructive interference, Ulrich Gerlach has shown in 1969 that the Eq. (10) are indeed fully equivalent to all ten Einstein field equations [32]; this approach to Einstein’s theory is one of the six routes to geometrodynamics presented in [33].

If one now looks for wave equations for a wave functional  $\Psi[h_{ab}(\mathbf{x})]$  which lead to (10) in the semiclassical limit, that is, when  $\Psi$  is of the WKB form

$$\Psi[h_{ab}] = C[h_{ab}] \exp\left(\frac{i}{\hbar} S[h_{ab}]\right), \quad (11)$$

with a slowly varying amplitude  $C$  and a rapidly varying phase  $S$ , one arrives at the quantum constraint Eqs. (7) and (8).

Independent of their status at the most fundamental level, therefore, one can argue that the Eqs. (7) and (8) should at least be valid approximately for energies below the Planck scale. This conclusion is based only on two rather conservative assumptions: the universality of the quantum framework (that is, the universal validity of the superposition principle) and the validity of Einstein’s equation in the classical limit. Both of these assumptions enjoy strong support: general relativity has passed all experimental and observational tests so far, and the same is true for quantum theory

<sup>2</sup> *wir wissen doch heute, daß unsere klassische Mechanik bei sehr kleinen Bahndimensionen und sehr starken Bahnkrümmungen versagt. Vielleicht ist dieses Versagen eine volle Analogie zum Versagen der geometrischen Optik ..., das bekanntlich eintritt, sobald die ‘Hindernisse’ oder ‘Öffnungen’ nicht mehr groß sind gegen die wirkliche, endliche Wellenlänge. ...Dann gilt es, eine ‘undulatorische Mechanik’ zu suchen—und der nächstliegende Weg dazu ist wohl die wellentheoretische Ausgestaltung des Hamiltonschen Bildes [30].*

where interference experiments can be extended far into the mesoscopic regime and where the emergence of classical behaviour is understood as arising from decoherence [34,35].

### 3.2 Semiclassical approximation

The discussion in the last subsection suggests that the semiclassical limit from quantum geometrodynamics is well understood at least at the level of the formal constraint Eqs.(7) and (8). This is indeed the case [3]. One can derive the limit of quantum field theory in an external spacetime through a kind of Born–Oppenheimer approximation scheme. This idea was first spelled out by Lapchinsky and Rubakov [36].

Starting point is the following ansatz for a general solution of (7) and (8):

$$|\Psi[h_{ab}]\rangle = C[h_{ab}]e^{im_P^2 S[h_{ab}]}|\psi[h_{ab}]\rangle, \tag{12}$$

where the bra-ket notation of the wave functional refers to the standard Hilbert space of non-gravitational degrees of freedom and where  $m_P$  is the Planck mass. Inserting this into (7) and (8) and performing an expansion with respect to the Planck mass, one finds in the highest-order approximations that  $S$  obeys (10) and that  $\psi[h_{ab}]\rangle$  obeys

$$\begin{aligned} \left( \hat{\mathcal{H}}_{\perp}^m - \langle \psi | \hat{\mathcal{H}}_{\perp}^m | \psi \rangle - iG_{abcd} \frac{\delta S}{\delta h_{ab}} \frac{\delta}{\delta h_{cd}} \right) |\psi[h_{ab}]\rangle &= 0, \\ \left( \hat{\mathcal{H}}_a^m - \langle \psi | \hat{\mathcal{H}}_a^m | \psi \rangle - \frac{2}{i} h_{ab} D_c \frac{\delta}{\delta h_{bc}} \right) |\psi[h_{ab}]\rangle &= 0. \end{aligned} \tag{13}$$

