

The Issue of Time in Quantum Geometrodynamics

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Abstract.

Standard techniques of canonical gravity quantization on the superspace of 3-metrics are known to cause insurmountable difficulties in the description of time evolution. We forward a new quantization procedure on the superspace of true dynamic variables – geometrodynamical quantization. This procedure takes into account the states that are “off-shell” with respect to the constraints and thus circumvents the notorious problems of time. In this approach quantum geometrodynamics, general covariance, and the interpretation of time emerge together as parts of the solution to the total problem of geometrodynamical evolution.

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The standard approach to canonical quantum gravity [1, 2] is based on the classical dynamic picture of the evolving 3-geometry of a slicing of a spacetime manifold described by the lapse function N and the shift functions N^i . The canonical variables are the 3-metric components g_{ik} on a spatial slice Σ of the foliation induced by the spacetime 4-metric, and their canonical conjugate momenta π^{ik} . The customary variational procedure applied to the Hilbert action expressed in terms of these canonical variables yields Hamilton dynamics that, after applying the canonical quantization procedure on the superspace of 3-metrics (in both Dirac’s and ADM square root Hamiltonian approaches), produces a quantum theory that appears to be incapable of providing a consistent description of time evolution for quantum gravitational systems. The source of the difficulties can be traced to mixing dynamical considerations with the requirements of general covariance and to restricting quantum states to the shell determined by constraints.

The situation changes dramatically if York’s analysis of gravitational degrees of freedom [4] is taken into account and actively utilized. According to York, the set of six parameters describing the slice 3-metric should be split into two subsets, $\{\beta_1, \beta_2\}$ (two functions) and $\{\alpha_1, \alpha_2, \alpha_3, \Omega\}$. The first of these is treated as the set of true gravitational degrees of freedom (the initial values for them can be given freely), while the second is considered to be the set of embedding variables. The α parameters are often referred to as coordinatization parameters, while Ω is called, depending on the context, the slicing parameter, the scale factor, or the many-fingered time parameter. Information relevant to dynamics is carried by β parameters, while α and Ω essentially describe time. The true dynamic variables form what we call a dynamic superspace while the embedding variables are treated as functional parameters.

The idea is to develop geometrodynamics from the very beginning on the dynamic superspace instead of the superspace of 3-metrics or 3-geometries. The variational principle on the dynamic superspace or its phase space (formed by true dynamic variables $\{\beta_1, \beta_2\}$ and their conjugate momenta $\{\pi_{\beta_1}, \pi_{\beta_2}\}$) yields the dynamic equations describing the evolution

of the true dynamic variables. All of these equations depend on lapse and shift and contain embedding variables as functional parameters. These are treated as an external field and are determined by additional equations that do not follow from the variational principle on dynamic superspace. The quantization procedure is performed on the dynamic superspace (only β -s are quantized, i. e. generate commutation relations, while the embedding variables form a classical field). The Schrödinger equation is obtained by a quantization procedure from the Hamilton–Jacobi equation on the dynamic superspace and describes the time evolution of the state functional on the true dynamic superspace coupled with the external classical field determined by the embedding variables. Such a coupling can be achieved via a procedure similar to that of Hartree–Fock.

In a more detailed and precise description that follows, we omit indices on variables β and α for the sake of notational simplicity. They can be recovered easily whenever necessary.

We start from the standard Lagrangian \mathcal{L} (written in terms of the 3–metric, shift and lapse) and the associated action (with appropriate boundary terms, as needed, to remove the second time derivatives terms) and we introduce the momenta conjugate to the true dynamic variables

$$\pi_\beta = \frac{\partial \mathcal{L}}{\partial \dot{\beta}}. \quad (1)$$

We then use these π_β 's to form the geometrodynamical Hamiltonian \mathcal{H}_{dyn} ,

$$\mathcal{H}_{dyn} = \pi_\beta \dot{\beta} - \mathcal{L}. \quad (2)$$

The arguments of the Hamiltonian \mathcal{H}_{dyn} are described by the expression

$$\mathcal{H}_{dyn} = \mathcal{H}_{dyn}(\beta, \pi_\beta; \Omega, \alpha). \quad (3)$$

The variables following the semicolon are treated as describing an external field, while the ones preceding the semicolon are the coordinates and momenta of the true gravitational degrees of freedom, i.e. of the true geometrodynamics. The variation of β and π_β produce Hamilton equations on the dynamic superspace, while variation of the ends leads to the Hamilton–Jacobi equation

$$\frac{\delta S}{\delta t} = -\mathcal{H}_{DYN} \left(\beta, \frac{\delta S}{\delta \beta}; \Omega, \alpha \right). \quad (4)$$

Here S is a functional of β and, in addition, a function of t ,

$$S = S[\beta; t]. \quad (5)$$

and $\frac{\delta}{\delta t}$ is defined by

$$\frac{\partial S}{\partial t} = \int \frac{\delta S}{\delta t} d^3x. \quad (6)$$

