

Clifford Geometrodynamics

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Abstract

Classical anti-commuting spinor fields and their dynamics are derived from the geometry of the Clifford bundle over space-time via the BRST formulation. In conjunction with Kaluza-Klein theory, this results in a geometric description of all the fields and dynamics of the standard model coupled to gravity and provides the starting point for a new approach to quantum gravity.

1 Introduction

In most approaches to the classical and quantum dynamics of spinor fields the fields and their properties are postulated ad-hoc, without any geometric motivation as to why they should exist, but only the rationale that they are necessary to represent fermions. The desired mathematical structures, such as complex valued matrix columns and the necessity that the field components anti-commute, are put in by hand along with the equations of motion, without any geometric justification. It is the purpose of this paper to propose a new geometric foundation and justification for the existence and dynamics of spinor fields. In so doing, all the fields and dynamics of the standard model may be derived from pure geometry.

The key to this construction is to begin with the Clifford algebra bundle (or the associated matrix bundle) as the fundamental geometric framework. This fibre bundle has a connection and curvature, and a frame provides a bundle map to the cotangent bundle and gives the metric on the pseudo-Riemannian base manifold. The dynamics of the Clifford connection and frame is given by extremizing the total scalar Clifford bundle curvature, the gravitational action [1]. The curvature is invariant under adjoint automorphisms of the Clifford bundle, and this gauge symmetry, when properly restricted using the BRST formulation [2], results in the appearance and familiar dynamics of a pair of anti-commuting, Clifford valued spinor fields. The result is the coupled system of gravitational and spinor fields. A Kaluza-Klein decomposition of the vielbein then provides the vector gauge fields, completing the picture.

2 Geometric framework

To describe the geometry of the physical universe, an n dimensional, pseudo-Riemannian differential manifold is used as the base space for the Clifford algebra fibre bundle. The n basis vector elements, $\{\gamma^\alpha\}$, for the Clifford bundle provide a local trivialization, $\partial_i \gamma^\alpha = \frac{\partial}{\partial x^i} \gamma^\alpha = 0$, and satisfy, under the symmetric product,

$$\gamma^\alpha \bullet \gamma^\beta = \frac{1}{2} (\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha) = \eta^{\alpha\beta} \quad (1)$$

The Clifford algebra, Cl , has a faithful representation in the complex matrices, $GL(2^{[n/2]}, C)$, with the Clifford product isomorphic to matrix multiplication, and it is possible and sometimes helpful, although not necessary, to write and manipulate Clifford elements as matrices, using (Dirac) matrices for the basis.

The fundamental Clifford algebra equation (1) is invariant under the Clifford group, $Cl^* = \{S \in Cl \mid \exists S^{-1}\}$, adjoint automorphism (similarity transformation), $\gamma^\alpha \mapsto \gamma'^\alpha = S \gamma^\alpha S^{-1}$, which may well change the grade of the Clifford basis elements. Through this automorphism the Cl^* elements serve as the transition functions for the Clifford bundle. For infinitesimal transformations, $S \simeq 1 + \frac{1}{2}C$, $C \in Cl$, this automorphism is

$$\gamma'^\alpha = S \gamma^\alpha S^{-1} \simeq \gamma^\alpha + C \times \gamma^\alpha \quad (2)$$

with the anti-symmetric (cross) product defined as $A \times B = [A, B] = \frac{1}{2} (AB - BA)$.

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In general, the Clifford product may be written as $AB = A \bullet B + A \times B$ and, since $\gamma^\alpha \gamma^\beta = \eta^{\alpha\beta} + \gamma^\alpha \times \gamma^\beta$, any Clifford element may be written out with real, anti-symmetric coefficients times anti-symmetric collections of the basis elements,

$$C = C_s + C_\alpha \gamma^\alpha + C_{\alpha\beta} [\gamma^\alpha, \gamma^\beta] + C_{\alpha\beta\delta} [\gamma^\alpha, \gamma^\beta, \gamma^\delta] + \dots + C_p \gamma$$

where the anti-symmetric bracket operator is, for example,

$$\begin{aligned} [A, B, C] &= \frac{1}{3!} (ABC + BCA + CAB - ACB - CBA - BAC) \\ C_{\alpha\beta\delta} &= C_{[\alpha\beta\delta]} = \frac{1}{3!} (C_{\alpha\beta\delta} + C_{\beta\delta\alpha} + C_{\delta\alpha\beta} - C_{\alpha\delta\beta} - C_{\delta\beta\alpha} - C_{\beta\alpha\delta}) \end{aligned}$$

