

# Geometrodynamical Quantization and Time Evolution in Quantum Gravity

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We advance here a new gravity quantization procedure that explicitly utilizes York’s analysis of the geometrodynamical degrees of freedom. This geometrodynamical procedure of quantization is based on a separation of the true dynamic variables from the embedding parameters and a distinctly different treatment of these two kinds of variables. While the dynamic variables are quantized following the standard quantum mechanical and quantum field theoretic procedures, the embedding parameters are determined by the “classical” constraint equations in which the expectation values of the dynamic variables are substituted in place of their classical values. This self-consistent procedure of quantization leads to a linear Schrödinger equation augmented by nonlinear “classical” constraints and supplies a natural description of time evolution in quantum geometrodynamics. In particular, the procedure sheds new light on the “problems of time” in quantum gravity.

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## I. INTRODUCTION.

The task of describing time evolution in quantum gravity appears to encounter numerous and seemingly insurmountable obstacles. The difficulties are so pervasive that some researchers have voiced a doubt whether the concept of time and time evolution can be adequately introduced in quantum gravity. [1] The issue of time evolution in quantum gravity has been reviewed by Kuchař [2] who formulated it as a set of “problems of time.” He concluded that the problems of time have not been successfully resolved in either the Dirac or the square-root ADM approaches to gravity quantization. In this paper we review and analyze both these procedures of quantization in light of a new approach to gravity quantization. [3]

The classical theory of gravity is a fully constrained theory. It is our thesis that this feature should not be mirrored in quantum gravity, and our observation that the source of the “time” difficulties in both the Dirac and ADM quantization approaches arises because the York split [4] of the dynamical variables from the time parameterization variables occurs too late. Here we propose to introduce such a split before the dynamical theory is developed. This, we suggest, will supply a more cogent quantization procedure as well as bypass the “problems of time.”

We provide in (Sec. II) a brief review of the standard techniques of canonical gravity quantization. In Sec. III we advance a new procedure of gravity quantization (geometrodynamical quantization). This procedure incorporates York’s analysis of the geometrodynamical degrees of freedom [4] and appears to avoid the difficulties of the standard procedures of quantization. In Sec. IV we provide two simple examples to illustrate our geometrodynamical procedure of quantization; namely, we quantize the axisymmetric Kasner and Taub cosmologies. Sec. V provides a brief discussion of the specific features of this quantization approach, while Secs. VI and VII provide a brief description of the “problems of time” in standard canonical quantum gravity and the solution of these problems within our new framework.

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## II. CANONICAL QUANTUM GRAVITY.

Both the Dirac and square-root Hamiltonian approaches to canonical quantization make use of the 3+1 split of the spacetime geometry by Arnowitt, Deser and Misner (hereafter referred to as ADM). [5] According to the ADM prescription of classical general relativity one considers a slicing of the spacetime by a family of spacelike hypersurfaces labeled by a parameter  $t$ . This parameter can be thought of as a time coordinate so that any slice is identified by the relation  $t = \text{const}$ . The remaining three spacetime coordinates  $x^i$  determine a coordinatization of each slice. The spacetime metric  $({}^{(4)}g_{\mu\nu})$  is parameterized by the shift  $(N^i)$ , lapse  $(N)$  and the 3-metric of the slice  $(g_{ik})$ .

$$\begin{pmatrix} ({}^{(4)}g_{00} & ({}^{(4)}g_{0k}) \\ ({}^{(4)}g_{i0} & ({}^{(4)}g_{ik}) \end{pmatrix} = \begin{pmatrix} (N_s N^s - N^2) & N_k \\ N_i & g_{ik} \end{pmatrix}. \quad (2.1)$$

The standard action of general relativity for the gravity field

$$I = \int \mathcal{L} d^4x = \int ({}^{(4)}R \sqrt{-({}^{(4)}g)} d^4x, \quad (2.2)$$

after its augmentation by appropriate boundary terms can be represented as

$$I = \int \left[ \pi^{ij} \frac{\partial g_{ij}}{\partial t} - N \mathcal{H}(\pi^{ij}, g_{ij}) - N_i \mathcal{H}^i(\pi^{ij}, g_{ij}) \right] d^4x. \quad (2.3)$$

In this equation  $\pi^{ij}$ ,  $\mathcal{H}$ , and  $\mathcal{H}^i$  are given as follows [7]:

$$\pi^{ij} = \begin{pmatrix} \text{“geometrodynamical} \\ \text{field momentum”} \\ \text{dynamically conjugate to} \\ \text{the “geometrodynamical”} \\ \text{field coordinate” } g_{ij} \end{pmatrix} = g^{\frac{1}{2}} (g^{ij} K - K^{ij}); \quad (2.4)$$

$$\mathcal{H} = (\text{“super-Hamiltonian”}) = g^{-\frac{1}{2}} \left( \text{Tr} \Pi^2 - \frac{1}{2} (\text{Tr} \Pi)^2 \right) - g^{\frac{1}{2}} R; \quad \text{and} \quad (2.5)$$

$$\mathcal{H}^i = (\text{“supermomentum”}) = -2\pi^{ik}{}_{|k}; \quad (2.6)$$

where  $K^{ij}$  is the extrinsic curvature tensor,  $K = K^i_i$  is the trace of the extrinsic curvature tensor, and  $\Pi$  is the matrix of  $\pi^{ij}$ . All the quantities in (2.4) – (2.6), including the covariant derivative in (2.6) are related to the 3-geometry of the slice.

Without any further refinement or redirection, the ADM approach treats all six components of the 3-metric  $g_{ik}$  as the gravitational “field coordinates” (with their conjugate momenta  $\pi^{ik}$ ), while the lapse  $N$  and the shift  $N^i$  are treated as Lagrange multipliers (no conjugate momenta for them are present in the expression for action). The expression

$$\mathcal{H}_{ADM}(\pi^{ij}, g_{ij}, N, N_i) = N \mathcal{H}(\pi^{ij}, g_{ij}) + N_i \mathcal{H}^i(\pi^{ij}, g_{ij}) \quad (2.7)$$

plays the part of Hamiltonian when variations of  $\pi^{ij}$  and  $g_{ij}$  are considered. These variations produce the twelve Hamilton equations.

$$\frac{\partial g_{ij}}{\partial t} = \frac{\partial \mathcal{H}_{ADM}}{\partial \pi^{ij}} = 2N g^{-\frac{1}{2}} \left( \pi_{ij} - \frac{1}{2} \text{Tr} \Pi \right) + N_{|i} j + N_{j|i} \quad (2.8)$$

$$\frac{\partial \pi^{ij}}{\partial t} = - \frac{\partial \mathcal{H}_{ADM}}{\partial g_{ij}} \quad (2.9)$$

We do not display this last equation in all detail as it is rather lengthy and is not necessary for this paper. In addition, there are three supermomentum constraints

$$\mathcal{H}^i(\pi^{ij}, g_{ij}) = 0 \quad (2.10)$$

and the super–Hamiltonian constraint

$$\mathcal{H}(\pi^{ij}, g_{ij}) = 0. \quad (2.11)$$

These four constraint equations are obtained by variations of  $N_i$  and  $N$  in the ADM action, respectively. The equations (2.8), (2.9), (2.9), (2.10) are not independent. The interplay between the Hamilton equations (2.8), (2.9) and the constraints (2.10), (2.11) is rather involved. In particular, if the constraints are satisfied on an initial slice then the Hamilton equations guarantee that they are also satisfied on all spacelike slices. On the other hand, if the constraints are satisfied on each slice of all possible spacelike foliations of a given spacetime then necessarily the Hamilton equations are satisfied. This last feature of general relativity has led people to refer to gravitational dynamics as a “fully-constrained theory.” While it is clear that this property can be valuable in the classical theory, it is also equally clear that it is implicitly based on the assumption of the uniqueness of the spacetime 4–geometry [6]. This last assumption of uniqueness becomes problematic in quantum geometrodynamics, and consequently, the fully-constrained property of geometrodynamics also becomes problematic. This situation is, in our opinion, the root of the notorious “problems of time” in quantum gravity.

A further investigation of the ADM picture of gravitational dynamics leads one to the interpretation of the super-momentum constraints (2.10) as expressing the 3–dimensional diffeomorphism invariance (the freedom of choice of coordinates on the slices) and the super–Hamiltonian constraint (2.11), together with the natural identification [7]

$$\pi^{ij} = \frac{\delta S}{\delta g_{ij}} \quad (2.12)$$

as the Hamilton–Jacobi equation of the theory [7,8]

$$\mathcal{H}\left(\frac{\delta S}{\delta g_{ij}}, g_{ij}\right) = 0. \quad (2.13)$$

Although this Hamilton–Jacobi equation involves all six components of the 3–metric, the 3–diffeomorphic invariance as expressed by the supermomentum constraints,

$$\left(\frac{\delta S}{\delta g_{ij}}\right)_{|j} = 0, \quad (2.14)$$

allows one to identify the Hamilton–Jacobi equation as an equation that describes the evolution of the 3–geometry rather than of the 3–metric. Equation (2.13) has numerous disturbing features. The most disturbing of them is that, according to it, the super–Hamiltonian  $\mathcal{H}$  cannot be interpreted as the generator of the time translation [9]. In addition, unlike the standard situation in mechanics, the super–Hamiltonian  $\mathcal{H}$  participating in the Hamilton–Jacobi equation (2.13) does not coincide with the  $\mathcal{H}_{ADM}$  Hamiltonian (2.7) which generates the dynamic evolution via Hamilton equations (2.8), (2.9). This  $\mathcal{H}_{ADM}$  Hamiltonian is related to the Lagrangian  $\mathcal{L}$  by the standard relation

$$\mathcal{H}_{ADM} = \pi^{ij} \frac{\partial g_{ij}}{\partial t} - \mathcal{L} \quad (2.15)$$

while, ordinarily,  $\mathcal{H}$  is not.

