

# Quantum Geometrodynamics I: Quantum-Driven Many-Fingered Time

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The classical theory of gravity predicts its own demise – singularities. We therefore attempt to quantize gravitation, and present here a new approach to the quantization of gravity wherein the concept of time is derived by imposing the constraints as expectation-value equations over the true dynamical degrees of freedom of the gravitational field – a representation of the underlying anisotropy of space. This self-consistent approach leads to qualitatively different predictions than the Dirac and the ADM quantizations, and in addition, our theory avoids the interpretational conundrums associated with the problem of time in quantum gravity. We briefly describe the structure of our functional equations, and apply our quantization technique to two examples so as to illustrate the basic ideas of our approach.

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## I. CLASSICAL DYNAMICS OF GENERAL RELATIVITY AND ITS QUANTIZATION.

It is reasonable to say that very few problems of modern theoretical physics have attracted as much attention and as much effort as gravity quantization. Simply stated, the classical theory of gravity predicts its own demise. Under quite general circumstances it has been shown that the classical theory evolves toward singularities — singularities in the fabric of spacetime. [1–3] Secondly, the vacuum fluctuations of spacetime, commonly referred to as spacetime foam, may exhibit “collective mode oscillations” (CMO’s) of a rich enough variety and spectrum so as to provide physics, once and for all, with a quantum geometric description of all matter and field interactions. [4] We are so fantastically far from solving such lofty problems; nevertheless, these goals provide us with the motivational foundations for our research in quantum geometrodynamics. However, modern attempts to quantize gravity are interwoven with contradictions, internal inconsistencies and conceptual ambiguities. In particular, the following three problems have received much attention and have plagued the development of a theory of quantum gravity: (1) the square-root Hamiltonian problem; (2) the problem of time; and (3) the Hilbert space problem. These difficulties are so pervasive that they persist, and their nature remains about the same even when restricted to the very simplest of models — quantum cosmology with a finite amount of degrees of freedom.

Upon a careful analysis of existing approaches to the gravity quantization we have come to the conclusion that the source of this state of affairs can be traced to a misinterpretation of the classical dynamical theory of general relativity. To appreciate this statement one should note that every single attempt of gravity quantization is based on the original ADM [5] picture of the gravitational field dynamics. In this picture, the dynamical evolution of the gravity field manifests itself as a change from one spacelike 3-geometry to another. In other words, the configuration space of geometrodynamics is believed to be Wheeler’s superspace. [6] Such an approach does not utilize York’s analysis of the gravitational field’s degrees of freedom and the initial-value formulation. [7,8] This is not surprising as the foundations of quantum gravity [9,10] were originally formulated before York completed his investigation.

The ADM analysis of Einstein equations leads to a natural split of these equations into (1) six evolution equations, and (2) four constraints that enforce the general covariance. [11] This situation is similar, in a sense, to that of the dynamics of gauge fields. In the case of a homogeneous cosmology, the situation closely resembles the dynamics of a relativistic particle. [12] The peculiarity of the gravitational field, as described by general relativity, is that its dynamics is maximally constrained, i. e., given the constraints on all possible spacelike slices of a given spacetime, it is possible to recover the full system of Einstein equations. [13] This particular property has led some researchers to

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the conclusion that the gravity field dynamics is determined completely by the constraints. [14] The last statement should be, in our opinion, handled with extreme caution. Its proper interpretation demands additional assumptions that are frequently made implicitly and, if not analyzed carefully, might lead to numerous paradoxes in both classical and quantum dynamics. A detailed analysis of this situation has been provided by U. Gerlach. [13] When applied to gravity quantization, this line of reasoning leads to the conclusion that, to quantize such a fully constrained field, one needs only to quantize the constraints. Such an approach leads to a wave equation, called the Wheeler–DeWitt equation, which is more akin to the Klein–Gordon equation rather than the Schrödinger equation. An equation of this type, as it is well known, cannot be interpreted in a one–particle (in case of gravity, one–Universe) representation.

The original ADM quantization proposal was different. According to it, one had to solve constraints before quantization and to extract a Schrödinger equation from the Hamilton–Jacobi classical equation. The peculiar feature of this approach was the emergence of the so called square–root Hamiltonian problem. [12,1] This difficulty has, to our knowledge, never been resolved.