One now evaluates  $|\psi[h_{ab}]\rangle$  along a solution of the classical Einstein equations,  $h_{ab}(\mathbf{x}, t)$ , corresponding to a solution,  $S[h_{ab}]$ , of the Hamilton–Jacobi equations (10); this solution is obtained from

$$\dot{h}_{ab} = NG_{abcd} \frac{\delta S}{\delta h_{cd}} + 2D_{(a}N_{b)}, \tag{14}$$

which is the analogue in relativity of the equation  $\dot{q} = m^{-1} \partial S / \partial q$  in classical mechanics. Defining a time parameter  $t$  by

$$\frac{\partial}{\partial t} |\psi(t)\rangle = \int d^3x \dot{h}_{ab}(\mathbf{x}, t) \frac{\delta}{\delta h_{ab}(\mathbf{x})} |\psi[h_{ab}]\rangle,$$

one can derive from (13) the following functional Schrödinger equation for the quantized non-gravitational fields in the chosen external classical gravitational field:

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle &= \hat{H}^m |\psi(t)\rangle, \\ \hat{H}^m &\equiv \int d^3x \left\{ N(\mathbf{x}) \hat{\mathcal{H}}_{\perp}^m(\mathbf{x}) + N^a(\mathbf{x}) \hat{\mathcal{H}}_a^m(\mathbf{x}) \right\}, \end{aligned} \tag{15}$$

where  $\hat{H}^m$  is the Hamiltonian for the non-gravitational fields in the Schrödinger picture, which depends parametrically on the (generally non-static) metric coefficients of the curved spacetime background. In this level of approximation, the “WKB time  $t$ ” controls the dynamics—time has been regained from timeless quantum gravity in an appropriate limit.

Together with the parameter  $t$ , the imaginary unit  $i$  has appeared in (15). This entails then the use of the complex wave functions in quantum theory, which are so essential for its formalism. But has this not been introduced by hand through the special ansatz (12)? In a certain sense, yes. However, one can show that superpositions of such complex wave functions become dynamically independent from each other through decoherence [34].

Consider, for example, a superposition of a state of the form (12) with its complex conjugate. Taking into account inhomogeneous degrees of freedom such as density fluctuations or weak gravitational waves, one can show that the resulting entangled state exhibits only a tiny interference factor between the  $\exp(iS/\hbar)$ - and the  $\exp(-iS/\hbar)$ -component of the total quantum state after the inhomogeneous degrees of freedom have been traced out. This is the effect of decoherence. In one example which I calculated some time ago, the decoherence factor responsible for this suppression of interference turned out to read [37]

$$\exp\left(-\frac{\pi m H_0^2 a^3}{128\hbar}\right) \sim \exp(-10^{43}),$$

where  $a$  is the scale factor of a Friedmann universe (see below),  $H_0$  the Hubble constant, and  $m$  the mass of a scalar field used in this model. The numerical value arises after some standard values for the parameters are inserted. The smallness of this number means that our present Universe can be treated as behaving classically to a high degree of accuracy.

One can interpret this result also as follows. The full quantum equations (7) and (8) are real equations and are therefore invariant under complex conjugation. The state (12), on the other hand, is complex, violating this symmetry. Since the time parameter  $t$  only follows from such a complex state (which can be interpreted as a decohered branch of a full real state), one can say that time itself emerges from symmetry breaking.

The situation is analogous to molecular physics where the chiral behaviour of molecules (e.g. sugar molecules) can emerge through a similar symmetry-breaking effect: while the fundamental equation (the Schrödinger equation with the Hamilton operator for the molecules) is parity-invariant, the chiral states are not. The dynamical reason for this symmetry breaking is again the process of decoherence, there caused by the scattering with light or air molecules.

### 3.3 Quantum gravitational corrections

If the functional Schrödinger equation can be recovered from full quantum gravity in an appropriate limit, the question arises whether one can go beyond this limit

and calculate quantum gravitational correction terms. This can be done at least at a formal level, that is, at the level where one treats the functional derivatives like partial derivatives.

The next order in the Born–Oppenheimer approximation then gives corrections to the Hamiltonian for the non-gravitational fields,

$$\hat{H}^m \rightarrow \hat{H}^m + \frac{1}{m_p^2} \text{ (various terms)}. \tag{16}$$

The detailed form of these terms can be found in [3,38,39]. Future investigations should deal with a concrete application of these terms in cosmology, for example, in the search for quantum gravitational effects in the anisotropy spectrum of the Cosmic Background Radiation.

