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Geometrodynamics Regained*

SERGIO A. HOJMAN,[†] KAREL KUCHAŘ,[‡] AND CLAUDIO TEITELBOIM[1]

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

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Einsteinian geometrodynamics is the only (time-reversible) canonical representation of the set of generators of deformations of a spacelike hypersurface embedded in a Riemannian spacetime, if the intrinsic metric of that hypersurface and a conjugate momentum are the sole canonical variables.

at our great Feast
I went into the Temple, there to hear
The Teachers of our Law, and to propose
What might improve my knowledge or thir own;

John Milton: *Paradise Regain'd*

And, without question, all those different planes, upon which Time, since I had regained it at this reception, had exhibited my life, by reminding me that in a book which gave the history of one, it would be necessary to make use of a sort of spatial psychology as opposed to the usual flat psychology, added a new beauty to the resurrections my memory was operating during my solitary reflections in the library, since memory, by introducing the past into the present without modifications, as though it were the present, eliminates precisely that great Time-dimension in accordance with which life is realized.

Marcel Proust: Time Regained

INTRODUCTION

Formal schemes have their own life. Geometrodynamics was originally derived by a laborious rearrangement of Hilbert's action principle as a preliminary to the canonical quantization of gravity. If such a program is ever completed, spacetime

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† On leave from Facultad de Ciencias Fisicas y Matematicas, Universidad de Chile, Santiago, Chile.

‡ Present address: Department of Physics, University of Utah, Salt Lake City, Utah 84112.

[1] On leave from Facultad de Ciencias, Universidad de Chile, Santiago, Chile.

dies and quantum geometrodynamics becomes its heir. Thus one would like to understand why geometrodynamics has the structure it does from its own conceptual framework.

We start by looking back instead of forward—looking back on how Einstein's law of gravitation was discovered and then placed on a pedestal of "first principles" (Section 1) and looking back on how geometrodynamics was derived from the Einstein law of gravitation in the late fifties (Section 2). After that we build a new pedestal. Its base is the set of deformations of a spacelike hypersurface embedded in an arbitrary Riemannian spacetime (Section 3), or rather a set of vector fields that generate those deformations. This set has a structure and the structure is mirrored by geometrodynamics. Strictly speaking, we find a set of vector fields on hyperspace (the space of all spacelike hypersurfaces) which generate the deformations. These vector fields have definite commutation relations. We represent the vector fields by canonical generators (called super-Hamiltonian and supermomentum) in such a way that the Poisson brackets of the canonical generators mirror the commutation relations of the vector fields on hyperspace. In this sense, we view geometrodynamics, pure or driven by sources, as providing different canonical representations of the generators of deformations (Section 4). The geometrodynamical data cannot be freely specified, but must be restricted by constraints; otherwise, the geometrodynamical evolution would not be path-independent (Section 5). To simplify further derivations, we investigate how to express the time-reversibility of Einstein spacetimes in the geometrodynamical language (Section 6). Then, finally, we put geometrodynamics on the new pedestal. The geometrodynamical supermomentum is regained because we know that its only function is to reshuffle the data given on a hypersurface (Section 7). The geometrodynamical super-Hamiltonian is regained as the unique canonical generator which satisfies the commutation relations and depends only on the intrinsic geometry of the hypersurface and a conjugate momentum (Section 8). At the end of our way, we look back at its turns and compare it with the spacetime route to Einstein's law of gravitation (Section 9).

We have tried to answer the question that we have heard so many times from John Wheeler [0]: "If one did not know the Einstein–Hamilton–Jacobi equation, how might one hope to derive it straight off from plausible first principles, without ever going through the formulation of the Einstein field equations themselves?" We know, of course, that John Wheeler will never be happy with our answer. He will either tell us that our first principles are actually second principles, or he will accept them as the first principles, but immediately start asking what the zeroth principles are. To honor his firm belief that not only people, but Nature herself must start from the beginning, we have started our numberings from 0.

1. Einstein's Law of Gravitation

In 1913, Einstein in collaboration with Grossman [1] finally found the correct mathematical expression of the principle of equivalence which, in its intuitive form, has been around since the 1907 Einstein review article Über das Relativitätsprinzip un die aus demselben gezogenen Folgerungen [2]. The idea, which is a part of common knowledge today, was to describe the gravitational field by 10 components of the spacetime metric tensor and cast all physical laws into a form invariant with respect to general transformations of spacetime coordinates. In the 1913 paper, however, Einstein did not succeed in finding a generally covariant law of gravitation. "We must stress that we do not have any basis for the general covariance of the equations of gravity," he admits. The search for such a law was perhaps the most painstaking part of the evolution of the general theory of relativity. At one stage Einstein believed that a generally covariant law of gravitation does not exist [3]: "It thus seems necessary that the differential equations for ${}^4g_{\mu\nu}$ also be *generally covariant*. We will show, however, that this assumption must be slightly restricted, if we want to satisfy fully the principle of causality. Namely, we shall prove that the laws determining the flow of events in the gravitational field cannot be *generally covariant*." It was only after more than 2 years of unsuccessful attempts that Einstein finally discovered in 1915 the correct form of the law according to which gravitation is produced by matter,

$$ {}^4R_{\iota\kappa} = \kappa(T_{\iota\kappa} - \tfrac{1}{2}T\,{}^4g_{\iota\kappa}), \qquad \kappa = 8\pi G c^{-4}, $$

or, as we prefer to write it today,

$$ {}^4R_{\iota\kappa} - \tfrac{1}{2}\,{}^4R\,{}^4g_{\iota\kappa} = \kappa T_{\iota\kappa}\,. \tag{1.1} $$

"This completes the construction of the general theory of relativity as a logical scheme. The postulate of relativity in its most general form which deprives the spacetime coordinates of any physical meaning, leads with iron necessity to a completely definite theory of gravitation, explaining the motion of the perihelion of Mercury," Einstein concluded. In his final summary of the 9 years of work leading to the general theory of relativity, however, Die Grundlage der allgemeinen Relativitätstheorie [5], which has since become a classic, Einstein explicitly mentions another assumption which points toward the law of gravitation (1.1): "The strongest argument in favor of the given equations is, however, that the conservation equations for the energy and momentum components of the total energy tensor follow from them."

Several people apparently felt at the same time that Einstein's derivation of the law of gravitation should be presented in a logically more compelling way. One of them was Cartan [6], who proved in 1922 that the most general second rank tensor:

(a) constructed in a coordinate-independent way from the metric tensor and its first and second partial derivatives,

(b) having a vanishing divergence,

(c) and linear in the second derivatives,

is a linear combination of Einstein's tensor ${}^4R_{\iota\kappa} - \frac{1}{2}\,{}^4R\,{}^4g_{\iota\kappa}$ and of the metric tensor ${}^4g_{\iota\kappa}$ itself,

$$G_{\iota\kappa} = \kappa^{-1}({}^4R_{\iota\kappa} - \tfrac{1}{2}\,{}^4R\,{}^4g_{\iota\kappa}) + \kappa^{-1}\lambda\,{}^4g_{\iota\kappa}\,, \qquad \kappa = \text{const.}, \quad \lambda = \text{const.} \tag{1.2}$$

Essentially the same statement appears in an Appendix to the fourth edition of Weyl's book "Space, Time, Matter" which, in its German version [7], was published in 1921. Weyl quotes Vermeil [8] as the source of his theorems. He prefers, however, to base his derivation of Einstein's law on a variational principle. Max von Laue, who mentions the 1921 edition of Weyl's book in the introduction to his own 1922 book on the general theory of relativity [9], proves essentially the same theorem as Cartan and uses it directly for the derivation of Einstein's law. Asking for the law of gravitation $G_{\iota\kappa} = T_{\iota\kappa}$ with the left-hand side $G_{\iota\kappa}$ satisfying assumptions (a), (b), and (c), he is led uniquely to the Einstein law with the cosmological term $\lambda\,{}^4g_{\iota\kappa}$ included,

$${}^4R_{\iota\kappa} - \tfrac{1}{2}\,{}^4R\,{}^4g_{\iota\kappa} + \lambda\,{}^4g_{\iota\kappa} = \kappa T_{\iota\kappa}\,. \tag{1.3}$$

Requirement (a) is motivated by the "general principle of relativity" or "the principle of general covariance," as different people prefer to call it (others carefully avoiding the term at all). Requirement (b) formalizes Einstein's statement that the laws of conservation of energy and momentum should follow from the field equations. In comparison with these natural requirements, assumption (c) seems rather ad hoc. Trying to make it plausible, one usually evokes the correspondence of Einstein's law with the Poisson equation of the Newtonian theory of gravitation. It took almost 50 years to prove that assumption (c) is in fact unnecessary. Lovelock [10] has recently found all second rank tensors $G_{\iota\kappa}$ having properties (a) and (b). Curiously enough, the result depends on the dimension n of space. For $n = 4$, which is the case of spacetime, Lovelock proved that the tensors $G_{\iota\kappa}$ form exactly the two-parameter family (1.2). The same result follows also for $n = 3$; we shall have the opportunity to use it later.

Thus, we may summarize the assumptions that lead to the derivation of Einstein's law of gravitation:

(1) The gravitational field is fully described by the spacetime metric tensor ${}^4g_{\iota\kappa}$ with signature $(-, +, +, +)$.

(2) The source of the gravitational field is the symmetrical energy-momentum tensor $T_{\iota\kappa}$ of all matter and nongravitational fields present in space. By the principle of equivalence, it satisfies the conservation law

$$T^{\iota\kappa}{}_{;\kappa} = 0. \tag{1.4}$$

(3) In any system of coordinates, the gravitational field is produced by the energy-momentum tensor $T_{\iota\kappa}$ according to the second-order field equations

$$G_{\iota\kappa}({}^4g_{\mu\nu}, {}^4g_{\mu\nu,\sigma}, {}^4g_{\mu\nu,\sigma\tau}) = T_{\iota\kappa}. \tag{1.5}$$

Here, $G_{\iota\kappa}$ is a tensor constructed entirely from the metric tensor ${}^4g_{\mu\nu}$ and its first and second derivatives in a way that does not depend on the system of coordinates.

(4) The divergence of $G_{\iota\kappa}$ vanishes identically so that the energy-momentum conservation (1.4) follows from the field equations (1.5).

Assumptions (1)–(4) are very plausible, but they are by no means necessary. Modifying them, one gets alternative theories of gravitation [11]. First of all, one can question whether the gravitational field is described by the symmetrical metric tensor and produced by the symmetrical energy-momentum tensor. Moving in one direction, one may think that more complicated objects are necessary. Some theories of gravitation assume that the basic gravitational variables are the tetrad vectors, which are produced by the generally asymmetrical canonical energy-momentum tensor and the spin tensor [12], or, as in the Jordan–Thiry–Brans–Dicke theory, one may think that a scalar in addition to the symmetrical tensor is needed to describe the gravitational field [13], or one may try to build a two-tensor theory of gravitation [14]. Another variation on this theme is the unified field theories [15]. The unified field is described by a sufficiently rich geometrical object (e.g., by a nonsymmetrical metric tensor plus a nonsymmetrical affine connection), the components of which are identified with the gravitational and electromagnetic variables. The unified field equations do not have an external source, but they may be cast into the form in which the electromagnetic variables are the source of the gravitational variables and the gravitational variables influence the behavior of the electromagnetic variables.

Moving in an opposite direction, one may think that a less complicated object than the full symmetrical metric tensor is sufficient to describe gravity. An example is the Nordström theory [16], in which the gravitational field is described by a scalar and produced by the trace of the energy-momentum tensor. Einstein and Fokker [17] pointed out that by reducing the 10 components of the metric tensor to one independent component of a conformally flat metric tensor, Nordström's theory may be interpreted both geometrically and physically within the framework of the general theory of relativity.

Even if one assumes that the components of the symmetrical metric tensor describe gravity and the components of the symmetrical energy-momentum tensor its source, one may still ask whether the law of gravitation must have the general form (1.5).

At the very outset, one can imagine that the field–source relationship may be nonlocal. The Newtonian theory of gravitation started as a theory of action at a distance. Attempts to base electrodynamics on a similar basis made by Ampère, Gauss, and Weber, actually preceded the Faraday–Maxwell electrodynamics. Action at a distance may be reconciled with the theory of relativity if one introduces retardation. Maxwell's electrodynamics was reformulated as a theory of retarded action at a distance by Fokker [18] and Wheeler and Feynman [19]. The emission electrodynamics of Ritz [20] was also formulated as a theory of retarded action at a distance, with retardation depending on the velocity of the source. In the same spirit, relativistic theories of retarded gravitational interaction were proposed by Poincaré [21] and Whitehead [22]. In particular, Whitehead's theory describes the gravitational field by a symmetrical tensor produced by the symmetrical energy–momentum tensor.

If one accepts a local field–source relationship, one can still ask how local it should be. Ordinary fields in flat spacetime are governed by equations of at most second differential order. The order of the equations depends, however, on the choice of variables. Maxwell's equations are of the first order in the field strengths, but of the second order in the electromagnetic potentials. Higher-order field equations were considered in electrodynamics by Podolski [23]. A similar situation exists in the theory of the gravitational field. The order of the equations depends on the choice of variables. For example, using both the metric tensor and the affine connection as independent variables (Palatini's method), one casts Einstein's law of gravitation into a set of first-order equations in the new variables. A higher-order law of gravitation for the metric tensor only is also conceivable. A long time ago, Eddington [24] investigated some fourth-order equations $G_{\iota\kappa} = T_{\iota\kappa}$ for the metric field with the left-hand side still satisfying the identity $G^{\iota\kappa}{}_{;\kappa} = 0$. Many people paid attention to such laws afterwards [25]. The Schwarzschild solution, on which the standard experimental tests of Einstein's law are based, satisfies Eddington's equations. Similarly, the field equations of the Rainich–Misner–Wheeler already unified field theory [26] (which are equivalent to the Einstein–Maxwell equations reexpressed exclusively by means of the metric) contain an equation of the fourth differential order.

Finally, let us mention why the clause "in a way that does not depend on the system of coordinates" is inserted into condition (3). If this condition is valid only in one system of coordinates (chosen either arbitrarily or by means of some coordinate conditions), one may, introducing supplementary variables characterizing this system of coordinates, obtain the form that the law of gravitation assumes in any system of coordinates. Such a form, of course, contains the supplementary variables in addition to the metric tensor (some two-tensor theories may be constructed in this way). The clause is thus designed to prevent additional variables from entering the gravitational law by the back door.

Granted assumptions (1)–(3), assumption (4) is necessary for the physical consistency of the theory. Otherwise, matter could produce the gravitational field according to the law $G_{\iota\kappa} = T_{\iota\kappa}$ only if it was distributed uniformly in space with a density unchanging in time [11].

One thus sees that Einstein's original statement about the "iron necessity" with which the law of gravitation follows from the general principle of relativity must be taken with a pinch of salt. One must actually complement the general principle of relativity by other, more specific, assumptions, before Einstein's law uniquely follows. Among these, the simultaneous restriction of the type and number of the gravitational variables and of the differential order of the field equations in these variables is vital. Playing with additional variables, one can reduce the order of the field equations, and, increasing the order of field equations, one can eliminate some of the variables. Finally, introducing a sufficient number of supplementary variables, one can bring almost any theory into harmony with the general principle of relativity—a point first made by Kretschmann [27] and more recently raised against the general principle of relativity by Fock [28]. In any proof of the inevitability of Einstein's law, one must thus appeal at some stage to people's conceptions about correct gravitational variables and about the differential order that the field equations are allowed to have. We have discussed this point in such detail to safeguard our own method of derivation of Einstein's law against possible objections. In it, as well as in more traditional methods, a limitation of the basic gravitational variables is necessary. The differential order of the equations, however, is limited in our method only indirectly, through the requirement that geometrodynamics be Hamiltonian. One can persuade the reader that he should accept such limitations only by appealing to his own beliefs, the method carrying the name *argumentum ad hominem*.