Dirac’s procedure of canonical quantization utilizes the standard prescription

$$\pi^{ij} \longrightarrow \frac{\delta S}{\delta g_{ij}} \longrightarrow \hat{\pi}_{ij} = \frac{\hbar}{i} \frac{\delta}{\delta g_{ij}} \quad (2.16)$$

and leads to four functional differential equations based solely on the four constraint equations of the classical theory

$$\hat{\mathcal{H}}^i(\hat{\pi}^{ij}, g_{ij}) \Psi = 0, \quad (2.17)$$

$$\hat{\mathcal{H}}(\hat{\pi}^{ij}, g_{ij}) \Psi = 0. \quad (2.18)$$

The three equations (2.17) are often interpreted as a requirement for the state functional to be a functional of the 3–geometry  $\Psi = \Psi^{(3)}[\mathcal{G}]$  rather than of the 3–metric. Equation (2.18) is the Wheeler–DeWitt equation [7,10] which

is considered to be a proper wave equation for quantum gravity. This equation reminds one more of a Klein–Gordon equation than a Schrödinger equation. As we have already mentioned, the state functional  $\Psi$  in this equation, although originally introduced as a functional of the 3–metric  $\Psi = \Psi[g_{ik}]$ , after imposing on it the requirement of 3-dimensional diffeomorphic invariance, is considered to be a functional of the underlying 3–geometry  $\Psi = \Psi[{}^{(3)}\mathcal{G}]$ . The commutation relations are imposed on all the components of the 3–metric, although proper care is taken to ensure that they are diffeomorphically invariant. The Wheeler–DeWitt equation, just as its classical counterpart, the Hamilton–Jacobi equation, does not admit an interpretation of the Hamiltonian as a generator of time translations.

An attempt to remedy this situation via introducing, instead of the super–Hamiltonian  $\mathcal{H}$  a Hamiltonian that, in a sense, does generate the time translation, has been undertaken in the ADM square-root procedure of quantization [2]. In this procedure the set of six 3–metric parameters is split in two subsets  $\{\beta_1, \beta_2\}$ , and  $\{\alpha_1, \alpha_2, \alpha_3, \Omega\}$  of parameters. The first of them is treated as the set of true gravitational degrees of freedom, and the second is considered as the set of embedding variables. The classical super–momentum and super–Hamiltonian constraints (2.10), (2.11) are then solved on the classical level with respect to the momenta conjugate to the embedding variables, which, after natural identifications (for the sake of simplifications in notations we omit indices on  $\beta$  and  $\alpha$  parameters; they can be recovered easily whenever it is necessary) [7,10]

$$\pi^\beta = \frac{\delta S}{\delta \beta}, \quad \pi^\alpha = \frac{\delta S}{\delta \alpha}, \quad \pi^\Omega = \frac{\delta S}{\delta \Omega} \quad (2.19)$$

leads to the following four equations:

$$\frac{\delta S}{\delta \alpha} = -h_\alpha \left( \frac{\delta S}{\delta \beta}, \beta, \alpha, \Omega \right) \quad (2.20)$$

$$\frac{\delta S}{\delta \Omega} = -h_\Omega \left( \frac{\delta S}{\delta \Omega}, \beta, \alpha, \Omega \right). \quad (2.21)$$

One of the functions  $h_\alpha, h_\Omega$  contains a square root operation in it (we assume from now on that it is  $h_\Omega$ ), which is why it is called the square-root Hamiltonian. The standard prescription for quantization,

$$\frac{\delta S}{\delta \alpha} \longrightarrow i\hbar \frac{\delta}{\delta \alpha}, \quad \frac{\delta S}{\delta \Omega} \longrightarrow i\hbar \frac{\delta}{\delta \Omega}, \quad \frac{\delta S}{\delta \beta} \longrightarrow \hat{\pi}^\beta = \frac{\hbar}{i} \frac{\delta}{\delta \beta}, \quad (2.22)$$

leads to the four quantum equations,

$$i\hbar \frac{\delta \Psi}{\delta \alpha} = \hat{h}_\alpha (\hat{\pi}^\beta, \beta, \alpha, \Omega) \Psi, \quad (2.23)$$

$$i\hbar \frac{\delta \Psi}{\delta \Omega} = \hat{h}_\Omega (\hat{\pi}^\beta, \beta, \alpha, \Omega) \Psi, \quad (2.24)$$

where  $\Psi = \Psi[\beta, \alpha, \Omega]$ . The choice of parameters  $\alpha$  and  $\Omega$  is frequently made in such a way that the three equations (2.23) are interpreted as a requirement of the diffeomorphism invariance, while equation (2.24) is considered to be the proper Schrödinger equation. In this case parameters  $\alpha$  can be interpreted as coordinatization parameters while the function  $\Omega$  can be thought of as the many-fingered time parameter [11,12] with  $h_\Omega$  treated as the Hamiltonian.

We wish to point out that the square-root Hamiltonian  $h_\Omega$  in the classical theory, just as it was the case with the super-Hamiltonian  $\mathcal{H}$ , is not related to the Lagrangian  $\mathcal{L}$  as it would be in a standard dynamic theory (cf. equation (2.15)).

Both the ADM square-root and Dirac quantization procedures admit, in principle, a time evolution prescription to be introduced in them, although this should be done on different stages. In Dirac’s quantization procedure the state functional  $\Psi$  is a functional of either the slice 3-metric or the slice 3-geometry, which can be written in terms of  $\beta, \alpha, \Omega$  functions as

$$\Psi = \Psi[{}^{(3)}g] = \Psi[\alpha, \Omega, \beta] \quad \text{or} \quad \Psi = \Psi[{}^{(3)}\mathcal{G}] = \Psi[\Omega, \beta], \quad (2.25)$$

where  $\alpha, \Omega$  and  $\beta$  are treated on equal footing. In particular, a computation of any quantity similar to an expectation function, if it were possible, ordinarily would involve functional integration over either all six functional parameters  $\alpha, \Omega, \beta$  or the three functional parameters  $\beta$  and  $\Omega$ . Along these lines we wish to remind the reader here that, in Dirac’s quantization approach, the commutation relations are imposed on all the components of the 3-metric, and in quantum

mechanics the evaluation of expectations involves integration over all the variables participating in the commutation relations. Such a calculation, however, would ordinarily exclude a possibility for one to introduce a functional time and a time evolution in the theory. A way to avoid such an unpleasant situation is to split the set of variables in two subsets, one of which ( $\beta$ ) is to be considered as a set of true functional arguments of the state functional (and to be integrated over during computing the expectations), while the others ( $\alpha, \Omega$ ) are to be considered as a set of functional parameters (one per slice) and to be interpreted as a representation of functional time. This split should be introduced only after the solution for the state functional is obtained and prior to the evaluation of expectation values. Such a procedure is thoroughly artificial and amounts to treating the embedding variables as quantum ones in one part of the theory yet as classical ones in another.

The square-root Hamiltonian quantization is based on the same classical picture of the evolution in geometrodynamics as a change of 3-metric or 3-geometry from one spacelike slice to another and leads one to a Hamilton–Jacobi equation. However, it introduces the split of variables just before quantization. From this moment on it assumes that there is only one function  $\Omega$  per slice. The state functional  $\Psi$  when restricted to one slice becomes essentially a functional of  $\beta$  only, which can be properly written as

$$\Psi = \Psi[\alpha, \Omega; \beta] \quad \text{or} \quad \Psi = \Psi[\Omega; \beta]. \quad (2.26)$$

The computation of expectation functions (or any other momenta) now assumes functional integration over the two parameters  $\beta$  only, with the result depending on  $\Omega$  as a parameter (of course, it will also depend on three other functional parameters  $\alpha$ , which we call coordinatization parameters). This functional parameter ( $\Omega$ ) plays the role of the many-fingered time parameter. To summarize, in the square-root Hamiltonian approach the switch from one picture of evolution to another occurs at the final stage of formulating classical geometrodynamics, and comes, just as in Dirac’s approach, as an afterthought aimed to meet the difficulties generated by the original picture of evolution (as evolution of the 3-geometry) in the quantized theory. Such a switch does not cause any problems in the classical theory, but comes at a price in quantum geometrodynamics by generating numerous difficulties, including the notorious “problems of time” (cf. Sec. VI).

The equations of quantum gravity, be it equations (2.17), (2.18) of Dirac’s quantization or equations (2.23), (2.24) of the square-root Hamiltonian quantization, do not provide for a natural split. The split is, to an extent, arbitrary. It reflects, for any particular gravitational system, our understanding of the system’s dynamics. In a sense, it is similar to the situation described by N. Bohr regarding the split of any considered phenomenon into its quantum and classical parts. Such a split depends not only on the object that we are considering but, also, on the questions that we are asking.

It appears to us that the equations of quantum gravity themselves do not imply the necessity of a picture of temporal evolution for quantum gravitational systems. The assumption of a temporal evolution should be added, at some level, so as to not contradict to the rest of the theory. The first step in this direction is to introduce a split of metric variables into the true dynamic variables  $\beta$  on one hand, and the coordinatization parameters  $\alpha$  and the many-fingered time parameter  $\Omega$  on the other hand. The variables  $\beta$  are the true quantum variables (calculation of expectation values involves functional integration of  $\Psi$  only over these functions), while  $\alpha$  and  $\Omega$  are merely functional parameters that allow one, after some manipulations, to introduce the concept of a temporal evolution. The two different approaches described in this section introduce the split of variables at different stages. This leads to difficulties of both the technical and conceptual nature in quantum geometrodynamics.

In the next section we advance an alternative procedure of quantization – a procedure that, from the very beginning, is based on a picture of geometrodynamics evolution induced by York’s analysis of the geometrodynamics degrees of freedom. This alternative approach does not involve a change in the paradigm of time evolution in geometrodynamics.

### III. GEOMETRODYNAMIC QUANTIZATION.

We propose here an alternative approach to the quantization of gravity. Our approach is based on the post-ADM achievements made in classical geometrodynamics. [13] In particular, we are referring to York’s solution of the initial–value problem and his analysis of the gravitational degrees of freedom. [4,14] This development was initially motivated by Wheeler’s semi–intuitive remark that the 3-geometry of a spacelike hypersurface has encoded within it the two gravitational degrees of freedom as well as its temporal location within spacetime. It is this notion that the 3-geometry is a carrier of information on time that has been referred to as “Wheeler’s *many-fingered time*.” [11,12,7] It was J. York who first made this thesis precise. He forwarded what has now become almost the canonical split of the 3-geometry into its underlying conformal equivalence class (its *shape* representing the two dynamic degrees of freedom of the gravitational field coordinate per space point) and the conformal scale factor (its *scale* representing Wheeler’s many-fingered time). Only the conformal 3-geometry is truly dynamic in the sense that it can be specified freely as

the initial data. The scale factor is non-dynamic and essentially specifies Wheeler's many-fingered time. The results of York have demonstrated that the true dynamic part of the gravitational field is not the 3-geometry but only its conformal part, and that the proper configuration space or "arena for geometrodynamics" should be the underlying conformal superspace (the space of all conformal 3-geometries) rather than Wheeler's superspace (the space of all 3-geometries). The conformal scale factor and three other functional parameters of the 3-metric (responsible for coordinate conditions) thus become external parameters. In what follows we associate the true dynamic variables  $\beta$  with conformal 3-geometry, the many fingered time variable  $\Omega$  with the 3-geometry scale factor, and the remaining three variables  $\alpha$  of the 3-metric with a coordinatization of a spacelike 3-surface. The many-fingered time variable and coordinatization variables are assumed to be fixed initially by some conditions, and then subsequently controlled by the shift and lapse.