A simple example is the calculation of the quantum gravitational correction to the trace anomaly in de Sitter space [40]. For a conformally coupled scalar field, the trace of the energy–momentum tensor, although being zero classically, is non-vanishing in the quantum theory; this “anomalous trace” is proportional to  $\hbar$ . It corresponds to the following expectation value,  $\epsilon$ , of the Hamiltonian density,

$$\epsilon = \frac{\hbar H_{\text{dS}}^4}{1440\pi^2 c^3}, \tag{17}$$

where  $H_{\text{dS}}$  is the constant Hubble parameter of de Sitter space. The first quantum gravitational correction calculated from the Born–Oppenheimer expansion discussed above reads

$$\delta\epsilon \approx -\frac{2G\hbar^2 H_{\text{dS}}^6}{3(1440)^2\pi^3 c^8}, \tag{18}$$

so that the ratio is given by

$$\frac{\delta\epsilon}{\epsilon} \approx -\frac{1}{2160\pi} \left( \frac{t_p}{H_{\text{dS}}^{-1}} \right)^2, \tag{19}$$

where  $t_p$  denotes the Planck time. One might perhaps have guessed for dimensional reasons that the ratio of the Planck time to the Hubble time enters, but this example shows that in principle exact results can be obtained from canonical quantum gravity. Numerically, the ratio (19) is, of course, small. Using values motivated by inflationary cosmology, one can assume that  $H_{\text{dS}}$  lies between  $10^{13}$  and  $10^{15}$  GeV, leading for the ratio (19) to values between roughly  $10^{-16}$  and  $10^{-22}$ . It is at present an open question whether there are relevant cases where the correction terms can be big enough to be observable. I finally note that the quantum geometrodynamical formalism and its semiclassical approximation can also be performed for supergravity, cf. [41] and the references therein.

## 4 Quantum black holes and quantum cosmology

### 4.1 Quantum black holes

According to the no-hair theorem of general relativity, stationary black holes are uniquely characterized by the three parameters mass, angular momentum, and electric charge. If all parameters are non-vanishing, the solution is given by the Kerr–Newman metric, which is axially symmetric. Most investigations into the quantum aspects have focused on the simple situation of vanishing angular momentum, because then the solutions are spherically symmetric. Still, the difficulties in performing the quantization are formidable.

The simplest case is the eternal Schwarzschild black hole without matter degrees of freedom. Such a black hole is fully characterized by its mass,  $M$ . Through a series of sophisticated transformations, Karel Kuchař was able to reduce the problem to a purely quantum mechanical one and give an explicit form of the resulting wave function [42]. If one extends this solution to include an electric charge  $q$ , the wave function reads (see e.g. [3, 43])

$$\Psi(\alpha, \tau, \lambda) = \chi(M, q) \exp \left[ \frac{i}{\hbar} \left( \frac{A(M, q)\alpha}{8\pi G} - M\tau - q\lambda \right) \right], \quad (20)$$

where  $\chi(M, q)$  is an arbitrary function of  $M$  and  $q$ ,  $A(M, q)$  is the area of the horizon as expressed through mass and charge,  $\lambda$  is a parameter conjugated to charge,  $\alpha$  a ‘rapidity parameter’ connected with the bifurcation sphere of the black-hole horizons in the Kruskal diagramme, and  $\tau$  denotes the Schwarzschild (Killing) time at asymptotic infinity. In contrast to the general case discussed above, such a time variable is available in the asymptotic regime of an asymptotically flat situation, that is, far away from the black hole. If additional matter is present, such a reduction to finitely many degrees of freedom is no longer possible and one has to deal with the full functional equations.