Note that, since the basis elements can change grade under the adjoint automorphism, the grade of a Clifford element, except for the invariant scalar part (grade 0 part, trace), is only a meaningful concept with respect to the local specification of the basis elements. The \cdot (dot) and \wedge (wedge) products familiar to disciples of Clifford algebra have grade dependent definitions and are therefore not used here, though they are equivalent to \bullet and \times for vector elements and useful within the context of a local vector basis.

The connection for the Cl bundle takes values in the Lie algebra of the transitions and is thus a Clifford element acting via the cross product, allowing the covariant derivative of the basis elements to be written as

$$\nabla_i \gamma^\alpha = \Omega_i \times \gamma^\alpha$$

giving, for any Clifford element, $\nabla_i C = \partial_i C + \Omega_i \times C$. Under an adjoint automorphism (2) this connection transforms as

$$\Omega_i \mapsto \Omega'_i = 2S\partial_i S^{-1} + S\Omega_i S^{-1} = 2S\nabla_i S^{-1} \simeq \Omega_i - \nabla_i C \quad (3)$$

in which the Dirac derivative operator is introduced as $\nabla_i = \partial_i + \frac{1}{2}\Omega_i$.

The Clifford basis is related to the metric via the fundamental frame (frame, soldering form, bundle map),

$$\begin{aligned} \hat{e} &= \gamma^\alpha (e_\alpha)^i \vec{\partial}_i = \gamma^\alpha \hat{e}_\alpha = \gamma^i \vec{\partial}_i \\ \underline{e} &= \underline{dx}^i (e^{-1}_i)^\alpha \gamma_\alpha \end{aligned} \quad (4)$$

which gives the metric through the relation for the orthonormal basis (vielbein, tetrad, ONB), $g_{ij} = (e^{-1}_i)^\alpha \eta_{\alpha\beta} (e^{-1}_j)^\beta$.

3 Dynamics

The curvature of the Clifford bundle, giving the parallel transport of Clifford elements around infinitesimal loops, is the Cl valued 2-form,

$$\begin{aligned} \underline{\underline{R}} &= \underline{dx}^i \underline{dx}^j \frac{1}{2} R_{ij} = \underline{d} \underline{\Omega} + \frac{1}{2} \underline{\Omega} \times \underline{\Omega} = \underline{d} \underline{\Omega} + \frac{1}{2} \underline{\Omega} \underline{\Omega} = \underline{\nabla} \underline{\Omega} \\ R_{ij} &= \partial_i \Omega_j - \partial_j \Omega_i + \Omega_i \times \Omega_j \end{aligned} \quad (5)$$

The scalar curvature of the Clifford bundle may be defined, using the frame (4), as

$$R = \left\langle \underline{\underline{R}} \hat{e} \hat{e} \right\rangle = \left\langle R_{ij} (e_\alpha)^i (e_\beta)^j \gamma^\alpha \gamma^\beta \right\rangle \quad (6)$$

with $\langle A \rangle$ giving the scalar part (trace) of a Clifford element.