Einstein's law (1.3) also may be derived from the principle of least action,

$$\begin{gathered} \delta S^{(T)} = 0, \\ S^{(T)} = S + S^{(M)} = (2\kappa)^{-1} \int d^4x \, ({}^4R + 2\lambda)(-{}^4g)^{1/2} + \int d^4x \, \mathscr{L}^{(M)}. \end{gathered} \tag{1.6}$$

Variation of the gravitational action S with respect to ${}^4g^{\iota\kappa}$ yields the left-hand side of Einstein's law (1.3), whereas the variation of the matter part $S^{(M)}$ of the total action $S^{(T)}$ with respect to ${}^4g^{\iota\kappa}$ yields the energy-momentum tensor. The Lagrangian density $\mathscr{L}^{(M)}$ depends, of course, on the matter field variables in addition to the metric tensor, and variation of these leads to the field equations for the sources.

The gravitational action was first written down by Hilbert [29]. Instead of arguing about the uniqueness of the left-hand side of Einstein's law, one can argue about the uniqueness of the gravitational action. As we have already noted, this was done by Weyl in the fourth edition of his book [30]. In the preface he writes that "there are a number of small changes and additions, the most important of which are: ... (2) We show that the reason that Einstein arrives necessarily at uniquely determined gravitational equations is that the scalar of curvature is the only invariant having a certain character in Riemann's space." The title of Appendix II of his book then explains what this certain character means: "Proof of the theorem that, in Riemann's space, ${}^4R$ is the sole invariant that contains the derivatives of the ${}^4g_{\iota\kappa}$ only to the second order, and those of the second order only linearly." Nowadays, using the Lovelock theorem, we may drop the assumption that ${}^4R$ depends linearly on the second-order derivatives of ${}^4g_{\iota\kappa}$. Though not as immediately intuitive as showing the plausibility of the structure of Einstein's law itself, the argument about the uniqueness of the gravitational action amounts essentially to the same thing.

Since the early days of Einstein, Cartan, and Weyl, many different ways of introducing Einstein's law were presented. We refer an interested reader to Box 17.2, Six Routes to Einstein's Geometrodynamic Law, in "Gravitation" by Misner, Thorne, and Wheeler [31]. The importance of alternate foundations of a basic physical theory cannot be overexaggerated. The conceptual reformulation of a theory may open a new path to its development or even lead to its modification. Thus, Feynman's path-integral approach to quantum field theory led to the implementation of powerful approximation techniques, and Faraday's reformulation of action-at-a-distance stationary electrodynamics in terms of the field concept developed into Maxwell's electrodynamics. In this spirit, believing in the potential fruitfulness of the canonical variational–differential approach to the general theory of relativity, we have undertaken the study of a Seventh Route to Einstein's law in this paper.

2. Geometrodynamics Reviewed

Einstein's law determines the spacetime geometry as a single entity. In itself, the spacetime geometry is as timeless as the Platonic world of ideas. It actually represents a complete and unabridged history of the gravitational field given all at once in a single volume. To see the history of the gravitational field unraveling, one must read this volume in one way or another.

Such a dynamical viewpoint is important for the quantum theory of gravitation, where one would like to let only the dynamical degrees of freedom of the gravitational field be quantized. The historical route to quantization of a dynamical system went along the line of the Hamiltonian formalism. In it, time plays a privileged role. To describe a field evolving in spacetime in Hamiltonian language, one should slice the spacetime by a spacelike hypersurface and observe how the canonical coordinates and momenta of the field change if we push the hypersurface forward or backward in time. To span the whole spacetime, it is sufficient to pick up a one-parameter family of hypersurfaces cutting spacetime like a loaf of bread into slices. Due to the arbitrariness of the spacetime coordinates, one may pick the time coordinate t so that each slice becomes a surface of a constant t. The question is: What are the canonical coordinates and momenta of the gravitational field?

An answer to this question was proposed almost simultaneously by Dirac [32] and Arnowitt, Deser, and Misner [33]. ADM split the spacetime metric tensor ${}^4g_{\iota\kappa}$ into the spatial metric tensor g_{ik} of the hypersurfaces of constant t, the lapse function N, and the shift functions N_i according to the scheme

$$ {}^4g_{\iota\kappa} = \left\| \begin{matrix} -N^2 + g^{mn}N_mN_n\,, & N_k \\ N_i \quad , & g_{ik} \end{matrix} \right\|, $$

$$ N = (-{}^4g^{00})^{-1/2}, \qquad N_i = g_{0i}\,. $$

Here, g^{ik} is the inverse of the matrix g_{ik}. It is used to raise the Latin indices throughout this paper. Then, discarding certain divergences, ADM brought the Hilbert gravitational action (1.6) into the form

$$ S = (2\kappa)^{-1} \int dt \int d^3x\, Ng^{1/2}(K_{ij}K^{ij} - K^2 + R - 2\lambda). \tag{2.2} $$

Here, K_{ij} is the extrinsic curvature of the slices $t = \text{const}$,

$$ K_{ij} = \tfrac{1}{2}N^{-1}(-g_{ij,0} + N_{i|j} + N_{j|i}). \tag{2.3} $$

The action (2.2) does not contain the time derivatives of the lapse and shift functions N and N_i ; such variables may be ignored when performing the Legendre transformation [34]. One passes from the Lagrangian form (2.2) of the action to the Hamiltonian form paying attention only to the remaining variables g_{ij}. At first, the momentum conjugate to the metric g_{ij} is introduced,

$$ \pi^{ij} = \frac{\delta S}{\delta g_{ij,0}} = -(2\kappa)^{-1}\, g^{1/2}(K^{ij} - Kg^{ij}), \tag{2.4} $$

and then the action (2.2) is transformed into

$$ S = \int dt \int d^3x\, (\pi^{ij}g_{ij,0} - N\mathscr{H} - N^i\mathscr{H}_i). \tag{2.5} $$

The gravitational Hamiltonian is determined by the expressions $\mathscr{H}$ and $\mathscr{H}_i$, which are constructed solely from the canonical variables g_{ij}, π^{ij},

$$\mathscr{H} = G_{ijkl}\pi^{ij}\pi^{kl} - (2\kappa)^{-1} g^{1/2}(R - 2\lambda), \tag{2.6}$$

$$G_{ijkl} = 2\kappa \cdot \tfrac{1}{2}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}), \tag{2.7}$$

$$\mathscr{H}_i = -2\pi_i{}^j{}_{|j} . \tag{2.8}$$

One calls $\mathscr{H}$ the super-Hamiltonian and $\mathscr{H}_i$ the supermomentum of the gravitational field. The supermomentum is linear in the canonical momentum π^{ij}. The super-Hamiltonian has a "potential term" $-(2\kappa)^{-1} g^{1/2}(R - 2\lambda)$, which depends only on the canonical coordinates g_{ij}, and a "kinetic term," which is a quadratic form of the canonical momentum. The coefficients G_{ijkl} of this quadratic form may be interpreted as a normal hyperbolic metric in the space Riem $(\mathscr{M})$ of all (positive definite) Riemannian metrics [35]. They have the appropriate symmetries

$$G_{ijkl} = G_{jikl} = G_{ijlk} = G_{klij} . \tag{2.9}$$

We shall use the term "supermetric" for G_{ijkl}.

The matter action $S^{(M)}$ is also easily cast into the general form (2.5) if the matter Lagrangian $\mathscr{L}^{(M)}$ does not depend on the derivatives of the metric tensor ${}^4g_{\iota\kappa}$. This holds, e.g., for the scalar or the electromagnetic fields. The canonical formalism for fields with derivative gravitational coupling deserves a closer investigation which we plan to undertake in the future. We are not considering such fields in the present paper. For the fields with nonderivative coupling,

$$S^{(M)} = \int dt \int d^3x \, (\pi_A \phi^A{}_{,0} - N\mathscr{H}^{(M)} - N^i\mathscr{H}^{(M)}{}_i), \tag{2.10}$$

where ϕ^A are the field variables and π_A their conjugate momenta. The super-Hamiltonian $\mathscr{H}^{(M)}$ and the supermomentum $\mathscr{H}^{(M)}{}_i$ of matter depend only on the field variables ϕ^A, π_A and the spatial metric tensor. For example, the scalar field described by the standard Lagrangian

$$\mathscr{L}^{(M)} = -\tfrac{1}{2}({}^4g^{\iota\kappa}\phi_{,\iota}\phi_{,\kappa} + m^2\phi^2)(-{}^4g)^{1/2} \tag{2.11}$$

leads to the super-Hamiltonian

$$\mathscr{H}^{(M)} = \tfrac{1}{2} g^{-1/2}\pi_\phi^2 + \tfrac{1}{2} g^{1/2}(g^{ij}\phi_{,i}\phi_{,j} + m^2\phi^2) \tag{2.12}$$

and the supermomentum

$$\mathscr{H}^{(M)}{}_i = \pi_\phi \phi_{,i} . \tag{2.13}$$

Note that the supermomentum (2.13) does not depend on the metric. This is not accidental, but holds for an arbitrary matter field.

The total action $S^{(T)}$ is the sum of the gravitational action S and the matter action $S^{(M)}$,

$$S^{(T)} = \int dt \int d^3x \, (\pi^{ij} g_{ij,0} + \pi_A \phi^A{}_{,0} - N\mathscr{H}^{(T)} - N^i \mathscr{H}^{(T)}{}_i), \tag{2.14}$$

with

$$\mathscr{H}^{(T)} = \mathscr{H} + \mathscr{H}^{(M)}, \tag{2.15}$$

$$\mathscr{H}^{(T)}{}_i = \mathscr{H}_i + \mathscr{H}^{(M)}{}_i . \tag{2.16}$$

The action (1.6) is to be varied with respect to the spacetime metric ${}^4g_{\iota\kappa}$ and the matter field variables. When the spacetime metric tensor is split according to Eq. (2.1) and the canonical momenta π^{ij} and π_A conjugate to the spatial metric tensor g_{ij} and to the field variables ϕ^A are introduced, we must vary the modified action (2.14) with respect to the lapse and shift functions N, N^i, the spatial metric g_{ij}, its conjugate momentum π^{ij}, and the conjugate field variables ϕ^A and π_A. Varying with respect to the field variables, we recover the field equations of the source. Varying with respect to the geometrical variables g_{ij}, π^{ij}, N, N_i, we recover the Einstein law of gravitation in the canonical form.

To eliminate an arbitrary t-labeling, we introduce the labeling-independent quantities

$$\begin{aligned} \delta g_{ij} &= g_{ij,0}\,\delta t, & \delta\pi^{ij} &= \pi^{ij}{}_{,0}\,\delta t, \\ \delta N &= N\,\delta t, & \delta N^i &= N^i\,\delta t, \\ \delta\phi^A &= \phi^A{}_{,0}\,\delta t, & \delta\pi_A &= \pi_{A,0}\,\delta t. \end{aligned} \tag{2.17}$$

Also, we condense our notation by extending the summation convention to continuous spatial lables: whenever x is written as an index, the integration over a repeated x is implied. For example,

$$\mathscr{H}_{ix}\,\delta N^{ix} = \int d^3x \, \mathscr{H}_i(x^j)\,\delta N^i(x^j).$$

With this understanding, the Hamilton equations take the form

$$\delta g_{ij}(x) = [g_{ij}(x), \mathscr{H}^{(T)}{}_{x'}]\,\delta N^{x'} + [g_{ij}(x), \mathscr{H}^{(T)}{}_{kx'}]\,\delta N^{kx'}, \tag{2.18}$$

$$\delta\pi^{ij}(x) = [\pi^{ij}(x), \mathscr{H}^{(T)}{}_{x'}]\,\delta N^{x'} + [\pi^{ij}(x), \mathscr{H}^{(T)}{}_{kx'}]\,\delta N^{kx'}, \tag{2.19}$$

$$\delta\phi^A(x) = [\phi^A(x), \mathscr{H}^{(T)}{}_{x'}]\,\delta N^{x'} + [\phi^A(x), \mathscr{H}^{(T)}{}_{kx'}]\,\delta N^{kx'}, \tag{2.20}$$

$$\delta\pi_A(x) = [\pi_A(x), \mathscr{H}^{(T)}{}_{x'}]\,\delta N^{x'} + [\pi_A(x), \mathscr{H}^{(T)}{}_{kx'}]\,\delta N^{kx'}. \tag{2.21}$$

Evaluating the Poisson brackets (2.18), (2.19), we may, of course, restrict ourselves to the gravitational variables g_{ij}, π^{ij}, treating the field variables ϕ^A, π_A as given functions of x. Similarly, in the Poisson brackets (2.20), (2.21), we may treat the gravitational variables g_{ij}, π^{ij} as given functions of x. Also, because $\mathscr{H}^{(M)}$ and $\mathscr{H}^{(M)}{}_i$ do not depend on the gravitional momentum, we may replace $\mathscr{H}^{(T)}$ and $\mathscr{H}^{(T)}{}_i$ in Eq. (2.18) by $\mathscr{H}$ and $\mathscr{H}_i$. We may conveniently summarize the content of Eqs. (2.18)–(2.21) by saying that the change of an arbitrary functional F of the canonical variables g_{ij}, π^{ij}, ϕ^A, π_A from one slice to another is given by the formula

$$\delta F = [F, \mathscr{H}^{(T)}{}_x]\, \delta N^x + [F, \mathscr{H}^{(T)}{}_{ix}]\, \delta N^{ix}. \tag{2.22}$$

We must still vary the action (2.14) with respect to the lapse and shift functions N, N^i. Doing so, we obtain constraints on the variables g_{ij}, π^{ij}, ϕ^A, π_A,

$$\mathscr{H}^{(T)} = 0, \qquad \mathscr{H}^{(T)}{}_i = 0. \tag{2.23}$$

The constraints (2.23) must be preserved from one slice to another by the Hamilton equations of motion (2.18)–(2.21). Due to the arbitrariness of δN and δN^i, this is possible only if the Poisson brackets of the constraints (2.23) are expressible as some combinations of the original constraints themselves. To see what this combination is, one may at first evaluate the Poisson brackets of the gravitational super-Hamiltonian (2.6) and supermomentum (2.8) by brute force, getting [36]

$$[\mathscr{H}(x), \mathscr{H}(x')] = \mathscr{H}^i(x)\, \delta_{,i}(x, x') - \mathscr{H}^i(x')\, \delta_{,i}(x', x), \tag{2.24}$$

$$[\mathscr{H}_i(x), \mathscr{H}(x')] = \mathscr{H}(x)\, \delta_{,i}(x, x'), \tag{2.25}$$

$$[\mathscr{H}_i(x), \mathscr{H}_j(x')] = \mathscr{H}_i(x')\, \delta_{,j}(x, x') + \mathscr{H}_j(x)\, \delta_{,i}(x, x'). \tag{2.26}$$

One may then argue that the matter super-Hamiltonian and supermomentum should follow suit, leading to exactly the same Poisson bracket relations (2.24)–(2.26) for the total quantities. To see how the argument works, let us first consider the Poisson bracket of two super-Hamiltonians,

$$\begin{aligned}[\mathscr{H}^{(T)}(x), \mathscr{H}^{(T)}(x')] = [\mathscr{H}(x), \mathscr{H}(x')] &+ [\mathscr{H}^{(M)}(x), \mathscr{H}(x')] \\ &+ [\mathscr{H}(x), \mathscr{H}^{(M)}(x')] + [\mathscr{H}^{(M)}(x), \mathscr{H}^{(M)}(x')].\end{aligned}$$

The first Poisson bracket on the right is given by Eq. (2.21). The next two Poisson brackets cancel each other, because

$$[\mathscr{H}^{(M)}(x), \mathscr{H}(x')] = \frac{\delta\mathscr{H}^{(M)}(x)}{\delta g_{ijx''}} \frac{\delta\mathscr{H}(x')}{\delta\pi^{ijx''}}$$

and $\mathscr{H}(x')$ is local in $\pi^{ijx''}$ and $\mathscr{H}^{(M)}(x)$ is local in $g_{ijx''}$, so that the variational derivatives yield expressions proportional to δ-functions. Because $\mathscr{H}^{(M)}$ does not contain the gravitational momentum π^{ij}, the only chance to make the Poisson bracket $[\mathscr{H}^{(T)}(x), \mathscr{H}^{(T)}(x')]$ proportional to $\mathscr{H}^{(T)}$ or $\mathscr{H}^{(T)}{}_i$ is that $\mathscr{H}^{(M)}$'s satisfy exactly the same relation (2.24) as $\mathscr{H}$'s do. This can be, of course, checked for any particular source, as, e.g., for the scalar field (2.11), (2.12).