It is possible to discuss a quantum state for the black hole in a one-loop approximation. Choosing such a state in accordance with the no-boundary state in quantum cosmology (see below), Barvinsky et al. have calculated the entanglement entropy arising from this state when all the degrees of freedom outside the horizon are traced out [44]. They found for the entropy the expression

$$S = -k_B \text{Tr}(\rho \ln \rho) = k_B \frac{A}{360\pi l^2}, \quad (21)$$

where  $\rho$  is the density matrix resulting from tracing out the exterior degrees of freedom, and  $l$  is a cutoff parameter denoting the proper distance to the horizon. One recognizes that this expression is divergent for  $l \rightarrow 0$ . This calculation is therefore not yet a complete one; on the other hand, it yields the expected proportionality between black-hole entropy and area.

In the attempt to recover the Bekenstein–Hawking entropy  $S_{\text{BH}}$  from an entanglement entropy, one has to keep in mind the universality of  $S_{\text{BH}}$ , that is, its independence



from the actual field content. What could give such a universality? One universal feature of a black hole is the spectrum of its quasi-normal modes, which are damped out when reaching the stationary black-hole state, but which could still play a role in the quantum theory. They stay entangled with the black hole and tracing them out could perhaps give  $S_{\text{BH}}$  [45]. However, any serious calculation is elusive.

Instead of an eternal black hole one can attempt to describe a black hole that results dynamically from a gravitational collapse. One example is a collapsing spherically symmetric dust shell. Classically, it collapses to form a black hole. In the quantum theory, interesting features can happen [46]. If the shell is described by a narrow wave packet, it turns out that this packet will first collapse, enter slightly inside the classical event horizon and then re-expand to infinity. In a sense, the quantum theory yields a superposition of a black-hole with a white-hole solution, resulting in a destructive interference of the total wave packet in the region of the classical singularity: for  $r \rightarrow 0$ , the wave function obeys  $\Psi \rightarrow 0$ . This is a consequence of constructing a unitary (with respect to asymptotic time) canonical quantum theory.

Instead of a dust shell, one can consider a spherically symmetric dust cloud—the Lemaître–Tolman–Bondi (LTB) model. Classically, this is a self-gravitating dust cloud with energy–momentum tensor  $T_{\mu\nu} = \epsilon(\tau, \rho)u_\mu u_\nu$  and is given by the line element

$$ds^2 = -d\tau^2 + \frac{(\partial_\rho R)^2}{1 + 2E(\rho)}d\rho^2 + R^2(\rho)(d\theta^2 + \sin^2\theta d\phi^2). \quad (22)$$

The canonical formalism and its quantization were developed by Vaz et al. in [47]. After some manipulations both the Wheeler–DeWitt equation and the diffeomorphism constraint (in the case of spherical symmetry there is only one such constraint) were presented in a simplified, but still functional, form.

In a series of paper, the following results were obtained (see [48] and the references therein). Firstly, exact quantum states of a particular type were found. This is possible because the dust shell can be imagined as being composed of infinitely many decoupled shells. The exact quantum states, which can be found only in a special factor ordering, can be interpreted as an infinite product of single-shell states. Although being exact solutions, they are of a WKB form. Secondly, it was possible to retrieve from these quantum gravitational states the standard expressions for the Hawking radiation *plus* explicit corrections due to greybody factors. For the BTZ black hole, which is a solution in  $2 + 1$  dimensions with negative cosmological constant  $\Lambda$ , it was possible to derive the Hawking temperature and to give a microscopic derivation of the black-hole entropy. In fact, it was found in this  $2 + 1$ -dimensional case that there is a discrete mass spectrum for the shells collapsing to the black hole.