The action for the Clifford bundle is

$$S = \int \underline{dx}^m |e| R = \int \underline{dx}^m |e| \left\langle 2 \left(\partial_i \Omega_j + \frac{1}{2} \Omega_i \Omega_j \right) (e_\alpha)^i (e_\beta)^j (\gamma^\alpha \times \gamma^\beta) \right\rangle \quad (7)$$

with volume scale $|e| = \det (e^{-1}_i)^\alpha$. Varying the vielbein and requiring the variation of this action to vanish gives Einstein's equation,

$$0 = G_i^\alpha = \left\langle \left(\partial_{[i} \Omega_{j]} + \frac{1}{2} \Omega_{[i} \Omega_{j]} \right) (e_\beta)^j (\gamma^\alpha \times \gamma^\beta) \right\rangle + \frac{1}{2} (e^{-1}_i)^\alpha R$$

and varying the Clifford connection gives a relationship with derivatives of the vielbein

$$\left[\partial_i |e| (e_\alpha)^i (e_\beta)^j \right] (\gamma^\alpha \times \gamma^\beta) = |e| (e_\alpha)^i (e_\beta)^j (\gamma^\alpha \times \gamma^\beta) \times \Omega_i$$

which holds if and only if the Clifford connection is equal to the torsionless spin connection bivector,

$$\Omega_i = \Omega_{i\alpha\beta} [\gamma^\alpha, \gamma^\beta] \quad , \quad \Omega_{i\alpha\beta} = \omega_{i\alpha\beta}$$

satisfying Cartan's structure equation, $\underline{d} \underline{e}^\alpha = \underline{\omega}_{\beta}^\alpha \underline{e}^\beta$, equivalent to $\partial_{[i} (e^{-1}_{j]})^\alpha = \omega_{[i\beta}^\alpha (e^{-1}_{j]})^\beta$.

4 BRST

The curvature scalar, and thus the action (7), is invariant under adjoint automorphisms of the frame,

$$\begin{aligned}\hat{e} &\mapsto \hat{e}' = S\hat{e}S^{-1} \simeq \hat{e} + C \times \hat{e} \\ \underline{\Omega} &\mapsto \underline{\Omega}' = 2S\underline{\nabla}S^{-1} \simeq \underline{\Omega} - \underline{\nabla}C \Rightarrow R \mapsto R' = R\end{aligned}$$

Via the BRST formulation, new ‘‘gauge ghost’’ fields, with real, anti-commuting coefficients (Grassmann number 1, satisfying $ab = -ba$ and $(ab)^* = -b^*a^* = -ba$), are introduced to properly restrict and account for this symmetry. The new variables are the anti-commuting Cl element fields, $\{C, B\}$, which have real coefficients with Grassmann number 1, and another ghost field, A , with Grassmann number 0. The infinitesimal BRST transformation corresponding to the Clifford group adjoint operation is

$$\begin{aligned}\delta_\Lambda \hat{e} &= C \times \hat{e} = \frac{1}{2}C\hat{e} - \frac{1}{2}\hat{e}C \\ \delta_\Lambda \underline{\Omega} &= -\underline{\nabla}C = -\underline{dx}^i (\partial_i C + \frac{1}{2}\Omega_i C - \frac{1}{2}C\Omega_i) \\ \delta_\Lambda \underline{C} &= \frac{1}{2}C \times C = \frac{1}{2}CC \\ \delta_\Lambda B &= iA \\ \delta_\Lambda A &= 0\end{aligned}$$

The nilpotence of the BRST operator, $\delta_\Lambda \delta_\Lambda = 0$, which has Grassmann number 1, is confirmed by calculation,

$$\begin{aligned}\delta_\Lambda \delta_\Lambda \hat{e} &= \frac{1}{2} [\frac{1}{2}CC] \hat{e} - \frac{1}{2}C [\frac{1}{2}C\hat{e} - \frac{1}{2}\hat{e}C] - \frac{1}{2} [\frac{1}{2}C\hat{e} - \frac{1}{2}\hat{e}C] C - \frac{1}{2}\hat{e} [\frac{1}{2}CC] = 0 \\ \delta_\Lambda \delta_\Lambda \underline{\Omega} &= -\underline{dx}^i (\partial_i [\frac{1}{2}CC] - \frac{1}{2} [\partial_i C + \Omega_i \times C] C + \frac{1}{2}\Omega_i [\frac{1}{2}CC] - \frac{1}{2} [\frac{1}{2}CC] \Omega_i - \frac{1}{2}C [\partial_i C + \Omega_i \times C]) = 0 \\ \delta_\Lambda \delta_\Lambda \underline{C} &= \frac{1}{2} [\frac{1}{2}CC] C - \frac{1}{2}C [\frac{1}{2}CC] = 0\end{aligned}$$