The same argument may be applied to the closing relation (2.26), the mixed Poisson brackets $[\mathscr{H}_i(x), \mathscr{H}^{(M)}{}_j(x')]$ vanishing because $\mathscr{H}_i(x)$ contains only the gravitational and $\mathscr{H}^{(M)}{}_j(x')$ only the field variables. The situation is somewhat more complicated for the closing relation (2.25), as $[\mathscr{H}_i(x), \mathscr{H}^{(M)}(x')] = -2 (\delta\mathscr{H}^{(M)}(x')/\delta g_{ij}(x))_{|j}$ does not vanish due to the dependence of $\mathscr{H}^{(M)}(x')$ on the metric. However, one may still argue that

$$[\mathscr{H}^{(M)}{}_i(x), \mathscr{H}^{(M)}(x')] = 2 \left(\frac{\delta\mathscr{H}^{(M)}(x')}{\delta g_{ij}(x)}\right)_{|j} + \mathscr{H}^{(M)}(x)\, \delta_{,i}(x, x') \tag{2.27}$$

in order that the Poisson bracket $[\mathscr{H}^{(T)}{}_i(x), \mathscr{H}^{(T)}(x')]$ yields a combination of the total expressions $\mathscr{H}^{(T)}(x)$, $\mathscr{H}^{(T)}{}_{,i}(x)$.

To compare Eqs. (2.18), (2.19), and (2.23) with the Einstein law in the spacetime notation (1.3), one uses at first Eq. (2.18) to express the gravitational momentum π^{ij} in terms of the "velocity" $g_{ij,0}$ and the lapse and shift functions N and N_i. In this way, one gets the same result as by using the formulas (2.3) and (2.4). When the expression for the momentum is substituted into the constraints (2.23), they become the $\iota = 0$, $\kappa = 0, k$ components of the law (1.3), and when it is substituted into the second set (2.19) of Hamilton's equations, these equations yield the remaining components $\iota = i$, $\kappa = k$ of the law (1.3).

The Einstein law in the form (1.3) characterizes the spacetime geometry ${}^4g_{\iota\kappa}$ as a single entity. On the other hand, the Hamilton equations (2.18), (2.19) tell us how the spatial metric g_{ij} of a slice changes if we push the slice forward by the amount δN and stretch it by the amount δN^i. Exploring all different slices means exploring all different ways in which the original slice may be deformed. The momentum that carries a three-geometry of a slice into a three-geometry of a deformed slice is not arbitrary, but it is subject to the constraints (2.23). The spacetime geometry must be reconstructed from the spatial geometries g_{ij} of the slices and their stacking δN, δN^i, just as a movie is reconstructed from individual stills. The Hamilton equations (2.18), (2.19), however, represent not a single movie, but a many-track movie, because of the different ways in which the slice may be deformed. Regarding the single spacetime geometry as a many-track movie of spatial geometries is a viewpoint that John Wheeler called geometrodynamics.

Geometrodynamics may be pure or driven by sources. In both cases, any functional of the canonical variables changes from one slice to another according to Eq. (2.22). In both cases, the super-Hamiltonian $\mathscr{H}^{(T)}$ and supermomentum $\mathscr{H}^{(T)}{}_i$ which generate such a change are afterwards constrained to vanish, Eq. (2.23). In both cases, the Poisson brackets of super-Hamiltonians and supermomenta close in the same way, (2.24)–(2.26). These equations provide the general rules according to which dynamics proceeds, with or without sources. But next comes the actual structure of the super-Hamiltonian and supermomentum. For pure geometrodynamics without sources, the super-Hamiltonian and supermomentum are quite specific functions (2.6)–(2.8) of the geometrodynamical variables g_{ij}, π^{ij}. Moreover, if we add a source that is coupled to the gravitational field in a non-derivative manner, the gravitational super-Hamiltonian and supermomentum do not change. Rather, the matter super-Hamiltonian and supermomentum describing the particular source are simply added to the gravitational super-Hamiltonian and supermomentum to yield the total expressions. Thus, one may conclude that the expressions (2.6)–(2.8) describe the gravitational field even in the presence of sources. The coupling comes only through the fact that the matter super-Hamiltonian contains the spatial metric g_{ij} which lowers and raises the indices carried by the field variables so that $\mathscr{H}^{(M)}$ is a scalar density with respect to the change of spatial coordinates on the slices. Giving the specific $\mathscr{H}^{(M)}$ and $\mathscr{H}^{(M)}{}_i$ constructed from specific field variables means giving a specific source of the gravitational field. For example, giving the expressions (2.12) and (2.13) constructed from ϕ and π_ϕ as field variables means that the gravitational field is produced by a scalar field which is subject to the linear "wave equation."

One would like, however, not only to write down the basic geometrodynamical equations, but one would also like to understand intuitively their geometrical and physical meaning. Of course, Eqs. (2.18)–(2.21), (2.23), and (2.6)–(2.8) are equivalent to Einstein's law of gravitation and the field equations for the source. They thus stem from the same basic postulates (1)–(4) which determine the form of Einstein's law. However, this way of deriving the geometrodynamical equations is long and tedious, some steps being carried more by force than by a feeling of purpose. The meaning of the original assumptions has thus been lost on the way.

This raises the problem of deriving geometrodynamics directly from some first principles rather than by a formal rearrangement of Einstein's law. This problem has been raised repeatedly by John Wheeler, who concentrated his attention on the gravitational super-Hamiltonian (which, as one can see, contains all pure geometrodynamics in a nutshell) and asked why this super-Hamiltonian has the structure that it actually has instead of some other structure. For example, what would happen if the super-Hamiltonian contained a potential term other than $-(2\kappa)^{-1} g^{1/2}(R - 2\lambda)$, say $\mu g^{1/2} R_{ij} R^{ij}$? If Einstein's law is inevitable, such a modified potential must be excluded. But what natural requirement, formulated directly in the geometrodynamical language, does exclude it?

A desire to have geometrodynamics derived from purely geometrodynamical principles is esthetical in its origin. There is, however, yet another motivation for undertaking such an enterprise. The language of geometrodynamics is much closer to the language of quantum dynamics than the original language of Einstein's law ever was. One could foresee, e.g., that the equations (2.22) arise directly from the quantum dynamics as the Heisenberg operator equations. Thus, one may hope that the first principles of geometrodynamics can be adapted more readily to the quantum theory and lead to a deeper understanding of how the macroscopic spacetime theory grows from the quantum geometrodynamical roots.

The key to the direct derivation of geometrodynamics lies in the Poisson bracket relations (2.24)–(2.26) between the super-Hamiltonian and supermomentum. In the usual derivation of geometrodynamics from the Hilbert action principle, these relations are merely checked after the structure of the super-Hamiltonian and supermomentum in the canonical variables is already known. We would like to put the horse before the cart and argue that the gravitational super-Hamiltonian is what it is because it must satisfy the Poisson bracket relations (2.24)–(2.25). But why are these relations themselves "inevitable"?

3. Deformations of a Hypersurface

The Poisson bracket relations (2.24)–(2.25) are inevitable because they express the fact that the dynamics inevitably takes place on spacelike hypersurfaces embedded in a Riemannian spacetime with signature $(-, +, +, +)$. To follow how a field develops when we prescribe it on a spacelike hypersurface and then push and deform this hypersurface through spacetime is simply what dynamics is all about. It does not matter if we are following a field changing in spacetime with a prescribed geometry, or the dynamics of the geometry itself, or finally the dynamics of a field curving the geometry and propagating on this geometry and together with it towards a common future. In the last resort we are always studying some field variables, extrageometrical or geometrical, on all slices across a spacetime that was given in advance or arose during the dynamical process. The deformations of hypersurfaces in a Riemannian spacetime observe a simple geometrical pattern. Any dynamics taking place in a Riemannian spacetime must reflect the structure of this pattern. The best way to start studying *dynamics* is to abstract from any *particular* dynamics and investigate the pattern of deformations of spacelike hypersurfaces in a Riemannian spacetime, i.e. the *kinematics* of spacelike slices.

One of the questions we would like to answer is how the signature of spacetime is mirrored by this pattern of deformations. We thus treat both signatures, $(-, +, +, +)$ and $(+, +, +, +)$, of the spacetime metric ${}^4g_{\iota\kappa}$ at the same time, introducing the indicator ϵ, which distinguishes between these two cases.

The geometrical nature of the closing relations (2.24)–(2.26) of the super-Hamiltonian and supermomentum was anticipated by Dirac when he derived them for a parametrized field dynamics in a prescribed Minkowskian spacetime [37]. The geometrical construction underlying Dirac's argument was recognized by

Teitelboim [38], who presented a general derivation of the closing relations (2.24)–(2.26) independent of Dirac's assumptions that the background is Minkowskian and that the parametrized field theory super-Hamiltonian and supermomentum have a particular form. Kuchař separated the geometrical construction from the language of Hamiltonian dynamics in which it was formulated and cast it [39] into a kinematical language borrowed from the theory of infinitely dimensional groups [40]. Our review follows the last method of presentation. For the figures illustrating the geometry of the closing relations, see the paper by Teitelboim [38].

A three-dimensional hypersurface embedded in a four-dimensional manifold is given when we know in which points ${}^4\mathscr{P}$ of the manifold the points ${}^3\mathscr{P}$ of the hypersurface lie, ${}^4\mathscr{P} = {}^4\mathscr{P}({}^3\mathscr{P})$. Our hypersurface is thus a "marked hypersurface," the points of which are individually identifiable. If it were a two-dimensional surface in a three-dimensional space, we could visualize it as a rubber membrane with points identified by pencil marks. If the membrane occupied the same position in space, but were differently stretched along this position, we would speak about two different surfaces. In other words, we use the word "hypersurface" to mean "an embedding of a three-dimensional manifold (space) in a four-dimensional manifold (spacetime)" [41]. Introducing an arbitrary system x^i of intrinsic coordinates in the hypersurface, and an equally arbitrary system of coordinates X^ι in the embedding manifold, the hypersurface is specified by four functions of three coordinates,

$$X^\iota = X^\iota(x^i). \tag{3.1}$$

Equation (3.1) tells us that the point of the hypersurface carrying the intrinsic label x^i is located in spacetime at the point carrying the spacetime label X^ι.

In geometrodynamics, we limit our attention to spacelike hypersurfaces. We thus assume that the spacetime is equipped by a four-dimensional metric ${}^4g_{\iota\kappa}$, given either a priori or developed by the dynamical process governed by the Einstein's equations. The spacetime metric ${}^4g_{\iota\kappa}$ induces the spatial metric

$$g_{ik} = {}^4g_{\iota\kappa}(X^\lambda(x))\, X^\iota_i(x)\, X^\kappa_k(x), \quad X^\iota_i \equiv X^\iota{}_{,i} \tag{3.2}$$

on the hypersurface $X^\lambda = X^\lambda(x)$. The hypersurface is spacelike if the metric (3.2) is positive definite.

The set of all spacelike hypersurfaces is so important that it deserves to be given a name; we will call it hyperspace. A spacelike hypersurface is a single point in hyperspace and the four functions $X^\iota(x^i)$ of three coordinates x^i in Eq. (3.1) limited by the condition that the metric (3.2) is positive definite are the coordinates of a point in hyperspace. In the following, "hypersurface" always means a spacelike hypersurface.

We can displace a marked hypersurface in two different ways. We can either leave its overall position in the embedding spacetime unchanged, but stretch it differently along the spacetime points which it occupies, or we can deform it into a new position. Is there a way of deciding whether such a deformation is a pure deformation without any unnecessary tangential stretching? Fortunately, the embedding spacetime has a Riemannian structure that helps us to distinguish (by definition) pure deformations from deformations plus stretchings. A deformation is pure if the marked point of the hypersurface moves along a geodesic of the embedding spacetime starting perpendicular to the original hypersurface (Fig. 1). Pure deformation is characterized by a single function: the proper time $\tau(x^i)$ which the point x^i of the hypersurface travels until it reaches its new position in spacetime.

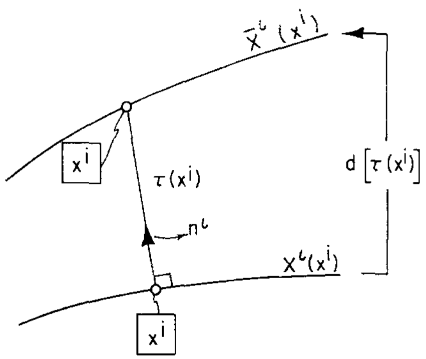

FIG. 1. Pure deformation. The pure deformation $d[\tau(x^i)]$ takes the point x^i of the initial hypersurface and displaces it by the proper time $\tau(x^i)$ along the geodesic (of the embedding spacetime) which starts normally to the hypersurface.

Similarly, pure stretching is characterized by three functions $\bar{x}^i(x^j)$. It is the operation which takes a point x^j of the hypersurface and displaces it along the hypersurface to a position which was previously occupied by a point $\bar{x}^i$. Pure stretching is nothing else but a diffeomorphism in the three-dimensional manifold $\mathscr{M}^3$; the stretchings thus form an infinitely dimensional group, Diff($\mathscr{M}^3$).

Every deformation may be decomposed into a pure deformation and a pure stretching. This is especially easy for infinitesimal deformations δX^ι,

$$\delta X^\iota = \delta N\, n^\iota + \delta N^i\, X^\iota_i\,. \tag{3.3}$$

Here, the displacement vector $\delta X^\iota(x)$ connecting two points with the same intrinsic label x on two neighboring hypersurfaces is decomposed into normal and tangential components (Fig. 2), The three vectors $X^\iota_i = X^\iota{}_{,i}$ are tangential to the hypersurface and n^ι is the unit normal to the hypersurface,

$$^4g_{\iota\kappa}n^\iota n^\kappa = \epsilon, \qquad ^4g_{\iota\kappa}n^\iota X^\kappa_k = 0. \tag{3.4}$$

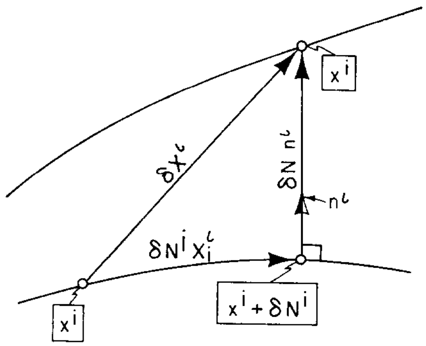

FIG. 2. Decomposition of deformations. Every deformation may be decomposed into a pure deformation and a pure stretching. Such a decomposition is illustrated here for infinitesimal deformations. The displacement vector $\delta X^\iota(x)$ connecting two points with the same intrinsic label x on two neighboring hypersurfaces is decomposed into a normal part $\delta N\, n^\iota$ and a tangential part $\delta N^i X_i^\iota$.

The coordinates of the point on the first hypersurface from which the normal must be erected to pierce the point with the coordinate x^i on the second hypersurface are $x^i + \delta N^i$, which gives the meaning to the "infinitesimal shift," δN^i. The proper time separation of the two hypersurfaces in the normal direction defines the "infinitesimal lapse," δN. The infinitesimal lapse and shift as calculated from the decomposition formula (3.2) are

$$\delta N = \epsilon n_\iota\, \delta X^\iota, \qquad \delta N^i = X^i_\iota\, \delta X^\iota.$$

Each deformation is characterized by the functions $\bar{x}^i(x^j)$ and $\tau(x^j)$ of the pure stretching $s[\bar{x}^i(x^j)]$ and the pure deformation $d[\tau(x^j)]$ into which the original deformation D may be decomposed. It seems plausible that the deformations thus form an infinitely dimensional manifold, though we do not attempt to check this statement in its strictly technical sense. One would further hope that the law of composition of the two deformations

$$D_{(1)} = D[\bar{x}^i_{(1)}(x^j),\ \tau_{(1)}(x^j)] \qquad \text{and} \qquad D_{(2)} = D[\bar{x}^i_{(2)}(x^j),\ \tau_{(2)}(x^j)]$$

turns this manifold into a $4\infty^3$-dimensional Lie group. This expectation is not fulfilled; due to the fact that the pure deformation requires for its definition the metric structure ${}^4g_{\iota\kappa}$ of spacetime, one cannot write the composition law in terms of the functions $\bar{x}^i(x^j)$, $\tau(x^j)$ alone, without reference to the hypersurfaces $X^\iota(x^i)$ on which the deformations act.