Following early suggestions by Jacob Bekenstein,<sup>3</sup> the black-hole entropy is there defined as the number of possible distributions of  $\mathcal{N}$  identical shells between these

<sup>3</sup> “It is then natural to introduce the concept of black-hole entropy as the measure of the *inaccessibility* of information (to an exterior observer) as to which particular internal configuration of the black hole is actually realized in a given case” [49].

levels. The result is

$$S_{\text{can}} \approx 2\pi k_{\text{B}} \sqrt{\left(1 - \frac{48lM_0}{\hbar}\right) \frac{lM}{6\hbar}}, \quad (23)$$

where  $l \equiv |\Lambda|^{-1/2}$ ,  $M$  is the mass of the BTZ black hole, and  $M_0$  is a free constant of the model. This entropy is equal to the Bekenstein–Hawking entropy if this constant is chosen as follows:

$$M_0 = -\frac{1}{16G} + \frac{\hbar}{48l}. \quad (24)$$

Actually,  $M_0$  can be related with the conformal charge of the effective conformal-field theory usually used to derive the entropy for the BTZ black hole, cf. [50]. All of these results are, of course, preliminary, but they demonstrate to which extent quantum geometrodynamics can be applied in the understanding of black holes.

## 4.2 Quantum cosmology

Quantum cosmology is one of the main applications of quantum geometrodynamics. Its purpose is twofold: On the one hand, it can serve as a toy model for full quantum gravity in which the mathematical difficulties disappear. On the other hand, it can be employed as a description for the real Universe, with the final goal to be tested by observation.

In this subsection, I shall focus on some recent work into which I was myself involved. More detailed overviews of quantum cosmology can be found, for example, in [3, 51–54].

The simplest model of quantum cosmology is the quantization of a Friedmann–Lemaître universe. The classical line element is taken to be of the form

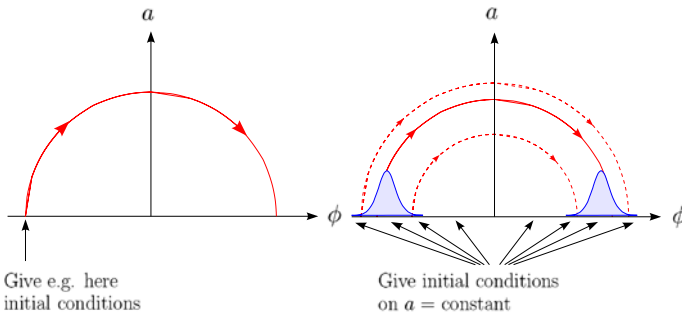
$$ds^2 = -N^2(t)dt^2 + a^2(t)d\Omega_3^2, \quad (25)$$

where  $N$  is the lapse function,  $a$  the scale factor, and we have chosen the three-dimensional space to be closed. In addition, we shall implement a homogeneous matter field  $\phi$  as a representative for matter. We are thus left with a two-dimensional configuration space (consisting of  $a$  and  $\phi$ ); because of the huge truncation of the infinite-dimensional superspace, such a space is called minisuperspace.

The diffeomorphism constraints are identically satisfied by this ansatz, and the Wheeler–DeWitt equation reads (with units  $2G/3\pi = 1$  and  $c = 1$ )

$$\frac{1}{2} \left( \frac{\hbar^2}{a^2} \frac{\partial}{\partial a} \left( a \frac{\partial}{\partial a} \right) - \frac{\hbar^2}{a^3} \frac{\partial^2}{\partial \phi^2} - a + \frac{\Lambda a^3}{3} + m^2 a^3 \phi^2 \right) \psi(a, \phi) = 0. \quad (26)$$

The factor ordering has been chosen to be of the Laplace–Beltrami form, which has the advantage that it guarantees covariance in minisuperspace.



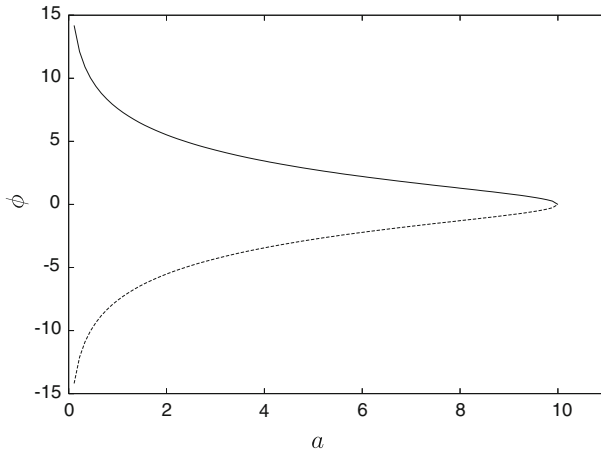
**Fig. 2** The classical and the quantum theory of gravity exhibit drastically different notions of determinism [3]

