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The Two-Body Problem in Geometrodynamics

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The problem of two interacting masses is investigated within the framework of geometrodynamics. It is assumed that the space-time continuum is free of all real sources of mass or charge; particles are identified with multiply connected regions of empty space. Particular attention is focused on an asymptotically flat space containing a "handle" or "wormhole." When the two "mouths" of the wormhole are well separated, they seem to appear as two centers of gravitational attraction of equal mass. To simplify the problem, it is assumed that the metric is invariant under rotations about the axis of symmetry, and symmetric with respect to the time $t = 0$ of maximum separation of the two mouths. Analytic initial value data for this case have been obtained by Misner; these contain two arbitrary parameters, which are uniquely determined when the mass of the two mouths and their initial separation have been specified. We treat a particular case in which the ratio of mass to initial separation is approximately one-half. To determine a unique solution of the remaining (dynamic) field equations, the coordinate conditions $g_{0\alpha} = -\delta_{0\alpha}$ are imposed; then the set of second order equations is transformed into a quasilinear first order system and the difference scheme of Friedrichs used to obtain a numerical solution. Its behavior agrees qualitatively with that of the one-body problem, and can be interpreted as a mutual attraction and pinching-off of the two mouths of the wormhole.

I. INTRODUCTION

Wheeler (*1*, *2*) has used the term "geometrodynamics" to characterize those solutions of the field equations for gravitation and electromagnetism[1]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}R = 2(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \tfrac{1}{4} g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}) \tag{1.1a}$$

$$F^{\mu\nu}{}_{;\nu} = 0 \tag{1.1b}$$

[1] Throughout this paper Greek subscripts and superscripts range from 0 to 3 and Latin ones from 1 to 3. Also, units are chosen so that G (universal gravitation constant) $= c = 1$.

which are free of singularities and of external sources. He pointed out that many of the properties one normally associates with classical ponderable bodies (e.g., mass and charge) can also be defined on a manifold which contains an *everywhere continuous* Lorentzian metric $g_{\mu\nu}$ and electromagnetic field tensor $F_{\mu\nu}$. In this view, particles are identified with *multiply connected topologies* in "empty" space; the space-time continuum is regarded, not as an arena populated with extraneous objects, but as a complete description of a classical physical system.[2]

Among the problems which can be treated from the point of view of geometrodynamics, one of the most interesting is the interaction between two concentrations of mass-energy: this situation provides a representation of the two-body problem in general relativity. In all attacks on this difficult and important problem made up to now, the representation of the particles themselves has posed severe difficulties. One either chose to work with the source-free field equations, in which case the metric became singular at two points in space, or else introduced "real" sources of mass whose complete description lay outside the scope of general relativity. Fortunately, the precise specification of the masses makes little difference as long as the two interacting regions are separated by a distance large in comparison with GM/c^2 (the distance at which the space surrounding the particles becomes strongly curved). Then the well-known approximation method of Einstein, Infeld, and Hoffmann (*3*, *4*) allows one to compute the equations of motion of the two mass centers. Such approximation techniques have provided considerable information on the behavior of two masses in the post-Newtonian approximation.[3] Since the extreme weakness of the gravitational interaction makes even the first correction term almost negligible, we may regard the two-body problem as essentially solved for all cases likely to be observed.

As a matter of principle, however, the solution to the two-body problem provided by these approximation methods cannot be deemed complete. Any complete solution would give a detailed description of the time evolution of the metric tensor in the immediate vicinity of the two mass centers, a description

[2] One must make a clear distinction between charge and mass as they occur in the real world (with a specifically quantum character) and concentrations of charge or mass so large that they can be treated on a purely *classical* level. Only the latter are legitimate objects of investigation within the framework of classical general relativity, and only the latter will be considered in this paper.

[3] The earliest attempts to solve the two-body problem in general relativity were made by de Sitter (*5*), using perturbation theory, and by Levi-Civita (*6*), using the linearized form of the field equations.

A subsequent, more careful treatment by Robertson (*7*), starting with the sixth order equations of motion obtained from the Einstein, Infeld, and Hoffman approximation, showed that the only nonperiodic relativistic effect to this order was the well-known procession of the perihelion. This result has been extended, and rederived using other methods, by several authors; recent developments are described in the books by Fock (*8*) and by Infeld and Plebanski (*9*), and in an article by Goldberg (*10*).

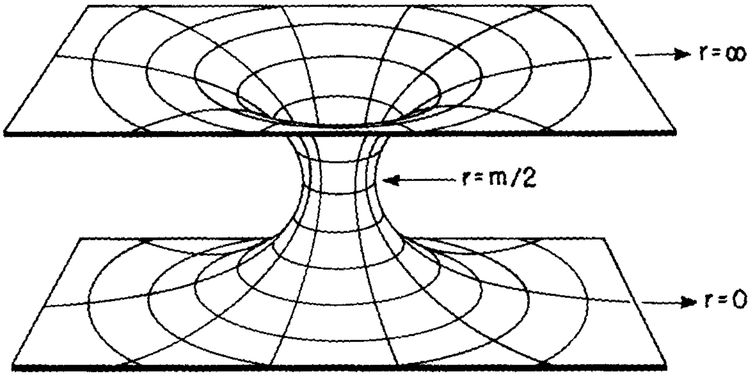

FIG. 1. A two-dimensional analogue of the Einstein-Rosen manifold for a single particle is shown imbedded in flat 3-space to suggest its curvature and topology. The sheets at the top and bottom of the funnel continue to infinity and represent the two asymptotically flat regions of the manifold.

which must necessarily depend on the particular model assumed for the structure of the masses. The guiding principles of geometrodynamics provide a natural definition of "mass centers" for such a deepseated analysis; here the field equations are given full reign to predict the subsequent evolution of the system.

Even within the framework of geometrodynamics the description of the masses is not unique. One can construct several different 3-manifolds, with different topologies, all of which can reasonably be interpreted as universes containing two "particles."

The starting point for each construction is the familiar one-body metric of Schwarzschild (expressed in "isotropic" coordinates):

$$ds^2 = -\left(1 - \frac{m}{2r}\right)^2 \left(1 + \frac{m}{2r}\right)^{-2} dt^2 + \left(1 + \frac{m}{2r}\right)^4 (dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2) \quad (1.2)$$

Einstein and Rosen (*11*) suggested that this metric could be interpreted most naturally on a two-sheeted space[4] joined at the Schwarzschild radius $r = m/2$ (see Fig. 1). One is thus led to consider manifolds with two such Einstein-Rosen bridges as models for the two-body problem. Three possible candidates are illustrated in Fig. 2. The last of these describes an asymptotically flat 3-space containing two Einstein-Rosen bridges whose "mouths" are joined smoothly together; it has the topology of a Euclidean 3-space containing a "handle." This

[4] Kruskal (*12*) has shown that one is led inevitably to this two-sheeted description when seeking the maximal analytic extension of (1.2) through its (removable) singularity at the Schwarzschild radius.

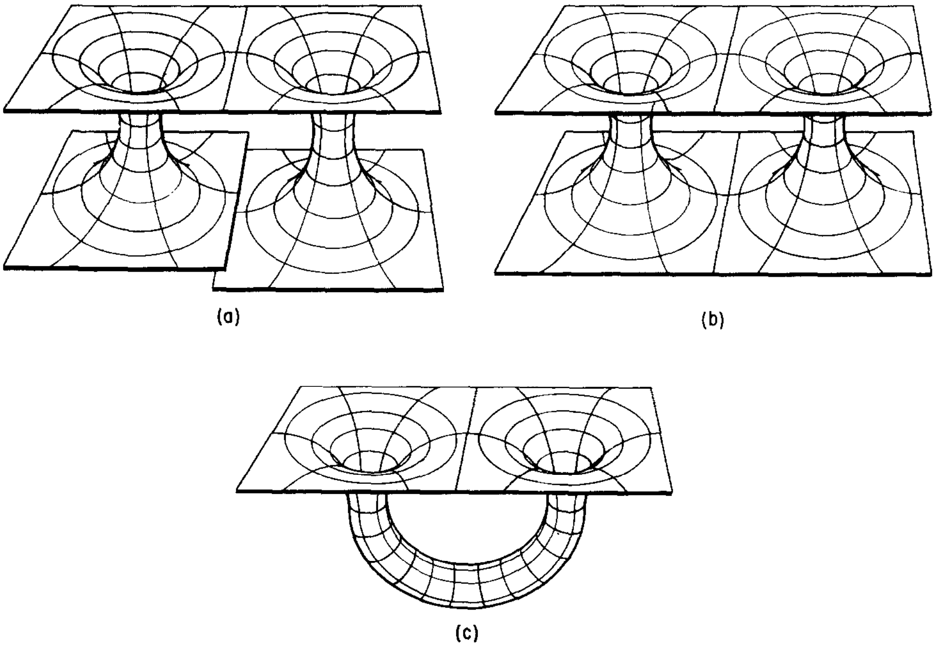

FIG. 2. Three models for the two-body problems are displayed as two-dimensional surfaces imbedded in flat 3-space. The first, (a), a direct generalization of Fig. 1, shows a pair of Einstein-Rosen bridges whose upper sheets are joined together. In (b) their lower sheets have been joined as well, to form a multiply connected space whose upper and lower sheets are equivalent (i.e. isometric). If the two masses represented by the two mouths are equal, the figure will also be symmetric under reflections in a vertical plane midway between them. These symmetries then allow one to identify corresponding points on the two sheets, and also to identify corresponding points at the two mouths; the space which results is shown in (c).

model was first proposed by Wheeler (*1*) and later called by him a "wormhole" manifold (*2*). In order that the two mouths shall match up sufficiently smoothly (i.e., analytically), they must be identical; thus this model describes a universe containing two particles of *equal mass*.[5]

While there is no compelling reason for preferring one model over another, we have focused attention exclusively on the wormhole picture. As far as the purely

[5] The present investigation has been restricted to the case of pure gravitation, for which the appropriate field equations are those given in Eq. (1.1a) with the right-hand side set equal to zero (i.e., with $F_{\mu\nu} = 0$). One can, however, also investigate solutions to the complete set of equations (1.1) on this manifold. In this more general approach it is also possible to trap electric lines of force in the "wormhole," giving the appearance of two particles with equal mass and *equal but opposite charge*. The time-symmetric initial value problem for this case has been solved by Lindquist (*13*).

mathematical problems of solving the field equations are concerned, the chief obstacles arise from the nonlinearity of the equations, and would be present regardless of the model used to represent the masses. Furthermore, one can expect the general behavior of the solution to be relatively insensitive to the particular model chosen, at least far from the regions of high curvature. Our analysis of the wormhole model should thus serve to illustrate many of the features common to all of them.