Our proposed procedure of gravity quantization is based, from the very beginning, on York's analysis of gravitational degrees of freedom. We suggest that one should interpret geometrodynamics as an evolution of the conformal 3-geometry  $\beta$  in an external field determined by the scale factor  $\Omega$  and coordinatization variables  $\alpha$ . Such an approach calls for a reformulation of geometrodynamics on the classical level. We start from the standard Lagrangian  $\mathcal{L}$  (written in terms of the 3-metric, shift and lapse) and the associated action (with appropriate boundary terms, as needed, to remove the second time derivatives terms) and we introduce the momenta conjugate to the true dynamic variables

$$\pi_\beta = \frac{\partial \mathcal{L}}{\partial \dot{\beta}}. \quad (3.1)$$

We then use these  $\pi_\beta$ 's to form the geometrodynamical Hamiltonian  $\mathcal{H}_{DYN}$  of our approach,

$$\mathcal{H}_{DYN} = \pi_\beta \dot{\beta} - \mathcal{L}. \quad (3.2)$$

The new Hamiltonian  $\mathcal{H}_{DYN}$  is distinctly different from  $\mathcal{H}_{ADM}$  and its arguments do not coincide with those of  $\mathcal{H}_{ADM}$ , namely

$$\mathcal{H}_{DYN} = \mathcal{H}_{DYN}(\Omega, \alpha; \beta, \pi_\beta). \quad (3.3)$$

The variables preceding the semicolon are treated as describing an external field, while the ones following the semicolon are the coordinates and momenta of the gravitational true degrees of freedom, i.e. of the conformal geometrodynamics. The variation of  $\beta$  and  $\pi_\beta$  leads to the equations of geometrodynamics, i.e. to two pairs of Hamilton equations,

$$\dot{\beta} = \frac{\partial \mathcal{H}_{DYN}}{\partial \pi_\beta}, \quad (3.4)$$

$$\dot{\pi}_\beta = -\frac{\partial \mathcal{H}_{DYN}}{\partial \beta}, \quad (3.5)$$

and, subsequently, to the Hamilton–Jacobi equation

$$\frac{\delta S}{\delta t} = -\mathcal{H}_{DYN} \left( \Omega, \alpha; \beta, \frac{\delta S}{\delta \beta} \right). \quad (3.6)$$

Here  $S$  is a functional of  $\beta$  and, in addition, depends on the same parameters as  $\mathcal{H}_{DYN}$ ,

$$S = S[\Omega, \alpha; \beta]. \quad (3.7)$$

Neither the Hamilton equations (3.4), (3.5) nor the Hamilton–Jacobi equation (3.6) are capable of providing any predictions as their solutions depend on the functional parameters  $\Omega$  and  $\alpha$  which are not yet known. One can complete the system of equations by adding to the Hamilton equations, or to the Hamilton–Jacobi equation, the standard constraint equations of general relativity. They should be satisfied when the solution for  $\beta$ ,  $\pi_\beta$  of equations of conformal geometrodynamics (with appropriate initial data) is substituted in them (we use symbols  $[\beta]_s$ ,  $[\pi_\beta]_s$  for such a solution)

$$\begin{aligned} \mathcal{H}^i(\Omega, \alpha, [\beta]_s, [\pi_\beta]_s) &= 0 \\ \mathcal{H}(\Omega, \alpha, [\beta]_s, [\pi_\beta]_s) &= 0 \end{aligned} \quad (3.8)$$

The resulting equations are equivalent to the standard equations of classical geometrodynamics. It should be emphasized that these constraint equations cannot be derived from variational principles in our theory.

For the purpose of quantization, we start from our Hamilton–Jacobi equation (3.6), describing effectively what we refer to as *conformal geometrodynamics*. That is the evolution of the dynamic variables corresponding to the conformal part of the 3–geometry, where this evolution is parameterized by the “external” field represented by (1) the scale parameter and (2) the coordinatization variables. Using the Hamilton–Jacobi equation (3.6) we may transition to the corresponding Schrödinger equation

$$i\hbar \frac{\delta \Psi}{\delta t} = \widehat{\mathcal{H}}_{DYN}(\Omega, \alpha; \beta, \widehat{\pi}_\beta) \Psi \quad (3.9)$$

where  $\widehat{\pi}_\beta = \frac{\hbar}{i} \frac{\delta}{\delta \beta}$ . The Schrödinger equation (3.9) treats the scale parameter and coordinatization functions as external classical fields and quantizes only the true dynamic variables,  $\beta$ . The state functional  $\Psi$  in this equation is a functional of  $\beta$  and also depends on the functional parameters  $\Omega$  and  $\alpha$ .

$$\Psi = \Psi[\Omega, \alpha; \beta] \quad (3.10)$$

This Schrödinger equation (with specific initial data) can be solved (cf., for instance the example of the Bianchi 1A cosmological model in Sec.V). The resulting solution  $\Psi_s$  of this Schrödinger equation is not capable of providing any definite predictions as it depends on four functional parameters  $\Omega, \alpha$  which remain at this stage undetermined. All expectations, such as, for instance the expectation values of  $\beta$

$$\langle \beta \rangle_s = \langle \Psi_s | \beta | \Psi_s \rangle = \int \Psi_s^* \beta \Psi_s \mathcal{D}\beta \quad (3.11)$$

or of  $\widehat{\pi}_\beta$

$$\langle \pi_\beta \rangle_s = \langle \Psi_s | \widehat{\pi}_\beta | \Psi_s \rangle = \int \Psi_s^* \widehat{\pi}_\beta \Psi_s \mathcal{D}\beta \quad (3.12)$$

also depend on these functional parameters. To specify these functions we resort to the constraint equations. As in case of classical geometrodynamics the constraints should be imposed on the solution of the initial-value problem of conformal geometrodynamics, and in this way, determine the unique values of  $\Omega$  and  $\alpha$ . It is possible that there are several ways to couple the constraints to the quantization of the true dynamic variables,  $\beta$ . We propose here that the four constraints be imposed only on the expectation values of the conformal dynamics

$$\begin{aligned} \mathcal{H}^i(\Omega, \alpha, \langle \beta \rangle_s, \langle \pi_\beta \rangle_s) &= 0 \\ \mathcal{H}(\Omega, \alpha, \langle \beta \rangle_s, \langle \pi_\beta \rangle_s) &= 0 \end{aligned} \quad (3.13)$$

i.e. only on measurable quantities. In so doing, we explicitly avoid the interpretational conundrums associated with the problems of time, and we form a “classical” gravitational clock driven by the quantized geometrodynamical system – i.e. *quantum-driven many-fingered time*.

This procedure of quantization utilizes explicitly the correct treatment of the geometrodynamical degrees of freedom and introduces a meaningful time parameterization utilizing the shift and lapse. The super-momentum and super-Hamiltonian constraints are not satisfied as strictly as in the Dirac and ADM square-root approaches. Our point of view is that the question of slicing independence of evolution is not a well posed question in quantum gravity and should be recovered only in the classical limit. It is clear that this weakening of the constraints, so that they hold only on the expectation values over the solution of the initial-value problem of the Schrödinger equation, makes the theory less restrictive and enlarges the set of possible solutions.

The goal of this paper is limited to a clarification of our basic thesis regarding the quantization of gravity — quantum geometrodynamics. [3] We believe that the best way to achieve this goal is to consider a couple of simple illustrative examples.

## IV. QUANTUM GEOMETRODYNAMICS: THE KASNER AND TAUB COSMOLOGIES

### A. Quantum Geometrodynamics of the Bianchi 1A Cosmology.

The Bianchi 1A cosmological model is commonly referred to as the axisymmetric Kasner model [15]. Its metric is determined by two parameters, the scale factor  $\Omega$  and the anisotropy parameter  $\beta$  (we choose  $N^i = 0$  and  $N = 1$  values of shift and lapse for this example)

$$ds^2 = -dt^2 + e^{-2\Omega} (e^{2\beta} dx^2 + e^{2\beta} dy^2 + e^{-4\beta} dz^2). \quad (4.1)$$

As this cosmology is homogeneous the two functions  $\Omega$  and  $\beta$  are the functions of the time parameter  $t$  only. The scalar 4-curvature can be expressed in terms of these two functions to yield the Hilbert action and, after subtracting the boundary term, the cosmological action,

$$I_C = I_H + \frac{3V}{8\pi} \dot{\Omega} e^{-3\Omega} \Big|_{t_0}^{t_f} = \frac{3V}{8\pi} \int_{t_0}^{t_f} (\dot{\beta}^2 - \dot{\Omega}^2) e^{-3\Omega} dt, \quad (4.2)$$

where  $V = \int \int \int dx dy dz$  is the spatial volume element.

We treat the scale factor  $\Omega(t)$  as the many-fingered time parameter and the anisotropy  $\beta(t)$  as the dynamic degree of freedom. The momentum conjugate to  $\beta$  is

$$p_\beta = \frac{\partial L}{\partial \dot{\beta}} = \frac{3V}{4\pi} e^{-3\Omega} \dot{\beta}. \quad (4.3)$$

(as it is usually the case with homogeneous cosmologies, we are working here with the momentum  $p_\beta$  rather than with the density  $\pi_\beta$ ). The Hamiltonian of the system in our approach can be expressed in terms of the momentum conjugate to  $\beta$  and the Lagrangian.

$$\begin{aligned} H_{DYN} &= p_\beta \dot{\beta} - L \\ &= \frac{4\pi}{3V} e^{3\Omega} p_\beta^2 - \frac{3V}{8\pi} \left( \frac{4\pi}{3V} \right)^2 e^{3\Omega} p_\beta^2 + \frac{3V}{8\pi} \dot{\Omega}^2 e^{-3\Omega} \\ &= \frac{2\pi}{3V} e^{3\Omega} p_\beta^2 + \frac{3V}{8\pi} \dot{\Omega}^2 e^{-3\Omega}. \end{aligned} \quad (4.4)$$

In the classical theory this Hamiltonian can be used to produce either one pair of Hamilton equations or the equivalent Hamilton–Jacobi equation. In any case, the dynamics picture derived in this way is incomplete. To complete it we impose the super-Hamiltonian constraint.

$$p_\beta^2 = \left( \frac{3V}{4\pi} \right)^2 e^{-6\Omega} \dot{\Omega}^2. \quad (4.5)$$

Using the Hamilton–Jacobi equation,

$$\frac{\partial S}{\partial t} = -H_{DYN} \left( \frac{\partial S}{\partial \beta}, \Omega(t), \dot{\Omega}(t) \right), \quad (4.6)$$

together with the expression (4.4) for the Hamiltonian  $H$  and the standard quantization prescription we obtain the Schrödinger equation for the axisymmetric Kasner model.