One can define, however, the generators of infinitesimal deformations. To do that, take an arbitrary functional F defined on hyperspace. To every hypersurface $X^\iota(x^i)$, this functional assings a number

$$F = F[X^\iota(x^i)].$$

We would like to know how this number changes if we deform the hypersurface by an amount $\delta N(x)$ and stretch it by an amount $\delta N^i(x)$. Using the decomposition (3.2), we get

$$\delta F = \delta X^{\iota x} \left(\frac{\delta}{\delta X^{\iota x}} \right) F = (\delta N^x \mathscr{H}_x + \delta N^{ix} \mathscr{H}_{ix}) F, \tag{3.5}$$

where we have introduced the operators

$$\mathscr{H}(x) \equiv n^\iota(x) \frac{\delta}{\delta X^\iota(x)}, \tag{3.6}$$

$$\mathscr{H}_i(x) \equiv X^\iota_i(x) \frac{\delta}{\delta X^\iota(x)}. \tag{3.7}$$

Using the language of manifolds and diffeomorphisms, we call $\mathscr{H}(x)$ the generator of the pure deformation, and $\mathscr{H}_i(x)$ the generator of the stretching.

To see how the generators act on a particular functional of the hypersurface, we take for F a component $g_{ik}(x)$ of the intrinsic metric of this hypersurface at a point x,

$$g_{ik}(x) = {}^4g_{\iota\kappa}(X^\lambda(x))\, X^\iota_i(x)\, X^\kappa_k(x), \tag{3.8}$$

and apply to it the operator $\mathscr{H}(x')$. We get

$$\begin{aligned} \mathscr{H}(x') \cdot g_{ik}(x) &= n^\lambda(x') \frac{\delta g_{ik}(x)}{\delta X^\lambda(x')} \\ &= n^\lambda(x')\, g_{\iota\kappa,\lambda}(X^\alpha(x))\, X^\iota_i(x)\, X^\kappa_k(x)\, \delta(x, x') \\ &\quad + n^\lambda(x')\, g_{\iota\kappa}(X^\alpha(x))(X^\iota_i(x)\, \delta^\kappa_\lambda \delta_{,k}(x, x') + \delta^\iota_\lambda X^\kappa_k(x)\, \delta_{,i}(x, x')). \end{aligned} \tag{3.9}$$

Using Eq. (3.4) and the identity

$$a(x')\, \delta_{,i}(x, x')\, b(x) = a(x')\, \delta_{,i}(x, x')\, b(x') - a(x)\, \delta(x, x')\, b_{,i}(x), \tag{3.10}$$

which holds for arbitrary test functions $a(x)$ and $b(x)$, we cast Eq. (3.9) to the form

$$\mathscr{H}(x') \cdot g_{ij}(x) = -2K_{ij}(x)\, \delta(x, x'), \tag{3.11}$$

where K_{ij} is the extrinsic curvature of the hypersurface,

$$K_{ij}(x) \equiv -X_{i\iota} \frac{{}^{(4)}\nabla n^\iota(x)}{\partial x^j}. \tag{3.12}$$

Equation (3.11) will turn out to be important later, showing that the super-Hamiltonian must be a purely local function of the geometrodynamical momentum.

The generators $\mathscr{H}(x)$ and $\mathscr{H}_i(x)$ span the tangent space to hyperspace at each hypersurface $X^\iota(x^i)$. However, unlike the coordinate-induced basis $\delta/\delta X^\iota(x^i)$, the basis $\mathscr{H}(x)$, $\mathscr{H}_i(x)$ is a nonholonomic one. Therefore, the Lie brackets between the generators $\mathscr{H}(x)$, $\mathscr{H}_i(x)$ do not vanish. We can easily evaluate these Lie brackets once we know how the normal of a hypersurface changes under the deformation of the hypersurface,

$$\begin{aligned} \delta n^\iota &= n^\iota(X^\kappa + \delta X^\kappa) - n^\iota(X^\kappa) \\ &= -\epsilon X^{i\iota}\, \delta N_{,i} - K_{ij} X^{i\iota}\, \delta N^j - {}^4\Gamma^\iota_{\kappa\lambda} X^\kappa_k n^\lambda\, \delta N^k - {}^4\Gamma^\iota_{\kappa\lambda} n^\kappa n^\lambda\, \delta N. \end{aligned} \tag{3.13}$$

The geometrical meaning of formula (3.13) is discussed in [39]. There it is also shown in detail how to calculate the Lie brackets. Here, we shall simply write the final answer which is

$$[\mathscr{H}(x), \mathscr{H}(x')] = \epsilon(\mathscr{H}^i(x)\, \delta_{,i}(x, x') - \mathscr{H}^i(x')\, \delta_{,i}(x', x)), \tag{3.14}$$

$$[\mathscr{H}_i(x), \mathscr{H}(x')] = -\mathscr{H}(x)\, \delta_{,i}(x, x'), \tag{3.15}$$

$$[\mathscr{H}_i(x), \mathscr{H}_k(x')] = -\mathscr{H}_i(x')\, \delta_{,k}(x, x') - \mathscr{H}_k(x)\, \delta_{,i}(x, x'). \tag{3.16}$$

The nonholonomic basis $\mathscr{H}(x)$, $\mathscr{H}_i(x)$ has an important advantage over the coordinate-induced basis $\delta/\delta X^\iota(x^i)$: It does not depend on the choice of spacetime coordinates X^ι. Thus, geometrokinematics may be described in terms intrinsic to the hypersurfaces themselves.

The spatial metric g_{ij} explicitly enters into the commutation relation (3.14), raising the index of the supermomentum. This once again reflects the fact that the generators $\mathscr{H}(x)$, $\mathscr{H}_i(x)$ do not belong to a true group, as the corresponding "structure constants" are not constants, but depend through g^{ij} on the point in hyperspace on which the deformation acts. Bergmann and Komar [42] proceeded to close the algebra (3.14)–(3.16) and arrive thus at a true group. In our approach, however, the original unenlarged structure plays the central role.

4. Canonical Representations

In Section 3, we confined our attention to the *kinematics* of deformations of a slice cut through a Riemannian spacetime. Now we are ready to return to *dynamics* again: Define a field on this slice and watch how it changes under pure deformations δN and stretchings δN^i. In Hamiltonian language, the field is described by a set of canonically conjugate variables. The dynamics of the field is governed by a super-Hamiltonian $\mathscr{H}^{(T)}$ and a supermomentum $\mathscr{H}^{(T)}{}_i$ constructed from these variables. The dynamical rule is simple: The change δF of an arbitrary functional F of the canonical field variables under the deformation δN^x and the stretching δN^{ix} of the slice is given by the Poisson brackets of F with $\mathscr{H}_x$ and $\mathscr{H}_{ix}$,

$$\delta F = [F, \mathscr{H}^{(T)}{}_x]\, \delta N^x + [F, \mathscr{H}^{(T)}{}_{ix}]\, \delta N^{ix}. \tag{4.1}$$

The super-Hamiltonian $\mathscr{H}^{(T)}{}_x$ and the supermomentum $\mathscr{H}^{(T)}{}_{ix}$ push the field by means of the Poisson brackets (4.1) just as the generators (3.6) and (3.7) push the hypersurface $X^\iota(x)$ by means of the operator action (3.5). In order that the dynamics of the field be consistent with the kinematics of the slice deformations, the field pushers should combine their actions in the same manner as the generators do. This motivates the basic postulate that we make in this paper: The super-Hamiltonian $\mathscr{H}^{(T)}$ and the supermomentum $\mathscr{H}^{(T)}{}_i$ should be constructed from the canonical field variables in such a way that their Poisson brackets close exactly as the commutators of the corresponding generators. In other words, thery should *represent* the generators of deformations.

When we say "close exactly as," we mean actually "close up to the sign as." Indeed, one can see in Section 2 that the super-Hamiltonian and supermomentum close according to the relations (2.24)–(2.26), which differ in a "hyperbolic" spacetime ($\epsilon = -1$) only by a sign from the commutation relations (3.14)–(3.16). The difference in sign is due to the convention that the generators act on the functionals of hypersurfaces from the left: Had we let them act from the right, the signs of the commutation relations and the closing relations would coincide. Under our convention, when the dynamical variable a represents the generator A, and the dynamical variable b represents the generator B, then the Poisson bracket $[a, b]$ represents the commutator $[B, A]$, not the commutator $[A, B]$. We thus see that the dynamical theories that we studied in Section 2 provide different canonical representations of the generators of deformations.

To construct a permissible relativistic field theory from scratch, without taking recourse to spacetime Lagrangians, one starts by representing the commutation relations (3.14)–(3.16). Depending on what variables one allows as the canonical field variables of the theory, i.e., which cotagent bundle of which configuration manifold one selects as the carrier of the canonical representation, one gets several different types of canonical representations. We describe them as they naturally follow one after another.

I. *Field Theories on a Prescribed Spacetime Background*

At the start, let us take the spacetime as given. It does not matter if it is flat or curved: We simply prescribe its spacetime metric ${}^4g_{\iota\kappa}(X^\lambda)$. Then we pick up a hypersurface (3.1) and calculate its intrinsic metric $g_{ik}(x)$ by Eq. (3.8). A field that propagates on our given spacetime background is described on this hypersurface by its canonical variables ϕ^A, π_A. We try to build up $\mathscr{H}^{(M)}$ and $\mathscr{H}^{(M)}{}_i$ from these variables in such a way that the Poisson brackets between the $\mathscr{H}^{(M)}$'s and between the $\mathscr{H}^{(M)}{}_i$'s satisfy the closing relations (2.24), (2.26). The expression for $\mathscr{H}^{(M)}$ may explicitly depend on the metric $g_{ik}(x)$; in fact, $\mathscr{H}^{(M)}$ *must* depend on it, because otherwise the metric could not appear on the right-hand side of the closing relation (2.24). The spatial metric, however, is treated as a prescribed function of x and not as a canonical variable when evaluating the Poisson brackets. This disturbs the remaining closing relation (2.25) when applied to $\mathscr{H}^{(M)}$ and $\mathscr{H}^{(M)}{}_i$ alone. One cannot represent the original closing relations (2.24)–(2.26) while keeping the metric fixed. In fact, we have seen that Eq. (2.25) must be replaced in this case by Eq. (2.27). An example of $\mathscr{H}^{(M)}$ and $\mathscr{H}^{(M)}{}_i$ which satisfy the modified closing relations (2.4), (2.26), (2.27) was given in Section 2, where we have treated the scalar field super-Hamiltonian and supermomentum (2.11) and (2.12). A systematic way of arriving at these expressions without referring to the spacetime Lagrangian is discussed in [43].

Once we possess $\mathscr{H}^{(M)}$ and $\mathscr{H}^{(M)}{}_i$, we may freely prescribe the field variables ϕ^A, π_A on the hypersurface $X^\iota = X^\iota(x)$ and evolve them to a deformed hypersurface by using equations (2.20), (2.21). The parameter equation of the deformed hypersurface is determined from Eq. (3.2), and the intrinsic metric of the deformed hypersurface again calculated from Eq. (3.8). Repeating this process step by step, we propagate the fields ϕ^A, π_A to any hypersurface.

II. *Parametrized Field Theories on a Prescribed Background*

To obtaine a true representation of the original commutation relations (3.14)–(3.16), one should include the metric among the canonical coordinates in one way or another. In parametrized field theories, we do not introduce all six components of the spatial metric $g_{ij}(x)$ as canonical coordinates. Rather, we introduce the four hypersurface variables $X^\iota(x)$ as canonical coordinates, prescribe the spacetime metric ${}^4g_{\iota\kappa}(X^\lambda)$, and express the spatial metric by means of $X^\iota(x)$ using Eq. (3.8). Conjugate to $X^\iota(x)$ are four momenta $\Pi_\iota(x)$. Parametrized theories were discussed in the past only on the flat background [44]. It turns out that the expressions

$$\mathscr{H} = n^\iota \Pi_\iota, \qquad \mathscr{H}_i = X^\iota_i \Pi_\iota \tag{4.2}$$

constructed entirely from the hypersurface variables X^ι and Π_ι satisfy the closing relations (2.24)–(2.26). Moreover, if we add to the expressions (4.2) the corresponding expressions constructed for a field propagating on the background (as discussed in Section 4.I), the resulting expressions still satisfy the closing relations (2.24)–(2.26). We shall study the parametrized field theories on a curved background in another paper.

III. *Pure Geometrodynamics*

The transition from the special theory of relativity (prescribed flat background) to the general theory of relativity is traditionally achieved by the "principle of equivalence." In this transition, the spacetime metric ${}^4g_{\iota\kappa}(X^\lambda)$, which is treated as prescribed in the special theory of relativity, is suddenly "unfrozen" and turned into a field variable by the general theory of relativity [45]. The transition from a field theory on a prescribed background to geometrodynamics is accomplished in a similar way.

Studying the commutation relations (3.14)–(3.16), one observes that the only quantity (except the super-Hamiltonian and supermomentum) that ever explicitly enters into them is the *spatial* metric tensor $g_{ij}(x)$. Despite the fact that the commutation relations were derived by drawing spacetime pictures, the full space*time* metric drops out of them and the *spatial* metric is the only entity left over. When seeking the canonical representations of the commutation relations (3.14)–(3.16), a natural idea presents itself: Is it not possible to represent them by using the spatial metric $g_{ij}(x)$ and a conjugate momentum as the *sole* canonical variables? In this sense, we are asking for a *minimal* representation, which would use as canonical coordinates only those variables that necessarily enter into the commutation relations. The conjugate momentum $\pi^{ij}(x)$ is, of course, needed as a subsidiary quantity to build a canonical formalism. As this stage, however, we do not ascribe to it any geometrical meaning. It is introduced through the single property of being canonically conjugate to the metric $g_{ij}(x)$, which means that it satisfies the Poisson bracket relations

$$[g_{ij}(x), \pi^{kl}(x')] = \delta^{kl}_{ij}\delta(x, x') = \tfrac{1}{2}(\delta^k_i\delta^l_j + \delta^l_i\delta^k_j)\,\delta(x, x'). \tag{4.3}$$

The true geometrical meaning of the momentum $\pi^{ij}(x)$, namely, its relation (2.4) to the extrinsic curvature, automatically emerges after the minimal representation for the super-Hamiltonian and supermomentum is actually found. Let us note, however, that $\pi^{ij}(x)$ must be a tensor density of weight 1 in order that the form $\pi^{ijx}\,\delta g_{ijx}$ be labeling independent.

The momentum $\pi^{ij}(x)$, as a matter of fact, is not uniquely determined by the Poisson brackets (4.3) with $g_{ij}(x)$. If the momentum $\pi^{ij}(x)$ satisfies the relations (4.3), the new momentum

$$\bar{\pi}^{ij}(x) = \pi^{ij}(x) + \frac{\delta \Lambda}{\delta g_{ij}(x)} \tag{4.4}$$

satisfies the same relations. Here, Λ is an arbitrary scalar functional of the metric. In order that the canonical transformation (4.4) does not depend on the labeling, the functional Λ itself should not depend on the labeling, i.e., it must be a functional of the spatial *geometry* ${}^3\mathscr{G}$ only: $\Lambda = \Lambda[{}^3\mathscr{G}]$. It is well known that the canonical transformation (4.4) is the only freedom that we have in picking up the momentum conjugate to g_{ij}.