II. WORMHOLE INITIAL DATA

Construction of Wormhole Manifold

One can construct a wormhole manifold W out of Euclidean 3-space by the following procedure: take a sample of Euclidean space, cut out two nonintersecting spheres S and S' (each of radius R), throw away their interiors, and sew the two boundaries smoothly together (i.e., identify corresponding points on the surfaces of the two spheres). One must check that the resulting space is in fact a manifold, and this requires that one specify a family of open sets covering W, each of which is homeomorphic to a Euclidean 3-cell. It is clear from this construction that the only difficulty lies in specifying a family of neighborhoods in W for points on the spheres S and S' (all other points have neighborhoods in E^3 lying in the complement of $S \cup S'$, which can be carried over directly to W).

Such a construction is most easily performed in some special coordinate system, and for this purpose the so-called "bispherical coordinates" are particularly appropriate. These are related to a set of standard Cartesian coordinates in E^3 through the transformation equations

$$\coth \mu = \frac{x^2 + y^2 + z^2 + a^2}{2az} \tag{2.1a}$$

$$\cot \eta = \frac{x^2 + y^2 + z^2 - a^2}{2a(x^2 + y^2)^{1/2}} \tag{2.1b}$$

$$\cot \varphi = \frac{x}{y} \tag{2.1c}$$

a being a parameter with dimension of a length which serves to fix the scale of the coordinate system. One checks readily that the surface $\mu = \text{const.}$ ($= \mu_0$, say) is a sphere, with its center on the z-axis at $z = a \coth \mu_0$ and with radius $R = a \operatorname{csch} \mu_0$. If one chooses the spheres S and S' to be given by

$$\mu = +\mu_0 \quad \text{and} \quad \mu = -\mu_0 \tag{2.2}$$

then the points in E^3 outside these spheres have coordinate values lying within the ranges

$$-\mu_0 < \mu < \mu_0, \qquad 0 \leqq \eta \leqq \pi, \qquad 0 \leqq \varphi < 2\pi \tag{2.3}$$

It is clear from (2.1) that both η and φ are to be interpreted as angular coordinates (with period 2π); if one also demands that μ be a *periodic coordinate*, with period $2\mu_0$—that is, if one identifies the points (μ, η, φ) and $(\mu + 2n\mu_0, \eta, \varphi)$ for all integers n—one achieves the desired match-up of S and S', and at the same time defines a suitable topology on W.

Figure 3 shows a two-dimensional cross section of the wormhole manifold (obtained by setting $\varphi =$ const.) imbedded in 3-space. It should be noted that the point with coordinates $\mu = \eta = 0$ actually corresponds to the point at infinity in a standard Cartesian system, and is thus not properly a part of the manifold.

Time-Symmetric Initial Value Data

Four of the ten Einstein field equations (1.1a) for a purely gravitational field:

$$G^0_\alpha \equiv R^0_\alpha - \tfrac{1}{2}\, \delta_{0\alpha} R = 0 \tag{2.4}$$

do not contain second time derivatives of the $g_{\mu\nu}$; hence they impose necessary conditions on the values of $g_{\mu\nu}$ and $\partial g_{\mu\nu}/\partial t$ on the initial hypersurface $\Sigma : t = 0$. If one requires that this be a hypersurface of time-symmetry[6]:

$$\left.\frac{\partial g_{\alpha\beta}}{\partial t}\right|_{t=0} = 0 \tag{2.5a}$$

$$g_{0i}\big|_{t=0} = 0 \tag{2.5b}$$

then these "initial value equations" reduce to

$$^3R = 0 \tag{2.6}$$

^3R being the curvature scalar computed from the three-dimensional metric ds_0^2 on the initial surface Σ.

The conservation conditions

$$G^\beta_{\alpha;\beta} = 0 \tag{2.7}$$

which can be derived from the Bianchi identities, assure that the solution of the remaining six field equations

$$G_{ij} \equiv R_{ij} - \tfrac{1}{2}\, g_{ij} R = 0 \tag{2.8}$$

will satisfy the initial value equations (2.4) for all times. It follows that, once

[6] See, e.g., Brill (*14*), Weber and Wheeler (*15*). It is necessary to invoke time symmetry in the present work to obtain explicit formulas for the initial data. This is a fundamental limitation, since it restricts us to the problem of two particles with *zero* angular momentum. In addition, it implies that no outgoing gravitational waves are present on the initial hypersurface. (While a symmetric mixture of incoming and outgoing waves would still be possible, the particular solution of the initial value equations which we adopt is free of such waves also. Cf. Arnowitt, Deser, and Misner (*16*).)

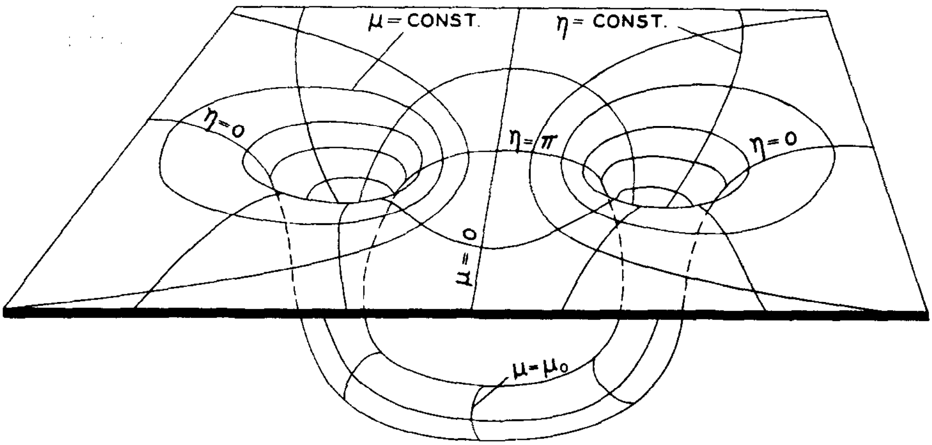

FIG. 3. Coordinate curves μ = const. and η = const. are drawn on a two-dimensional section of the wormhole manifold (cf. Fig. 2(c)). At large distances these coordinate curves become arcs of circles.

appropriate initial data have been chosen, one has only to solve Eq. (2.8), or equivalently

$$R_{ij} = 0 \tag{2.9}$$

while Eq. (2.4) can be used for checking purposes.

Equation (2.6) is a single nonlinear elliptic equation connecting the metric components g_{ij} on Σ. If one assumes that

$$ds_0^2 = \psi^4 \, d\bar{s}^2 \tag{2.10}$$

with $d\bar{s}^2$ some suitably chosen base metric, Eq. (2.6) becomes a linear elliptic equation for ψ:

$$\bar{\Delta}\psi - \tfrac{1}{8} \bar{R}\psi = 0 \tag{2.11}$$

where $\bar{\Delta}$ and $\bar{R}$ are the Laplacian and scalar curvature, respectively, computed from $d\bar{s}^2$.

The appropriate solution of (2.11) for the wormhole manifold has been obtained by Misner (*17*). He chooses the base metric $d\bar{s}^2$ to have the form

$$d\bar{s}^2 = d\mu^2 + d\eta^2 + \sin^2\eta \, d\varphi^2 \tag{2.12}$$

and then gets for (2.11)

$$\left[\frac{\partial^2}{\partial\mu^2} + \frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left(\sin\eta\frac{\partial}{\partial\eta}\right) + \frac{1}{\sin^2\eta}\frac{\partial^2}{\partial\varphi^2} - \frac{1}{4}\right]\psi = 0 \tag{2.13}$$

To formulate suitable boundary conditions for ψ, he compares (2.12) with the *flat-space* metric tensor expressed in bispherical coordinates:

$$ds_F^2 = a^2(\cosh \mu - \cos \eta)^{-2}(d\mu^2 + d\eta^2 + \sin^2\eta\, d\varphi^2) \tag{2.14}$$

and requires that ds_0^2 be asymptotically flat, i.e., that it approach ds_F^2 as μ and η tend to 0:

$$\psi(\mu, \eta, \varphi) \underset{\substack{\mu\to 0\\ \eta\to 0}}{\longrightarrow} \left(\frac{a}{\cosh \mu - \cos \eta}\right)^{1/2} \tag{2.15}$$

In order that $ds_0^2 = \psi^4\, d\bar{s}^2$ shall be an acceptable metric on W, $\psi(\mu, \eta, \varphi)$ must be analytic throughout the region $-\mu_0 \leqq \mu \leqq \mu_0$, $0 \leqq \eta \leqq \pi$, $0 \leqq \varphi < 2\pi$, except for a pole at $\mu = 0$, $\eta = 0$ of the form (2.15), and must also be periodic in each of its arguments (with periods $2\mu_0$, 2π, and 2π, respectively). Furthermore, since the manifold possesses rotational symmetry about the z-axis (the axis joining the centers of the two spheres $\mu = \pm\mu_0$), one must have

$$\partial\psi/\partial\varphi = 0 \tag{2.16a}$$

while from the condition that the two masses be identical (mirror symmetry in the plane $z = 0$) it follows that

$$\psi(-\mu, \eta, \varphi) = \psi(\mu, \eta, \varphi) \tag{2.16b}$$

As Misner shows, the (unique) solution of Eq. (2.13) satisfying all the above conditions is

$$\psi = a^{1/2} \sum_{n=-\infty}^{\infty} [\cosh(\mu + 2n\mu_0) - \cos \eta]^{-1/2} \tag{2.17}$$

The parameters μ_0 and a can be related to the total mass M of the system and to the length L_0 of the minimal closed 3-geodesic connecting the two mouths through the formulas[7]

$$M = 4a \sum_{n=1}^{\infty} \operatorname{csch} n\mu_0 \tag{2.18}$$

$$L_0 = 2a\left[1 + 2\mu_0 \sum_{n=1}^{\infty} n \operatorname{csch} n\mu_0\right] \tag{2.19}$$

For the individual mass m associated with each mouth separately one finds[8]

$$m = 2a \sum_{n=1}^{\infty} n \operatorname{csch} n\mu_0 \tag{2.20}$$

[7] See Misner (*17*). Analogous formulas for the case of a charged wormhole were found by Lindquist (*13*). Note that L_0 gives an invariant characterization of the distance of separation between the two masses; it is defined as the length of the curve (on Σ)

$$\eta = \pi, \qquad \varphi = 0 \qquad\qquad (-\mu_0 \leqq \mu \leqq \mu_0).$$

[8] The total mass M is not twice the mass of each particle, but rather the sum of their masses and their (negative) gravitational interaction energy. See Lindquist (*13*) and also Brill and Lindquist (*18*).