The Hamilton–Jacobi equation (4) is incapable of providing any predictions as its solutions depend on the functional parameters Ω and α which are not yet known. One can complete the picture by adding the standard constraint equations of general relativity, obtained by variations of shift and lapse. These constraints should be satisfied once the solution for the geometrodynamical variables β, π_β (with appropriate initial data) is obtained and substituted. Using the symbols $[\beta]_s, [\pi_\beta]_s$ for such a solution, we have

$$\mathcal{H}^i([\beta]_s, [\pi_\beta]_s, \Omega, \alpha) = 0, \quad (7)$$

$$\mathcal{H}([\beta]_s, [\pi_\beta]_s, \Omega, \alpha) = 0. \quad (8)$$

These constraint equations should be treated as additional symmetries, or the equations for an external field. They do follow from the shift and lapse invariance of the action but their

derivation in this new setting depends on the structure of the whole action integral. As a result, they cannot replace the full set of equations for geometrodynamics evolution (which is usually done on the superspace of 3-metrics). However, the resulting complete system of equations is equivalent to that of the standard geometrodynamics on the superspace of 3-geometries [6].

For the purpose of quantization, we make a transition to the corresponding Schrödinger equation based entirely on dynamics and ignoring the system symmetries

$$i\hbar \frac{\delta \Psi}{\delta t} = \widehat{\mathcal{H}}_{dyn}(\beta, \widehat{\pi}_\beta; \Omega, \alpha) \Psi \quad \text{where} \quad \widehat{\pi}_\beta = \frac{\hbar}{i} \frac{\delta}{\delta \beta}. \quad (9)$$

The Schrödinger equation (9) implies that commutation relations are imposed only on the true dynamic variables and treats the embedding variables as external classical fields. The state functional Ψ in this equation is a functional of β and a function of t ,

$$\Psi = \Psi[\beta, t]. \quad (10)$$

This Schrödinger equation (with specific initial data) can be solved (cf., for instance the example of the Bianchi 1A cosmological model [5, 6]). The resulting solution Ψ_s of this Schrödinger equation is not capable of providing any definite predictions as it depends on four functional parameters Ω, α which remain at this stage undetermined. All expectations, such as the expectation values of β

$$\langle \beta \rangle_s = \langle \Psi_s | \beta | \Psi_s \rangle = \int \Psi_s^* \beta \Psi_s \mathcal{D}\beta \quad (11)$$

or of $\widehat{\pi}_\beta$

$$\langle \pi_\beta \rangle_s = \langle \Psi_s | \widehat{\pi}_\beta | \Psi_s \rangle = \int \Psi_s^* \widehat{\pi}_\beta \Psi_s \mathcal{D}\beta \quad (12)$$

also depend on these functional parameters. To specify these functions we resort to the constraint equations. The treatment of the constraints has nothing to do with the quantization of geometrodynamics. It merely introduces the coupling between the already quantized geometrodynamics and the classical field determined by the embedding variables. In other words, the constraints take care of the symmetries which are classical in nature to the extent that they are capable of doing so.

As in case of classical geometrodynamics, we impose the constraints on the solution of the dynamic equations (Schrödinger equation) with appropriate initial data and in this way, determine the unique values of Ω and α . It is possible that there are several ways to couple the constraints to the quantization of the true dynamic variables, β . Here we impose the four constraints only on the expectation values of the conformal dynamics

$$\mathcal{H}^i(\langle \beta \rangle_s, \langle \pi_\beta \rangle_s, \Omega, \alpha) = 0 \quad (13)$$

$$\mathcal{H}(\langle \beta \rangle_s, \langle \pi_\beta \rangle_s, \Omega, \alpha) = 0.$$

Lapse and shift are assumed to be given either explicitly or by additional conditions.

Evolution can be described as follows. Initial data at $t = t_0$ consist of the initial state functional $\Psi = \Psi_0$ and the initial values (functions) of embedding variables. In addition, lapse and shift are supposed to be given either explicitly or by additional conditions. Equations (11), (12) yield the expectation values (functions) of the true dynamic variables and their conjugate momenta. The results are substituted into the constraints (13). After this, the constraints are solved with respect to the time derivatives of embedding variables. A step forward in time (say, with the increment Δt) is performed by integration of the constraints to evolve the embedding variables and by integration of the Schrödinger equation (9) to evolve the state functional. This concludes one step forward in time. The next step is performed by repeating the same operations in the same order.

One can be referred to [5, 6] for two particular examples illustrating such geometrodynamical evolution for the Bianchi 1A and Taub cosmologies respectively. The first one can and has been solved analytically, while the latter one has been solved numerically.

The resulting canonical gravity quantization procedure circumvents all the standard problems of time and removes all the obstacles for describing the time evolution of quantum gravitational systems. This has been achieved by including “off-shell” quantum states and imposing the constraints only on the expectation values of the dynamic variables.

It should be stressed that all three components of the evolution procedure described for quantum geometrodynamical systems — quantum dynamics itself, constraints enforcing the symmetries (general covariance), and the interpretation of time — emerge together as the solution to the total problem of geometrodynamical evolution.

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