The dynamics of the gauge and ghost degrees of freedom are determined by the choice of a BRST potential; a good choice is

$$\Psi = \int \underline{d}^n x |e| \langle B \hat{e} \underline{\Omega} \rangle$$

which gives the new action,

$$S' = S - i\delta_\Lambda \Psi = \int \underline{d}^n x |e| \left\{ R \left[\hat{e}, \underline{\Omega} \right] + \langle A \hat{e} \underline{\Omega} \rangle - i \langle B \left[\gamma^\alpha (e_\alpha)^i \partial_i C + (\hat{e} \underline{\Omega}) \times C \right] \rangle \right\}$$

The ghost field A appears in this action as a Lagrange multiplier, constraining the connection to satisfy $\hat{e} \underline{\Omega}' = 0$. With this restricted connection the effective action for the remaining fields, $\{\hat{e}, \underline{\Omega}', C, B\}$, is

$$S_{eff} = \int \underline{d}^n x |e| \left\{ R \left[\hat{e}, \underline{\Omega}' \right] - i \langle B \gamma^\alpha (e_\alpha)^i \partial_i C \rangle \right\} \quad (8)$$

an Einstein-Weyl-like action for anti-commuting spinor field, anti-field, vielbein, and restricted connection. Note that the constraint on the connection, a result of the choice of BRST potential, insures that the connection vanishes from the Dirac operator.

The equations of motion from the new action, S' , or, after removing A and restricting to Ω'_i , from S_{eff} , are

$$\begin{aligned}G_i^\alpha &= \frac{i}{2} \langle B \gamma^\alpha \partial_i C \rangle - \frac{i}{2} (e^{-1})^\alpha \langle B \gamma^\beta (e_\beta)^j \partial_j C \rangle = T_i^\alpha \\ A &= -2 \frac{(n-2)}{(n-1)} |e| \omega^\beta{}_{\alpha\beta} \gamma^\alpha - 4 |e| \omega_{[\alpha\beta\delta]} [\gamma^\alpha, \gamma^\beta, \gamma^\delta] \\ \Omega'_\delta &= (e_\delta)^i \Omega'_i = \Omega'_{\delta\alpha\beta} [\gamma^\alpha, \gamma^\beta] \\ \Omega'_{\delta\alpha\beta} &= \omega_{\delta\alpha\beta} - \omega_{[\delta\alpha\beta]} + \frac{2}{(n-1)} \eta_{\delta[\alpha} \omega^\gamma{}_{\beta]\gamma} \\ 0 &= \gamma^\alpha \partial_i |e| B (e_\alpha)^i \\ 0 &= \gamma^\alpha (e_\alpha)^i \partial_i C\end{aligned}$$

The restricted connection, satisfying $\gamma^\alpha \Omega'_\alpha = 0$, is hence a Clifford bivector with gauge degrees of freedom removed—the coefficients constrained to have vanishing trace $\Omega'_{\delta\alpha}{}^\delta = 0$ and vanishing fully anti-symmetric part $\Omega'_{[\delta\alpha\beta]} = 0$. Note that if one begins with an arbitrary connection, the restricted connection may be obtained by applying an adjoint transformation (3) with an S such that $0 = \hat{e}' \underline{\Omega}' = 2S\gamma^\alpha (e_\alpha)^i \nabla_i S^{-1}$ —so the geometric interpretation of a solution to the curved spacetime Weyl equation is that it provides a transformation to a restricted connection. The BRST formulation balances the gauge restriction

with the C and B fields and their (Weyl) equations of motion and stress energy tensor. By using the equation for the restricted Clifford connection in terms of the spin connection it is also possible to write the Clifford scalar curvature (6), and hence a new effective action, purely in terms of derivatives of the vielbein,