III. TIME DEPENDENT SYSTEM

Coordinate and Boundary Conditions

To determine the ten terms of the metric tensor for all times only the six equations (2.9) are available. Four of these components, namely $g_{0\alpha}$, can be arbitrarily prescribed, provided that they have continuous second derivatives and maintain the proper signature.

The simplest possible choice for the three mixed components is

$$g_{0i} = 0 \tag{3.1}$$

(Note that this is an extension of the initial condition (2.5b) for all times t.) Any set of coordinates in which Eq. (3.1) is satisfied will be called *normal.* These coordinates have the property that the timelike curves x^i = const. are everywhere orthogonal to the spacelike surfaces t = const. Furthermore if the space coordinates are periodic on the initial surface (for example, μ, η, φ on the wormhole manifold), they will remain periodic, with the *same* period, on any later hypersurface as well.

One can always choose the orientation of the t = const. hypersurfaces in such a way that the orthogonal trajectories are geodesics. From the geodesic equations it follows that

$$\partial g_{00}/\partial x^i = 0,$$

so that $g_{00} = f(t)$. By a proper choice of the parameter t, the function f can always be set to -1. This special coordinate frame is called *normal Gaussian* (or geodesic normal); it is thus described by the four conditions

$$g_{0\alpha} = -\delta_{0\alpha} \tag{3.2}$$

These coordinates admit a direct physical interpretation: the curves x^i = const. describe the path of a set of freely falling observers, with t the proper time measured by each observer along his world line. Since Eq. (3.2) imposes a set of *algebraic* conditions on the arbitrary functions $g_{0\alpha}$ (in contrast to the so-called De Donder or "harmonic" coordinates (*8*), also frequently used, for which the $g_{0\alpha}$ satisfy a system of first order *differential* equations), they are particularly suitable for numerical work.

To complete the specification of coordinates, we use the bispherical system $[\mu, \eta, \varphi]$ already introduced on the initial hypersurface $t = 0$; thus, the point (μ, η, φ, t) in space-time lies on the geodesic orthogonal to $t = 0$ and passing through $(\mu, \eta, \varphi, 0)$, and is located at proper distance t from the latter point. Since this construction preserves the periodicity in the coordinate variables μ, η, φ, the three-dimensional metric g_{ij} on any later hypersurface t = const. will obey the same periodicity conditions as the initial value function ψ:

$$g_{ij}(\mu, \eta, \varphi, t) = g_{ij}(\mu + 2\mu_0, \eta, \varphi, t) = g_{ij}(\mu, \eta + 2\pi, \varphi, t) \tag{3.3}$$

as well as the condition of mirror symmetry (analogous to Eq. (2.16b)):

$$ds^2(\mu, \eta, \varphi, t) = ds^2(-\mu, \eta, \varphi, t) \tag{3.4a}$$

Also,

$$ds^2(\mu, \eta, \varphi, t) = ds^2(\mu, -\eta, \varphi, t) \tag{3.4b}$$

Moreover, the condition of rotational symmetry about the z-axis implies

$$\partial g_{ij}/\partial\varphi = 0 \tag{3.5a}$$

$$g_{13} = g_{23} = 0 \tag{3.5b}$$

Equation (3.5a) allows the elimination of φ as an independent variable; Eq. (3.5b) then implies

$$R_{13} \equiv 0 \qquad R_{23} \equiv 0 \tag{3.6}$$

Thus, both the number of field equations and dependent variables are reduced to four.

Because of Eqs. (3.3) and (3.4) it is sufficient to work within the domain

$$0 \leqq \mu \leqq \mu_0, \qquad 0 \leqq \eta \leqq \pi$$

with the following boundary conditions:

$$g_{12}\big|_{\mu=0,\mu_0} = 0, \qquad \left.\frac{\partial g_{ii}}{\partial \mu}\right|_{\mu=0,\mu_0} = 0 \quad \text{(no summations)} \tag{3.7a}$$

$$g_{12}\big|_{\eta=0,\pi} = 0, \qquad \left.\frac{\partial g_{ii}}{\partial \eta}\right|_{\eta=0,\pi} = 0 \tag{3.7b}$$

Finally, by analogy with spherical coordinates, we demand that

$$\lim_{\eta\to 0} \frac{g_{22}\sin^2\eta}{g_{33}} = \lim_{\eta\to\pi} \frac{g_{22}\sin^2\eta}{g_{33}} = 1 \tag{3.8}$$

if the metric is to remain regular on the polar axis (i.e., on $x = y = 0$).

Quasi-Linear First Order System

To solve the four remaining equations

$$R_{11} = R_{12} = R_{22} = R_{33} = 0 \tag{3.9}$$

numerically, we make use of Friedrichs' difference scheme (*19*) for first order systems. The first step is to convert the second order system (3.9) into one of first order, by introducing new dependent variables. A natural choice for such new variables are the Christoffel symbols:

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left(\frac{\partial g_{\alpha\delta}}{\partial x^{\beta}} + \frac{\partial g_{\beta\delta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\delta}} \right) \tag{3.10}$$

The normal Gaussian coordinate conditions (3.2) cause seven of the forty Christoffel symbols to vanish:

$$\Gamma^{\alpha}_{00} = 0 \qquad \Gamma^{0}_{0\alpha} = 0 \tag{3.11a}$$

and nine others to be expressible as linear combinations of the rest:

$$\Gamma^{j}_{0i} = g^{jk}\Gamma^{0}_{ik} \tag{3.11b}$$

Furthermore, due to condition (3.5b), one has

$$\Gamma^{\alpha}_{i3} = 0 \qquad (\alpha, i \neq 3) \tag{3.12a}$$

$$\Gamma^{3}_{ij} = 0 \qquad (i, j \neq 3) \tag{3.12b}$$

$$\Gamma^{3}_{33} = 0 \tag{3.12c}$$

$$\Gamma^{i}_{33} = -\frac{g^{ij}}{g^{33}}\,\Gamma^{3}_{j3} \qquad (i, j \neq 3) \tag{3.12d}$$

These four equations eliminate twelve more Christoffel symbols, leaving twelve to be determined.[9]

In terms of the new dependent variables the four field equations can be written as

$$\frac{\partial\Gamma^{0}_{ij}}{\partial t} = \frac{\partial\Gamma^{k}_{ik}}{\partial x^{j}} - \frac{\partial\Gamma^{k}_{ij}}{\partial x^{k}} + \Gamma^{l}_{ik}\Gamma^{k}_{jl} - \Gamma^{k}_{ij}\Gamma^{l}_{kl} + g^{kl}(2\Gamma^{0}_{ik}\Gamma^{0}_{jl} - \Gamma^{0}_{ij}\Gamma^{0}_{kl}) \tag{3.13}$$

The eight additional equations necessary to complete the system can be obtained by differentiating the defining equations (3.10) for Γ^{k}_{ij} with respect to time:

$$\frac{\partial\Gamma^{k}_{ij}}{\partial t} = g^{kl}\left(\frac{\partial\Gamma^{0}_{il}}{\partial x^{j}} + \frac{\partial\Gamma^{0}_{jl}}{\partial x^{i}} - \frac{\partial\Gamma^{0}_{ij}}{\partial x^{l}} - 2\Gamma^{m}_{ij}\Gamma^{0}_{lm}\right) \tag{3.14}$$

Removal of Singularities

Before the system (3.13), (3.14) is converted into a set of difference equations, further changes must be made. The metric has a singularity at the point $\mu = 0$,

[9] Note that the remaining $\Gamma^{\gamma}_{\alpha\beta}$ split naturally into two groups: (1) eight Christoffel symbols Γ^{k}_{ij}, which involve only the metric components g_{ij} and their space derivatives, and which are thus completely determined by the geometry on a given spacelike hypersurface $t =$ const.; (2) the four components Γ^{0}_{ij}, which describe the curvature of this hypersurface as seen in the full four-dimensional manifold in which it is imbedded. These are related to the *second fundamental form* K_{ij} of classical differential geometry (see, e.g., Eisenhart (*20*)); indeed, in our coordinate system $\Gamma^{0}_{ij} = -K_{ij}$. The advantages of using the components of the second fundamental form as dependent variables (rather than, say, the time derivatives $\partial g_{ij}/\partial t$, to which they are related) have been pointed out by Arnowitt, Deser, and Misner (*16*).