The main aim of our approach is the proof that the super-Hamiltonian (2.6), (2.7), and the supermomentum (2.8) of the Einsteinian geometrodynamics in vacuum provide the *only* representation up to the canonical transformation (4.4) of the generators of deformations of spacelike hypersurfaces embedded in a Riemannian spacetime by means of the spatial metric g_{ij} and a conjugate momentum π^{ij} as the sole canonical variables [43, 46].

IV. *Driven Geometrodynamics*

Nobody can prevent us, however, from using *more* canonical variables than the metric $g_{ij}(x)$ and a conjugate momentum $\pi^{ij}(x)$. In fact, once we possess $\mathscr{H}^{(M)}$ and $\mathscr{H}^{(M)}{}_i$ constructed from the additional field variables ϕ^A, π_A (and the undifferentiated metric $g_{ij}(x)$ treated as a prescribed function of x) and satisfying the modified closing relations (2.24), (2.25), (2.27), we may add them to the gravitational super-Hamiltonian (2.6) and supermomentum (2.8) and arrive thus at the total quantities that represent the commutation relations (3.14)–(3.16) when both g_{ij}, π^{ij} and ϕ^A, π_A are considered as canonical variables. Knowing how to generate systematically the super-Hamiltonians and supermomenta of various fields with nonderivative gravitational coupling propagating on a *prescribed* spacetime and representing the commutation relations (2.24), (2.26), (2.27), we know that adding them to the unique gravitational super-Hamiltonian and supermomentum, we switch these fields on as the sources of the metric field on which they propagate. We thus obtain geometrodynamics driven by these source-fields. The described process of constructing the field super-Hamiltonians and supermomenta on a prescribed background with a subsequent switching on mechanism working by the simple addition of $\mathscr{H}^{(M)}$ and $\mathscr{H}^{(M)}{}_i$ to the purely geometrodynamical expressions may be taken as a formulation of a "geometrodynamical principle of equivalence."

5. Constraints

Our requirement that the super-Hamiltonian and supermomentum represent the generators of deformations was motivated by the desire that they propagate the field consistently from an initial hypersurface and create thus the field in the whole spacetime. What is meant by a consistent propagation is best summarized by the *principle of path independence*: If the same final marked hypersurface is reached from an initial marked hypersurface by two different sequences of intermediate marked hypersurfaces (by two different paths), the final field calculated from the initial field by means of formula (4.1) along each of these two paths is the same.

If deformations formed a true group, the representation requirement would be a straightforward formal expression of the principle of path-independence. But the generators $\mathscr{H}(x)$, $\mathscr{H}_i(x)$ do not form a true group, which is reflected by the fact the spatial metric explicitly enters into the commutation relation (3.14). The relation of the representation requirement to path-independence thus becomes more tricky. It depends vitally on our reluctance or willingness to include the spatial metric tensor among the dynamical variables. In field theories on a prescribed background, the path-independence of the field evolution is equivalent to the requirement that $\mathscr{H}^{(M)}$ and $\mathscr{H}^{(M)}{}_i$ represent the modified commutation relations. In pure or driven geometrodynamics, the representation requirement in itself does not ensure the path-independence of the geometrodynamical evolution. (This is true for parametrized field theories as well.) One must restrict the evolution to that submanifold ("reduced phase space") of the full phase space on which the total super-Hamiltonian and supermomentum vanish, $\mathscr{H}^{(T)} = 0$, $\mathscr{H}^{(T)}{}_i = 0$. Otherwise, the data cannot be propagated in a path-independent way. The fact that the generators of deformations do not form a true group and the "structure constants" are not true constants turns out to be really important in this context—it points toward the necessity of imposing the initial value constraints [38].

The logical interconnection among the representation requirement, the principle of the path independence, and the constraints, which we are going to prove, is the following: We assume that the representation requirement is fulfilled, namely, that the Poisson brackets between the super-Hamiltonians and the supermomenta close in the same way as the commutators (3.14)–(3.16) between the generators of the normal and tangential deformations of a hypersurface. Under this assumption, we prove that the geometrodynamical data g_{ij}, π^{ij} evolve in a path-independent way only if they satisfy the initial value constraints $\mathscr{H}^{(T)} = 0 = \mathscr{H}^{(T)}{}_i$. In the proof, we use the infinitesimal version of the path-independence principle, passing from the initial hypersurface to the final hypersurface along two different two-step paths, and evaluating the changes in g_{ij}, π^{ij} to the second order in the infinitesimal lapse function δN.

In the proof, we concentrate on the commutation relation (3.14), which is the only one containing the metric. This commutation relation describes two elementary paths by which the same final marked hypersurface is reached from the initial hypersurface (Fig. 3). The first path is a sequence of two deformations: a normal deformation $\delta N_{(1)}$ followed by a normal deformation $\delta N_{(2)}$. The second path is built from the same normal deformations performed in the reversed order, $\delta N_{(2)}$ first and $\delta N_{(1)}$ second, followed by the stretching

$$\delta N^i = -\epsilon g^{ij}(\delta N_{(1)}\, \delta N_{(2),j} - \delta N_{(2)}\, \delta N_{(1),j}). \tag{5.1}$$

The commutation relation (3.14) tells us that the stretching (5.1) is exactly what is needed to compensate for the reversed order of normal deformations and thus to arrive (up to terms quadratic in δN) at the same final marked hypersurface. Indeed, multiplying the commutation relation (3.14) by $\delta N_{(1)}(x)\,\delta N_{(2)}(x')$ and integrating it over x and x', we get the operator equations

$$\delta N^x_{(1)}\,\mathscr{H}_x \cdot \delta N^{x'}_{(2)}\,\mathscr{H}_{x'} = \delta N^{x'}_{(2)}\,\mathscr{H}_{x'} \cdot \delta N^x_{(1)}\,\mathscr{H}_x + \delta N^{ix}\,\mathscr{H}_{ix}\,, \tag{5.2}$$

with δN^{ix} given by Eq. (5.1). Realize now that the operators $\delta N^x\,\mathscr{H}_x$ and $\delta N^{ix}\,\mathscr{H}_{ix}$ applied to special functionals $F[X^\lambda]$ of X^λ, namely, to X^λ themselves, yield the changes of X^λ under normal deformations and stretchings, respectively. Applying the operator equation (5.2) to X^λ thus leads immediately to our pictorial representation of the commutation relation (3.14). Similar pictorial representations may be given for the other two commutation relations, (3.15) and (3.16). Actually, one may start from the composition pictures like Fig. 3 and, looking at their geometry, derive the commutation relations (3.14)–(3.16), as was done by Teitelbeim [38].

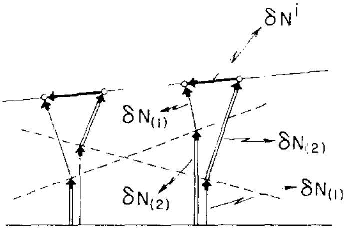

FIG. 3. Composition picture. The stretching $\delta N^i = -\epsilon g^{ij}(\delta N_{(1)}\delta N_{(2),j} - \delta N_{(2)}\delta N_{(1),j})$ is needed to compensate for the reversed order of the pure deformations $d[\delta N_{(1)}]$, $d[\delta N_{(2)}]$ and thus to arrive (up to terms quadratic in δN) to the same final marked hypersurface.

Now take a field prescribed on the initial hypersurface and propagate it to the final hypersurface along each of these two paths. The change of an arbitrary functional F of the canonical field variables under each step is given by Eq. (4.1). Composing the individual steps and consistently keeping all terms up to the second order in δN, we get

$$\delta F = (\delta N^x_{(1)} + \delta N^x_{(2)})[F, \mathscr{H}^{(T)}{}_x] + \tfrac{1}{2}(\delta N^x_{(1)}\,\delta N^{x'}_{(2)} + 2\delta N^x_{(1)}\,\delta N^{x'}_{(2)} + \delta N^x_{(2)}\,\delta N^{x'}_{(2)}) \times [[F, \mathscr{H}^{(T)}{}_x], \mathscr{H}^{(T)}{}_{x'}] \tag{5.3}$$

for the change in F when going along the first path. Similarly, going along the second path,

$$\delta F = \text{expression (5.3) with } \delta N_{(1)} \text{ and } \delta N_{(2)} \text{ interchanged} + \delta N^{ix}[F, \mathscr{H}^{(T)}{}_{ix}]. \tag{5.4}$$

The evolution is path independent if the changes (5.3) and (5.4) are equal. Using the Jacobi identity and the arbitrariness of $\delta N_{(1)}(x)$ and $\delta N_{(2)}(x')$, we get

$$[F, [\mathscr{H}^{(T)}(x), \mathscr{H}^{(T)}(x')]] = \delta_{,j}(x, x')\, g^{ij}(x)[F, \mathscr{H}^{(T)}{}_i(x)] \\ - \delta_{,j'}(x', x)\, g^{ij}(x')[F, \mathscr{H}^{(T)}{}_i(x')] \tag{5.5}$$

as a condition for path-independence. Similar conditions follow from the other two commutation relations, (3.15) and (3.16).

In Eq. (5.5), the metric g^{ij} stands outside the Poisson brackets. This is due to the fact that the displacements δN^x and δN^{ix} stand outside the Poisson brackets in the propagation equation (4.1), so that δN^{ix} given by Eq. (5.1) also appears outside the Poisson bracket in Eq. (5.4). This is a fundamental point in the argument and we will discuss it later in detail.

In geometrodynamics, the metric is a canonical variable and cannot be taken inside the Poisson brackets without compensation. Even if the representation requirement is fulfilled, the geometrodynamical evolution is path-dependent unless

$$\delta_{,j}(x, x')(\mathscr{H}^{(T)}{}_i(x)[F, g^{ij}(x)] + \mathscr{H}^{(T)}{}_i(x')[F, g^{ij}(x')]) = 0 \tag{5.6}$$

for every F. Choosing for F the geometrodynamical momentum $\pi^{kl}(x^n)$, we conclude from (5.6) that

$$\mathscr{H}^{(T)}{}_i(x) = 0. \tag{5.7}$$

The same conclusion may be reached in parametrized field theories, where $g^{ij}(x)$ is expressed through the hypersurface variables $X^\iota(x)$ as canonical coordinates. It is sufficient then to take for F the conjugate momenta $\Pi_\iota(x'')$.

The constraint (5.7), of course, should hold on any hypersurface. It must therefore be preserved under the pure deformation of the hypersurface,

$$\delta\mathscr{H}^{(T)}{}_i(x) = [\mathscr{H}^{(T)}{}_i(x), \mathscr{H}^{(T)}{}_{x'}]\, \delta N^{x'} = 0. \tag{5.8}$$

This forces on us another constraint,

$$\mathscr{H}^{(T)}(x) = 0, \tag{5.9}$$

through the closing relation (2.25). The path independence leads therefore in geometrodynamics and in parametrized field theories to the constraints (5.7) and (5.9).

Return now to the fundamental point on which the whole argument rests, namely, that the infinitesimal lapse and shift should be written outside the Poisson brackets in the propagation equation (4.1). If they were written inside, no need for imposing the constraints would arise. Surely, life would be simple if the lapse and shift could be always considered as given functions of x. Unfortunately, they cannot be considered as such in the path-independence argument, because the shift (5.1), which is necessary to close the path, depends on the metric, which is a canonical variable in geometrodynamics. Quite independent of that, one sometimes wants to specify how to deform the hypersurface by looking at its geometrical features, the intrinsic geometry and extrinsic curvature. This often happens in the general theory of relativity when one imposes coordinate conditions. In cases like these, the lapse and shift functions depend on the canonical variables and one should know whether to put them inside or outside the Poisson brackets in the propagation equation (4.1).

To make the decision, one should simply stick to the interpretation of this equation: It tells us how the functional F changes if we deform and stretch the hypersurface by the amounts $\delta N(x)$ and $\delta N^i(x)$. The only thing that matters when calculating such a change is clearly the numerical value of $\delta N(x)$ and $\delta N^i(x)$ at the point x and not a possible functional dependence of the lapse and shift on canonical variables. The functional F on the deformed and stretched hypersurface is what it is and does not care about whether we specify the hypersurface in a way that depends on the geometrical properties of the initial hypersurface or not. For example, the change of the metric $g_{ij}(x)$ under the stretching $\delta N^k(x)$ should always be given by the Lie derivative

$$\pounds_{\delta N^k} g_{ij} = g_{ij,k}\, \delta N^k + g_{ik}\, \delta N^k{}_{,j} + g_{kj}\, \delta N^k{}_{,i}\,, \tag{5.10}$$

even if the infinitesimal shift δN^k depends on the metric, as in Eq. (5.1), or even if it is constructed entirely from the metric, e.g., $\delta N^i \sim g^{ij} R_{ij}$. One can check directly that in such cases Eq. (5.10) follows from the evolution equation (4.1) with the supermomentum (2.8) only when δN^i is written outside the Poisson brackets.

Once the decision to write the lapse and shift outside the Poisson brackets is made, the constraints (5.8), (5.9) follow from our path-independence argument. And once the constraints are imposed, it does not actually matter if we put the lapse and shift outside or inside the Poisson brackets—the difference between these two options is automatically killed by the constraints. The whole formalism becomes nicely self-consistent.

The line of our argument shows very clearly that for a parameterized field theory the constraints arise from the fact that the hypersurface variables X^ι (the many-fingered time) are included among the canonical variables. The constraints themselves prove to be definitions of the energy and momentum densities Π_ι through the dynamical degrees of freedom ϕ^A, π_A of the field which evolves in the given spacetime. In geometrodynamics, all six components g_{ij} of the metric tensor are taken as canonical variables on the same footing, no clear distinction being made between the four hypersurface variables and the two gravitational degrees of freedom. In the constraints themselves, one does not know what the energy and momentum densities are. For an analysis of this hidden nature of time, see Kuchař [47].

6. Geometrodynamics and Reversibility

Through Sections 6 to 8, we limit our attention to pure geometrodynamics. At the end, we would like to regain the super-Hamiltonian (2.6), (2.7) and the supermomentum (2.8) as canonical representation of the generators of deformations in terms of the metric $g_{ij}(x)$ and a conjugate momentum $\pi^{ij}(x)$. Certain features of the gravitational super-Hamiltonian and supermomentum, however, may be conveniently discussed without knowing their exact structure in terms of g_{ij} and π^{ij}. In this section, we shall study one such feature: The time-reversibility of the geometrodynamical evolution.

Let us start by noting that if the functionals $\mathscr{H}(x)[g_{ij}, \pi^{ij}]$ and $\mathscr{H}_k(x)[g_{ij}, \pi^{ij}]$ satisfy the closing relations (2.24)–(2.26), then the functionals

$$\begin{aligned} \bar{\mathscr{H}}(x)[g_{ij}, \pi^{ij}] &\equiv \mathscr{H}(x)[g_{ij}, -\pi^{ij}], \\ \bar{\mathscr{H}}_k(x)[g_{ij}, \pi^{ij}] &\equiv -\mathscr{H}_k(x)[g_{ij}, -\pi^{ij}] \end{aligned} \tag{6.1}$$

also satisfy them. Thus, if we were to find a realization of the closing relations (2.24)–(2.26) that did not have a definite parity in the momentum, we would know that the realization of these closing relations is not unique. The barred functionals (6.1) would provide a different realization.