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{2\pi\hbar^2}{3V} e^{3\Omega} \frac{\partial^2 \Psi}{\partial \beta^2} + \frac{3V}{8\pi} \dot{\Omega}^2 e^{-3\Omega} \Psi. \quad (4.7)$$

The constant  $\hbar$  in this equation should be understood as the square of Planck’s length scale, rather than the standard Planck constant. We wish to stress here that the scale factor  $\Omega$  in the Schrödinger equation is so far an unknown function of time. This means that the equation does not describe completely the quantum dynamics of the axisymmetric Kasner model. To complete the dynamics picture we follow our prescription and impose, in addition to equation (4.7), the super-Hamiltonian constraint.

$$\langle p_\beta \rangle_s^2 = \left( \frac{4\pi}{3V} \right)^2 e^{-6\Omega} \dot{\Omega}^2. \quad (4.8)$$

Here  $\langle p_\beta \rangle_s$  is the expectation value of the momentum  $\hat{p}_\beta = \frac{\hbar}{i} \frac{\partial}{\partial \beta}$

$$\langle p_\beta \rangle_s = \langle \Psi_s | \hat{p}_\beta | \Psi_s \rangle = \int_{-\infty}^{\infty} \Psi_s^*(\beta, t) \hat{p}_\beta \Psi_s(\beta, t) d\beta \quad (4.9)$$



where  $\Psi_s$  is the solution of the Schrödinger equation with specified initial data. The system of equations (4.7), (4.8) provide us with a complete quantum dynamic picture of the axisymmetric Kasner model and, when augmented by appropriate initial and boundary conditions, can be solved analytically.

Before discussing the initial value conditions we will find the general solution of the Schrödinger equation considering the scale factor  $\Omega$  as a function of time generating an external potential. For this we separate variables,

$$\Psi(\beta, t) = \phi(\beta)T(t). \quad (4.10)$$

After substituting (4.10) in the Schrödinger equation (4.7) we obtain,

$$i\hbar\phi\dot{T} = -\frac{2\pi\hbar^2}{3V}e^{3\Omega}T\phi'' + \frac{3V}{8\pi}\dot{\Omega}^2e^{-3\Omega}T\phi, \quad (4.11)$$

where the prime means differentiation with respect to  $\beta$ . Rewriting it as

$$\frac{2\pi\hbar^2}{3V}\frac{\phi''}{\phi} = -i\hbar e^{-3\Omega}\frac{\dot{T}}{T} + \frac{3V}{8\pi}e^{-6\Omega}\dot{\Omega}^2 = -\lambda, \quad (4.12)$$

where  $\lambda$  is the constant of separation, we obtain the equations for  $\phi(\beta)$  and  $T(t)$ .

$$\phi'' + \frac{3V}{2\pi\hbar^2}\lambda\phi = 0 \quad (4.13)$$

$$\frac{\dot{T}}{T} = -\frac{i}{\hbar}e^{3\Omega}\left(\frac{3V}{8\pi}e^{-6\Omega}\dot{\Omega}^2 + \lambda\right) \quad (4.14)$$

Equation (4.13) admits only positive eigenvalues for  $\lambda$ . Introducing the notation  $\frac{3V\lambda}{2\pi} = k^2$  we can write the solutions  $\phi_k(\beta)$ ,  $T_k(t)$  for  $k \in (-\infty, \infty)$ .

$$\begin{aligned} \phi_k(\beta) &= A_k e^{\frac{i}{\hbar}k\beta} \\ T_k(t) &= B_k \exp\left\{-\frac{i}{\hbar}\int_{t_0}^t \left(\frac{2\pi}{3V}k^2 + \frac{3V}{8\pi}e^{-6\Omega}\dot{\Omega}^2\right) e^{3\Omega} dt\right\} \end{aligned} \quad (4.15)$$

Using the superposition of these solutions we come up with the general solution of the Schrödinger equation (4.7).

$$\Psi(\beta, t) = \int_{-\infty}^{\infty} A_k e^{\frac{i}{\hbar}k\beta} \exp\left\{-\frac{i}{\hbar}\int_{t_0}^t \left(\frac{2\pi}{3V}k^2 + \frac{3V}{8\pi}e^{-6\Omega}\dot{\Omega}^2\right) e^{3\Omega} dt\right\} dk \quad (4.16)$$

To specify a particular problem one has to furnish appropriate initial data.

$$\Psi(\beta, t)|_{t_0} = \Psi(\beta, t_0) = \int_{-\infty}^{\infty} A_k e^{\frac{i}{\hbar}k\beta} dk \quad (4.17)$$

It can be done either by specifying a function  $\Psi(\beta, t_0)$  and then recovering  $A_k$  from the equation

$$\Psi(\beta, t_0) = \int_{-\infty}^{\infty} A_k e^{\frac{i}{\hbar}k\beta} dk, \quad (4.18)$$

using Fourier transforms, or by assigning  $A_k$  as a function of  $k$ , depending on the type of the problem to be formulated. In this section we consider the simplest example comparable with the quantum mechanics of a particle, namely a wave packet. To describe a Gaussian wave packet centered initially at the value  $k_0$  of  $k$  (we will describe the meaning of  $k_0$  later) we assign

$$A_k = C e^{-a(k-k_0)^2}, \quad (4.19)$$

where the constant  $a$  effectively determines the initial width of the wave packet in momenta and  $C$  is the normalization constant. This leads to the following expression for the initial values of the wave function:

$$\Psi(\beta, t_0) = C \int_{-\infty}^{\infty} e^{-a(k-k_0)^2} e^{\frac{i}{\hbar} k \beta} dk = C \sqrt{\frac{\pi}{a}} e^{\frac{i}{\hbar} \beta k_0} e^{-\frac{\beta^2}{4a\hbar^2}}. \quad (4.20)$$

The value of the normalization constant  $C$  is determined by the condition

$$\langle \Psi | \Psi \rangle = C^2 \frac{\pi}{a} \int_{-\infty}^{\infty} e^{-\frac{\beta^2}{2a\hbar^2}} d\beta = C^2 \hbar \pi^{\frac{3}{2}} \sqrt{\frac{2}{a}} = 1 \quad (4.21)$$

which leads to the value of  $C^2$

$$C^2 = \frac{\sqrt{a}}{\hbar \pi^{\frac{3}{2}} \sqrt{2}}. \quad (4.22)$$

Using expression (4.19) for  $A_k$  and introducing notations for  $f$  and  $g$ ,

$$f = f(t) = \frac{2\pi}{3V} \int_{t_0}^t e^{3\Omega} dt, \quad (4.23)$$

$$g = g(t) = \frac{3V}{8\pi} \int_{t_0}^t \dot{\Omega}^2 e^{-3\Omega} dt,$$

we can write down the solution  $\Psi_s(\beta, t)$  for the wave packet.

$$\Psi_s(\beta, t) = C e^{-\frac{i}{\hbar} g} \int_{-\infty}^{\infty} e^{-a(k-k_0)^2} e^{\frac{i}{\hbar} \beta k} e^{\frac{i}{\hbar} f k^2} dk. \quad (4.24)$$

After a simple transformation this expression can be rewritten in the form,

$$\Psi_s(\beta, t) = C \exp \left\{ \frac{i}{\hbar} [(\beta - k_0 f) k_0 - g] \right\} \int_{-\infty}^{\infty} e^{-ak^2} e^{-\frac{i}{\hbar} f k^2} e^{\frac{i}{\hbar} (\beta - 2k_0 f) k} dk. \quad (4.25)$$

The integral on the right hand side of (4.25) can be evaluated. The final expression for the solution describing a Gaussian wave packet may be written in the following form which will prove to be convenient for future calculations:

$$\Psi_s(\beta, t) = C \sqrt{\pi} \left( a^2 + \frac{f^2}{\hbar^2} \right)^{-\frac{1}{4}} \cdot \exp \left\{ -\frac{a}{4 \left( a^2 + \frac{f^2}{\hbar^2} \right)} \frac{(\beta - 2k_0 f)^2}{\hbar^2} \right\} \times$$

$$\exp \left\{ \frac{i}{\hbar} (\beta - k_0 f) k_0 \right\} \cdot \exp \left\{ i \frac{\frac{f}{\hbar}}{4 \left( a^2 + \frac{f^2}{\hbar^2} \right)} \frac{(\beta - 2k_0 f)^2}{\hbar^2} \right\} \cdot \exp \left\{ -\frac{i}{\hbar} g - i\theta \right\}; \quad (4.26)$$

where,

$$\cos(2\theta) = a / \sqrt{a^2 + f^2 / \hbar^2}, \quad \sin(2\theta) = (f / \hbar) / \sqrt{a^2 + f^2 / \hbar^2} \quad (4.27)$$

Although expression (4.25) looks quite involved the last three exponential factors are phase factors and do not complicate the determination of the expectation values of the observables.

It is clear that this solution of the Schrödinger equation describing the wave packet cannot provide any definite predictions as it contains the two functions of time  $f(t)$  and  $g(t)$  which are themselves related to the as yet undetermined scale factor  $\Omega$ . To find  $\Omega(t)$  we need to (1) compute the expectation  $\langle p_\beta \rangle_s$  of the momentum  $\hat{p}_\beta = \frac{\hbar}{i} \frac{\partial}{\partial \beta}$ , (2) substitute this expectation value into the constraint (4.8) and (3) solve the resulting equation with respect to  $\Omega$ . We start from computing  $\langle p_\beta \rangle_s$ .

$$\langle p_\beta \rangle_s = \langle \Psi_s | \hat{p}_\beta | \Psi_s \rangle = C^2 \pi \left( a^2 + \frac{f^2}{\hbar^2} \right)^{-\frac{1}{2}} k_0 \int_{-\infty}^{\infty} \exp \left\{ -\frac{a}{2 \left( a^2 + \frac{f^2}{\hbar^2} \right)} \frac{(\beta - 2k_0 f)^2}{\hbar^2} \right\} d\beta = k_0. \quad (4.28)$$

In other words the expectation value of the momentum  $\langle p_\beta \rangle_s$  does not change with time. It is determined by the  $k$ -center of the packet at  $t = t_0$ . Substitution of this result in (6.5) yields

$$k_0^2 = \left( \frac{3V}{4\pi} \right)^2 e^{-6\Omega} \dot{\Omega}^2. \quad (4.29)$$

This equation and the classical equations are identical. Therefore, we need not describe it in detail. We only wish to point out once more that after the solution of this equation is substituted in (4.26) the geometrodynamics problem (4.7), (4.8) for the wave packet (4.19) is solved completely. To summarize, the many-fingered time of quantum geometrodynamics in case of a Gaussian wave packet of axisymmetric Kasner spacetimes coincides with its classical counterpart if the expectation value of the momentum of the packet is identified with the (conserved) value of the momentum of the classical solution.