$$\begin{aligned} R'[\hat{e}] &= \left\langle 2 \left(\partial_i \Omega'_j + \frac{1}{2} \Omega'_i \Omega'_j \right) (e_\alpha)^i (e_\beta)^j (\gamma^\alpha \times \gamma^\beta) \right\rangle \\ &= \frac{1}{3} \omega_{\beta\alpha\delta} \omega^{\alpha\delta\beta} - \frac{1}{3} \omega_{\beta\alpha\delta} \omega^{\beta\alpha\delta} + \frac{1}{(n-1)} \omega^\delta_{\beta\delta} \omega_\gamma^{\beta\gamma} \\ &= -\frac{2}{3} F_{\delta(\beta\alpha)} F^{\delta(\beta\alpha)} + \frac{1}{(n-1)} F^\delta_{\beta\delta} F_\gamma^{\beta\gamma} \end{aligned}$$

in which the field strength (anholonomy) for the vielbein is defined as $F_{\beta\gamma}{}^\alpha = (e_\beta)^i (e_\gamma)^j 2\partial_{[i} (e^{-1}{}_{j]}{}^\alpha)$.

5 Conclusion

The existence and dynamics of anti-commuting spinor fields have been derived and given a firm geometric foundation by starting with the Clifford algebra fibre bundle and curvature and applying the BRST formulation to the adjoint automorphism gauge symmetry. The BRST construction was carried out in the relativistic Lagrangian framework, but may be carried through to a Hamiltonian formulation as well.

The ultimate goal is to obtain all the fields and dynamics of the standard model of particle physics from this geometric foundation. To do this, the methods of Kaluza-Klein theory may be employed by assuming the dimensions greater than the four of spacetime are wrapped up in a spatial, highly symmetric, compact manifold. By using a vielbein of the form

$$(e_\alpha)^i = \begin{bmatrix} (e_\alpha^S)^i & A_\alpha^A \xi_A^i \\ 0 & \frac{1}{\rho} (e_\alpha^K)^i \end{bmatrix}$$

in which $\vec{\xi}_A$ are Killing vector fields of the compact space, K , one obtains the Yang-Mills action and dynamics for the gauge fields, A^A . By expanding these gauge and spinor fields in terms of resonant modes of the compact space one may get all the fields and dynamics of the standard model along with gravity. The details of how these higher dimensional, Clifford element spinor fields can be broken up into the familiar fermion multiplets is laid out in a beautiful exposition by Trayling [3]. Note that the spinor fields derived here are not originally relegated to ideals of the algebra, but rather are “full” Clifford element (matrix) spinors—the ideals (matrix columns) emerge only in the fermion multiplet decomposition.

It is important to note that not all the gauge symmetries of the system have yet been addressed. The other two symmetries, transformations of the frame that leave the action invariant, are diffeomorphisms (coordinate changes) and local Lorentz rotations of the vielbein. These symmetries may also be handled via the BRST formulation, resulting in an effective action for a restricted vielbein. Used in conjunction with Kaluza-Klein theory, the BRST formulation for diffeomorphisms parallels and reproduces the BRST formulation for Yang-Mills gauge theory, with the relevant diffeomorphism, producing the familiar gauge field transformation, being $x^i \mapsto x'^i = x^i + \xi_A^i \phi^A(x)$.

The BRST formulation for Yang-Mills theory, and the appearance and utility of gauge ghosts, is familiar to researchers in quantum field theory, where the ghosts play a crucial role in facilitating quantization and renormalization. It is hoped that the spinor ghost fields and dynamics introduced here (8) may play a similar role in the quantization of gravity.

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References

- [1] P. Peldan, *Actions for Gravity, with Generalizations: A Review*, gr-qc/9305011
- [2] J.W. van Holten, *Aspects of BRST Quantization*, hep-th/0201124
- [3] G. Trayling and W. E. Baylis, *A geometric basis for the standard-model gauge group*, hep-th/0103137

¹<http://www.lyx.org>