In Section 2, we have seen how to split the spacetime metric ${}^4g_{\iota\kappa}$ into the spatial metric g_{ik} and the lapse and shift functions. The splitting formula (2.1) tells us that time reversal

$$t \to \bar{t} = -t \tag{6.2}$$

leaves the spatial metric and the lapse unchanged, but reverses the sign of the shift. We can now clarify the meaning of the barred quantities $\bar{\mathscr{H}}$ and $\bar{\mathscr{H}}_i$. Namely, when $\mathscr{H}$, $\mathscr{H}_i$ generate the spacetime

$$g_{ij}(x, t),\ N(x, t),\ N^i(x, t), \tag{6.3}$$

then $\bar{\mathscr{H}}$, $\bar{\mathscr{H}}_i$ generate the time-reversed spacetime

$$\begin{aligned} \bar{g}_{ij}(x, \bar{t}) &\equiv g_{ij}(x, t), \\ \bar{N}(x, \bar{t}) &\equiv N(x, t), \\ \bar{N}^i(x, \bar{t}) &\equiv -N^i(x, t). \end{aligned} \tag{6.4}$$

To show this, we actually prove a slightly more elaborate statement which is: When

$$g_{ij}(x, t), \qquad \pi^{ij}(x, t) \tag{6.5}$$

solve the Hamilton equations of motion

$$\frac{d}{dt} g_{ij}(x) = [g_{ij}(x), N^{x'}\mathscr{H}_{x'} + N^{kx'}\mathscr{H}_{kx'}], \tag{6.6}$$

$$\frac{d}{dt} \pi^{ij}(x) = [\pi^{ij}(x), N^{x'}\mathscr{H}_{x'} + N^{kx'}\mathscr{H}_{kx'}] \tag{6.7}$$

generated by the super-Hamiltonian and supermomentum

$$\mathscr{H}(x)[g_{ij}, \pi^{ij}], \qquad \mathscr{H}_k(x)[g_{ij}, \pi^{ij}] \tag{6.8}$$

with the prescribed lapse and shift functions

$$N(x, t), \qquad N^i(x, t), \tag{6.9}$$

then

$$\bar{g}_{ij}(x, \bar{t}) = g_{ij}(x, t), \qquad \bar{\pi}^{ij}(x, \bar{t}) = -\pi^{ij}(x, t) \tag{6.10}$$

solve the Hamilton equations of motion

$$\frac{d}{d\bar{t}} \bar{g}_{ij}(x) = [\bar{g}_{ij}(x), \bar{N}^{x'}\bar{\mathscr{H}}_{x'} + \bar{N}^{kx'}\bar{\mathscr{H}}_{kx'}], \tag{6.11}$$

$$\frac{d}{d\bar{t}} \bar{\pi}^{ij}(x) = [\bar{\pi}^{ij}(x), \bar{N}^{x'}\bar{\mathscr{H}}_{x'} + \bar{N}^{kx'}\bar{\mathscr{H}}_{kx'}] \tag{6.12}$$

generated by the super-Hamiltonian and supermomentum

$$\begin{aligned} \bar{\mathscr{H}}(x)[\bar{g}_{ij}, \bar{\pi}^{ij}] &= \mathscr{H}(x)[g_{ij}, \pi^{ij}], \\ \bar{\mathscr{H}}_k(x)[\bar{g}_{ij}, \bar{\pi}^{ij}] &= -\mathscr{H}_k(x)[g_{ij}, \pi^{ij}] \end{aligned} \tag{6.13}$$

with the lapse and shift functions

$$\bar{N}(x, \bar{t}) = N(x, t), \qquad \bar{N}^i(x, \bar{t}) = -N^i(x, t). \tag{6.14}$$

The proof itself is a simple check that the transformation of time (6.2) coupled with the transformations (6.10) of the canonical variables, the transformation (6.14) of the lapse and shift functions, and the change (6.13) of the super-Hamiltonian and supermomentum carry the Hamilton equations (6.6), (6.7) into the Hamilton equations (6.11), (6.12).

We say that geometrodynamics is *reversible* when the time reversed spacetime is generated by the same super-Hamiltonian and supermomentum as the original spacetime. Stirctly speaking, we could allow the two super-Hamiltonians and supermomenta to differ by a canonical transformation (4.4), but let us disregard this trivial complication. From Eqs. (6.1) we see that reversible geometrodynamics must be generated by a super-Hamiltonian which is an even and a supermomentum which is an odd functional of the momentum π^{ij}.

This helps us to understand why the ADM super-Hamiltonian (2.6) is quadratic, and the ADM supermomentum (2.8) is linear in the momentum π^{ij}. Conversely, if we believe strongly that geometrodynamics should be reversible, we may impose reversibility as an additional postulate and thereby simplify the arguments leading to the recovery of the ADM super-Hamiltonian. We shall follow this line in Section 9. However, it seems more open to doubt today than before the discovery of the C–P violation that geometrodynamics must a priori be reversible. It is thus comforting to learn [43] that the closing relations (2.24)–(2.25) themselves, without any additional assumption about reversibility, inevitably lead back again to Einsteinian geometrodynamics, which *is* time-reversible. Therefore, no irreversible geometrodynamics exists!

7. Supermomentum Regained

The first stage in recovering pure geometrodynamics is to find the gravitational supermomentum $\mathscr{H}_i$ that would satisfy the closing relations (2.26). The second stage then proceeds to the reconstruction of the super-Hamiltonian that would satisfy the closing relations (2.24) and (2.25). When it comes to the question of determining the super-Hamiltonian, the closing relations (2.24) and (2.25) themselves must do the job. On the other hand, we have a direct handle on the supermomentum. The difference is due to the fact that the super-Hamiltonian predicts what we shall find on a deformed hypersurface, and we have never before been on the deformed hypersurface to see what is there, whereas supermomentum merely reshuffles the data on the initial hypersurface, and we can easily foresee what data we shall find after the reshuffling.

In Fig. 4, a marked hypersurface fixed in spacetime is displayed. At each spacetime point along the hypersurface, a field variable grows, symbolized by a flower. The stretching of the marked hypersurface that takes the point carrying the label x^i and displaces it to the point that previously carried the label $x^i + \delta N^i$ is accomplished by the operator $\delta N^{ix}\mathscr{H}_{ix}$. The field variable that one finds at the new position is the one that grew there before, but one refers it to the stretched system of original labels x^i. In formal language, the variable $F + \delta F$, which is found after the stretching at the point carrying the label x differs from the variable F, which was found before the stretching at the point carrying the same label x by the Lie derivative with respect to δN^k,

$$\delta F = \pounds_{\delta N^k} F. \tag{7.1}$$

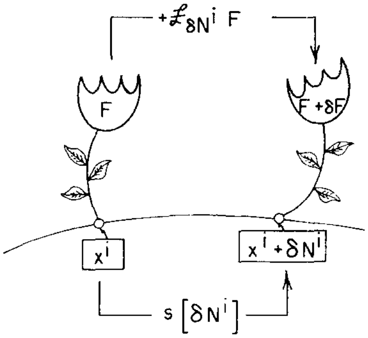

FIG. 4. Supermomentum regained. Field variables growing along a hypersurface are symbolized by flowers. The stretching $s[\delta N^i]$ takes an old point carrying the label x^i into a new point which previously carried the label $x^i + \delta N^i$. The field variable $F + \delta F$ found at the new point and referred to the stretched hypersurface differs by the Lie derivative $\pounds_{\delta N^i} F$ from the variable F found at the old point before the stretching.

Equation (7.1) is a kinematical equation, which must hold for an arbitrary field F defined along the hypersurface. In a dynamical theory, the generator of the stretching is represented by the supermomentum and the same change in F may be calculated from the propagation equation (4.1) with $\delta N = 0$. The comparison with (7.1) yields the condition

$$\pounds_{\delta N^k} F = \delta N^{kx'} [F, \mathscr{H}^{(T)}{}_{kx'}]. \tag{7.2}$$

Equation (7.2) holds for a field ϕ^A, π_A growing on a given spacetime background as well as for the geometrodynamical field g_{ij}, π^{ij} itself. Here we apply it to the geometrodynamical variables g_{ij}, π^{ij}. We already know that g_{ij} is a tensor and π^{ij} a tensor density of weight 1 and so we easily write down the appropriate Lie derivatives,

$$\begin{aligned} \pounds_{\delta N^k} g_{ij} &= g_{ij,k}\, \delta N^k + g_{ik}\, \delta N^k{}_{,j} + g_{kj}\, \delta N^k{}_{,i}\,, \\ \pounds_{\delta N^k} \pi^{ij} &= (\pi^{ij}\, \delta N^k)_{,k} - \pi^{ik}\, \delta N^j{}_{,k} - \pi^{kj}\, \delta N^i{}_{,k}\,. \end{aligned} \tag{7.3}$$

Because Eq. (7.2) holds for an arbitrary $\delta N^k(x)$, we extract from it the functional derivatives of $\mathscr{H}_k(x')$ with respect to $g_{ij}(x)$ and $\pi^{ij}(x)$:

$$\frac{\delta \mathscr{H}_k(x')}{\delta g_{ij}(x)} = -\pi^{ij}{}_{,k}(x)\, \delta(x, x') - \pi^{ij}(x)\, \delta_{,k}(x, x') + \pi^{il}(x)\, \delta^j_k \delta_{,l}(x, x') + \pi^{lj}(x)\, \delta^i_k \delta_{,l}(x, x'), \tag{7.4}$$

$$\frac{\delta \mathscr{H}_k(x')}{\delta \pi^{ij}(x)} = g_{ij,k}(x)\, \delta(x, x') + g_{ik}(x)\, \delta_{,j}(x, x') + g_{kj}(x)\, \delta_{,i}(x, x'). \tag{7.5}$$

Equations (7.4), (7.5) have a solution if their right-hand sides satisfy the integrability conditions following from the interchangeability of variational derivatives,

$$\frac{\delta^2 \mathscr{H}_k(x')}{\delta \pi^{mn}(x'')\, \delta g_{ij}(x)} = \frac{\delta^2 \mathscr{H}_k(x')}{\delta g_{ij}(x)\, \delta \pi^{mn}(x'')}.$$

If they do, the solution $\mathscr{H}_k(x')$ is unique up to an "integration constant" $h_k(x)$ independent of $g_{ij}(x)$ and $\pi^{ij}(x)$. It is easy to check that the integrability conditions are identities in the δ-functions, and it is equally easy to check that the solution of Eqs. (7.4)–(7.5) is

$$\mathscr{H}_k(x) = -2 g_{kl}(x)\, \pi^{lm}{}_{|m}(x) + h_k(x). \tag{7.6}$$

Finally, if expressions (7.6) are to satisfy the closing relations (2.26), we must put the function $h_k(x)$ equal to zero. This we would do anyway, because we do not want to introduce any new function (in addition to the canonical variables $g_{ij}(x)$ and $\pi^{ij}(x)$ themselves) into the theory. We have thus regained the gravitational supermomentum (2.8).

8. Super-Hamiltonian Regained

Here we finally come to the core of our derivation of Einsteinian geometrodynamics. We have seen that certain basic kinematical relations are satisfied in an arbitrary Riemannian spacetime. In particular, the Lie brackets between the generators of the normal and tangential deformations of a hypersurface close in a definite way, Eqs. (2.24) and (2.25). These equations hold in hyperspace, the Lie bracket operations taking place between two vector fields in hyperspace. The *representation requirement* states that the Poisson brackets between the super-Hamiltonian $\mathscr{H}(x)$ and the supermomentum $\mathscr{H}_i(x)$, which are considered as functionals defined over the geometrodynamical phase space $\{g_{ij}(x), \pi^{ij}(x)\}$, close in the same way as the Lie brackets between the corresponding generators in hyperspace. Further, we know that in an arbitrary Riemannian spacetime, the normal deformation of a hypersurface induces the change (3.11) of the intrinsic metric $g_{ij}(x)$. We require that the intrinsic geometry $g_{ij}(x)$, considered now as a canonical coordinate in the geometrodynamical phase space $\{g_{ij}(x), \pi^{ij}(x)\}$, be changed through its Poisson bracket with the super-Hamiltonian according to the same equation, in which the extrinsic curvature is considered as some undetermined functional defined over the geometrodynamical phase space (the *locality requirement*). We will prove that the only time-reversible super-Hamiltonian $\mathscr{H}(x)[g_{ij}(x), \pi^{ij}(x)]$ which satisfies the representation requirement and the locality requirement is the ADM super-Hamiltonian (2.6), (2.7).

We start with the closing relation (2.25). Its meaning was deciphered a long time ago by Dirac [44]. If we multiply it by an arbitrary shift $\delta N^i(x)$ and integrate over x, we get

$$[\mathscr{H}(x'), \mathscr{H}_{ix}]\, \delta N^{ix} = (\mathscr{H}(x')\, \delta N^i(x'))_{,i'}\,. \tag{8.1}$$

The left-hand side of Eq. (8.1) gives the change of $\mathscr{H}(x')$ under the stretching δN^{ix}. The right-hand side of Eq. (8.1) is just the Lie derivative $\pounds_{\delta N^i}\mathscr{H}(x')$ of a scalar density of weight 1. Equation (8.1) tells us therefore that $\mathscr{H}(x)$ must be constructed from the canonical variables g_{ij}, π^{ij} in such a way that it transforms like a scalar density of weight 1 under the relabeling of the hypersurface.

Another important piece of information about the structure of $\mathscr{H}(x)$ may be extracted from Eq. (3.11) which specifies the change of the metric $g_{ij}(x)$ produced by the action of the generator $\mathscr{H}(x')$ of a pure deformation. In geometrodynamics, the metric becomes a canonical variable and the same change is produced by its Poisson bracket with the super-Hamiltonian. Equation (3.11) then reads

$$\frac{\delta\mathscr{H}(x')}{\delta\pi^{ij}(x)} = [g_{ij}(x), \mathscr{H}(x')] = -2K_{ij}(x)\, \delta(x, x'). \tag{8.2}$$

The extrinsic curvature $K_{ij}(x)$ in the canonical theory becomes a functional of the canonical variables g_{ij}, π^{ij}. At this stage we do not have the slightest idea what functional it might be. However, the δ-function on the right-hand side of Eq. (8.2) tells us something important, that the super-Hamiltonian $\mathscr{H}(x')$ must be a strictly local functional of the momentum $\pi^{ij}(x)$, i.e., a *function* of $\pi^{ij}(x')$ taken at the same point x' at which the super-Hamiltonian $\mathscr{H}(x')$ is evaluated. This still does not mean that geometrodynamics must be a local theory, because $\mathscr{H}(x')$ may be nonlocal in the metric $g_{ij}(x')$. Nevertheless, the pure locality of $\mathscr{H}(x')$ in the momentum $\pi^{ij}(x')$ substantially simplifies all further arguments.

We are now finally prepared to analyze the last remaining closing relation, namely, that between two super-Hamiltonians (Eq. (2.24)). The first thing to observe is that this relation may determine π^{ij} only up to the canonical transformation (4.4). This transformation does not change the supermomentum, because the variational derivative $\delta\Lambda/\delta g_{ij}(x)$ of a labeling-independent functional $\Lambda[^3\mathscr{G}]$ is automatically divergence-free. The right-hand side of the closing relation

(2.24) thus remains untouched by the canonical transformation (4.3). On the other hand, any canonical transformation, by its very definition, leaves all Poisson brackets unchanged. Therefore, if a super-Hamiltonian $\mathscr{H}(x)[g_{ij}, \pi^{ij}]$ satisfies the closing relation (2.24), the new super-Hamiltonian obtained from it by the substitution $\pi^{ij}(x) \to \pi^{ij}(x) + \delta \Lambda[{}^3\mathscr{G}]/\delta g_{ij}(x)$ also satisfies it. The substitution changes the functional form of the super-Hamiltonian, but leaves the physical content of the theory unaffected. What one ultimately wants to prove is that the canonical transformation (4.3) is the only freedom left to the super-Hamiltonian by the closing relation (2.24), with the supermomentum already fixed to the form (2.8) by the considerations of Section 7 [46].

The proof of this statement is naturally carried through in the Lagrangian version of geometrodynamics obtained from the Hamiltonian version by a functional Legendre transformation. The role of velocity is played there by the extrinsic curvature K_{ij}. In the transformed closing relation, the even and odd velocity parts of the geometrodynamical super-Lagrangian decouple and may be determined separately. The even velocity part corresponds to the ADM super-Hamiltonian (2.6). The odd velocity part leads to the gauge degree of freedom (4.3). In this paper, we stick to the Hamiltonian geometrodynamics and derive the ADM super-Hamiltonian (2.6) from the closing relation (2.24) under the simplifying assumption of time reversibility. This assumption automatically removes the odd momenta terms which couple with the even momenta terms in the Hamiltonian method and enables us to explore the structure of the super-Hamiltonian by a simple recursive procedure. We expand the super-Hamiltonian in (even) powers of the momenta π^{ij} and substitute this expansion into the closing relation (2.24). The comparison of the lowest order terms (those linear in the momenta) already fixes the super-Hamiltonian up to the quadratic term into its standard form (2.6), (2.7). Comparing the cubic and higher-order terms in the closing relation (2.24), we get a set of recursive equations for the coefficients of the quartic and higher-order terms in the expansion of $\mathscr{H}$. All of these equations have the same basic form and it is easy to prove by induction that the higher-order coefficients must actually vanish, which leaves us finally with the purely quadratic gravitational super-Hamiltonian.