$\eta = 0$, of the form

$$(\cosh \mu - \cos \eta)^{-2} \approx 4(\mu^2 + \eta^2)^{-2}.$$

To avoid excessive errors in the difference approximation close to this point, it is advisable to factor out the singularity explicitly from each metric component, and it is convenient to use the periodic singular function $\sigma^2 \equiv \psi^4$, defined by Eq. (2.17), for this purpose.[10] In addition, the boundary condition (3.8) suggests the factoring out of $\sin^2 \eta$ from g_{33}. Thus we set

$$g_{ij} = \sigma^2 f_{ij}, \qquad g^{ij} = \sigma^{-2} f^{ij} \qquad (i, j \neq 3) \tag{3.15a}$$

$$g_{33} = (\sigma \sin \eta)^2 f_{33}, \qquad g^{33} = (\sigma \sin \eta)^{-2} f^{33} \tag{3.15b}$$

Let γ^0_{ij}, γ^k_{ij} be the Christoffel symbols formed from the f_{ij}; then

$$\Gamma^0_{ij} = \sigma^2 \gamma^0_{ij} \quad (i, j \neq 3), \qquad \Gamma^0_{33} = (\sigma \sin \eta)^2 \gamma^0_{33} \tag{3.16a}$$

$$\Gamma^k_{ij} = \gamma^k_{ij} + \frac{1}{\sigma}\left(\delta_{ik}\frac{\partial \sigma}{\partial x^j} + \delta_{jk}\frac{\partial \sigma}{\partial x^i} - f_{ij} f^{kl}\frac{\partial \sigma}{\partial x^l}\right) \quad (i, j, k, l \neq 3) \tag{3.16b}$$

$$\Gamma^3_{13} = \gamma^3_{13} + \frac{1}{\sigma}\frac{\partial \sigma}{\partial \mu}, \qquad \Gamma^3_{23} = \gamma^3_{23} + \cot \eta + \frac{1}{\sigma}\frac{\partial \sigma}{\partial \eta} \tag{3.16c}$$

Next put

$$\begin{aligned} U_{ij} = {} & f^{kl}(2\gamma^0_{ik}\gamma^0_{jl} - \gamma^0_{ij}\gamma^0_{kl}) - \frac{1}{\sigma^2}(\gamma^k_{ij}\gamma^l_{kl} + \gamma^2_{ij}\cot \eta) \\ & + \frac{1}{\sigma^3}\left\{\frac{\partial^2 \sigma}{\partial x^i \partial x^j} + f_{ij} f^{kl}\frac{\partial^2 \sigma}{\partial x^k \partial x^l} - \frac{1}{\sigma}\frac{\partial \sigma}{\partial x^i}\frac{\partial \sigma}{\partial x^j}\right. \\ & \left. - [\gamma^k_{ij} + f_{ij}(f^{lm}\gamma^k_{lm} - f^{kl}\gamma^3_{l3} - f^{k2}\cot \eta)]\frac{\partial \sigma}{\partial x^k}\right\} \qquad (i, j, m \neq 3) \end{aligned} \tag{3.17}$$

The four equations (3.13), when expressed in terms of the new variables, become

$$\begin{aligned} \frac{\partial \gamma^0_{11}}{\partial t} = U_{11} + \frac{1}{\sigma^2}\Bigg[& \frac{\partial \gamma^2_{12}}{\partial \mu} + \frac{\partial \gamma^3_{13}}{\partial \mu} - \frac{\partial \gamma^2_{11}}{\partial \eta} \\ & + (\gamma^1_{11})^2 + 2\gamma^2_{11}\gamma^1_{12} + (\gamma^2_{12})^2 + (\gamma^3_{13})^2\Bigg] \end{aligned} \tag{3.18a}$$

$$\begin{aligned} \frac{\partial \gamma^0_{12}}{\partial t} = U_{12} + \frac{1}{\sigma^2}\Bigg[& -\frac{\partial \gamma^1_{12}}{\partial \mu} + \frac{\partial \gamma^1_{11}}{\partial \eta} + \frac{\partial \gamma^3_{13}}{\partial \eta} \\ & + \gamma^1_{11}\gamma^1_{12} + \gamma^2_{11}\gamma^1_{22} + \gamma^2_{12}(\gamma^1_{12} + \gamma^2_{22}) + \gamma^3_{13}(\gamma^3_{23} + \cot \eta)\Bigg] \end{aligned} \tag{3.18b}$$

[10] The introduction of $\sigma \equiv \psi^2$ instead of ψ reduces the number of terms in later equations.

$$\frac{\partial \overset{0}{\gamma}_{22}}{\partial t} = U_{22} + \frac{1}{\sigma^2}\left[-\frac{\partial \overset{1}{\gamma}_{22}}{\partial \mu} + \frac{\partial \overset{1}{\gamma}_{12}}{\partial \eta} + \frac{\partial \overset{3}{\gamma}_{23}}{\partial \eta} - 1 \right. \\ \left. + (\overset{1}{\gamma}_{12})^2 + 2\overset{2}{\gamma}_{12}\overset{1}{\gamma}_{22} + (\overset{2}{\gamma}_{22})^2 + \overset{3}{\gamma}_{23}(\overset{3}{\gamma}_{23} + 2 \cot \eta) \right] \tag{3.18c}$$

$$\frac{\partial \overset{0}{\gamma}_{33}}{\partial t} = \frac{f_{33}}{\sigma^2}\left[f^{ij}\left(\frac{\partial \overset{3}{\gamma}_{i3}}{\partial x^j} - \overset{k}{\gamma}_{ij}\overset{3}{\gamma}_{k3} + \overset{3}{\gamma}_{i3}\overset{3}{\gamma}_{j3} \right) + (2f^{i2}\overset{3}{\gamma}_{i3} - f^{ij}\overset{2}{\gamma}_{ij}) \cot \eta - f^{22} \right] \\ + \overset{0}{\gamma}_{33}(f^{33}\overset{0}{\gamma}_{33} - f^{ij}\overset{0}{\gamma}_{ij}) \\ + \frac{f^{33}}{\sigma^3}\left[f^{ij}\frac{\partial^2 \sigma}{\partial x^i \partial x^j} - (f^{ij}\overset{k}{\gamma}_{ij} - 2f^{ik}\overset{3}{\gamma}_{i3} - 2f^{k2}\cot \eta)\frac{\partial \sigma}{\partial x^k} \right] \\ (i, j, k \neq 3) \tag{3.18d}$$

On the other hand, Eq. (3.14) remains unchanged in form:

$$\frac{\partial \overset{k}{\gamma}_{ij}}{\partial t} = f^{kl}\left(\frac{\partial \overset{0}{\gamma}_{il}}{\partial x^j} + \frac{\partial \overset{0}{\gamma}_{jl}}{\partial x^i} - \frac{\partial \overset{0}{\gamma}_{ij}}{\partial x^l} - 2\overset{m}{\gamma}_{ij}\overset{0}{\gamma}_{lm} \right) \tag{3.19}$$

Initial values for the new dependent variables, in accordance with Eqs. (2.10), (2.12) and (2.17), are

$$f_{ij}\,|_{t=0} = f^{ij}\,|_{t=0} = \delta_{ij} \tag{3.20a}$$

$$\overset{\alpha}{\gamma}_{ij}\,|_{t=0} = 0 \tag{3.20b}$$

The boundary conditions (3.7) require on $\mu = 0, \mu_0$ the vanishing of

$$f_{12}, \quad \overset{0}{\gamma}_{12}, \quad \overset{i}{\gamma}_{1i}, \quad \overset{1}{\gamma}_{22},$$

$$\frac{\partial \sigma}{\partial \mu}, \quad \frac{\partial f_{ii}}{\partial \mu}, \quad \frac{\partial \overset{0}{\gamma}_{ii}}{\partial \mu}, \quad \frac{\partial \overset{2}{\gamma}_{11}}{\partial \mu}, \quad \frac{\partial \overset{i}{\gamma}_{2i}}{\partial \mu}, \quad \frac{\partial \overset{0}{\gamma}_{12}}{\partial \eta}, \quad \frac{\partial \overset{i}{\gamma}_{1i}}{\partial \eta}, \quad \frac{\partial \overset{1}{\gamma}_{22}}{\partial \eta} \quad \text{(no summations)}$$

and on $\eta = 0, \pi$ the vanishing of

$$f_{12}, \quad \overset{0}{\gamma}_{12}, \quad \overset{2}{\gamma}_{11}, \quad \overset{i}{\gamma}_{2i},$$

$$\frac{\partial \sigma}{\partial \eta}, \quad \frac{\partial f_{ii}}{\partial \eta}, \quad \frac{\partial \overset{0}{\gamma}_{12}}{\partial \mu}, \quad \frac{\partial \overset{2}{\gamma}_{11}}{\partial \mu}, \quad \frac{\partial \overset{i}{\gamma}_{2i}}{\partial \mu}, \quad \frac{\partial \overset{0}{\gamma}_{ii}}{\partial \eta}, \quad \frac{\partial \overset{i}{\gamma}_{1i}}{\partial \eta}, \quad \frac{\partial \overset{1}{\gamma}_{22}}{\partial \eta} \quad \text{(no summations).}$$

Furthermore, from Eq. (3.8),

$$f_{22} = f_{33}, \qquad \overset{0}{\gamma}_{22} = \overset{0}{\gamma}_{33}, \qquad \overset{2}{\gamma}_{12} = \overset{3}{\gamma}_{13} \quad \text{at} \quad \eta = 0, \pi \tag{3.21}$$

On the boundaries $\eta = 0, \pi$, Eq. (3.19) remains unchanged, while Eqs. (3.18) have apparent singularities, which can be removed by applying L'Hospital's rule. Using the identities

$$\frac{\partial f^{ij}}{\partial x^k} = -(f^{il}\overset{j}{\gamma}_{kl} + f^{jl}\overset{i}{\gamma}_{kl}) \tag{3.22}$$

one gets from Eq. (3.17)

$$U_{ij}\,|_{\eta=0,\pi} = f^{kl}(2\overset{0}{\gamma}_{ik}\overset{0}{\gamma}_{jl} - \overset{0}{\gamma}_{ij}\overset{0}{\gamma}_{kl}) - \frac{1}{\sigma^2}\left(\overset{k}{\gamma}_{ij}\overset{l}{\gamma}_{kl} + \frac{\partial \overset{2}{\gamma}_{ij}}{\partial\eta}\right) + \frac{1}{\sigma^3}\left\{\frac{\partial^2\sigma}{\partial x^i\partial x^j} + f_{ij}\left(f^{kl}\frac{\partial^2\sigma}{\partial x^k\partial x^l} + f^{22}\frac{\partial^2\sigma}{\partial\eta^2}\right) - \frac{1}{\sigma}\frac{\partial\sigma}{\partial x^i}\frac{\partial\sigma}{\partial x^j} - [\overset{1}{\gamma}_{ij} + f_{ij}(f^{lm}\overset{1}{\gamma}_{lm} - f^{l1}\overset{3}{\gamma}_{l3} + f^{11}\overset{2}{\gamma}_{12} + f^{22}\overset{1}{\gamma}_{22})]\frac{\partial\sigma}{\partial\mu}\right\} \quad (i, j, m \neq 3) \tag{3.23}$$