It is evident that equations such as (26) do not possess the mathematical problems of the full functional equation (7). One can thus focus attention on physical applications. One important application is the imposition of boundary conditions. Popular proposals are the no-boundary condition [19] and the tunneling condition [55]. The no-boundary proposal makes essential use of the connection between covariant and canonical quantum gravity discussed in Sect. 2.6: it is defined conceptually by a Euclidean path integral, but also relies on solving a minisuperspace Wheeler–DeWitt equation such as (26). Other important applications include the discussion of wave packets, the validity of the semiclassical approximation, the origin of classical behaviour and the arrow of time, and the possible quantum avoidance of classical singularities [3, 18].

Before picking out one particular model, I want to emphasize one important conceptual point which is relevant for the problem of time discussed above, see Fig. 2.

Consider a two-dimensional minisuperspace model with the variables  $a$  and  $\phi$  as above. The figure on the left shows the classical trajectory in configuration space for a universe which is expanding and recollapsing. Classically, one can give initial conditions, for example, on the left end of the trajectory for small  $a$  and then determine the whole trajectory. In this sense, the recollapsing part of the trajectory is the deterministic successor of the expanding part. One could, of course, also start from the right end of the trajectory because there is no distinguished direction; but the important point is that a trajectory exists. Not so in the quantum theory where both the trajectory and the time parameter  $t$  are absent! If one wants to find a solution of the Wheeler–DeWitt equation which describes a wave packet following the classical trajectory, one has to specify two packets at the would-be ends of the classical trajectory, see the right figure. The reason is that (26) is a hyperbolic equation with respect to intrinsic time  $a$ , and the natural formulation of boundary conditions is to impose the wave function (and its derivative) at constant  $a$ . If one imposed only one of the two wave packets, the full solution would be a smeared-out wave function which does not resemble anything like a wave packet following the classical trajectory. In this sense, the “recollapsing” wave packet must be present “initially”.

Quantum geometrodynamics thus provides us with crucial insights into the nature of time in quantum gravity. And the consequences of this new concept of time are independent of any particular scale, that is, independent of possible modifications of the theory at the Planck scale.



**Fig. 3** Classical trajectory in configuration space [56]

Let us now turn to a specific example [56]: a cosmological model with a “big brake”. Classically, the model is characterized by an equation of state of the form  $p = A/\rho$ , where  $A > 0$  (“anti-Chaplygin gas”). This can be realized by a scalar field  $\phi$  with the following potential (with  $\kappa^2 = 8\pi G$ ):

$$V(\phi) = V_0 \left( \sinh(\sqrt{3\kappa^2}|\phi|) - \frac{1}{\sinh(\sqrt{3\kappa^2}|\phi|)} \right); \quad V_0 = \sqrt{A/4}. \quad (27)$$

This model universe develops a pressure singularity at the end of its evolution where it comes to an abrupt halt:  $\dot{a}$  remains finite there but  $\ddot{a}(t)$  tends to minus infinity; this is why it is called a “big brake”. Since this model does not describe an accelerating universe, it is as such in conflict with present observations. However, it can easily be generalized in order to accommodate such an acceleration, without modifying the following discussion. The total lifetime of this universe is

$$t_0 \approx 7 \times 10^2 \frac{1}{\sqrt{V_0 \left[ \frac{\text{g}}{\text{cm}^3} \right]}} \text{ s,}$$

which is much bigger than the current age of our Universe for

$$V_0 \ll 2.6 \times 10^{-30} \frac{\text{g}}{\text{cm}^3}.$$

The classical trajectory in configuration space is shown in Fig. 3. The big-brake singularity is at  $\phi = 0$ . In addition, there are the usual big-bang and big-crunch singularities at  $a = 0$  and  $\phi \rightarrow \pm\infty$ .