The expectation value for the anisotropy parameter  $\beta$ , where  $\beta$  is the only quantum dynamic variable in this model, is given by:

$$\langle \beta \rangle_s = \langle \Psi_s | \beta | \Psi_s \rangle = C^2 \pi \left( a^2 + \frac{f^2}{\hbar^2} \right)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \beta \exp \left\{ -\frac{a}{2 \left( a^2 + \frac{f^2}{\hbar^2} \right)} \frac{(\beta - 2k_0 f)^2}{\hbar^2} \right\} d\beta = 2k_0 f(t). \quad (4.30)$$

Thus “the center” of the wave packet evolves as the classical Kasner universe determined by the momentum value equal to  $k_0$  would evolve. The spread of the wave packet with time is the variance in  $\beta$ .

$$C^2 \pi \left( a^2 + \frac{f^2}{\hbar^2} \right)^{-\frac{1}{2}} \int_{-\infty}^{\infty} (\beta - 2k_0 f)^2 \exp \left\{ -\frac{a}{2 \left( a^2 + \frac{f^2}{\hbar^2} \right)} \frac{(\beta - 2k_0 f)^2}{\hbar^2} \right\} d\beta = \frac{\hbar^2 a^2 + f^2}{a} \langle (\beta - \langle \beta \rangle)^2 \rangle_s \quad (4.31)$$

It is obvious from (4.31) that the spread of the packet increases with time. The result is similar to that of the quantum mechanics of a free particle; after all the Bianchi I cosmology is the free-particle analogue of quantum cosmology.

## B. Quantum Geometrodynamics of the Taub Cosmology.

We present within this section a second application of our approach to quantum gravity. In addition to illustrating our theory on a relatively simple model, we show that very concept of time emerges by imposing the principle of general covariance as weakly as possible. [3] In particular, the Hamiltonian constraint is imposed as an expectation-value equation over the true dynamic degree of freedom of the Taub cosmology – a representation of the underlying anisotropy of the 3-space. In this way the concept of time appears to be inextricably intertwined and woven to the initial conditions as well as to the quantum dynamics over the space of all conformal 3-geometries. This quantum geometrodynamics approach will ordinarily lead to quantitatively different predictions than either the Dirac or ADM quantizations, and in addition, our approach appears to avoid the interpretational conundrums associated with the “problems of time.” [2] So without further ado, in this subsection we apply our quantum geometrodynamics approach to the Taub cosmology [15] and numerically solve the coupled Schrödinger and expectation-value constraint equations.

The Taub cosmology is an axisymmetric homogeneous cosmology parameterized by a scale factor  $\Omega(t)$ , and an anisotropy parameter  $\beta(t)$ . The line element may be expressed as

$$ds^2 = -dt^2 + a_o^2 e^{2\Omega} (e^{2\beta})_{ij} \sigma^i \sigma^j, \quad (4.32)$$

where  $(\beta) = \text{diag}(\beta, \beta, -2\beta)$ , and the one forms as,  $\sigma^1 = \cos \psi d\theta + \sin \psi d\phi$ ,  $\sigma^2 = \sin \psi d\theta - \cos \psi d\phi$ , and  $\sigma^3 = d\psi + \cos \theta d\phi$ . The scalar 4-curvature is expressed in terms of  $\Omega$  and  $\beta$  and yields the action,

$$I_c = \frac{3\pi a_o^3}{4} \int \{ (\dot{\beta}^2 - \dot{\Omega}^2) - \frac{1}{6} {}^{(3)}R \} e^{3\Omega} dt. \quad (4.33)$$

Here  ${}^{(3)}R = \frac{e^{-2\beta}}{2a_o^2 e^{2\Omega}}(4 - e^{-6\beta})$  represents the scalar 3-curvature.

We treat here the scale factor  $\Omega(t)$  as the many-fingered time parameter [11] and the anisotropy  $\beta(t)$  as the dynamic degree of freedom. [4] The momentum conjugate to  $\beta$  is obtained from the Lagrangian,  $\mathcal{L}$ .

$$p_\beta = \frac{\partial \mathcal{L}}{\partial \dot{\beta}} = \frac{3}{2} \pi a_o^3 e^{3\Omega} \dot{\beta} = m \dot{\beta} \quad (4.34)$$

The dynamical Hamiltonian for this cosmology can be expressed in terms of this momentum and the Lagrangian,

$$\mathcal{H}_{DYN} = p_\beta \dot{\beta} - \mathcal{L} = \frac{1}{2m} p_\beta^2 + \underbrace{\frac{m}{2} \left( \dot{\Omega}^2 - \frac{1}{6} {}^{(3)}R \right)}_V. \quad (4.35)$$

In the classical theory  $\mathcal{H}_{DYN}$  can be used to construct either the Hamilton-Jacobi equation or the two Hamilton equations; however, to complete the dynamics we must in addition impose the super-Hamiltonian constraint,

$$p_\beta^2 = m^2 \left( \dot{\Omega}^2 + \frac{1}{6} {}^{(3)}R \right). \quad (4.36)$$

Using this dynamical Hamiltonian (Eq. (4.35)), the corresponding Hamilton-Jacobi equation and the standard quantization prescription we obtain the Schrödinger equation for the Taub cosmology.

$$-i\hbar \frac{\partial \Psi(\beta, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(\beta, t)}{\partial \beta^2} + V \Psi(\beta, t) \quad (4.37)$$

The scale factor  $\Omega$  in  $V$  and  $m$  in this equation should be treated as an unknown function of time,  $t$ . To complete the quantization we impose, in addition, Eq. (4.36) as an expectation-value equation over  $\beta$  (where  $\langle \bullet \rangle_s = \int \Psi_s^* \bullet \Psi_s d\beta$  and  $\Psi_s$  is the solution of (4.37) augmented by appropriate initial data, cf. Eq. (4.39)).

$$\dot{\Omega}^2 = \left( \frac{\langle p_\beta \rangle_s}{m} \right)^2 + \frac{1}{6} \langle {}^{(3)}R \rangle_s \quad (4.38)$$

This system of equations Eq. (4.37) and Eq. (4.38) provide us with a complete quantum-dynamic picture of the axisymmetric Taub model, and when augmented by the appropriate boundary conditions can be solved numerically.

We display in Figs. 1-2 a solution of the these two equations for an initially Gaussian wave packet.

$$\Psi_0 = C e^{i(\beta - \beta_o)p_o/\hbar} e^{-(\beta - \beta_o)^2/\Delta\beta^2} \quad (4.39)$$

with  $C^2 = \sqrt{2/\pi}/\Delta\beta$ ,  $p_o = -100$ ,  $\hbar = 1$ ,  $\beta_o = 0$ ,  $\Delta\beta = 0.1$  and  $\Omega(t=0) = 1$ . The solution was obtained using a 2nd-order split operator unitary integrator which preserves the norm exactly (up to round off errors). [16]

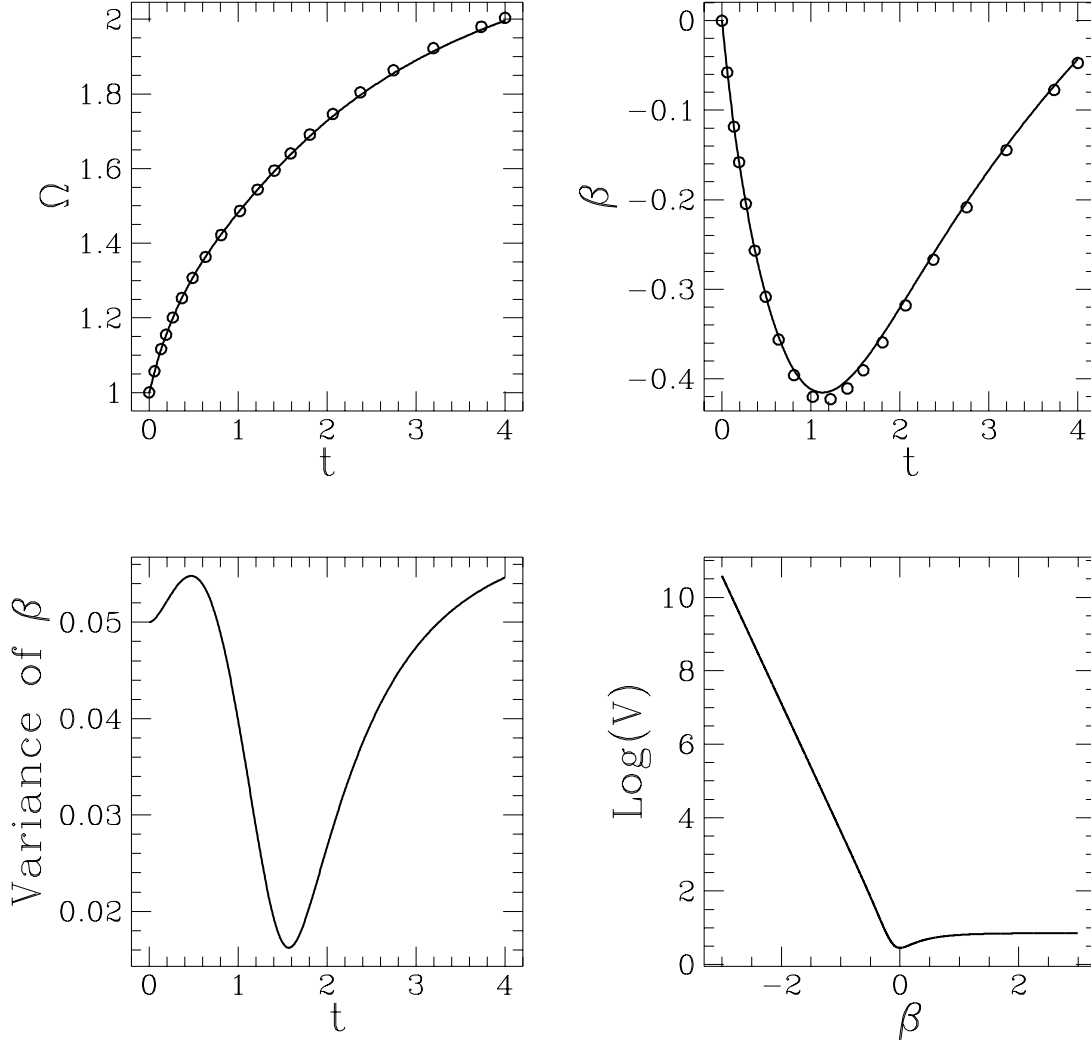


FIG. 1. *The quantum geometrodynamics of the Taub cosmology.* The solid lines in each of the four graphs represent the quantum solution, while the small circular dots in the upper two graphs represent the classical trajectory. The scale factor's ( $\Omega$ ) dependence on time ( $t$ ) is shown in the upper left, while the expectation of the anisotropy ( $\langle \beta \rangle$ ) is displayed in the upper right. The lower left shows the variance in  $\Psi_s$  throughout its evolution, and the lower right graph is a snapshot of the potential ( $V$ ) at the end of the simulation.