Equations (3.18) are then modified in the following way:

$$\frac{\partial\overset{0}{\gamma}_{11}}{\partial t}\,|_{\eta=0,\pi} = U_{11}\,|_{\eta=0,\pi} + \frac{1}{\sigma^2}\left[\frac{\partial\overset{2}{\gamma}_{12}}{\partial\mu} + \frac{\partial\overset{3}{\gamma}_{13}}{\partial\mu} - \frac{\partial\overset{2}{\gamma}_{11}}{\partial\eta} + (\overset{1}{\gamma}_{11})^2 + 2(\overset{2}{\gamma}_{12})^2\right] \tag{3.24a}$$

$$\frac{\partial\overset{0}{\gamma}_{12}}{\partial t}\,|_{\eta=0,\pi} = 0 \tag{3.24b}$$

$$\frac{\partial\overset{0}{\gamma}_{22}}{\partial t}\,|_{\eta=0,\pi} = U_{22}\,|_{\eta=0,\pi} + \frac{1}{\sigma^2}\left[-\frac{\partial\overset{1}{\gamma}_{22}}{\partial\mu} + \frac{\partial\overset{1}{\gamma}_{12}}{\partial\eta} + 3\,\frac{\overset{3}{\gamma}_{23}}{\partial\eta} - 1 + 2\overset{2}{\gamma}_{12}\overset{1}{\gamma}_{22}\right] \tag{3.24c}$$

$$\frac{\partial\overset{0}{\gamma}_{33}}{\partial t}\,|_{\eta=0,\pi} = \frac{\partial\overset{0}{\gamma}_{22}}{\partial t}\,|_{\eta=0,\pi} \tag{3.24d}$$

Note that, by Eqs. (3.24b) and (3.24d), the boundary conditions $\overset{0}{\gamma}_{12} = 0$, $\overset{0}{\gamma}_{22} = \overset{0}{\gamma}_{33}$ (on $\eta = 0, \pi$) are preserved in time.

Behavior at the Origin

Since the metric components become singular at $\mu = 0$, $\eta = 0$, f_{11} and f_{22} cannot be computed there. (Of course, $f_{12}(0, 0, t) = 0$ and $f_{33}(0, 0, t) = f_{22}(0, 0, t)$ from the boundary conditions.) One would expect $f_{ij}(0, 0, t) = \delta_{ij}$, in order that space-time remain flat at infinity, but it is necessary to check that this choice is consistent with the field equations.

From (3.23), (3.24a), and (3.24c) it follows that

$$\frac{\partial\overset{0}{\gamma}_{11}}{\partial t}\,|_{\mu=\eta=0} = \overset{0}{\gamma}_{11}\left(\frac{1}{f_{11}}\overset{0}{\gamma}_{11} - \frac{2}{f_{22}}\overset{0}{\gamma}_{22}\right) \tag{3.25a}$$

$$\frac{\partial\overset{0}{\gamma}_{22}}{\partial t}\,|_{\mu=\eta=0} = -\frac{1}{f_{11}}\overset{0}{\gamma}_{11}\overset{0}{\gamma}_{22} \tag{3.25b}$$

If one divides through by f_{11} in Eq. (3.25a) and by f_{22} in Eq. (3.25b), and expresses the $\overset{0}{\gamma}_{ii}$ in terms of the f_{ii}, one finds

$$\frac{\partial^2}{\partial t^2}\log f_{ii}\,|_{\mu=\eta=0} = -\frac{1}{2}\frac{\partial}{\partial t}\log f_{ii}\left(\frac{\partial}{\partial t}\log f_{11} + 2\,\frac{\partial}{\partial t}\log f_{22}\right) \quad (i \neq 3) \tag{3.26}$$

On setting

$$\frac{\partial}{\partial t}\log f_{11}(0, 0, t) = u(t), \qquad \frac{\partial}{\partial t}\log f_{22}(0, 0, t) = v(t),$$

one obtains a first order system of ordinary differential equations

$$u' = -\tfrac{1}{2}\, u(u + 2v) \tag{3.27a}$$

$$v' = -\tfrac{1}{2}\, v(u + 2v) \tag{3.27b}$$

whose solution is

$$u = \frac{2(c_1 - 1)}{c_1 t + c_2} \qquad v = \frac{1}{c_1 t + c_2} \tag{3.28}$$

Here c_1 and c_2 are independent of t. Since $u(0) = 0$ and $v(0) = 0$ by (3.20a), c_2 must be infinite and thus $u(t) \equiv 0$ and $v(t) \equiv 0$. It follows that, for all times,

$$f_{ij}\,|_{\mu=\eta=0} = \text{const.} = \delta_{ij} \qquad \gamma_{ij}^{\alpha}\,|_{\mu=\eta=0} = \text{const.} = 0 \tag{3.29}$$

IV. NUMERICAL SOLUTION

Friedrichs' Difference Scheme

The first order system (3.18), (3.19) can be written in vector form as

$$\frac{\partial \gamma}{\partial t} = A_1 \frac{\partial \gamma}{\partial \mu} + A_2 \frac{\partial \gamma}{\partial \eta} + B(\gamma) \tag{4.1}$$

where γ is the column vector with the twelve components γ_{ij}^{α}, A_1 and A_2 are 12×12 matrices depending only on σ and f^{ij}, and B is a vector with twelve components depending on σ, f^{ij}, and γ_{ij}^{α}, but not on derivatives of γ_{ij}^{α}.

One way to convert the system (4.1) into a set of difference equations is to apply Friedrichs' method (*19*). In this procedure one replaces space derivatives by centered differences

$$\frac{\partial \gamma_{ij}^{\alpha}}{\partial x^k} \approx \frac{\gamma_{ij}^{\alpha}(x^k + h_k) - \gamma_{ij}^{\alpha}(x^k - h_k)}{2h_k} \qquad (k = 1, 2) \tag{4.2a}$$

where h_k is the mesh length in the x^k-direction, and time derivatives by forward differences, using an average of the four neighboring points for the lower time level:

$$\frac{\partial \gamma_{ij}^{\alpha}}{\partial t} \approx \frac{\gamma_{ij}^{\alpha}(x^k, t + \Delta t) - \tfrac{1}{4}\sum_{k=1}^{2} \gamma_{ij}^{\alpha}(x^k \pm h_k, t)}{\Delta t} \tag{4.2b}$$

(The sum over k includes terms with both the $+$ and the $-$ sign.) Employing these differences one obtains for Eq. (4.1)

$$\gamma(\mu, \eta, t + \Delta t) = \left(\frac{I}{4} \pm \frac{\Delta t}{2h_1} A_1\right) \gamma(\mu \pm h_1, \eta, t) + \left(\frac{I}{4} \pm \frac{\Delta t}{2h_2} A_2\right) \gamma(\mu, \eta \pm h_2, t) + B \tag{4.3}$$

Friedrichs' stability condition stipulates that the coefficient matrices

$$C_k \equiv \frac{I}{4} \pm \frac{\Delta t}{2h_k} A_k \qquad (k = 1, 2) \tag{4.4}$$

be nonnegative, i.e., that their eigenvalues, denoted by λ, be nonnegative:

$$\lambda(C_k) = \frac{1}{4} \pm \frac{\Delta t}{2h_k} \lambda(A_k) \geqq 0 \qquad (k = 1, 2) \tag{4.5}$$

This yields the following requirement for Δt:

$$\Delta t \leqq \frac{1}{2} \min_k \frac{h_k}{|\lambda(A_k)|} \tag{4.6}$$

The eigenvalues of A_k for the system (4.1) are 0 (with multiplicity 10) and $\pm(f^{kk}/\sigma)^{1/2}$; hence

$$\Delta t \leqq \frac{1}{2} \min_k h_k \sqrt{\frac{\sigma}{f^{kk}}} \tag{4.7}$$

Since $\max_k f^{kk}$ increases with time, Δt will decrease.

To solve the problem numerically, the values of the $\gamma^\alpha_{ij}(x^k, t + \Delta t)$ are first determined from $\gamma^\alpha_{ij}(x^k, t)$ and $f^{ij}(x^k, t)$; next $f_{ij}(x^k, t + \Delta t)$ is computed from

$$\frac{\partial f_{ij}}{\partial t} = 2\gamma^0_{ij} \tag{4.8}$$

using simple forward time differences[11]

$$f_{ij}(t + \Delta t) = f_{ij}(t) + 2\Delta t\, \gamma^0_{ij}(t) \tag{4.9}$$

[11] An alternative approach would be to use

$$f_{ij}(x^k, t + \Delta t) = \tfrac{1}{4} \sum_{k=1}^{2} f(x^k \pm h_k, t) + 2\Delta t\, \gamma^0_{ij}(x^k, t).$$

Although this does not violate stability, its error term, as seen from its Taylor series expansion, is

$$O(\Delta t^2) + O(h_1^2) + O(h_2^2),$$

whereas that of (4.9) is only $O(\Delta t^2)$.

Finally, the inverse elements f^{ij} are calculated from the f_{ij}, and the next time increment Δt is determined from Eq. (4.7).

The numerical calculations were carried out on an IBM 7090 electronic computer. The parameters a and μ_0 were both set equal to unity; the mesh lengths were assigned the values $h_1 = 0.02$, $h_2 = \pi/150 \approx 0.021$, yielding a 51×151 mesh. The calculations of all unknown functions, including a great number of input-output operations and some built-in checking procedures, took approximately four minutes per time step. Different check routines indicated that results close to the point $\mu = 0$, $\eta = 0$ lost accuracy fairly quickly. Since these would, in the long run, influence meshpoints further away, the computations were stopped after the 50th time step, when the total time elapsed was approximately 1.8. Some of the results are shown in Table I.