Both the ADM and Dirac approaches in any formulation (canonical, path integrals, Euclidean path integrals, etc.) lead to difficulties of about the same nature, which were captured so dramatically by K. Kuchař when he reformulated them as “the problem of time.” [14] Roughly speaking, both approaches eliminate a possibility of including in the theory a natural concept of time or an observer (cf. also a discussion of the issue by W. Unruh [15]). We prefer a slightly different formulation. In our opinion, the source of such difficulties arises when one treats the whole 3–geometry of spacelike slices as dynamical and quantizes the entire 3–geometry. Mathematically, it is expressed via imposing the commutation relations [10] on all the components of the 3–metric. One should keep in mind, however, that in general relativity the system described by the 3–metric or even the 3–geometry includes in itself the observer and his clock. In standard quantum mechanics, or even in the quantum field theory, an observer is external with respect to the quantum system. The observer is classical and has an external classical clock. There cannot be an external observer in the description of the gravitational field because the gravitational system is the Universe itself. Both ADM and Dirac’s approaches essentially quantize the observer and his clock on equal footing with the rest of the system (J. A. Wheeler would say that many–fingered time is directly quantized). Whether such a quantized observer and his clock can function in a fashion providing an opportunity to describe the system consistently is not clear. A discussion of such a possibility, however exciting, clearly would lead us far beyond the scope of this paper. We only wish to mention here that this difficulty has been noticed by some researchers, most notably by Gell–Mann and Hartle and has led them to propose a generalized form of quantum theory based on the ideas of histories and decoherence functionals. [16]. What we propose here is quite a different in nature than the Gell–Mann–Hartle theory.

## II. QUANTUM GEOMETRODYNAMICS: BASIC IDEAS.

We propose an alternative approach to the quantization of gravity. Our approach is based on the post-ADM achievements made in classical geometrodynamics. [17] In particular, we are referring to York’s solution of the initial–value problem and his analysis of the gravitational degrees of freedom. [7,8] This development was initially motivated by Wheeler’s semi–intuitive remark that the 3–geometry of a spacelike hypersurface has encoded within it the two gravitational degrees of freedom as well as its temporal location within spacetime. It is this notion that the 3–geometry is a carrier of information on time that has been referred to as “Wheeler’s *many–fingered time*.” [18,19,11] It was J. York who first made this thesis precise. He forwarded what has now become almost the canonical split of the 3–geometry into its underlying conformal equivalence class (its *shape* representing the two dynamical degrees of freedom of the gravitational field coordinate per space point) and the conformal scale factor (its *scale* representing Wheeler’s many–fingered time). Only the conformal 3–geometry is dynamical in the sense that it can be specified freely as the initial data. The scale factor is nondynamical and essentially specifies Wheeler’s many–fingered time. It is determined by both a slicing condition (the fixation of a field of observers) and the constraints (enforcing general covariance). The results of York have demonstrated that the true dynamical part of the gravitational field is not the 3–geometry but only its conformal part, and that the proper configuration space or “arena for geometrodynamics” should be the underlying conformal superspace rather than Wheeler’s superspace. The conformal scale factor, York’s representation of Wheeler’s many–fingered time, thus becomes an external parameter and should be explicitly treated as such in any viable quantization scheme.

None of the existing attempts of gravity quantization truly adopts York’s analysis of the gravitational degrees of freedom. We propose, on the contrary, to design a consistent quantization procedure by taking York’s construction as a priori. In the classical theory, we start from the standard Lagrangian and the associated action (with appropriate boundary terms as needed), develop the standard variational dynamical picture over conformal superspace and treat the scale factor (and, in more general setting, the coordinatization parameters) as an external parameter. It is clear that the Hamiltonian of such a theory will not be the usual super-Hamiltonian. Nevertheless, one can develop an

entire dynamical picture based on this Hamiltonian by deriving either the Hamiltonian equations or the Hamilton–Jacobi equation. However, these equations are incomplete. They contain as yet unknown functions related to the scale parameter. One can complete the system of equations by adding to the Hamilton equations, or to the Hamilton–Jacobi equation, the standard constraint equations of general relativity (they cannot be derived from variational principles in such a theory). The resulting equations are equivalent to the standard equations of classical geometrodynamics.

For the purpose of quantization, we start from our Hamilton–Jacobi equation, describing effectively what we refer to as *conformal geometrodynamics* (evolution of the dynamical variables corresponding to the conformal part of the 3–geometry) in a scale parameter dependent external field. We augment this with the four constraint equations so as to recover the relationship between the scale parameter and the dynamical variables. Using the Hamilton–Jacobi equation we write down the Schrödinger equation treating the scale parameter as an external classical field and quantizing only the true dynamical variables. This Schrödinger equation can be solved (cf., for instance the example of the Bianchi 1A cosmological model in Sec.IV). The solution will ordinarily depend on the many–fingered time parameter (as yet unspecified) as well as on the coordinatization parameters. To specify these functions we use the constraint equations. The procedure for this follows from our interpretation of the constraint equations. Whereas the approach of ADM attempted to isolate the dynamical degrees of freedom of the gravitational field by imposing the constraints at a classical level prior to quantization, we propose here a weaker condition that the four constraints be imposed only on the expectation values of the conformal dynamics. That is, we impose the constraint equations as expectation–value equations using the wave functional obtained from our Hamiltonian. In so doing we explicitly avoid the interpretational conundrums associated with the problem of time as well as square–root Hamiltonians, and we form a “classical” gravitational clock driven by the quantized geometrodynamical system — i.e. *quantum–driven many–fingered time*.