In the quantum theory, one encounters the following Wheeler–DeWitt equation:

$$\frac{\hbar^2}{2} \left( \frac{\kappa^2}{6} \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \phi^2} \right) \Psi(\alpha, \phi) + V_0 e^{6\alpha} \left( \sinh(\sqrt{3\kappa^2}|\phi|) - \frac{1}{\sinh(\sqrt{3\kappa^2}|\phi|)} \right) \Psi(\alpha, \phi) = 0, \tag{28}$$

where  $\alpha = \ln a$ , and a Laplace–Beltrami factor ordering has again been employed. In order to study the behaviour near the region of the classical singularity, it is sufficient to study the limit of small  $\phi$ . One can then use the approximate equation

$$\frac{\hbar^2}{2} \left( \frac{\kappa^2}{6} \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \phi^2} \right) \Psi(\alpha, \phi) - \frac{\tilde{V}_0}{|\phi|} e^{6\alpha} \Psi(\alpha, \phi) = 0, \tag{29}$$

where  $\tilde{V}_0 = V_0/3\kappa^2$ . A crucial input is now the choice of boundary conditions. Firstly, we have to impose the condition that the wave function go to zero for large  $a$ ; this is because the classical evolution stops at finite  $a$ . Secondly, we demand normalizability with respect to  $\phi$ . The resulting solutions are then of the form

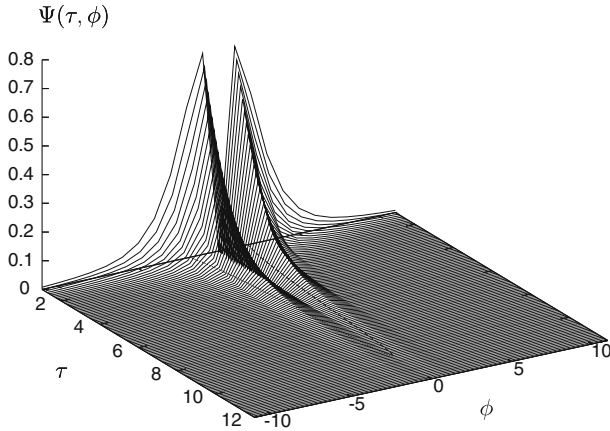
$$\Psi(\alpha, \phi) = \sum_{k=1}^{\infty} A(k) k^{-3/2} K_0 \left( \frac{1}{\sqrt{6}} \frac{V_\alpha}{\hbar^2 k \kappa} \right) \left( 2 \frac{V_\alpha}{k} |\phi| \right) e^{-\frac{V_\alpha}{k|\phi|}} L_{k-1}^1 \left( 2 \frac{V_\alpha}{k} |\phi| \right), \tag{30}$$

where  $K_0$  is a Bessel function,  $L_{k-1}^1$  denotes the Laguerre polynomials, and  $V_\alpha \equiv \tilde{V}_0 e^{6\alpha}$ . Inspection of this solution shows that it *vanishes* at  $\phi = 0$ , that is, at the classical big-brake singularity. Therefore, this singularity is avoided in the quantum theory. In fact, the normalization condition with respect to  $\phi$  also guarantees that the big-bang singularity is absent. One is thus left with a singularity-free quantum universe.

A wave-packet solution following the classical solution of Fig. 3 and approaching zero when  $\phi \rightarrow 0$  (that is, when approaching the region of the classical big-brake singularity), is shown in Fig. 4.

A somewhat related model with a quantum avoidance is phantom cosmology [57]. Classically, one has there a universe with scale factor  $a(t)$  containing a scalar field with negative kinetic term (“phantom”), which develops a “big-rip singularity”:  $\rho$  and  $p$  diverge as  $a$  goes to infinity at a *finite time*. An investigation of the Wheeler–DeWitt equation demonstrates that wave-packet solutions *disperse* in the region of the classical big-rip singularity. Therefore, time and the classical evolution come to an end before the singularity would be reached. Only a stationary quantum state is left. This, again, presents an example where quantum gravitational effects are important for large scale factor—much bigger than the Planck length. Quantum geometrodynamics is able to cope with this situation.