Checking Procedures

Friedrichs has proved (*19*) that the difference scheme used above is stable and convergent for first order linear symmetric hyperbolic systems with Lipschitz continuous coefficients, while Lax (*21*) has established convergence and stability for all first order hyperbolic systems with constant coefficients. Equation (4.1) does not fall in either category: it is quasi-linear, and the matrices A_1, A_2 are nonsymmetric. Unfortunately, no rigorous proof of stability and convergence has yet been found for a system of this generality. The scheme used, however, seems to indicate stability since the values obtained remain fairly monotonic, even after fifty time steps. The three different methods which were used to check the solution, and which are described below, suggest that the results in the asymptotic region (i.e., in the neighborhood of $\mu = 0$, $\eta = 0$) are the least accurate. This behavior can be accounted for by examining the truncation error.

We first write down a truncated Taylor series for the g_{ij} in powers of t. Since the metric is time-symmetric,

$$g_{ij}(t) = g_{ij}(0) + \frac{1}{2}\frac{\partial^2 g_{ij}}{\partial t^2}\Big|_{t=0} t^2 + O(t^4) \tag{4.10}$$

From the field equations it follows that

$$\frac{\partial^2 g_{ij}}{\partial t^2}\Big|_{t=0} = -2\,{}^3R_{ij}(0) \tag{4.11}$$

hence

$$g_{ij}(t) = g_{ij}(0) - {}^3R_{ij}(0)t^2 + O(t^4) \tag{4.12}$$

In terms of the singularity-free variables defined in (3.15), this equation can be written as

$$f_{ij}(t) = \delta_{ij} + h_{ij}(\sigma)t^2 + O(t^4) \tag{4.13}$$

where the h_{ij} are functions of σ and its partial derivatives. In particular, near

$\mu = 0$, $\eta = 0$, Eq. (4.13) becomes approximately

$$f_{ij}(t) \approx \delta_{ij} + a_{ij}(\mu, \eta)(\mu^2 + \eta^2)^{1/2}t^2 + O(t^4) \tag{4.14}$$

where $a_{ij}(\mu, \eta)$ is homogeneous in μ and η of degree 2.

Rewriting formula (4.2a) in more detail, one gets

$$\frac{\partial \gamma_{ij}^l}{\partial x^k} = \frac{\gamma_{ij}^l(x^k + h_k) - \gamma_{ij}^l(x^k - h_k)}{2h_k} - \frac{h_k^2}{6}\frac{\partial^3 \gamma_{ij}^l}{\partial (x^k)^3} + \cdots \tag{4.15}$$

By differentiating (4.14) one finds that, for $x^k \approx h_k$,

$$\frac{\partial^3 \gamma_{ij}^l}{\partial (x^k)^3} = O\left(\frac{t^2}{h_k}\right)$$

which, when substituted in Eq. (4.15), yields

$$\frac{\partial \gamma_{ij}^l}{\partial x^k}\Big|_{x^k \approx h_k} = \frac{\gamma_{ij}^l(x^k + h_k) - \gamma_{ij}^l(x^k - h_k)}{2h_k}\Big|_{x^k \approx h_k} + O(h_k) \tag{4.16}$$

Thus the truncation error in evaluating $\partial \gamma_{ij}^l / \partial x^k$ is increased from $O(h_k^2)$ to $O(h_k)$ close to the origin.

1. *Direct Evaluation of* γ_{ij}^l

The simplest and most straightforward method for checking is to compute γ_{ij}^l from f_{ij}, using a difference form of Eq. (3.10), and to compare the results with those obtained from the Friedrichs scheme. As one might expect, the differences between the two sets of numbers increase with time, and are largest close to the origin. In particular, at the fifth time step these differences range from 1×10^{-6} to 7×10^{-6} near $\mu = 0$, $\eta = 0$; at the fiftieth time step they vary between 1×10^{-3} and 3×10^{-3} close to the origin, and between 1×10^{-4} and 8×10^{-4} further away.

2. *Initial Value Equations*

The second approach makes use of the initial value equations (2.4), which may be written as

$$G_0^0 \equiv \frac{1}{2}\left[g^{ij}g^{kl}\Gamma_{ik}^0\Gamma_{jl}^0 - (g^{ij}\Gamma_{ij}^0)^2 - {}^3R\right] = 0 \tag{4.17a}$$

$$G_i^0 \equiv g^{jk}\left(\frac{\partial \Gamma_{jk}^0}{\partial x^i} - \frac{\partial \Gamma_{ik}^0}{\partial x^j} + \Gamma_{il}^0\Gamma_{jk}^l - \Gamma_{jl}^0\Gamma_{ik}^l\right) = 0 \tag{4.17b}$$

The formula for G_0^0, when combined with

$$R \equiv g^{ij}R_{ij} \equiv {}^3R + g^{ij}\left[\frac{\partial \Gamma_{ij}^0}{\partial t} + g^{kl}(\Gamma_{ij}^0\Gamma_{kl}^0 - 2\Gamma_{ik}^0\Gamma_{jl}^0)\right] = 0 \tag{4.18}$$

TABLE I

COMPARISON OF NUMERICAL AND POWER SERIES SOLUTIONS AT SELECTED MESH POINTS

μ	η	Numerical solution of difference equations				Truncated power series solution				$\bar{G}_0^0$	G_1^0	G_2^0
		f_{11}	f_{12}	f_{22}	f_{33}	f_{11}	f_{12}	f_{22}	f_{33}			
1. Fifteenth time step; total time elapsed 0.52907												
0	$\pi/30$	0.9999	0	1.0002	0.9999	0.9999	0	1.0002	0.9999	0.00000001	0	0.001
0	$3\pi/5$	1.0039	0	0.9985	0.9976	1.0039	0	0.9985	0.9976	0.000006	0	−0.000001
0	π	1.0057	0	0.9971	0.9971	1.0057	0	0.9972	0.9972	−0.0000003	0	0
0.1	0	1.0002	0	0.9999	0.9999	1.0002	0	0.9999	0.9999	−0.0000004	0.002	0
0.1	$\pi/30$	1.0001	0.00029	1.0001	0.9998	1.0001	0.00030	1.0001	0.9998	−0.00000009	0.0009	0.0002
0.6	0	1.0039	0	0.9981	0.9981	1.0039	0	0.9981	0.9981	−0.00001	0.0002	0
0.6	$3\pi/5$	1.0044	0.00025	0.9982	0.9975	1.0044	0.00025	0.9982	0.9975	0.000007	0.000002	−0.000001
0.6	π	1.0058	0	0.9971	0.9971	1.0058	0	0.9971	0.9971	0.00001	0.0000009	0.0000009
1.0	0	1.0053	0	0.9974	0.9974	1.0053	0	0.9974	0.9974	0.0000008	0	0
1.0	$3\pi/5$	1.0046	0	0.9980	0.9974	1.0046	0	0.9980	0.9974	0.000008	0	−0.000001
1.0	π	1.0058	0	0.9971	0.9971	1.0058	0	0.9971	0.9971	0.00001	0	0.0000006
2. Thirtieth time step; total time elapsed 1.07528												
0	$\pi/30$	0.9997	0	1.0009	0.9994	0.9996	0	1.0008	1.0008	0.00000006	0	0.004
0	$3\pi/5$	1.0163	0	0.9938	0.9900	1.0161	0	0.9938	0.9900	0.00001	0	−0.000007
0	π	1.0238	0	0.9882	0.9882	1.0235	0	0.9882	0.9882	−0.00002	0	0
0.1	0	1.0008	0	0.9996	0.9996	1.0007	0	0.9996	0.9996	−0.000003	0.007	0
0.1	$\pi/30$	1.0005	0.00120	1.0005	0.9990	1.0006	0.00122	1.0004	0.9992	−0.0000008	0.004	0.0003
0.6	0	1.0161	0	0.9920	0.9920	1.0160	0	0.9920	0.9920	−0.0001	0.0005	0
0.6	$3\pi/5$	1.0182	0.00103	0.9924	0.9896	1.0180	0.00103	0.9924	0.9896	0.00001	0.000005	−0.000004
0.6	π	1.0240	0	0.9881	0.9881	1.0238	0	0.9881	0.9881	−0.00001	−0.000009	0
1.0	0	1.0219	0	0.9892	0.9892	1.0217	0	0.9891	0.9891	−0.00005	0	0
1.0	$3\pi/5$	1.0192	0	0.9917	0.9893	1.0190	0	0.9917	0.9893	0.00001	0	−0.000003
1.0	π	1.0242	0	0.9880	0.9880	1.0240	0	0.9880	0.9880	−0.000009	0	0

3. Fiftieth time step; total time elapsed 1.79798

0	$\pi/30$	0.9993	0	1.0025	0.9982	0.9989	0	1.0022	0.9989	0.0000001	0	0.007
0	$3\pi/5$	1.0461	0	0.9827	0.9723	1.0451	0	0.9828	0.9721	0.00002	0	−0.00003
0	π	1.0677	0	0.9672	0.9672	1.0658	0	0.9671	0.9671	−0.00005	0	0
0.1	0	1.0025	0	0.9987	0.9987	1.0020	0	0.9990	0.9990	−0.00001	0.02	0
0.1	$\pi/30$	1.0015	0.00328	1.0014	0.9971	1.0011	0.00342	1.0012	0.9977	0.000004	0.01	−0.0007
0.6	0	1.0453	0	0.9778	0.9778	1.0049	0	0.9776	0.9776	−0.0004	0.001	0
0.6	$3\pi/5$	1.0515	0.00291	0.9788	0.9710	1.0504	0.00289	0.9788	0.9708	0.00002	0.000008	−0.00001
0.6	π	1.0685	0	0.9668	0.9668	1.0666	0	0.9667	0.9667	−0.00003	−0.00007	0
1.0	0	1.0620	0	0.9699	0.9699	1.0608	0	0.9696	0.9696	−0.0002	0	0
1.0	$3\pi/5$	1.0544	0	0.9767	0.9703	1.0532	0	0.9767	0.9701	0.00002	0	−0.000004
1.0	π	1.0689	0	0.9666	0.9666	1.0670	0	0.9665	0.9665	−0.00002	0	0

yields a simpler expression for checking purposes:

$$\bar{G}_0^0 \equiv 2G_0^0 + R \equiv g^{ij}\left(\frac{\partial \Gamma_{ij}^0}{\partial t} - g^{kl}\Gamma_{ik}^0\Gamma_{jl}^0\right) = 0 \tag{4.19}$$

$\bar{G}_0^0$, G_1^0, and G_2^0 were computed at each time step and at selected meshpoints: their magnitude was a measure of the accumulated error. Some representative values are listed in Table I.