The goal of this paper is to clarify our basic thesis regarding the quantization of gravity — quantum geometrodynamics. As we have emphasized, our idea implies a change in the quantization procedure of constrained dynamical systems. Such a procedure can be applied to other more familiar constrained Hamiltonian systems. In order to illustrate the salient features of our novel approach we cannot think of a more suitable example than the relativistic particle. Therefore, in the next section we apply our quantization procedure to a free relativistic particle. We do not discuss the issue of its usefulness for particle dynamics, as this issue should be a topic of a separate research project. We use this example merely as a simple illustration of our approach and a test of our procedure. The utility of this example lies entirely with its simplicity, clarity in illustrating our procedure, and in demonstrating that this procedure can be used in geometrodynamics.

### III. TIME AS AN EXTERNAL FIELD IN THE QUANTUM DYNAMICAL PICTURE OF A FREE RELATIVISTIC PARTICLE.

We start from the standard dynamics of a particle in the super-Hamiltonian [11] formulation and recover its Lagrangian. Then we change the dynamical picture via removing the time coordinate out of the set of the dynamical variables. This results in a new Hamiltonian. The time coordinate and its functions are treated as external fields. This procedure leads to an incomplete system of the equations of motion. The system is completed via imposing additionally the super-Hamiltonian constraint — a constraint that does not follow from the equations of motion in this new approach. It is considered merely as a relation between the dynamical variables and the external field. The resulting description is equivalent to the standard one within the classical theory, but leads to a considerably different quantum theory.

The super-Hamiltonian for a free particle is given by

$$\mathcal{H} = \frac{1}{2m} (m^2 + p^2) = \frac{1}{2m} (m^2 + p_\mu p^\mu), \quad (1)$$

where  $\mu = 0, 1, 2, 3$  and  $p_\mu$  is the momentum conjugate to the coordinate  $x^\mu$  and  $x^0$  is the time coordinate ( $x^0 = t$ ). The super-Hamiltonian constraint  $\mathcal{H} = 0$  implies

$$m^2 + p^2 = m^2 + p_\mu p^\mu = 0. \quad (2)$$

The Hamilton equations

$$\begin{aligned} \dot{x}^\mu &= \frac{\partial \mathcal{H}}{\partial p_\mu} = \frac{p^\mu}{m}, \quad \text{and} \\ \dot{p}_\mu &= -\frac{\partial \mathcal{H}}{\partial x^\mu} = 0, \end{aligned} \quad (3)$$

where the dot means the derivative with respect to the affine parameter  $\lambda$ , provide the expression for the momenta

$$p^\mu = m\dot{x}^\mu, \quad (4)$$

and the second order equations of motion

$$\ddot{x}^\mu = 0. \quad (5)$$

The super-Hamiltonian constraint provides a relation between the affine parameter  $\lambda$  and the proper time  $\tau$  along the world line of the particle.

The Lagrangian is related to the super-Hamiltonian  $\mathcal{H}$  via

$$\mathcal{H} = p_\mu \dot{x}^\mu - L, \quad (6)$$

and can be explicitly obtained in the following simple way:

$$L = p_\mu \dot{x}^\mu - \mathcal{H} = m\dot{x}^\mu \dot{x}^\mu - \frac{m}{2} (1 + \dot{x}^\mu \dot{x}_\mu) = \frac{m}{2} (\dot{x}_\mu \dot{x}^\mu - 1). \quad (7)$$

After separating the dynamical variables  $x^i, i = 1, 2, 3$  from the external parameter  $t = x^0$ , we obtain an expression for the Lagrangian.

$$L = \frac{m}{2} (\dot{x}_i \dot{x}^i - \dot{t}^2 - 1). \quad (8)$$

The standard definition of the momenta conjugate to the dynamical degrees of freedom  $x^i$ ,

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}_i, \quad (9)$$

reproduces three (not four) Hamilton equations.

$$\dot{x}_i = \frac{p_i}{m}. \quad (10)$$

Using them, we construct the Hamiltonian  $H$  of our theory,

$$H = p_i \dot{x}^i - L = \frac{1}{m} p_i p^i - \frac{1}{2m} (p_i p^i - m^2 \dot{t}^2 - m^2), \quad (11)$$

which, after simplifications, leads to

$$H = \frac{1}{2m} (p_i p^i + m^2 + m^2 \dot{t}^2). \quad (12)$$

The Hamilton equations are obtained as usual, i.e.

$$\begin{aligned} \dot{x}^i &= \frac{\partial H}{\partial p_i} = \frac{p_i}{m}, \\ \dot{p}_i &= -\frac{\partial H}{\partial x^i} = 0. \end{aligned} \quad (13)$$

The super-Hamiltonian constraint is imposed in addition to the Hamilton equations,

$$p_i p^i - m^2 \dot{t}^2 = -m^2, \quad (14)$$

and completes the system of equation describing the particle motion. It can be written (using Hamilton equations) as

$$dx_i dx^i - dt^2 = -d\lambda^2. \quad (15)$$

The parameter  $\lambda$  again coincides with the proper time on the particle world line.