Quantum cosmology extends well beyond the minisuperspace limit of homogeneity [3]. In order to understand structure formation, it is crucial to implement inhomogeneous perturbations [58]. The tensor part of these perturbations then describes weak



**Fig. 4** The wave packet for the big-brake model. The packet follows the classical trajectory but becomes zero at the classical singularity [56]

quantized gravitational waves. It is also of interest to investigate a quantum analogue of the Belinski–Khalatnikov–Lifshitz analysis of approaching a spacelike singularity. It has been argued that this leads, in addition to the disappearance of time, to an effective de-emergence of space [59]. The classical singularity would then be fully dissolved in quantum gravity.

All models of quantum cosmology discussed so far are based on the assumption that the total quantum state (the “wave function of the universe”) is a pure state. Recently the idea arose to start instead with a fundamental density matrix of a microcanonical ensemble [60,61]. If defined by a Euclidean path integral, it was found that such a state is dynamically preferred compared to the “no-boundary state” of [19]. An interesting result of this investigation is that the cosmological constant would be limited to a bounded range.

Quantum geometrodynamics can also be successfully applied to lower-dimensional gravity. In  $2 + 1$  dimensions, the gravitational theory is of a purely topological nature and one thus only has to deal with finitely many degrees of freedom, similar to quantum cosmology [62]. One thereby gets important insights in both the role of boundary conditions and the structure of the Wheeler–DeWitt equation.

## 5 Conclusions and outlook

“There is no experimental evidence for the quantization of the gravitational field, but we believe quantization should apply to all the fields of physics. They all interact with each other, and it is difficult to see how some could be quantized and others not.” This is, in Dirac’s words ([29, p. 539]), the main motivation for dealing with quantum gravity. Because there is no experimental evidence so far, it is not surprising that several different approaches are being seriously discussed. In my contribution, I have addressed one of them, quantum geometrodynamics, which is a direct quantization of Einstein’s theory by canonical means and choosing the three-metric as its canonical

configuration variable. As I have tried to argue, quantum geometrodynamics is still a viable field because it gives intuitive insights into many conceptual and technical questions and because it is able to address quantum aspects of black holes and cosmology. And independent of its status as a fundamental theory (which it is probably not) it should be valid at least approximately for length scales bigger than the Planck length—just because it can be constructed from the condition that it give the correct semiclassical limit.

The final decision about quantum gravity will, of course, be made by experiment. Before that state will be reached, it is important to be open minded and to investigate as many approaches as possible and to study both mathematical and conceptual aspects. I would like to close with a remark by Einstein, who emphasized the non-trivial nature of the relation between theory and experience in clear words:

The concepts and sentences only get “sense” and “content” through their relation with the sensual experiences. The connection of the latter with the former is purely intuitive, not itself of logical nature. The degree of certainty, with which this relation resp. intuitive connection can be undertaken, and nothing else, distinguishes the queer illusion from the scientific “truth”.<sup>4</sup>

**Acknowledgments** I thank the organizers of the 405th Heraeus seminar: *Quantum Gravity: Challenges and Perspectives* for inviting me to such an interesting and stimulating meeting. I am grateful to Andrei Barvinsky, Domenico Giulini, Alexander Kamenshchik, and Barbara Sandhöfer for their comments on my manuscript.

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<sup>4</sup> Die Begriffe und Sätze erhalten “Sinn” bzw. “Inhalt” nur durch ihre Beziehung zu den Sinnenerlebnissen. Die Verbindung der letzteren mit den ersteren ist rein intuitiv, nicht selbst von logischer Natur. Der Grad der Sicherheit, mit der diese Beziehung bzw. intuitive Verknüpfung vorgenommen werden kann, und nichts anderes, unterscheidet die leere Phantasterei von der wissenschaftlichen “Wahrheit” [63].

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