Note that $\bar{G}_0^0$ remains rather uniform over the entire mesh, and seldom exceeds 0.0001, while values of G_1^0 and G_2^0 close to the origin are greater than those far away by a factor of 10^3. (The largest single error invariably occurred in G_1^0, at the point $\mu = 0.02$, $\eta = 0$; it amounted to 0.04 at the fiftieth time step.) This extreme variation in the G_i^0 can be traced to the fact that they, unlike $\bar{G}_0^0$, involve space derivatives of the Christoffel symbols, and thus their difference approximation includes errors of the order shown in Eq. (4.16).

3. *Power Series in t*

Comparing the numerical results with the truncated power series (4.13) affords the most precise check, at least for small values of t. However, there is an obvious discrepancy between the numbers obtained in these two different approaches, even for small time values, since the solution of the difference equations satisfies the initial conditions

$$f_{ij}(\Delta t_1) = f_{ij}(0) = \delta_{ij} \tag{4.20}$$

Better agreement between the two methods can be achieved by setting

$$f_{ij}(t_{n+1}) = f_{ij}(t_n) + f'_{ij}(t_n)\Delta t_{n+1} + \tfrac{1}{2} f''_{ij}(t_n)(\Delta t_{n+1})^2 \tag{4.21}$$

and using

$$f'_{ij}(t_n) = 2h_{ij}t_n \qquad f''_{ij}(t_n) = 2h_{ij} \tag{4.22}$$

to derive an appropriate difference approximation to Eq. (4.13). One finds

$$f_{ij}(t_n) = \delta_{ij} + h_{ij}\left[\left(\sum_{k=1}^{n} \Delta t_k\right)^2 - n(\Delta t_1)^2\right] \tag{4.23}$$

as may be verified by induction.[12]

[12] If the time steps were constant ($=\Delta t$), Eq. (4.23) would reduce to

$$f_{ij}(t_n) \equiv f_{ij}(n\Delta t) = f_{ij} + n(n-1)(\Delta t)^2 h_{ij}(\sigma).$$

The replacement of t^2 in the exact solution by the factor $n(n-1)(\Delta t)^2$ is a common feature of numerical solutions to second order equations when an explicit forward difference scheme is used. Even though the time steps are not constant in the present problem, their change is so slight that the above formula is a fairly good approximation.

Up to the twenty-fifth time step ($t \approx 0.894$) the discrepancy between the computed solution and the power series approximation (4.23) amounted to 0.0001 at most. Even at the fiftieth time step ($t \approx 1.798$) the difference was surprisingly small; it ranged from 0.0001 to 0.0019 (the maximum occurring in the value of f_{11}, at the point $\mu = 1$, $\eta = \pi$). Rough estimates of the t^4 term in Eq. (4.13) suggest that most of the discrepancy arises from the neglect of this term.

V. DESCRIPTION OF RESULTS

An examination of the results in Table I shows that the metric changes most rapidly along the curves $\mu = \pm\mu_0$, $0 \leqq \eta \leqq \pi$ and $\eta = \pi$, $-\mu_0 \leqq \mu \leqq \mu_0$. These form, respectively, the "throat" and the "spine" of the wormhole (cf. Fig. 3). Along these curves the component f_{11} appears to diverge ($f_{11} > 1$), while both f_{22} and f_{33} approach zero ($f_{22}, f_{33} < 1$). This behavior is to be expected; indeed, as has been pointed out by Landau, Raychaudhuri (*22*), Komar (*23*), and more recently by Khalatnikov and Lifshitz (*24*), in any curved space-time satisfying the Einstein equations the normal geodesics will ultimately intersect, thus giving rise to an apparent singularity whenever normal Gaussian coordinates are employed. We argue that in the present, very specialized problem, this singularity is almost certainly real, and corresponds to the real singularity of the Schwarzschild metric, at which the throat shrinks to zero circumference.

Collapse of Throat

The collapse of the throat of the wormhole can be illustrated very simply by computing its area at various times:

$$A(t) = 2\pi \int_0^\pi \sigma^2(\mu_0, \eta)[f_{22}(\mu_0, \eta, t) f_{33}(\mu_0, \eta, t)]^{1/2} \sin \eta \, d\eta \qquad (5.1)$$

Values of $A(t)$ at selected time steps are listed in Table II. Figure 4 compares the dimensionless ratio $A(t)/m^2$ in the present case with that for the Schwarzschild solution, also expressed in normal Gaussian coordinates. Here m, the mass associated with each individual "particle," is given by Eq. (2.20); with the choice of parameters $a = 1$, $\mu_0 = 1$ it has approximately the value 3.9348. One sees that the agreement between the two solutions is remarkably close over the entire range of time steps taken, and it is not unreasonable to assume that this close agreement should persist for later times as well. Thus one would expect the two-body solution to develop a real singularity at the throat $\mu = \pm\mu_0$ after a time comparable to that for the Schwarzschild-Kruskal metric, namely

$$t_{\text{crit}} \approx \pi m \approx 12.362.$$

As a measure of the distance of separation between the two "particles" one

TABLE II

Time step	Time elapsed	Area of throat	Distance of separation
1	0	777.765	9.8696
5	0.16334	777.597	9.8723
10	0.34645	777.001	9.8818
15	0.52907	776.008	9.8980
20	0.71143	774.591	9.9210
25	0.89352	772.761	9.9507
30	1.07528	770.523	9.9872
35	1.25667	767.881	10.0305
40	1.43762	764.840	10.0805
45	1.61808	761.404	10.1374
50	1.79798	757.581	10.2011

might take the length of the 3-geodesic $\eta = \pi, -\mu_0 \leqq \mu \leqq \mu_0$:

$$L(t) = 2 \int_0^{\mu_0} \sigma(\mu, \pi)[f_{11}(\mu, \pi, t)]^{1/2} \, d\mu \tag{5.2}$$

Only at $t = 0$ is this curve also a 4-geodesic (cf. Eq. (2.19)); its physical significance at other times, therefore, is questionable, especially since there exists no corresponding quantity in the Schwarzschild problem with which it can be compared. Table II shows that $L(t)$ increases after the moment of time symmetry. This is hardly surprising, for the two mouths are collapsing while they are interacting gravitationally, and $L(t)$ measures a sum of both effects. A more reasonable quantity to examine is the distance between centers, obtained by adding to $L(t)$ a suitable average "radius" for each mouth, but even then it is not legitimate to compare the calculated distances with the predictions of Newtonian theory. The coordinate conditions we have employed are such that a family of observers, performing measurements on various $t =$ const. hypersurfaces at fixed values of μ and η, are in a freely-falling, *noninertial* frame. The proper distance measured by these accelerating observers will be greater than the corresponding distance measured relative to a single inertial frame, and the discrepancy between the two sets of measurements will increase with time. Thus it does not seem unreasonable that the separation between the two mouths, as computed in normal Gaussian coordinates, should appear to increase.

Investigation with Light Rays

One knows, from the analysis of the problem of motion by Einstein, Infeld, and Hoffman (*3*), that other coordinate conditions (for example, "harmonic" or De Donder coordinates) approximate inertial frames and do lead to agreement with Newtonian theory to first order. One could, therefore, transform to such

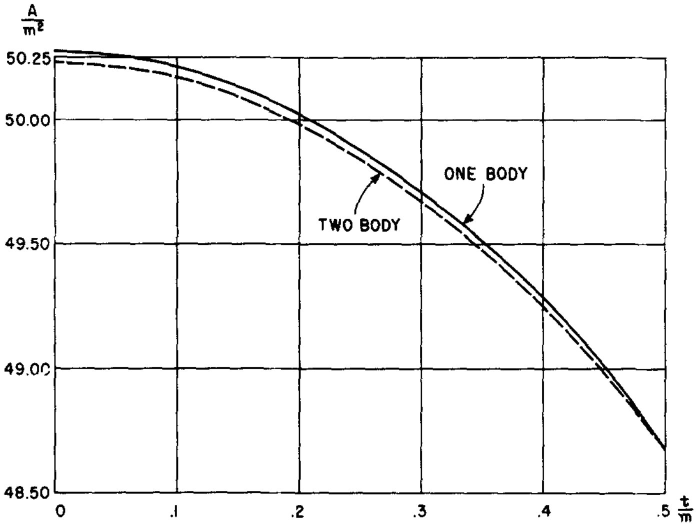

FIG. 4. The dimensionless parameters A/m^2 and t/m are plotted for both the one- and two-body solutions, when expressed in normal Gaussian coordinates. The one-body curve is given by the parametric equations

$$A/m^2 = 16\pi u^4, \qquad t/m = 2u(1 - u^2)^{1/2} + 2 \cos^{-1} u,$$

while the two-body curve has been drawn from the data in Table II (with $m \approx 3.9348$).

coordinates and repeat the analysis. Alternatively, one could try to reformulate the problem in invariant terms. Plebanski (*25*) has discussed one such method, which uses light rays to perform scattering experiments on the system under investigation. A particular version of this approach, especially appropriate for the present problem, is to place a light source at the throat of the wormhole and aim a narrow beam out through each mouth to an observer far away. It is convenient to adjust the starting direction for both beams so that they are asymptotically parallel to the midplane of symmetry (the plane $z = 0$); then the distance between them is a direct measure of the separation between the two mouths. Since these measurements are performed in the asymptotic region, where space-time is flat, there can be no ambiguity about their interpretation.