Although our new Hamiltonian  $H$  of a free particle is conserved, i. e. it is an integral of motion

$$\frac{dH}{d\lambda} = m\ddot{t} = 0, \quad (16)$$

it is, nevertheless, time dependent.

$$H = H(x^i, p_i, t(\lambda)). \quad (17)$$

The Hamilton–Jacobi equation is obtained from this Hamiltonian.

$$\frac{\partial S}{\partial \lambda} = -H \left( x^i, \frac{\partial S}{\partial x^i}, t(\lambda) \right). \quad (18)$$

For this we use the expression 12 for the Hamiltonian rewritten in the following form:

$$H = \frac{1}{2m} g^{ik} p_i p_k + \frac{m}{2} (1 + \dot{t}^2); \quad (19)$$

which leads to the Hamilton–Jacobi equation.

$$\frac{\partial S}{\partial \lambda} = -\frac{1}{2m} g^{ik} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} - \frac{m}{2} (1 + \dot{t}^2). \quad (20)$$

The Hamilton–Jacobi equation 20 does not provide a complete description of the particle dynamics as was the case for the Hamilton equations 13. It too should be augmented by the constraint 14 in order to recover general Lorentz covariance.

The standard prescription of transition from the Hamilton–Jacobi equation to the Schrödinger equation,

$$\frac{\partial S}{\partial \lambda} \longrightarrow i\hbar \frac{\partial}{\partial \lambda}; \quad \frac{\partial S}{\partial x^i} \longrightarrow \hat{p}_i = \frac{\hbar}{i} \frac{\partial}{\partial x^i}, \quad (21)$$

leads to the Schrödinger equation of a relativistic free particle,

$$i\hbar \frac{\partial \psi}{\partial \lambda} = \frac{\hbar^2}{2m} \Delta \psi - \frac{m}{2} (1 + \dot{t}^2) \psi. \quad (22)$$

Here  $\psi = \psi(x^i, \lambda)$  is the wave function of the free particle and  $\Delta = g^{ik} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^k}$  is the Laplacian. The equation is a linear differential equation with variable coefficients. It supports the superposition principle and can be solved using the standard techniques of quantum mechanics. Moreover, it can be interpreted as the Schrödinger equation for a particle in an external spatially-homogeneous but time-dependent potential field. This time-dependent field is determined by the second term of the right hand side ( $\dot{t}$  is assumed to be a function of  $\lambda$  only) of 22.

As usual, the general solution of the equation 22 depends on a constant of integration (it is determined by the condition of normalization of the  $\psi$ -function) as well as the initial conditions (those, roughly speaking, specify the initial shape of the wave packet). An unusual feature of the solution is its dependence on a potential term that has not been specified yet. This feature is related to the fact that in our formulation of the classical theory, the Hamilton–Jacobi equation does not provide a complete dynamic picture. To complete the dynamic picture we need to use an analog of the constraint 14. Our proposal is to use literally the same constraint substituting in it instead of the classical values of momenta  $p_i$  their expectation values,

$$\langle p_i(\lambda) \rangle = \langle \psi | \hat{p}_i | \psi \rangle = \int \psi^* (x^i, t(\lambda)) \hat{p}_i \psi (x^i, t(\lambda)) d^3x. \quad (23)$$

The Schrödinger equation 22 together with the constraint 14 obviously provide a complete dynamic description of the free relativistic particle.

It is clear that the procedure of quantization described here avoids the square-root Hamiltonian problem on both the quantum and classical levels. This problem on the quantum level disappears due to the elimination of  $t$  from the set of dynamical variables. On the classical level the constraint equation 14 is solved for  $\dot{t}$  which is a square root of nonnegative expression; therefore, it avoids such square-root operators. The parameter  $\lambda$  in the quantum mechanical picture can be interpreted as the proper time of an averaged quantum-driven distribution “classical” observer. The last sentence should not be interpreted too literally. Ordinarily, the observer is not classical as his motion is not described by the classical equations of motion. [20]

#### IV. QUANTUM GEOMETRODYNAMICS OF THE BIANCHI 1A COSMOLOGY.

The Bianchi 1A cosmological model is commonly referred to as the axisymmetric Kasner model. Its metric is determined by two parameters, the scale factor  $\Omega$  and the anisotropy parameter  $\beta$ .

$$ds^2 = -dt^2 + e^{-2\Omega} (e^{2\beta} dx^2 + e^{2\beta} dy^2 + e^{-4\beta} dz^2). \quad (24)$$

As this cosmology is homogeneous the two functions  $\Omega$  and  $\beta$  are the functions of the time parameter  $t$  only. The scalar 4-curvature can be expressed in terms of these two functions to yield the Hilbert action and, after subtracting the boundary term, the cosmological action,

$$I_C = I_H + \frac{3V}{8\pi} \dot{\Omega} e^{-3\Omega} \Big|_{t_0}^{t_f} = \frac{3V}{8\pi} \int_{t_0}^{t_f} (\dot{\beta}^2 - \dot{\Omega}^2) e^{-3\Omega} dt, \quad (25)$$

where  $V = \int \int \int dx dy dz$  is the spatial volume element.