So, let us restrict ourselves to the time-reversible Hamiltonian geometrodynamics. We know then that the super-Hamiltonian $\mathscr{H}$ must be, after a possible canonical transformation (4.3), an even functional of the momenta π^{ij}. We have also just learned that it must be a function of the momenta rather than a functional. Therefore, if we take this function and expand it in terms of the momenta, only the even powers will survive,

$$\mathscr{H}(x) = \sum_{n=0}^{\infty} {}^{(2n)}G_{i_1 j_1 i_2 j_2 \cdots i_{2n} j_{2n}}(x)[g_{kl}]\; \pi^{i_1 j_1}(x)\; \pi^{i_2 j_2}(x) \cdots \pi^{i_{2n} j_{2n}}(x). \tag{8.3}$$

The coefficients ${}^{(2n)}G_{i_1j_1i_2j_2\cdots i_{2n}j_{2n}}$ are assumed to have the appropriate symmetries: with respect to an interchange of indices $i_a \leftrightarrow j_a$ within an $i_a j_a$ pair, and with respect to an interchange $i_a j_a \leftrightarrow i_b j_b$ of pairs. They are some unknown functionals of the metric g_{kl}. The superscript $(2n)$ indicates the order of the term; for most of this section, however, we shall simply omit it, as the order of the term is recognized by the number of indices which the coefficient carries. Because $\mathscr{H}(x)$ must be a scalar density of weight 1 and $\pi^{ij}(x)$ is a tensor density of weight 1, the coefficient ${}^{(2n)}G_{i_1j_1i_2j_2\cdots i_{2n}j_{2n}}$ must be a tensor density of weight $1 - 2n$.

Substitute the expansion (8.3) into the Poisson bracket

$$[\mathscr{H}(x), \mathscr{H}(x')] = \frac{\delta\mathscr{H}(x)}{\delta g_{ijx''}} \frac{\delta\mathscr{H}(x')}{\delta \pi^{ijx''}} - (x \leftrightarrow x'),$$

(the symbol $(x \leftrightarrow x')$ denotes "the same terms with the lables x and x' interchanged") and evaluate it up to the terms cubic in π^{ij}. This yields

$$[\mathscr{H}(x), \mathscr{H}(x')] = 2\,\frac{\delta G(x)}{\delta g_{ij}(x')}\, G_{ijkl}(x')\, \pi^{kl}(x') + 2\left(2\,\frac{\delta G(x)}{\delta g_{ij}(x')}\, G_{ijklmnpq}(x') + \frac{\delta G_{klmn}(x)}{\delta g_{ij}(x')}\, G_{ijpq}(x)\right) \pi^{kl}(x')\, \pi^{mn}(x')\, \pi^{pq}(x') - (x \leftrightarrow x'). \tag{8.4}$$

Write the right-hand side of the closing relation (2.24) in the form

$$2 \int d^3x''\; \pi^{kl}(x'')(\delta(x, x'')\, \delta_{|kl}(x, x') - \delta(x', x'')\, \delta_{|k'l'}(x', x))$$

and compare it with the terms linear in π^{kl} in Eq. (8.4). Introduce the abbreviation

$$F_{kl}(x, x') \equiv \frac{\delta G(x)}{\delta g_{ij}(x')}\, G_{ijkl}(x'), \tag{8.5}$$

and get the equation

$$\begin{aligned} &F_{kl}(x, x'')\, \delta(x', x'') - F_{kl}(x', x'')\, \delta(x, x'') \\ &\quad = \delta(x, x'')\, \delta_{|kl}(x, x') - \delta(x', x'')\, \delta_{|k'l'}(x', x). \end{aligned} \tag{8.6}$$

When handling the δ-functions and changing the partial derivatives into covariant derivatives with respect to the spatial metric, it is good to remember that $\delta(x, x')$ is a scalar in the first and a scalar density of weight 1 in the second argument.

Integrating Eq. (8.6) with respect to x, we can express $F_{kl}(x', x'')$ in terms of the integrated function

$$F_{kl}(x') \equiv \int d^3x\; F_{kl}(x, x') \tag{8.7}$$

and the δ-functions as follows:

$$F_{kl}(x', x'') = -\delta_{|k''l''}(x'', x') + F_{kl}(x')\,\delta(x'', x). \tag{8.8}$$

Substituting the expression (8.8) back into the original equation (8.6), we get an identity. Equation (8.8) is thus the only conclusion we may draw from the closing relation (2.24) by comparing the terms linear in the momenta.

Recalling the original meaning (8.5) of the function $F_{kl}(x, x')$, we can determine from Eq. (8.8) the variational derivative $\delta G(x)/\delta g_{ij}(x')$. To do that, define the inverse $G^{klmn}(x)$ of the supermetric $G_{ijkl}(x)$ by the formula

$$G_{ijkl}(x)\,G^{klmn}(x) = \delta^{mn}_{ij} = \tfrac{1}{2}(\delta^m_i\delta^n_j + \delta^n_i\delta^m_j). \tag{8.9}$$

Such an inverse must exist in order that Eq. (8.2) can be used to calculate the geometrodynamical momentum π^{ij} in terms of the extrinsic curvature K_{ij} (which plays the role of the goemetrodynamical velocity). Notice that G^{ijkl} *does not* arise from G_{ijkl} through raising of the indices by the contravariant metric tensor g^{ij}. In fact, we shall find it convenient to raise the symmetrical pairs of indices by this "covariant supermetric" according to the rule

$$A_{mn}(x) \to A^{kl}(x) = G^{klmn}(x)\,A_{mn}(x). \tag{8.10}$$

This we do in Eqs. (8.8), (8.5), writing them in the form

$$\begin{aligned} \frac{\delta G(x)}{\delta g_{ij}(x')} &= -G^{ijkl}(x')\,\delta_{|k'l'}(x', x) + F^{ij}(x)\,\delta(x, x'), \\ F^{ij}(x) &\equiv \frac{\delta}{\delta g_{ij}(x)} \int d^3x'\, G(x'). \end{aligned} \tag{8.11}$$

Equation (8.11) tells us that the functional $G(x)[g_{ij}]$ is in fact a function of the metric $g_{ij}(x)$ and its partial derivatives to the second order; if it depended on the higher-order derivatives of the metric, the variational derivative with respect to $g_{ij}(x')$ would yield a higher derivative of the δ-function than the second. Further, because the zero-order coefficient $G(x)$ is a scalar density of weight 1 constructed in a form-invariant way from the metric tensor only, we may invoke the well known theorem of Riemannian geometry [9] and conclude that $G(x)$ can depend on $g_{ij}(x)$ and its derivatives only through the quantities $g_{ij}(x)$ and $R_{ijkl}(x)$. Let us further simplify the calculations and use the fact that the physical space is three-dimensional. The Riemann tensor R_{ijkl} of a three-dimensional space is expressible by means of the Ricci tensor R_{ij} and the metric g_{ij}, so that we may finally write

$$G(x) = G(g_{ij}(x), R_{ij}(x)). \tag{8.12}$$

The form of the potential $G(x)$ may be completely determined from Eq. (8.11). Varying Eq. (8.12), we get

$$\delta G(x) = \varphi^{ij}(x)\,\delta g_{ij}(x) + \Phi^{mn}(x)\,\delta R_{mn}(x), \tag{8.13}$$

where

$$\varphi^{ij}(x) = \frac{\partial G(g_{kl}(x), R_{kl}(x))}{\partial g_{ij}(x)}$$

and (8.14)

$$\Phi^{mn}(x) = \frac{\partial G(g_{kl}(x), R_{kl}(x))}{\partial R_{mn}}$$

are symmetrical contravariant tensor densities of weight 1 constructed locally from $g_{kl}(x)$ and $R_{kl}(x)$. The variation of the Ricci tensor may be written in the form

$$\delta R_{mn} = S_{mn}{}^{ijkl}\,\delta g_{ij|kl} + S_{mn}{}^{ij}\,\delta g_{ij}\,. \tag{8.15}$$

The coefficients

$$S_{mn}{}^{ijkl} = \tfrac{1}{4}(g^{jk}\delta^{il}_{mn} + g^{ik}\delta^{jl}_{mn} + g^{jl}\delta^{ik}_{mn} + g^{il}\delta^{jk}_{mn} - 2g^{ij}\delta^{kl}_{mn} - 2g^{kl}\delta^{ij}_{mn}) \tag{8.16}$$

and

$$S_{mn}{}^{ij} = \tfrac{1}{8}(R^i_m\delta^j_n + R^i_n\delta^j_m + R^j_m\delta^i_n + R^j_n\delta^i_m + 2R^i{}_{mn}{}^j + 2R^i{}_{nm}{}^j) \tag{8.17}$$

have the following symmetries: $S_{mn}{}^{ijkl}$ is symmetrical in $m \leftrightarrow n$, $i \leftrightarrow j$, and $k \leftrightarrow l$, and symmetrical in the interchange of pairs $ij \leftrightarrow kl$, whereas $S_{mn}{}^{ij}$ is symmetrical in $m \leftrightarrow n$, and $i \leftrightarrow j$. Substituting for δR_{mn} into Eq. (8.13), we get the variation of G,

$$\begin{aligned}\delta G = {} & (\varphi^{ij} + \tfrac{1}{2}R^i{}_{kl}{}^j\Phi^{kl} + \tfrac{1}{4}R^i_k\Phi^{kj} + \tfrac{1}{4}R^j_k\,\Phi^{ki})\,\delta g_{ij} \\ & + \tfrac{1}{4}(\Phi^{ik}g^{jl} + \Phi^{il}g^{jk} + \Phi^{jk}g^{il} + \Phi^{jl}g^{ik} - 2\Phi^{ij}g^{kl} - 2\Phi^{kl}g^{ij})\,\delta g_{ij|kl}\,.\end{aligned} \tag{8.18}$$

The coefficient of δg_{ij} in Eq. (8.18) is symmetrical in $i \leftrightarrow j$ and, due to the symmetries of the expression (8.16), the coefficient of $\delta g_{ij|kl}$ has all the symmetries of a supermetric: $i \leftrightarrow j$, $k \leftrightarrow l$, and $ij \leftrightarrow kl$.

On the other hand, the variation of G may also be determined from Eq. (8.11),

$$\begin{aligned}\delta G(x) &= \frac{\delta G(x)}{\delta g_{ijx'}}\,\delta g_{ijx'} = -(G^{ijkl}\,\delta g_{ij})_{|lk} + F^{ij}\,\delta g_{ij} \\ &= (-G^{ijkl}{}_{|lk} + F^{ij})\,\delta g_{ij} - 2G^{ijkl}{}_{|l}\,\delta g_{ij|k} - G^{ijkl}\,\delta g_{ij|kl}\,.\end{aligned} \tag{8.19}$$

When comparing Eqs. (8.18) and (8.19), it is important to realize that, at a given point x, δg_{ij} and $\delta g_{ij|k}$ are completely arbitrary. The antisymmetric part $\delta g_{ij|\underset{\vee}{kl}}$ of $\delta g_{ij|kl}$, of course, depends on δg_{ij} through

$$\delta g_{ij|\underset{\vee}{kl}} \equiv \tfrac{1}{2}(\delta g_{ij|kl} - \delta g_{ij|lk}) = \tfrac{1}{2}R^m{}_{ikl}\,\delta g_{mj} + \tfrac{1}{2}R^m{}_{jkl}\,\delta g_{im}\,.$$

The antisymmetric part of $\delta g_{ij|kl}$ is, however, automatically killed by the symmetry of the coefficients in Eqs. (8.18) and (8.19); this was our motivation for writing δR_{mn} in the form (8.15)–(8.17) rather than in other conceivable equivalent forms. It is vital that the *symmetric* part $\delta g_{ij|\underline{kl}}$ of $\delta g_{ij|kl}$ does not depend on δg_{ij} and $\delta g_{ij|k}$ and may be chosen arbitrarily. This allows us to extract from the comparison of expressions (8.18) and (8.19) three sets of equations:

$$G^{ijkl} = \tfrac{1}{4}(2g^{ij}\Phi^{kl} + 2g^{kl}\Phi^{ij} - g^{jk}\Phi^{il} - g^{jk}\Phi^{il} - g^{ik}\Phi^{jl} - g^{jl}\Phi^{ik} - g^{il}\Phi^{jk}), \tag{8.20}$$

$$G^{ijkl}{}_{|l} = 0, \tag{8.21}$$

$$F^{ij} = \varphi^{ij} + \tfrac{1}{4}(R^i_m\Phi^{jm} + R^j_m\Phi^{im} - 2R^i{}_{mn}{}^j\Phi^{mn}). \tag{8.22}$$

Equation (8.20) tells us that the supermetric G^{ijkl} is determined by the potential G through the quantity $\Phi^{ij} \equiv \partial G/\partial R_{ij}$. According to the next equation, Eq. (8.21), the supermetric has a vanishing divergence. We show in a moment that this condition fixes Φ^{ij} and with it the potential G. The last equation, Eq. (8.22), then becomes an identity.

Substituting the supermetric (8.20) into Eq. (8.21), we get the condition

$$2\Phi^{ij|k} - \Phi^{ik|j} - \Phi^{jk|i} + 2g^{ij}\Phi^{kl}{}_{|l} - g^{jk}\Phi^{il}{}_{|l} - g^{ik}\Phi^{jl}{}_{|l} = 0. \tag{8.23}$$

Equation (8.23) is equivalent to its antisymmetrized form in the induces jk:

$$\Phi^{ij|k} - \Phi^{ik|j} + g^{ij}\Phi^{kl}{}_{|l} - g^{ik}\Phi^{jl}{}_{|l} = 0. \tag{8.24}$$

Contracting Eq. (8.24) in the indices ij, we obtain the relation

$$(\Phi^{kl} + \Phi g^{kl})_{|l} = 0. \tag{8.25}$$

Equation (8.25) tells us that the divergence of a certain tensor density constructed in an invariant way from the metric tensor and its first and second partial derivatives vanishes. We may thus apply Lovelock's theorem [10] and conclude that

$$\Phi^{kl} + \Phi g^{kl} = \mu g^{1/2}(R^{kl} - \tfrac{1}{2}Rg^{kl}) - (2\kappa)^{-1}\,g^{1/2}g^{kl}, \tag{8.26}$$

κ and μ being some constants.

Determining Φ^{kl} from Eq. (8.26) and substituting it back into the original equation (8.24), we get the condition

$$\mu g^{1/2}((R^{ij} - \tfrac{1}{4}Rg^{ij})^{|k} - (R^{ik} - \tfrac{1}{4}Rg^{ik})^{|j}) = 0.$$

It is impossible to satisfy it for a general 3-geometry unless $\mu = 0$. Thus

$$\Phi^{kl} = -(2\kappa)^{-1} g^{1/2} g^{kl}. \tag{8.27}$$

Recalling that $\Phi^{kl} = \partial G/\partial R_{kl}$, we easily integrate Eq. (8.27) and obtain G,

$$G = -(2\kappa)^{-1} g^{1/2}(R - 2\lambda), \tag{8.28}$$

picking up an integration constant λ. Finally, substituting the expression (8.27) into Eq. (8.20), we recover the "covariant" supermetric

$$G^{ijkl} = (2\kappa)^{-1} \tfrac{1}{2} g^{1/2}(g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl}). \tag{8.29}$$

DeWitt's supermetric (2.7) is just the "contravariant" form of the supermetric (8.29), as one can easily check using the formula (8.9).

Returning to Eq. (8.22), one sees that all terms containing the Riemann tensor cancel each other when the expression (8.27) for Φ^{ij} is used. One calculates F^{ij} and φ^{ij} from the potential (8.28) according to their definitions (8.11) and (8.14), and checks that Eq. (8.22) turns into an identity.