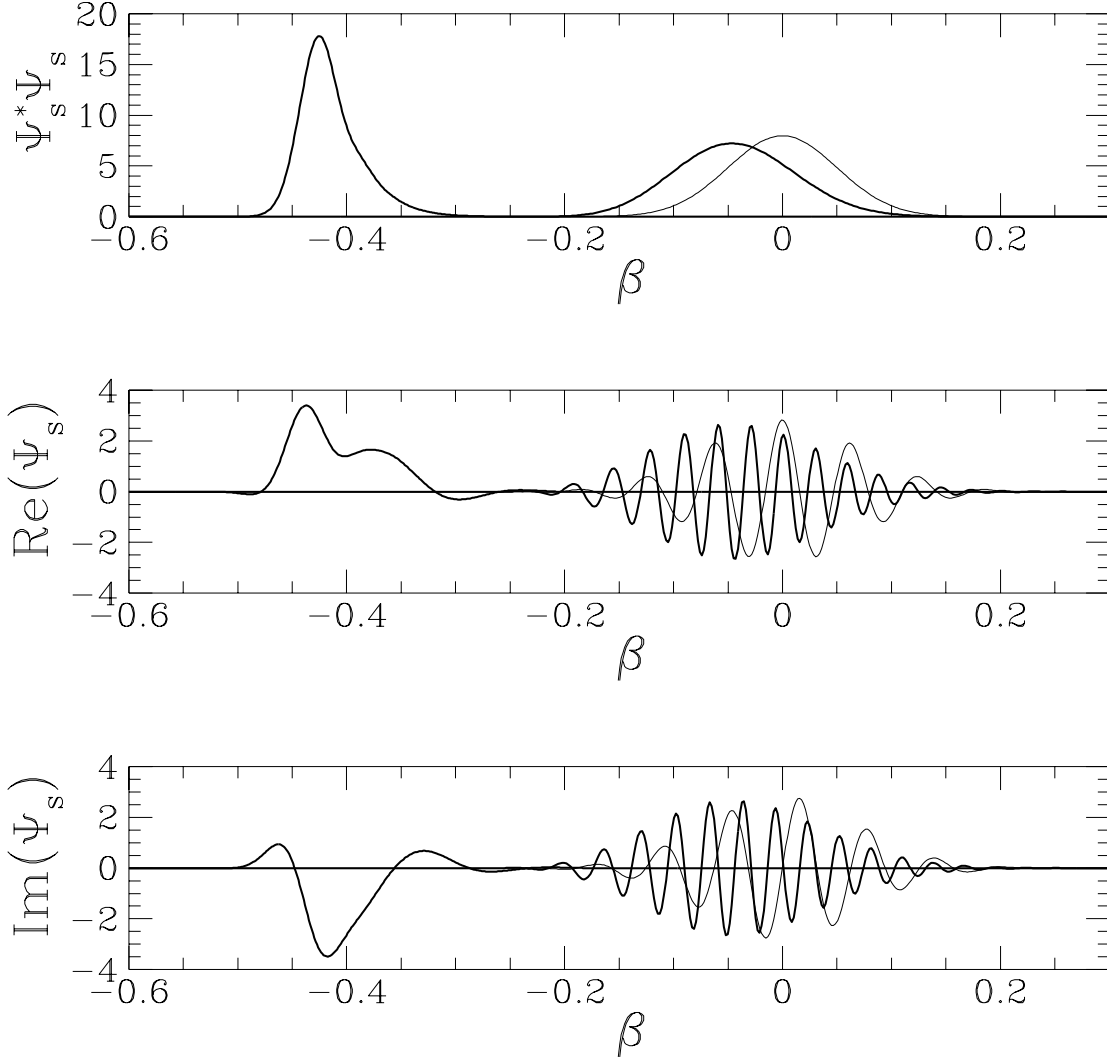


FIG. 2. *The Taub wave function.* A snapshot of the wave function at three distinct times throughout the evolution is represented on each of the three graphs. The first row plots  $\Psi_s^* \Psi_s$ , while the second and third are the real and imaginary parts of  $\Psi_s$ , respectively. The right-most wave function (plotted in a slightly narrower line width) shows the initial ( $t = 0$ ) gaussian wave function. The left-most and higher-peaked curve represents the wave function near the bounce at  $t = 1$ , while the remaining curve displays the right-going wave function well after the bounce at  $t = 4$ . This reflected wave function is spreading in time and therefore appears, and will continue to appear, lower in amplitude than the initial wave function.

We have introduced in this section an application of our approach to the quantization of gravity (this approach need not be restricted to quantum cosmology). When the Taub cosmology was quantized our theory generated a quantum picture based on a post-ADM treatment [4] of the gravitational field dynamics and appears to be free from the conceptual difficulties related with time usually associated with both the Dirac and ADM quantization procedures. Only the dynamical part, the underlying anisotropy ( $\beta$ ) has been quantized. The Hamiltonian participating in the quantization ( $\mathcal{H}_{DYN}$ ) is not a square-root Hamiltonian. This absence of a square-root Hamiltonian is a generic feature of our quantum geometrodynamics procedure.

We have demonstrated here using the Taub cosmology that the concept of time may very well be inextricably intertwined and woven to the initial conditions as well as to the quantum dynamics over the space of all conformal 3-geometries.

## V. THE CHARACTERISTIC FEATURES OF THE QUANTUM GEOMETRODYNAMIC PROCEDURE.

The quantum dynamical picture described in the two previous sections has two features that are not encountered typically in most of the more common quantum dynamical schemes. They should be kept in mind in order to avoid errors and misinterpretations in applying our procedure.

The first such feature is related to the structure of the Hamiltonian. Formally, the Hamiltonian appears as a Hamiltonian of a system placed in external time-dependent field, at least when the Schrödinger equation is analyzed. However, the external field is determined via the constraints by the quantum state of the system. Ordinarily, the Hamiltonian depends on the initial data. This feature is not unique for our approach as it is encountered elsewhere in standard quantum mechanics of Hartree–Fock systems.

The second feature is a split of the system parameters in two groups: (1) the truly dynamic variables (to be quantized); and (2) the descriptors of a “natural observer” associated to the system, driven as they are by the quantum geometrodynamics. Such a split is introduced prior to quantization, and is crucial to our theory. This feature can be clearly observed in both the general description of the geometrodynamical quantization procedure (Sec. III) and in the examples of Sec. IV. Essentially, this split is a part of our solution of the time problems. Most of the difficulties related to the problems of time are caused by an attempt to enforce general covariance at the quantum level. Such an attempt for gravitational systems is bound to fail. It leads to the questions that are not well defined unless they are referred to a background spacetime. Standard attempts to introduce a background spacetime tend to use a fixed spacetime with properties generally not related to the properties of the system itself, which is commonly considered as an unsatisfactory feature (we quite agree with this conclusion). Our procedure, on the other hand, can be considered as a universal prescription for defining a unique background spacetime structure driven by, and back reacting on, the quantum system. All questions concerning covariance should be referred to this spacetime structure, and this spacetime structure is ordinarily not classical. It cannot be considered as a result of a 3–geometry evolution (the classical evolution equations do not take a part in our procedure). In particular, we recover the classical evolution only through an appropriately–peaked wave function and only through constructive interference over conformal superspace.

The split of the metric variables into quantized geometrodynamical variables and “classical” or non-quantized variables amounts to a representation of a quantum gravitational system as two interacting subsystems. One of the subsystems is quantum while the other one is classical. Evolving the many-fingered time variable and coordinatization variables by the shift and lapse is just another aspect of such a subdivision of the quantum gravitational system. From this point of view the functional evolution problem and the multiple choice problem are simply ill posed. Different time evolutions or different quantum theories referred to in the description of these problems are essentially created by introducing different physical systems, or by introducing different models of what is believed to be the same system. In other words, quantum geometrodynamics yields a finer resolution of different systems than classical geometrodynamics. The reason that this split occurs is related to the key features of gravitational systems. In particular, the observer in gravitational systems is a part of the system, and the measurement process (coupling of a measuring device with the quantum subsystem) cannot be switched on and off at will. In quantum cosmology a particular split of variables into quantum and classical essentially reflects our view concerning the nature of time in the Universe. Whether this view is correct or not can be decided only on the basis of observations.

We wish to emphasize here that the square-root Hamiltonian quantization procedure essentially works in a way similar to our geometrodynamical quantization procedure. It starts from the split of variables in embedding parameters (they are to become the functional parameters of the state functional, essentially classical) and dynamic variables, that are to be quantized. A computation of the expectation values of observables involves functional integration over the dynamic variables while the embedding parameters are not integrated over. However the approach does not contain a meaningful prescription of time parameterization. An attempt to reintroduce such a prescription using shift and lapse parameterization of time evolution (cf. Sec. VI) leads to the theory demonstrating the same features as our procedure. The spectral analysis problem characteristic to this approach is a consequence of an attempt of one to transfer to quantum gravity the fully-constrained property of classical geometrodynamics. Our procedure forfeits this feature and instead introduces a system of equations that is not overdetermined, and for which the concept of being fully constrained loses its meaning. Nevertheless, this property is recovered in the classical limit only. As a result, the geometrodynamical approach does not introduce the square-root Hamiltonian.

Not every assignment of dynamical variables leads to a consistent quantum dynamical picture. For example, a bad choice of the dynamical component of the 3–geometry might lead to a non-elliptic differential operator on the right hand side of Schrödinger equation. Furthermore, under such conditions not every shape of the initial wave packet will agree with the constraints. In any case, only an appropriate choice of the set of dynamic variables will lead to a reasonable quantum mechanical picture. The properties of the ADM procedure together with the results of York indicate that there is at least one such reasonable choice. Is there more than one possible quantum dynamical picture for the same system? From our point of view, the existing theory is ill equipped to ask this question. The notion of

equivalence of quantum gravitational systems is simply not developed. Extreme care should be exercised when one attempts to discuss this question naively.