We treat the scale factor  $\Omega(t)$  as the many-fingered time parameter and the anisotropy  $\beta(t)$  as the dynamical degree of freedom. The momentum conjugate to  $\beta$  is

$$p_\beta = \frac{\partial L}{\partial \dot{\beta}} = \frac{3V}{4\pi} e^{-3\Omega} \dot{\beta}. \quad (26)$$

The Hamiltonian of the system in our approach can be expressed in terms of the momentum conjugate to  $\beta$  and the Lagrangian.

$$\begin{aligned} H &= p_\beta \dot{\beta} - L \\ &= \frac{4\pi}{3V} e^{3\Omega} p_\beta^2 - \frac{3V}{8\pi} \left( \frac{4\pi}{3V} \right)^2 e^{3\Omega} p_\beta^2 + \frac{3V}{8\pi} \dot{\Omega}^2 e^{-3\Omega} \\ &= \frac{2\pi}{3V} e^{3\Omega} p_\beta^2 + \frac{3V}{8\pi} \dot{\Omega}^2 e^{-3\Omega}. \end{aligned} \quad (27)$$

In the classical theory this Hamiltonian can be used to produce either one pair of Hamilton equations or the equivalent Hamilton–Jacobi equation. In any case, the dynamical picture derived in this way is incomplete. To complete it we impose the super-Hamiltonian constraint.

$$p_\beta^2 = \left( \frac{3V}{4\pi} \right)^2 e^{-6\Omega} \dot{\Omega}^2. \quad (28)$$

Using the Hamilton–Jacobi equation,

$$\frac{\partial S}{\partial t} = -H \left( \frac{\partial S}{\partial \beta}, \Omega(t), \dot{\Omega}(t) \right), \quad (29)$$

together with the expression 27 for the Hamiltonian  $H$  and the standard quantization prescription we obtain the Schrödinger equation for the axisymmetric Kasner model.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{2\pi\hbar^2}{3V} e^{3\Omega} \frac{\partial^2 \psi}{\partial \beta^2} + \frac{3V}{8\pi} \dot{\Omega}^2 e^{-3\Omega} \psi. \quad (30)$$

The constant  $\hbar$  in this equation should be understood as the square of Planck’s length scale, rather than the standard Planck constant. We wish to stress here that the scale factor  $\Omega$  in the Schrödinger equation is so far an unknown function of time. This means that the equation does not describe completely the quantum dynamics of the axisymmetric Kasner model. To complete the dynamical picture we follow our prescription and impose, in addition to equation 30, the super-Hamiltonian constraint.

$$\langle p_\beta \rangle^2 = \left( \frac{4\pi}{3V} \right)^2 e^{-6\Omega} \dot{\Omega}^2. \quad (31)$$

Here  $\langle p_\beta \rangle$  is the expectation value of the momentum  $\hat{p}_\beta = \frac{\hbar}{i} \frac{\partial}{\partial \beta}$

$$\langle p_\beta \rangle = \langle \psi | \hat{p}_\beta | \psi \rangle = \int_{-\infty}^{\infty} \psi^*(\beta, t) \hat{p}_\beta \psi(\beta, t) d\beta \quad (32)$$

The system of equations 30, 31 provide us with a complete quantum dynamical picture of the axisymmetric Kasner model and, when augmented by appropriate initial and boundary conditions, can be solved analytically.

Before discussing the initial value conditions we will find the general solution of the Schrödinger equation considering the scale factor  $\Omega$  as a function of time generating an external potential. For this we separate variables,

$$\psi(\beta, t) = \phi(\beta)T(t). \quad (33)$$

After substituting 33 in the Schrödinger equation 30 we obtain,

$$i\hbar\phi\dot{T} = -\frac{2\pi\hbar^2}{3V}e^{3\Omega}T\phi'' + \frac{3V}{8\pi}\dot{\Omega}^2e^{-3\Omega}T\phi, \quad (34)$$

where the prime means differentiation with respect to  $\beta$ . Rewriting it as

$$\frac{2\pi\hbar^2}{3V}\frac{\phi''}{\phi} = -i\hbar e^{-3\Omega}\frac{\dot{T}}{T} + \frac{3V}{8\pi}e^{-6\Omega}\dot{\Omega}^2 = -\lambda, \quad (35)$$

where  $\lambda$  is the constant of separation, we obtain the equations for  $\phi(\beta)$  and  $T(t)$ .

$$\phi'' + \frac{3V}{2\pi\hbar^2}\lambda\phi = 0 \quad (36)$$

$$\frac{\dot{T}}{T} = -\frac{i}{\hbar}e^{3\Omega}\left(\frac{3V}{8\pi}e^{-6\Omega}\dot{\Omega}^2 + \lambda\right) \quad (37)$$

Equation 36 admits only positive eigenvalues for  $\lambda$ . Introducing the notation  $\frac{3V\lambda}{2\pi} = k^2$  we can write the solutions  $\phi_k(\beta)$ ,  $T_k(t)$  for  $k \in (-\infty, \infty)$ .