The closing relation (2.24) taken to the terms linear in the momentum π^{ij} thus ensures that the super-Hamiltonian has the standard form (2.6), (2.7) to the terms quadratic in the momentum. It remains to be proved that $G_{ijklmnpq}$ and, in fact, all further expansion coefficients ${}^{(2n)}G$, $n \geqslant 2$, must vanish. Go back to the expression (8.4) and consider the terms cubic in π^{ij}. Because we already know that the supermetric is purely local in the spatial metric $g_{ij}(x)$, the cubic term containing G_{ijkl} vanishes after the commutation of the lables x and x'. The remaining term has a familiar structure. Just as we have passed from the closing relation (2.24) in the first order to Eq. (8.6), we pass in the third order to the equation

$$F_{klmnpq}(x, x')\, \delta(x', x'') - F_{klmnpq}(x', x)\, \delta(x, x'') = 0, \tag{8.30}$$

with $F_{klmnpq}(x, x')$ introduced as an abbreviation

$$F_{klmnpq}(x, x') \equiv \frac{\delta G(x)}{\delta g_{ij}(x')}\, G_{ijklmnpq}(x'). \tag{8.31}$$

Integrating Eq. (8.30) with respect to x', we conclude that the expression $F_{klmnpq}(x, x')$ is proportional to the δ-function,

$$\begin{aligned} F_{klmnpq}(x, x') &= F_{klmnpq}(x)\, \delta(x, x'), \\ F_{klmnpq}(x) &\equiv \int d^3x'\, F_{klmnpq}(x', x). \end{aligned} \tag{8.32}$$

On the other hand, substituting Eq. (8.11) into Eq. (8.31) we learn that the same expression contains the term

$$-G^{ijrs}(x')\, G_{ijklmnpq}(x')\, \delta_{|r's'}(x', x)$$

with the second derivative of the δ-function. This term must therefore vanish, which means that, due to the invertibility of the supermetric, the coefficient $G_{ijklmnpq}$ must vanish.

Complete the proof by induction. Assume that the coefficients ${}^{(4)}G$, ${}^{(6)}G$,..., ${}^{(2n-2)}G$ already vanish. Calculate the term of the order $2n - 1$ in the Poisson bracket $[\mathscr{H}(x), \mathscr{H}(x')]$. This term is equal to

$$\frac{\delta G(x)}{\delta g_{i_1 j_1}(x')}\; {}^{(2n)}G_{i_1 j_1 i_2 j_2 \cdots i_{2n} j_{2n}}(x')\; \pi^{i_2 j_2}(x') \cdots \pi^{i_{2n} j_{2n}}(x') - (x \leftrightarrow x')$$

and it must vanish in order that the closing relation for the $\mathscr{H}$'s be satisfied. By the same reasoning which led us to the equation ${}^{(4)}G = 0$, we conclude that ${}^{(2n)}G = 0$ for $n > 2$. This shows that the Einsteinian geometrodynamics in vacuum is the only purely geometrical time-reversible representation of the generators of deformations of a spacelike hypersurface embedded in a Riemannian spacetime.

We have seen in Section 3 that the only difference between the indefinite and positive definite spacetimes is the indicator ϵ on the right-hand side of the commutation relation (3.14). If we had repeated the argument of this section with $\epsilon = +1$, we would have arrived at the same supermetric, but the potential term would reverse its sign, becoming

$$+(2\kappa)^{-1}\, g^{1/2}(R - 2\lambda).$$

The signature of spacetime is thus reflected in the relative sign of the "kinetic" and "potential" terms in the super-Hamiltonian. This may be important for the proper identification of the internal time and energy variables.

9. Discussion

We are now in a position to look back and see how our reconstruction of Einsteinian geometrodynamics proceeded. We summarize the basic postulates and compare their function within our scheme with the function of the Weyl–Cartan–Lovelock postulates.

The leading idea of our approach is to carry certain structures defined over hyperspace into corresponding structures defined over the geometrodynamical phase space. Hyperspace is defined here as a collection of all spacelike hypersurfaces considered as embeddings of space in a Riemannian spacetime. The tangent space to hyperspace at a hypersurface is spanned by the generators of the normal and tangential deformations of the hypersurface (pure deformations and pure stretchings). These generators form a moving frames basis. The Lie brackets between the generators close in a characteristic way, Eqs. (2.24)–(2.26). Because of the normalization of the generator of pure deformation, the closing relation between the generators contains explicitly the intrinsic metric of the hypersurface. The differential geometry in hyperspace is discussed in detail in the separate paper [48].

While hyperspace is constructed over a given Riemannian spacetime, geometrodynamics wants to generate this spacetime by an evolution of geometry in the goemetrodynamical phase space. The principles of evolution are summarized in the

1. Evolution Rule. *Hamiltonian geometrodynamics describes how the geometrodynamical variables change when we deform the hypersurface by the amount* $\delta N(x)$ *and stretch it by the amount* $\delta N^i(x)$. *In pure geometrodynamics, the metric* $g_{ij}(x)$ *and its conjugate momentum* $\pi^{ij}(x)$ *are regarded as the sole canonical variables. There exists a super-Hamiltonian* $\mathscr{H}(x)[g_{ij}, \pi^{ij}]$ *and a supermomentum* $\mathscr{H}_k(x)[g_{ij}, \pi^{ij}]$ *constructed entirely from the canonical variables* g_{ij}, π^{ij}, *which generate the change of an arbitrary functional* $F[g_{ij}, \pi^{ij}]$ *of the canonical variables according to the formula*

$$\delta F = [F, \mathscr{H}_x]\, \delta N^x + [F, \mathscr{H}_{ix}]\, \delta N^{ix}. \tag{9.1}$$

The super-Hamiltonian and supermomentum are functionals defined over the geometrodynamical phase space. We can then form their Poisson brackets. At this point, we are able to formulate the basic correspondence of structures defined over hyperspace and the geometrodynamical phase space. We require that there exists a one-to-one correspondence between the generators of the normal and tangential deformations in the tangent space to hyperspace on the one hand, and the super-Hamiltonian and supermomentum defined over the geometrodynamical phase space on the other hand, which brings the Lie brackets between the generator fields in hyperspace into the Poisson brackets between the super-Hamiltonian and supermomentum fields in the geometrodynamical phase space. In short, we impose the

2. Representation Postulate. *The Poisson brackets between the super-Hamiltonian and the supermomentum close in the same way as the commutators between the corresponding generators of the deformations of a spacelike hypersurface embedded in a Riemannian spacetime with the signature* $(-, +, +, +)$.

The next two postulates inform us that geometrodynamics preserves certain kinematical relations, which are valid on arbitrary spacelike hypersurfaces in an arbitrary Riemannian spacetime:

3. INITIAL DATA RESHUFFLING. *The change* (9.1) *of an arbitrary functional F of the canonical variables under the stretching* $\delta N^i(x)$ *is given by the Lie derivative of F with respect to* δN^i.

4. LOCALITY. *In an aribitrary spacetime, the change of the metric* g_{ij} *under a normal deformation is purely local, Eq.* (3.11). *The dynamical rule* (9.1) *respects this kinematical property, Eq.* (8.2).

The two postulates that follow summarize the properties of reversibility and path-independence of the geometrodynamical evolution:

5. REVERSIBILITY. *The time-reversed spacetime is generated by the same super-Hamiltonian and supermomentum as the original spacetime.*

6. PATH INDEPENDENCE. *If the same final marked hypersurface is reached from an initial marked hypersurface by two different sequences of intermediate marked hypersurfaces (by two different paths), the final geometrodynamical state calculated from a permissible initial state by means of the evolution rule* (9.1) *along each of these two paths should be the same.*

Finally, if we want to include the nongravitational fields as sources of geometry, we need a rule that tells us how to proceed in the geometrodynamical language. The situation for general tensor fields is complicated [49], so we confine ourselves to the fields with a nonderivative gratitational coupling:

7. GEOMETRODYNAMICAL PRINCIPLE OF EQUIVALENCE. *Simple fields (with a nonderivative gravitational coupling) may be included as sources of the gravitational field by a two-step process. First, a super-Hamiltonian* $\mathscr{H}^{(M)}$ *and a supermomentum* $\mathscr{H}^{(M)}{}_i$ *are constructed which satisfy the modified closing relations* (2.24), (2.26), (2.27) *appropriate for a fixed geometrical background. Second, these expressions are added to the* ADM *super-Hamiltonian and supermomentum.*

Let us now discuss how these postulates function in building the Hamiltonian geometrodynamics. The evolution rule (1) explains the basic scheme of a many-fingered time Hamiltonian dynamics. It also limits the number and character of "purely geometrodynamical variables." We have pointed out in Section 1 that such a limitation is vital in any derivation of Einstein's law. Through the requirement that the basic equation (9.1) has the form of Hamilton's equation for the canonical variables g_{ij}, π^{ij}, the "order" of equations is indirectly limited. Such a limitation is far from the straightforward limitation (3), Section 1, of the Weyl–Cartan–Lovelock method. There, the exact locality of $G_{\iota\kappa}$ is explicitly stated: $G_{\iota\kappa}$ depends on ${}^4g_{\iota\kappa}$ and its first and second partial derivatives at a given point. On the other hand, the super-Hamiltonian $\mathscr{H}$ and the supermomentum $\mathscr{H}_i$, which play in geometrodynamics a similar role to $G_{\iota\kappa}$ in spacetime relativity, are not limited at this level at all. In principle, they may be arbitrary nonlocal functionals of the canonical variables g_{ij}, π^{ij}.

Restricting our configuration variables to the metric g_{ij}, we effectively exclude other alternative theories of gravitation which are in principle compatible with a Riemannian structure of spacetime. For example, the Jordan–Thiry–Brans–Dicke theory is excluded because its canonical formulation requires an enrichment of the configuration space by a scalar field variable. Similarly, the higher-order laws of gravitation [24, 25] for the single spacetime metric tensor ${}^4g_{\iota\kappa}$ are excluded, because their canonical formulation requires an introduction of supplementary variables which are, basically, new names for the higher order derivatives of ${}^4g_{\iota\kappa}$. The restriction of the configuration space thus represents an important input into our derivation of Einstein's law and plays a similar role as the restriction to the spacetime metric does in the Weyl–Cartan–Lovelock argument.

The next three postulates, (2)–(4), are all in a broader sense "representation" postulates. They pick up certain "kinematical relations"—relations which are valid in an arbitrary Riemannian spacetime—and state that the evolution rule (9.1) must respect them. In (2), the kinematical relations are the commutation relations (3.14)–(3.16). In (3), we have the kinematical formula (7.1). In (4), the relation (3.11) between the intrinsic metric and the extrinsic curvature is kinematical. On the dynamical level, the functionals of a hypersurface are replaced by the functionals of the canonical variables and the action of the generators is replaced by the Poisson brackets. The kinematical relations are thus translated into strong equations (in Dirac's terminology) containing the Poisson brackets of the super-Hamiltonian and supermomentum among themselves or with the canonical variables g_{ij}, π^{ij}. These equations then determine both the locality and the detailed structure of the super-Hamiltonian and the supermomentum. The "order" of the geometrodynamical equations is thus fixed in a natural way rather than by decree.

Among requirements (2)–(4), Eqs. (2.24) and (2.25) correspond to the contracted Bianchi identities [33]. These are exactly the equations which (apart from the locality requirement) fix the structure of the super-Hamiltonian. This links our derivation with the postulate (4) of the Weyl–Cartan–Lovelock method.

The determination of the supermomentum, on the other hand, has nothing to do with the Bianchi identities. It follows from the reshuffling argument which specifies the behavior of the initial data under the relabeling of the hypersurface. The closing relation (2.26) plays an entirely passive role in our derivation and may be easily dropped off.

After the structure of $\mathscr{H}$ and $\mathscr{H}_i$ is established, the path-independence leads to the necessity of imposing the constraints. Its relation to the representation postulate (2) is rather subtle. While the representation postulate was imposed as a strong equation, the path-independence should hold only weakly, i.e., for the data restricted by the initial value constraints. In fact, we showed that when the representation equations hold strongly, the data *must* be restricted by the initial value constraints if they are to be propagated in a path-independent way. One would like, of course, to have a better reason why some equations are to be imposed as strong equations and other equations as weak equations.

Our attention in this paper was concentrated on the reconstruction of pure geometrodynamics. Much work remains to be done before one learns how to build up all possible field equations on the geometrodynamical level and how to include the fields as sources of the gravitational field. Our discussion of the "geometrodynamical principle of equivalence" for the simple fields was meant as an illustration of a basic idea how to proceed in such cases and not as a final answer to the problem of sources.

We did not attempt to reduce our "postulates" to the bare minimum and sweep away any possible redundancy in the system. The closing relation (2.26) was, of course, made redundant by the reshuffling postulate. The reversibility assumption (5) was added simply for convenience, enabling a simple derivation of the super-Hamiltonian within the Hamiltonian formalism. Its redundancy is shown in [43]. One may guess that the locality requirement (4) is also superfluous and that the closing relations (2.24), (2.25) themselves carry the necessary information about the locality in the momenta. We were, however, unable to extract this information from them. (For a possible line of attack, see Teitelboim's thesis, quoted in [46].) More significantly, the exact relationship between the representation requirement and path-independence remains to be clarified, together with the role played by the strong and weak equations. One would also like to carry the derivation for any dimension n of the spacelike hypersurface, not only for $n = 3$. We have tried to work in this direction, but the simplification of the argument for $n = 3$ was too attractive to be resisted. We believe that our postulates conveniently summarize the spirit underlying the geometrodynamical foundations of geometrodynamics, even if their further reduction seems possible.

Turn from the assumptions to the results. In the Lovelock version of the traditional method one does not assume how the metric ${}^4g_{\iota\kappa}$ and its first and second derivatives enter into $G_{\iota\kappa}$. One gets as a result of the Lovelock theorem that the second derivatives enter linearly, with the coefficients which depend merely on the undifferentiated metric (quasilinearity of Einstein's equations), and the first derivatives enter in quadratic combinations. Similarly, we do not assume that the super-Hamiltonian is quadratic in the momenta; it follows as a result, together with the fact that the supermetric is purely local in g_{ij}. This is directly related to the quasilinearity of the dynamical Einstein's equations. Also, we did not fix the potential term in the super-Hamiltonian in advance, but it turned out that this term must be (apart from the cosmological constant) equal to the scalar curvature density, containing thus the proper combination of terms linear in the second spatial derivatives of the metric and terms quadratic in the first derivatives. This is related to the quasilinearity of the 00 component of Einstein's law of gravitation. The locality of the supermetric may turn out to be very important in quantum geometrodynamics, especially for the factor ordering. It also expresses a tendency of the gravitational field to behave like any other field would behave on a prescribed geometrical background— a tendency that Misner dubbed "super-equivalence principle."

These features are, of course, only some details of the complete structure of $\mathscr{H}$ and $\mathscr{H}_i$ which we get as our final result. At the end, all the geometrodynamical elements neatly interlock into these projections of a single spacetime tensor: the Einstein tensor $G_{\iota\kappa}$.

By and large, the same ingredients which went into the cooking of the Einstein law in the Weyl–Cartan–Lovelock pot—covariance, Bianchi identities, limitation of the basic variables, and the limitation of the order of equations—go into our derivation as well, though in a modified form and in a different context. We think, for example, that the limitation of the order of Einstein's equations is achieved more naturally in geometrodynamics than in the spacetime approach.

We believe that the new derivation is primarily important because the geometrodynamical equations are closer to the quantum formalism and thus more likely to preserve their form in the quantum theory of gravitation. Classically, our assumptions ensure that a single spacetime may be built up by a geometrodynamical evolution of a three-geometry. Quantum mechanically, as stressed by John Wheeler [50], three-geometries cannot be stacked into a single spacetime. According to one witty definition, education is what remains when everything what one has been taught is forgotten. We hope that at least some of our assumptions convey what remains from spacetime when spacetime itself falls into oblivion by quantization.

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