## VI. THE PROBLEMS OF TIME.

The issues related to time evolution in gravity have been summarized recently by Kuchař. [2] He calls them the “problems of time” in quantum gravity.

The first of the time problems is formulated as a dependence of the final state  $\Psi_{\Sigma_{FIN}}$  at a given  $\Psi_{\Sigma_{IN}}$  on the foliation that connects the spacelike hypersurfaces  $\Sigma_{IN}$  and  $\Sigma_{FIN}$ . This problem is called the problem of functional evolution.

The second problem is observed by Kuchař in the context of the square-root Hamiltonian quantization. It stems from the fact that the split of the metric variables into the true dynamic variables  $\beta$  as opposed to the coordinatization parameters  $\alpha$  and the slicing parameter  $\Omega$  is not unique. Different splits might lead to different and possibly inequivalent quantum theories. Kuchař calls this the multiple-choice problem.

The third difficulty is the so called Hilbert space problem and is particular to the Dirac quantization procedure. In the square-root Hamiltonian approach, the Schrödinger equation automatically determines an inner product and a possibility to form, at given values of  $\alpha$  and  $\Omega$  meaningful observables, for which expectation values can be evaluated in a way similar to that of standard quantum mechanics (modulo some technical difficulties caused by the functional nature of variables). Nothing similar is granted by the Wheeler–DeWitt equation of the Dirac approach. The Wheeler–DeWitt equation is a second order functional differential equation for which the problem of supplying the space of its solution with the structure of a Hilbert space encounters numerous difficulties. Roughly speaking, the equation does not admit a one–Universe interpretation.

We wish to add to these three problems one more (also described by Kuchař), namely the spectral-analysis problem that shows up if the square-root Hamiltonian is involved. It is well known that, if the operator expression under the square root is not positively definite, the spectral analysis procedure yields, as the result, a Hamiltonian that is not self-adjoint. This implies that the Schrödinger equation with such a Hamiltonian does not produce unitary evolution.

The functional-evolution problem and the multiple-choice problem are, essentially, two aspects of the same problem. Both these problems in quantum gravity are caused by (1) the ADM treatment of the whole 3–geometry as dynamic, and (2) the fully-constrained feature of the dynamics of the gravitational field. In both procedures of quantization (Dirac and ADM) these two requirements are expressed via the demand for the classical constraints to yield operator equations that should be satisfied for “any slicing” (or, as they frequently say, for any parameterization). The concept of “any parameterization”, however, is essentially classical, which is especially clear when it is thought of as a slicing. The concept of slicing works quite well in the classical theory where the 4–geometry of spacetime is unique, but does not have any analogue in the quantum theory unless we introduce one and give it an appropriate meaning. We believe that both problems emerge as a result of different assumptions concerning the split of a gravitational system into its classical and quantum parts which interact with each other. Different splits ordinarily provide physically different models of the gravitational system and cannot be compared to each other in the absence of a common spacetime. In other words, the concept of equivalent systems is absent in the theory and should be developed prior to posing these questions.

Although there are many different ways to specify coordinatization parameters and slicing parameters, not all of them are meaningful in the sense that the resulting time can be related to measurements. A standard procedure to introduce a meaningful time in general relativity uses the lapse and shift functions. In this approach the coordinatization and slicing conditions are given arbitrarily only at the initial slice. Subsequently, both the coordinatization and slicing conditions are propagated via specifying shift  $N^i$  and lapse  $N$  functions. In order to facilitate an explanation of the consequences of such a description for the square-root Hamiltonian quantization procedure, it is useful to rewrite the action principle and Hamilton equations in terms of the  $\beta$ ,  $\alpha$ , and  $\Omega$  parameters.

The expression (2.3) for the action can be written symbolically in the form

$$I = \int \left[ \pi^\beta \dot{\beta} + \pi^\alpha \dot{\alpha} + \pi^\Omega \dot{\Omega} - N\mathcal{H}(\pi^\beta, \pi^\alpha, \pi^\Omega, \beta, \alpha, \Omega) - N_i \mathcal{H}^i(\pi^\beta, \pi^\alpha, \pi^\Omega, \beta, \alpha, \Omega) \right] d^4x, \quad (6.1)$$

where

$$\pi^\beta = \frac{\partial \mathcal{L}}{\partial \dot{\beta}}, \quad \pi^\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\alpha}}, \quad \pi^\Omega = \frac{\partial \mathcal{L}}{\partial \dot{\Omega}}, \quad (6.2)$$

are the momenta conjugate to  $\beta$ ,  $\alpha$ ,  $\Omega$  respectively. Expression (2.7) takes the form



$$\mathcal{H}_{ADM}(\pi^\beta, \pi^\alpha, \pi^\Omega, \beta, \alpha, \Omega, N, N_i) = N\mathcal{H}(\pi^\beta, \pi^\alpha, \pi^\Omega, \beta, \alpha, \Omega) + N_i\mathcal{H}^i(\pi^\beta, \pi^\alpha, \pi^\Omega, \beta, \alpha, \Omega). \quad (6.3)$$

The Hamilton equations (2.8), (2.9) become

$$\dot{\beta} = \frac{\partial\mathcal{H}_{ADM}}{\partial\pi^\beta}, \quad (6.4)$$

$$\dot{\alpha} = \frac{\partial\mathcal{H}_{ADM}}{\partial\pi^\alpha}, \quad (6.5)$$

$$\dot{\Omega} = \frac{\partial\mathcal{H}_{ADM}}{\partial\pi^\Omega}, \quad (6.6)$$

and

$$\dot{\pi}^\beta = -\frac{\partial\mathcal{H}_{ADM}}{\partial\beta}, \quad (6.7)$$

$$\dot{\pi}^\alpha = -\frac{\partial\mathcal{H}_{ADM}}{\partial\alpha}, \quad (6.8)$$

$$\dot{\pi}^\Omega = -\frac{\partial\mathcal{H}_{ADM}}{\partial\Omega}. \quad (6.9)$$

The first three equations are merely kinematic relations between the parameters  $\beta$ ,  $\alpha$ ,  $\Omega$  and their conjugate momenta (they can be introduced in any spacetime, not necessarily a solution of Einstein equations). The role played by these relations (6.5), (6.6) is to provide a parameterization (by shift and lapse) of the slicing and coordinatization conditions given on the initial hypersurface to the whole spacetime. It is important to realize that this parameterization depends on the initial-value problem and is interwoven with the dynamics of the problem. Parameterizing the slicing and coordinatization conditions can be achieved by different means. However, one can convince oneself that any reasonable set of slicing and coordinatization conditions can be translated into the language of shift and lapse, and that such a translation provides a proper interpretation of the conditions in terms of measurements.

Equations (6.5), (6.6) can be transferred to the square-root Hamiltonian quantization procedure. One should notice, however, that the right hand sides of these equations depend on both  $\alpha$ ,  $\Omega$  and the true dynamic variables  $\beta$  that are given, after quantization, by distributions. The situation can be remedied if their expectation values over the solution of the Schrödinger equation (augmented by the initial-value data) are substituted in (6.5), (6.6). It is clear that no contradictions can be introduced this way (which follows from the simple count of equations and unknowns). The relations (6.5), (6.6) can be thought of as definitions of  $N^i$ ,  $N$  in quantum gravity. In practice, this procedure can become rather complicated. In particular, the parameterization contains functions (the expectation functions of dynamic variables as well as of their conjugate momenta) that can be completely determined only after the Schrödinger equation has been solved. This means that parameterization of coordinatization functions and the many-fingered time scale parameter by the shift and lapse results in an implicit procedure in which even the parameterization itself remains undetermined until the final solution of the entire problem has been obtained. Nevertheless, it seems to be unavoidable if one is to interpret the solutions of the quantum gravity equations in terms of observations. Appropriate care should be taken, as, in general, equation (6.4) does not have to be satisfied anymore even for the expectations, which means that, in general, the expectation values of momenta components are not related to the extrinsic curvature of the spacetime determined by the expectation values of the 3-metric, shift, and lapse. The spacetime itself is not a solution of Einstein equations.

A brief summary of the this section is that the problems of time in quantum gravity apparently originate from two mutually related sources.

The first source is a treatment of the classical gravitational field dynamics as dynamics of the 3-metric or 3-geometry of a spacelike slice. Both the Dirac and the square-root Hamiltonian quantization procedures are essentially based on the original ADM [5] picture of the gravitational field dynamics. In this picture, the dynamic evolution of the gravity field manifests itself as a change from one spacelike 3-geometry to another. In other words, the configuration space of geometrodynamics is believed to be Wheeler's superspace. [17] Such an approach does not utilize York's analysis of the gravitational field's degrees of freedom and the proper initial-value problem formulation. [4,14] This is not surprising as the foundations of quantum gravity [8,10] were originally formulated before York completed his investigation.

The second source of difficulties, internally related to the first one, is the demand that the quantum gravitational dynamics should be fully constrained, just as the dynamics of classical general relativity. This line of reasoning leads to the conclusion that, to quantize the gravity field, one needs only to quantize the constraints.

The picture of the dynamic evolution of the gravity field as a change from one 3-geometry to another leads in classical geometrodynamics to a peculiar situation wherein the Hamiltonian of the Hamilton-Jacobi equation (super-Hamiltonian  $\mathcal{H}$ ) do not ordinarily coincide with the Hamiltonian participating in Hamilton evolution equations ( $\mathcal{H}_{ADM} = N\mathcal{H} + N_i\mathcal{H}^i$ ). It can be argued that other pieces of  $\mathcal{H}_{ADM}$  are contained in the equations of the super-momentum constraint. Nevertheless, the shift and the lapse are thrown out of the picture, and, together with them, a meaningful description of a time evolution of gravitational systems. This does not cause any difficulties in the classical theory as one can always reintroduce such a description either by retrieving all the Hamilton evolution equations (that are compatible with the constraints) or just four of them (the ones that provide kinematic relations between the time derivatives of the coordinatization and the many-fingered time functions and the conjugate momenta). One should keep in mind, however, that these relations, ordinarily, involve all the conjugate momenta, including those of truly dynamic variables. The classical theory on this level treats all the variables in essentially the same way (as functions on a spacelike slice). The situation is entirely different for a quantum theory. Some of the variables become the arguments of the state functional, others become the functional parameters of the state functional. The arguments of the state functional form the superspace. A computation of the expectation value of an observable implies integration over the superspace, while the functional parameters are given according to some principles that are not a part of the dynamic picture. Ordinarily, the superspace is formed by the dynamic variables. In Dirac's quantization all the variables are treated on equal footing and superspace is formed by all the components of the 3-metric. There are no functional parameters to be used to fix coordinatization conditions or the many-fingered time. Sometimes, as an afterthought, the superspace is reduced to the space of 3-geometries, in which case there are three functional parameters to fix coordinatization, but then, in order to satisfy the super-momentum constraints, the observables are demanded to be independent of these three functional parameters. The many-fingered time parameter is not in this picture. If one decides to treat the many-fingered time parameter as a part of the answer rather than of a question, one can try to introduce it as an expectation function of one of the 3-geometry parameters. Questions and difficulties related to such a procedure have been outlined in the previous section.