The paths of the light rays (null geodesics) are the solutions of the well-known equations

$$\frac{d^2x^\rho}{d\lambda^2} + \Gamma^\rho_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\rho}{d\lambda} = 0 \tag{5.3a}$$

with the first integral

$$g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} - \left(\frac{dt}{d\lambda}\right)^2 = 0 \tag{5.3b}$$

(λ being path parameter). One chooses initial values for $dx^i/d\lambda$ arbitrarily (the initial value of $dt/d\lambda$ is then determined by (5.3b)) and integrates Eqs. (5.3a) numerically until a point is reached where the metric is nearly static and Schwarzschildian, then matches the numerical solution to the analytic solution of these equations for the Schwarzschild metric. The latter curve has an asymptote which will not, in general, be parallel to the plane $z = 0$; if not, one chooses new initial data and repeats the calculation until the desired solution has been obtained.

This method has a major drawback in that it requires a knowledge of the two-body metric well into the future—much further, in fact, than we have been able to get in fifty time steps of machine calculation. It is not possible, therefore, to analyze our limited data in this way. A rough description could be found by using instead the truncated power series (4.10); while the numerical values for the asymptotic separation so obtained would not be accurate, one might expect changes in this separation with time to be qualitatively similar to the exact solution. An even cruder, but simpler, approximation is to ignore time variations in g_{ij} completely, and to solve the geodesic equations for a static metric. One then repeats the calculation using for the g_{ij} their values at different times, and examines the changes in separation. (This approximation would not be too unreasonable if the metric changed relatively slowly in the time required for a light ray to reach the asymptotic region.) We observe the expected behavior even for this extremely rough approach: the radius of each individual mouth shrinks from a maximum at the moment of time-symmetry, while the distance between centers decreases as t^2.

One might raise the objection that, in using the method of null geodesics, we have no *a priori* guarantee that a beam of light sent from the throat will ever reach the asymptotic region. This objection gains strong support when one examines the corresponding behavior in the Schwarzschild problem.[13] The analysis is most easily carried out using Kruskal's coordinate system (*12*), in which the light rays are represented as straight lines of unit slope with respect to the coordinate axes. One sees readily that only those rays emitted from the throat between the time $t = -\pi m$ at which the throat is formed and the moment $t = 0$ of time-symmetry can ever escape to infinity. (The null geodesics passing through the throat at $t = 0$ are asymptotic to the real singularity $r = 0$.) Similar behavior can also be expected in the two-body problem, especially when the mass of either "particle" is much less than their initial separation (or equivalently, when $\mu_0 \gg 1$), so that they behave like two isolated Schwarzschild solutions. This would not invalidate the method of light rays and asymptotic observers,

[13] See, in particular, Fuller and Wheeler (*26*) for a careful analysis of geodesics in the Schwarzschild-Kruskal metric, especially with regard to the problem of causality in multiply connected spaces.

but it would imply that these observers could only receive signals from the wormhole which were emitted before the moment of time-symmetry.

THE PROBLEM OF CAUSALITY

Closely related to the above problem is the question whether one can send a light signal once around the wormhole and back to its starting point before the throat pinches off. Such a procedure, if possible, would appear to violate causality, since two observers near each mouth could convey information to each other through the wormhole much more quickly than they could send it across the intervening space between them. Fuller and Wheeler (*26*) have argued that for two well-separated masses ($\mu_0 \gg 1$) this is impossible. In the case of two strongly interacting masses ($\mu_0 \lesssim 1$), however, it is readily shown from the approximate solution (4.10) that this can in fact be done.

One first notes that for $\mu_0 \lesssim 1$ the initial value function (defined by Eq. (2.17)) is very nearly constant over the curve $\eta = \pi$ joining the two mouths; in particular,[14]

$$\psi(\mu, \pi) \approx \frac{\pi}{\sqrt{2}} \frac{1}{\mu_0} \tag{5.4}$$

Furthermore, Eq. (4.13) in this case yields

$$f_{11}(\mu, \pi, t) \approx 1 + 2\left(\frac{\mu_0}{\pi}\right)^4 t^2 + O(t^4) \tag{5.5}$$

Let us consider a light ray which is emitted from the throat ($\mu = \mu_0$, $\eta = \pi$) at $t = 0$, and which passes along this curve until it returns to its starting point ($\mu = -\mu_0$, $\eta = \pi$) at some later time $t = t_1$. The equation for the null geodesic (5.3b) becomes

$$\frac{d\mu}{dt} = \frac{1}{\sqrt{g_{11}(\mu, \pi, t)}} = \frac{1}{\psi^2(\mu, \pi)\sqrt{f_{11}(\mu, \pi, t)}}.$$

Upon inserting the approximate values (5.4) and (5.5) and integrating, one obtains

$$t_1 \approx \frac{1}{\sqrt{2}} \frac{\pi^2}{\mu_0^2} \sinh(\sqrt{2}\mu_0) \tag{5.6}$$

[14] To prove this one first shows, using standard formulas, that

$$\psi(\mu, \pi) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \operatorname{sech}\left(\frac{\mu}{2} + n\mu_0\right) = \frac{\sqrt{2}}{\mu_0} K(k)\, dn\left(\frac{K\mu}{\mu_0}\right),$$

where K is the complete elliptic function of the first kind, dn is a Jacobian elliptic function and the modulus k is related to μ_0 by $\mu_0 = \pi K(k)/K(\sqrt{1-k^2})$. For $\mu_0 \lesssim 1$, $k \ll 1$ and, therefore, $K(k) \approx \pi/2$, $dn(K\mu/\mu_0) \approx 1$, from which (5.4) follows.

This time is to be compared with the time t_{crit} at which the throat collapses to zero circumference:

$$t_{crit} \approx \pi m \approx \frac{\pi^3}{2\mu_0^2}.$$

We have used here a limiting form for Eq. (2.20), $m \approx \pi^2/2\mu_0^2$, which is valid for small μ_0. By choosing μ_0 sufficiently small, one can evidently make the ratio t_1/t_{crit} as small as one pleases.

This rather surprising result does not contradict the analysis of Fuller and Wheeler, for it applies only to the case of strongly interacting masses. There are good reasons for regarding the latter system as a single composite particle with an internal structure[15]; violations of causality within this super-particle would not be at variance with commonly accepted physical principles. Instead one should ask whether one can send a light ray from the asymptotic region through the wormhole and back out to infinity without encountering a singularity in the metric. We believe this to be impossible, although we have not been able to prove it conclusively.

In summary, the numerical solution of the Einstein field equations presents no insurmountable difficulties. Much still remains to be done, however, in the investigation both of stable difference schemes (a proof of stability being one of the outstanding unsolved problems) and of coordinate conditions that are well suited to numerical work. The practical impossibility of carrying the numerical solution sufficiently far into the future limits the conclusions which can be drawn about the dynamical behavior of the wormhole system. Nevertheless, one sees evidence for a gravitational collapse of each mouth, analogous to that of the Schwarzschild metric, together with an interaction between the two of them. These two effects can only be properly disentangled through measurements in the asymptotic region; with the limited data at our disposal, such an analysis has not been possible.

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REFERENCES

1. J. A. WHEELER, *Phys. Rev.* **97**, 511 (1955).
2. C. W. MISNER AND J. A. WHEELER, *Ann. Phys.* (*N.Y.*) **2**, 525 (1957).

[15] See, e.g., Brill and Lindquist (*18*).

3. A. EINSTEIN, L. INFELD, AND B. HOFFMAN, *Ann. Math.* **39,** 65 (1938).
4. A. EINSTEIN AND L. INFELD, *Ann. Math.* **41,** 455 (1940).
5. W. DE SITTER, *Monthly Notices Roy. Astron. Soc.* **77,** 155 (1916).
6. T. LEVI-CIVITA, *Am. J. Math.* **59,** 225 (1937).
7. H. P. ROBERTSON, *Ann. Math.* **39,** 101 (1938).
8. V. FOCK, "The Theory of Space, Time and Gravitation," Appendix B. Pergamon Press, London, 1959.
9. L. INFELD AND J. PLEBANSKI, "Motion and Relativity." PWN, Warsaw, and Pergamon Press, New York, 1960.
10. J. N. GOLDBERG, The equations of motion, *in* "Gravitation: An Introduction to Current Research," L. Witten, ed. Wiley, New York, 1962.
11. A. EINSTEIN AND N. ROSEN, *Phys. Rev.* **48,** 73 (1935).
12. M. D. KRUSKAL, *Phys. Rev.* **119,** 1743 (1960).
13. R. W. LINDQUIST, *J. Math. Phys.* **4,** 938 (1963).
14. D. R. BRILL, *Ann. Phys.* (*N.Y.*) **7,** 466 (1959).
15. J. WEBER AND J. A. WHEELER, *Rev. Mod. Phys.* **29,** 509 (1957).
16. R. ARNOWITT, S. DESER, AND C. W. MISNER, *Phys. Rev.* **118,** 1100 (1960); *ibid.* **120,** 313 (1960).
17. C. W. MISNER, *Phys. Rev.* **118,** 1110 (1960).
18. D. R. BRILL AND R. W. LINDQUIST, *Phys. Rev.* **151,** 471 (1963).
19. K. O. FRIEDRICHS, *Commun. Pure Appl. Math.* **7,** 345 (1954).
20. L. P. EISENHART, "Riemannian Geometry," p. 146 ff. Princeton Univ. Press, Princeton, New Jersey, 1960.
21. P. D. LAX, *Commun. Pure Appl. Math.* **11,** 175 (1958).
22. A. RAYCHAUDHURI, *Phys. Rev.* **98,** 1123 (1955); *ibid.* **106,** 172 (1957).
23. A. KOMAR, *Phys. Rev.* **104,** 544 (1956).
24. I. M. KHALATNIKOV AND E. M. LIFSHITZ, *Advances in Phys.* **12,** 185 (1963).
25. J. PLEBANSKI, *Phys. Rev.* **118,** 1396 (1960).
26. R. FULLER AND J. A. WHEELER, *Phys. Rev.* **128,** 919 (1962).