$$\begin{aligned} \phi_k(\beta) &= A_k e^{\frac{i}{\hbar}k\beta} \\ T_k(t) &= B_k \exp\left\{-\frac{i}{\hbar}\int_{t_0}^t \left(\frac{2\pi}{3V}k^2 + \frac{3V}{8\pi}e^{-6\Omega}\dot{\Omega}^2\right) e^{3\Omega} dt\right\} \end{aligned} \quad (38)$$

Using the superposition of these solutions we come up with the general solution of the Schrödinger equation 30.

$$\psi(\beta, t) = \int_{-\infty}^{\infty} A_k e^{\frac{i}{\hbar}k\beta} \exp\left\{-\frac{i}{\hbar}\int_{t_0}^t \left(\frac{2\pi}{3V}k^2 + \frac{3V}{8\pi}e^{-6\Omega}\dot{\Omega}^2\right) e^{3\Omega} dt\right\} dk \quad (39)$$

To specify a particular problem one has to furnish appropriate initial data.

$$\psi(\beta, t)|_{t_0} = \psi(\beta, t_0) = \int_{-\infty}^{\infty} A_k e^{\frac{i}{\hbar}k\beta} dk \quad (40)$$

It can be done either by specifying a function  $\psi(\beta, t_0)$  and then recovering  $A_k$  from the equation

$$\psi(\beta, t_0) = \int_{-\infty}^{\infty} A_k e^{\frac{i}{\hbar}k\beta} dk, \quad (41)$$

using Fourier transforms, or by assigning  $A_k$  as a function of  $k$ , depending on the type of the problem to be formulated. In this section we consider the simplest example comparable with the quantum mechanics of a particle, namely a wave packet. To describe a gaussian wave packet centered initially at the value  $k_0$  of  $k$  (we will describe the meaning of  $k_0$  later) we assign

$$A_k = C e^{-a(k-k_0)^2}, \quad (42)$$

where  $C$  is the normalization constant. This leads to the following expression for the initial values of the wave function:

$$\psi(\beta, t_0) = C \int_{-\infty}^{\infty} e^{-a(k-k_0)^2} e^{\frac{i}{\hbar}k\beta} dk = C \sqrt{\frac{\pi}{a}} e^{\frac{i}{\hbar}\beta k_0} e^{-\frac{\beta^2}{4a\hbar^2}}. \quad (43)$$

The value of the normalization constant  $C$  is determined by the condition

$$\langle \psi | \psi \rangle = C^2 \frac{\pi}{a} \int_{-\infty}^{\infty} e^{-\frac{\beta^2}{2a\hbar^2}} d\beta = C^2 \hbar \pi^{\frac{3}{2}} \sqrt{\frac{2}{a}} = 1 \quad (44)$$

which leads to the value of  $C^2$

$$C^2 = \frac{\sqrt{a}}{\hbar \pi^{\frac{3}{2}} \sqrt{2}}. \quad (45)$$

Using expression 42 for  $A_k$  and introducing notations for  $f$  and  $g$ ,

$$f = f(t) = \frac{2\pi}{3V} \int_{t_0}^t e^{3\Omega} dt, \quad (46)$$

$$g = g(t) = \frac{3V}{8\pi} \int_{t_0}^t \dot{\Omega}^2 e^{-3\Omega} dt,$$

we can write down the solution  $\psi(\beta, t)$  for the wave packet.

$$\psi(\beta, t) = C e^{-\frac{i}{\hbar}g} \int_{-\infty}^{\infty} e^{-a(k-k_0)^2} e^{\frac{i}{\hbar}\beta k} e^{\frac{i}{\hbar}f k^2} dk. \quad (47)$$

After a simple transformation this expression can be rewritten in the form,

$$\psi(\beta, t) = C \exp \left\{ \frac{i}{\hbar} [(\beta - k_0 f) k_0 - g] \right\} \int_{-\infty}^{\infty} e^{-ak^2} e^{-\frac{i}{\hbar}f k^2} e^{\frac{i}{\hbar}(\beta - 2k_0 f)k} dk. \quad (48)$$

The integral on the right hand side of 48 can be evaluated. The final expression for the solution describing a gaussian wave packet may be written in the following form which will prove to be convenient for future calculations:

$$\psi(\beta, t) = C \sqrt{\pi} \left( a^2 + \frac{f^2}{\hbar^2} \right)^{-\frac{1}{4}} \cdot \exp \left\{ -\frac{a}{4 \left( a^2 + \frac{f^2}{\hbar^2} \right)} \frac{(\beta - 2k_0 f)^2}{\hbar^2} \right\} \times$$

$$\exp \left\{ \frac{i}{\hbar} (\beta - k_0 f) k_0 \right\} \cdot \exp \left\{ i \frac{\frac{f}{\hbar}}{4 \left( a^2 + \frac{f^2}{\hbar^2} \right)} \frac{(\beta - 2k_0 f)^2}{\hbar^2} \right\} \cdot \exp \left\{ -\frac{i}{\hbar} g - i\theta \right\}; \quad (49)$$

where,

$$\cos(2\theta) = a / \sqrt{a^2 + f^2/\hbar^2}, \quad \sin(2\theta) = (f/\hbar) / \sqrt{a^2 + f^2/\hbar^2} \quad (50)$$

Although expression 49 looks quite involved the last three exponential factors are phase factors and do not complicate the determination of the expectation values of the observables.