We wish to emphasize here that this entire situation is created by the treatment of the 3-geometry as the dynamic object and an attempt to quantize the 3-geometry. Mathematically, it is expressed via imposing the commutation relations [10] on all the components of the 3-metric. One should keep in mind, however, that in general relativity the system described by the 3-metric or even the 3-geometry includes in itself the observer and his clock. In standard quantum mechanics, or even in the quantum field theory, an observer is external with respect to the quantum system. The observer is classical and has an external classical clock. A measuring device can be coupled to a quantum system and uncoupled from it at will. There cannot be an external observer in the description of the gravitational field because the gravitational system is the Universe itself. An observer cannot switch on and off the coupling of a classical measuring device to the system. Dirac's approach essentially quantizes the observer and his clock on equal footing with the rest of the system (J. A. Wheeler would say that many-fingered time is quantized). Whether such a quantized observer and his clock can function in a fashion providing an opportunity to describe the system consistently is not clear to us. A discussion of such a possibility, however exciting, clearly would lead us far beyond the scope of this paper.

Another resolution to the time evolution problem in Dirac's quantization approach would be to first solve the Wheeler-DeWitt equation and subsequently split all six variables into (1) the two true dynamic variables  $\beta$ , and (2) the remaining four embedding variables  $\alpha$  and  $\Omega$ . The embedding variables could then be reserved for fixing the coordinates as well as the many-fingered time parameter, while the true dynamic variables should be integrated over only when the expectations of the observables are computed. Unfortunately, such a procedure would contradict the entire ideology behind the Dirac quantization procedure.

The ADM square-root Hamiltonian procedure of quantization avoids the impossibility of introducing a functional time via solving the super-momentum and super-Hamiltonian constraints on the classical level and quantizing the resulting equations. The superspace of the theory is thus reduced to the space of true dynamic variables with four functional parameters left as coordinatization parameters and a many-fingered time parameter. This procedure, however, leads to a Schrödinger equation with a square-root Hamiltonian, which poses well known technical difficulties briefly discussed above. That no prescription for a meaningful choice of the coordinatization and many-fingered time variables, similar to the shift and lapse parameterization of classical general relativity, is supplied in this approach presents another difficulty of a conceptual nature. An attempt to augment the procedure by such a prescription is described above. The description of the evolution becomes implicit with a parameterization being finalized only after the entire problem of evolution has been solved. We wish to stress at this point that the ADM square-root approach to gravity quantization starts, as Dirac's approach, from a treatment of classical gravity field dynamics as the dynamics of a 3-geometry. Exactly as in Dirac's approach, the fully-constrained property of classical general relativity is

introduced as a key feature that is to be transferred to the quantum theory literally. This is accomplished by turning the constraint equations into operator equations which restrict the admissible states of the gravitational systems. The only difference is that the constraints are solved before quantization in order to introduce a Hamiltonian that, in a sense, can be interpreted as a time evolution generator. The (functional) dimension of the superspace is reduced to the correct dimension as determined by a proper analysis of gravitational degrees of freedom. [4,14] However, this achievement comes almost as an afterthought, a fix of a deficiency that has been created by the interpretation of the 3-geometry as the principal dynamic object, which, in turn, has forced on the theory an identification of the constraints with the Hamilton-Jacobi equation. The square-root Hamiltonian participating in this equation again differs from the Hamiltonian  $\mathcal{H}_{ADM}$  participating in Hamilton equations (2.8), (2.9) and is not related to the Lagrangian by a relation similar to (2.15). Such a switch of Hamiltonians is possible only because the classical dynamics of general relativity is fully constrained. In transition to the quantum theory, the achievements of this entire procedure come at a price, creating difficulties on both the technical and the conceptual levels.

## VII. DISCUSSION.

In this paper we have reviewed the issue of time evolution in quantum gravity. Both the Dirac and the ADM square-root Hamiltonian quantization procedures create difficulties in introducing a meaningful concept of time evolution. The difficulties in both approaches, we conclude, stem from their common tendency to transfer to quantum theory an interpretation of classical geometrodynamics as an evolution of a spacelike hypersurface 3-geometry together with the fully-constrained property of classical geometrodynamics. These two features essentially lead in classical geometrodynamics to an identification of Hamilton-Jacobi equation with the constraints and thereby removing a meaningful concept of time. This procedure does not present a serious problem in the classical theory of gravity as time evolution can be reinserted back. However, it leads to two different Hamiltonians in the theory. One of them participates in Hamilton evolution equation, the other one is the Hamiltonian of Hamilton-Jacobi equation.

These two features of classical general relativity, when transferred to quantum gravity, motivate quantization of constraints (in original form in Dirac's procedure, or resolved with respect to the momenta conjugate to the embedding variables in the ADM square-root Hamiltonian procedure) via imposing their operator versions on the state functional. The resulting quantum theories naturally counter any meaningful concept of time evolution. They lead to what Kuchař has identified as the "problems of time," including (1) the problem of functional evolution, (2) the problem of multiple choice, (3) the Hilbert-space problem of Dirac's approach and (4) the spectral-analysis problem of the square-root Hamiltonian approach.

We have discussed the problems of time mainly for the ADM square-root Hamiltonian quantization, as Dirac's quantization, when interpreted literally does not admit a functional many-fingered time. Presumably, it can be somehow reinterpreted (cf. Sec. II), but one such avenue leads back to the ADM square-root Hamiltonian approach. This approach has all the time problems in it except the Hilbert-space problem. The way the ADM approach is ordinarily formulated leaves little room for an analysis of the functional-evolution problem and the multiple-choice problem. At first sight, these problems appear to be genuine. The spectral-analysis problem seems to be unavoidable for the square-root Hamiltonian quantization. In addition, we conclude that the approach suffers from the lack of a meaningful prescription for time parameterization (although it admits functional time). Our attempt to reintroduce into the ADM quantization a time parameterization of evolution by shift and lapse has led to an implicit set of equations with a parameterization depending on the initial values. This last feature does not destroy the procedure but it gives one an idea of what one is to expect if he introduces time in a way accessible to measurements.

The new idea presented in this paper is to reconsider the classical picture of geometrodynamics via representing it as a dynamic theory of two interacting subsystems, one being described by the dynamic components of the 3-metric, and the other being described by the many-fingered time variable and coordinatization variables. If one is to keep the concept of classical time in the theory, one should quantize only the first subsystem, while treating the second one as classical yet generated by, and back reacting on, the first (quantum) subsystem. In such an approach classical geometrodynamics can be described by two sets of equations. Equations describing the evolution of the first subsystem treat the second subsystem as an external field. Having in mind that, eventually, we are to quantize the first subsystem, we describe its evolution by an appropriately formulated Hamilton-Jacobi equation. For the second system we use as field equations super-momentum and super-Hamiltonian constraints rewritten in appropriate variables. The full system of equations is equivalent to the ten Einstein equations. However all the equations of this system are independent.

Quantization of such a system is achieved via turning the Hamilton-Jacobi equation into a Schrödinger equation. The constraint equations describe the second subsystem as classical but include the dynamic variables as a source. We replace them by the expectation values of these variables over the solution of our Schrödinger equation (with

appropriate initial data). It is clear that in such a procedure of quantization there is no Hilbert space problem. The Hamiltonian of the Schrödinger equation does not contain square roots which eliminates the spectral-analysis problem as well.

The problem of functional evolution and the multiple-choice problem obtain an interesting interpretation in our proposed approach. In our geometrodynamics procedure of quantization they turn into a statement that the solution of a given problem depends on the split of the original system into its quantum and classical subsystems. In particular, different splits generate essentially different systems. In other words, quantum geometrodynamics resolves gravitational systems finer than classical geometrodynamics.

We have illustrated our procedure using a Bianchi 1A cosmology and an axisymmetric Taub model. When these models were quantized, our theory generated quantum geometrodynamics pictures based on a post-ADM treatment of the gravitational field dynamics and was free from the conceptual difficulties usually associated with the Dirac and ADM procedure of quantization. The variables describing the gravity field in these cases have been split into the true dynamic variables and a parameter related to Wheeler’s many-fingered time. Only the dynamical part, the underlying anisotropy, has been quantized. The Hamiltonian participating in the Schrödinger equation is not a square-root Hamiltonian. This absence of a square-root Hamiltonian is generic for our quantum geometrodynamics procedure. The effective “background spacetime” determined by the expectation values of dynamic variables together with the “observers” (related to Wheeler’s many-fingered time) allows us to pose unambiguously the questions of covariance.

The problems outlined in Sec.VI become all but eliminated by our quantum geometrodynamics approach. The nontrivial part of gravity quantization appears to shift from such conceptual problems too the problem of (1) the choice of an appropriate model to quantize, and to the related problem of (2) the choice of an appropriate initial condition for the wave functional. Both choices are crucial if one is to attempt using quantum geometrodynamics to better comprehend the properties of gravitational systems. It is our understanding that the success of gravity quantization rests on such meaningful choices. Furthermore, the choice of models should not be determined by the structure of quantum geometrodynamics; rather, it should be determined by observational data and our general understanding of gravitational phenomena.

It is clear that the procedure of quantizing the dynamical part of constrained systems described in this paper can be performed in the general case of geometrodynamics without any complications in principle, although it may become quite involved computationally as compared to our simple model examples. [18] The procedure differs only in two respects from the simplistic examples presented here. The first difference arises when the 3-metric is (1) parameterized by three coordinatization parameters, (2) the many-fingered time parameter, and (3) the two dynamic variables; then all four constraints should be solved with respect to the coordinatization parameters and the scale factor. In all four constraints the expectation values of the true dynamic variables should be used. The second difference is caused by the functional nature of the gravitational field dynamics in the general case. The operation of functional integration is involved, which might lead to analytic difficulties. Such difficulties are not specific for our approach as they are common for the canonical formulations of all field theories. Quantum geometrodynamics, in particular, does not seem to generate any specific new difficulties. In this paper we forwarded the beginnings of a quantization scheme consistent with York’s analysis of the gravitational degrees of freedom. Our particular imposition of the four constraint equations leads to a weaker theory that in turn avoids the problems of time.

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