It is clear that this solution of the Schrödinger equation describing the wave packet cannot provide any definite predictions as it contains the two functions of time  $f(t)$  and  $g(t)$  which are themselves related to the as yet undetermined scale factor  $\Omega$ . To find  $\Omega(t)$  we need to (1) compute the expectation  $\langle p_\beta \rangle$  of the momentum  $\hat{p}_\beta = \frac{\hbar}{i} \frac{\partial}{\partial \beta}$ , (2) substitute this expectation value into the constraint 31 and (3) solve the resulting equation with respect to  $\Omega$ . We start from computing  $\langle p_\beta \rangle$ .

$$\langle p_\beta \rangle = \langle \psi | \hat{p}_\beta | \psi \rangle = C^2 \pi \left( a^2 + \frac{f^2}{\hbar^2} \right)^{-\frac{1}{2}} k_0 \int_{-\infty}^{\infty} \exp \left\{ -\frac{a}{2 \left( a^2 + \frac{f^2}{\hbar^2} \right)} \frac{(\beta - 2k_0 f)^2}{\hbar^2} \right\} d\beta = k_0. \quad (51)$$



In other words the expectation value of the momentum  $\langle p_\beta \rangle$  does not change with time. It is determined by the  $k$ -center of the packet at  $t = t_0$ . Substitution of this result in 31 yields

$$k_0^2 = \left( \frac{3V}{4\pi} \right)^2 e^{-6\Omega} \dot{\Omega}^2. \quad (52)$$

This equation and the classical equations are identical. Therefore, we need not describe it in detail. We only wish to point out once more that after the solution of this equation is substituted in 49 the geometrodynamical problem 30, 31 for the wave packet 42 is solved completely. To summarize, the many-fingered time of quantum geometrodynamics in case of a gaussian wave packet of axisymmetric Kasner spacetimes coincides with its classical counterpart if the expectation value of the momentum of the packet is identified with the (conserved) value of the momentum of the classical solution.

The expectation value for the anisotropy parameter  $\beta$ , where  $\beta$  is the only quantum dynamical variable in this model, is given by:

$$\langle \beta \rangle = \langle \psi | \beta | \psi \rangle = C^2 \pi \left( a^2 + \frac{f^2}{\hbar^2} \right)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \beta \exp \left\{ -\frac{a}{2 \left( a^2 + \frac{f^2}{\hbar^2} \right)} \frac{(\beta - 2k_0 f)^2}{\hbar^2} \right\} d\beta = 2k_0 f(t). \quad (53)$$

Thus “the center” of the wave packet evolves as the classical Kasner universe determined by the momentum value equal to  $k_0$  would evolve. The spread of the wave packet with time is the variance in  $\beta$ .

$$\begin{aligned} & \langle (\beta - \langle \beta \rangle)^2 \rangle = \\ & C^2 \pi \left( a^2 + \frac{f^2}{\hbar^2} \right)^{-\frac{1}{2}} \int_{-\infty}^{\infty} (\beta - 2k_0 f)^2 \exp \left\{ -\frac{a}{2 \left( a^2 + \frac{f^2}{\hbar^2} \right)} \frac{(\beta - 2k_0 f)^2}{\hbar^2} \right\} d\beta = \frac{\hbar^2 a^2 + f^2}{a} \end{aligned} \quad (54)$$

It is obvious from 54 that the spread of the packet increases with time. The result is similar to that of the quantum mechanics of a free particle; after all the Bianchi I cosmology is the free-particle analogue of quantum cosmology.

## V. CHARACTERISTIC FEATURES OF THE QUANTUM GEOMETRODYNAMICAL PROCEDURE.

The quantum dynamical picture described in the previous sections has two features that are not encountered typically in most of the more common quantum dynamical schemes. They should be kept in mind in order to avoid errors and misinterpretations in applying our procedure.

The first such feature is related to the structure of the Hamiltonian. Formally, the Hamiltonian appears as a Hamiltonian of a system placed in external time-dependent field, at least when the Schrödinger equation is analyzed. However, the external field is determined via constraints by the quantum state of the system. Ordinarily, the Hamiltonian depends on the initial data. This feature is not unique for our approach as it is encountered in standard quantum mechanics such as Hartree-Fock-like systems.

The second feature is a split of the system parameters in two groups: (1) the truly dynamical variables (to be quantized); and (2) the descriptors of a “natural observer” associated to the system, driven as they are by the quantum geometrodynamics. Such a split is introduced prior to quantization, and is crucial to our theory. This feature can be clearly observed in both our examples. Essentially, this split is a part of our solution of the time problem. Most of the difficulties related to the problem of time are caused by an attempt to enforce general covariance at the quantum level. Such an attempt for gravitational systems is bound to fail. It leads to the questions that are not well defined unless they are referred to a background spacetime. Standard attempts to introduce a background spacetime tend to use a fixed spacetime with properties generally not related to the properties of the system itself, which is commonly considered as an unsatisfactory feature (we quite agree with this conclusion). Our procedure, on the other hand, can be considered as a universal prescription for defining a unique background spacetime structure driven by the quantum system. All questions concerning covariance should be referred to this spacetime structure, and this spacetime structure is ordinarily not classical. It cannot be considered as a result of a 3-geometry evolution (the evolution equations do not take a part in our procedure). In particular, we recover the classical evolution only through an appropriately-peaked wave function and only through constructive interference over conformal superspace.

Not every assignment of dynamical variables leads to a consistent quantum dynamical picture. For example, a bad choice of the dynamical component of the 3-geometry might lead to a nonelliptic differential operator on the right hand side of Schrödinger equation. Furthermore, under such conditions not every shape of the initial wave packet

will agree with the constraints. In any case, only an appropriate choice of the set of dynamical variables will lead to a reasonable quantum mechanical picture. The properties of the ADM procedure together with the results of J. York indicate that there is at least one such reasonable choice. Is more than one possible quantum dynamical picture for the same system, and if so, whether different dynamical pictures lead to the same predictions? This is a difficult question to answer, notwithstanding the fact that we have been unable to even properly formulate this question, as it is not only the question of rewriting the equations but also an appropriate reformulation of the initial conditions. The initial conditions are inextricably intertwined with the quantum dynamics.

## VI. DISCUSSION.

We have introduced in this paper a new approach for the quantization of constrained systems and illustrated its application to two examples. When the Bianchi 1A cosmology was quantized, our theory generated a quantum geometrodynamics picture based on a post-ADM treatment of the gravitational field dynamics and was free from the conceptual difficulties usually associated with the Dirac and ADM procedure of quantization. The variables describing the gravity field in this case have been split into the true dynamical variables and a parameter related to Wheeler’s many-fingered time. Only the dynamical part, the underlying anisotropy, has been quantized. The Hamiltonian participating in the Schrödinger equation is not a square-root Hamiltonian. This absence of a square-root Hamiltonian is generic for our quantum geometrodynamics procedure. The effective “background spacetime” determined by the expectation values of dynamic variables together with the “observers” (related to Wheeler’s many-fingered time) allows us to pose unambiguously the questions of covariance, which in turn eliminates the problem of time in quantum gravity. Furthermore, the Hilbert space problem does not appear to be the generic feature in our view of quantum geometrodynamics.

The problems outlined in Sec.I become all but eliminated by our quantum geometrodynamics approach. The nontrivial part of gravity quantization appears to shift from such conceptual problems too the problem of (1) the choice of an appropriate model to quantize, and to the related problem of (2) the choice of an appropriate initial condition for the wave functional. Both choices are crucial if one is to attempt using quantum geometrodynamics to better comprehend the properties of gravitational systems. It is our understanding that the success of gravity quantization rests on such meaningful choices. Furthermore, the choice of models should not be determined by the structure of quantum geometrodynamics; rather, it should be determined by observational data and our general understanding of gravitational phenomena.

It is clear that the procedure of quantizing the dynamical part of constrained systems described in this paper can be extended to the general case of geometrodynamics without any complications in principle, although it may become quite involved computationally. The procedure differs only in two respects from the simplistic examples presented here. The first difference arises when the 3-metric is (1) parametrized by three coordinatization parameters, (2) the many-fingered time parameter, and (3) the two dynamical variables; then all four constraints should be solved with respect to the coordinatization parameters and the scale factor. In all four constraints the expectation values of the true dynamic variables should be used. The second difference is caused by the functional nature of the gravitational field dynamics in the general case. The operation of functional integration is involved, which might lead to analytic difficulties. Such difficulties are not specific for our approach as they are common for the canonical formulations of all field theories. Quantum geometrodynamics, in particular, does not seem to generate any specific new difficulties. In this paper we forwarded the beginnings of a quantization scheme consistent with York’s analysis of the gravitational degrees of freedom. Although we parallel the original motivations of Misner and the ADM quantization procedure, our particular imposition of the four constraint equations leads to a weaker theory that in turn avoids the problem of time. As a more mathematically-intricate description of our theory would be beyond the scope of this introductory paper, we will publish the mathematical foundations in a separate paper. [21]

We have not discussed here the inclusion of matter into quantum geometrodynamics. One simply should follow the pattern of matter inclusion in the initial-value problem as outlined by J. York. However, this inclusion could be important, especially in the cases where the matter degrees of freedom are coupled with the true dynamical gravitational degrees of freedom<sup>11</sup>. Nevertheless, we do not include it in the present paper, as it does not contribute to the clarity of presentation of our ideas concerning the quantization of constrained system in general, and quantum geometrodynamics in particular.

We have demonstrated that the concept of time is inextricably intertwined and woven to the initial conditions as well as to the quantum dynamics over the space of all conformal 3-geometries.

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