

Constraints in Quantum Geometrodynamics

Adrian P. Gentle*, Nathan D. George†, Arkady Kheifets‡ and
Warner A. Miller§

December 18, 2018

Abstract

We compare different treatments of the constraints in canonical quantum gravity. The standard approach on the superspace of 3-geometries treats the constraints as the sole carriers of the dynamic content of the theory, thus rendering the traditional dynamical equations obsolete. Quantization of the constraints in both the Dirac and ADM square root Hamiltonian approaches leads to the well known problems of time evolution. These problems of time are of both an interpretational and technical nature. In contrast, the geometrodynamical quantization procedure on the superspace of the true dynamical variables separates the issues of quantization from the enforcement of the constraints. The resulting theory takes into account states that are off-shell with respect to the constraints, and thus avoids the problems of time. We develop, for the first time, the geometrodynamical quantization formalism in a general setting and show that it retains all essential features previously illustrated in the context of homogeneous cosmologies.

*Department of Mathematics, University of Southern Indiana, Evansville, IN 47712 and Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545

†DAMTP, University of Cambridge, Cambridge CB3 0WA, United Kingdom and Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545

‡Department of Mathematics, North Carolina State University, Raleigh, NC 27695

§Department of Physics, Florida Atlantic University, Boca Raton, FL 33431 and Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545

1 Introduction.

The standard approach to the canonical quantization of gravity[1, 2] is based on the classical dynamic picture of the evolving 3–geometry of a slicing of a spacetime manifold. The slicing is essentially a reference foliation of the spacetime manifold (endowed with a 4–geometry) with respect to which canonical variables are assigned. It is usually parametrized by a time coordinate t and tied to the enveloping spacetime by the lapse function N and the shift vector N^i . The canonical variables are the 3–metric g_{ik} on a spatial slices of the foliation induced by the spacetime 4–metric, and the matrix π^{ik} of their canonically conjugate momenta. The latter is related to the extrinsic curvature of Σ when it is considered to be embedded in the spacetime.

The customary variational procedure, applied to the Hilbert action expressed in terms of the canonical variables, produces Hamilton equations which describe the time evolution of the canonical variables. The Hamiltonian is given by $N\mathcal{H} + N_i\mathcal{H}^i$, where \mathcal{H} and \mathcal{H}^i are functions of the canonical variables and their spatial derivatives. The procedure is not extended to the derivation of the Hamilton–Jacobi equation in the usual manner, as such an equation with the chosen set of canonical variables[3] appears to become meaningless when the general covariance of the theory is implemented (cf. section 5 for a more precise description).

Alternatively, general covariance can be introduced in the variational principle from the outset, by requiring that the action is invariant with respect to variations of the lapse and shift. This leads to the constraint equations (to simplify notations, we omit indices on components of g and π in all equations in this section)

$$\mathcal{H}(g, \pi; x) = 0 \tag{1}$$

and

$$\mathcal{H}^i(g, \pi; x) = 0 \tag{2}$$

which are imposed on the canonical variables of each slice. An important feature of general relativity is that its dynamics is fully constrained. It can be shown that if the geometry of spacetime is such that the constraints are satisfied on all slices of all spatial foliations of spacetime, then the canonical variables necessarily satisfy the Hamilton evolution equations. This feature is often referred to as a key property of general relativity[1] and is used to argue that the entire theory is encoded in the constraints, with the conclusion

that the Hamilton equations are redundant and can be ignored in dynamical considerations. Substitution of $\delta S/\delta g$ in the place of p in the constraint equations leads to a new set of equations

$$\mathcal{H}\left(g, \frac{\delta S}{\delta g}; x\right) = 0 \quad (3)$$

and

$$\mathcal{H}^i\left(g, \frac{\delta S}{\delta g}; x\right) = 0, \quad (4)$$

the first of which is considered to be the Hamilton–Jacobi equation (see section 5). Arguments appealing to the variational principle on the superspace of 3–geometries support this assertion. Detailed arguments and the interpretation of other equations may be found elsewhere[4].

Dirac’s procedure of canonical gravity quantization is based directly on this Hamilton–Jacobi equation, and produces a quantum theory that consists of the Wheeler–DeWitt equation together with commutation relations imposed on all canonical variables.

The ADM square root quantization procedure is also based entirely on constraints, but in this procedure the set of canonical variables is split into two subsets: the embedding variables (four of them altogether; one slicing parameter Ω and three coordinatization parameters α) and the true dynamical variables β (two of them)[5, 6, 7]. The constraints are then solved with respect to the momenta conjugate to the embedding variables. After substituting $\delta S/\delta\Omega$, $\delta S/\delta\alpha$ for p_Ω , p_α , (where S is the principal Hamilton function) one of the resulting equations (the equation for the momentum conjugate to the slicing parameter) is identified with the Hamilton–Jacobi equation, and its right hand side yields an expression for a new (square root) Hamiltonian. The quantization is based on this equation, yielding a quantum theory that consists of the Schrödinger equation and commutation relations imposed on the true dynamical variables and their conjugate momenta.

Describing the time evolution of quantized gravitational fields is extremely troublesome in both formulations. Any attempt to introduce time that can be used in a way similar to time in quantum mechanics, or in quantum field theory on a flat background, invariably leads to the notorious problems of time[5]. Attempts to introduce time in a universal way externally (such as the readings of specially designed clock) have been unsuccessful. There are reasons to believe that this is impossible[1], whether the clock is believed

to be gravitationally defined (i.e. the readings depend only on the variables describing the gravitational field), or a matter clock (the readings depend on both gravitational and matter variables).

The conceptual difficulties (such as the problem of functional evolution and the multiple choice problem, in Kuchař's terminology) emerge due to the dual nature of time parametrization in general relativity. If spacetime is considered as a manifold, it can be coordinatized and sliced in an arbitrary manner. However, this is not sufficient for the description of geometrodynamical evolution, since both slicing and coordinatization need to be tied to the metric of spacetime. The standard approach in classical geometrodynamics involves lapse and shift. These quantities determine the slicing of spacetime (i.e. the slicing condition) and its coordinatization. On a physical basis the lapse (the slicing) represents the reading of the clock of the observer whose worldline is perpendicular to the spacelike hypersurface. While the shift represents the displacement of the spatial coordinates in time away from such an observer. Here, the dynamics is determined solely in terms of the constraints only if there exists a unique spacetime metric. This metric might not be known until the geometrodynamical problem is solved. While its existence is not a problem in classical general relativity, there is, in general, no possibility of assigning such a unique metric of spacetime in canonical quantum gravity.

Given these problems, continuing to use the constraint equations as the foundation of geometrodynamics becomes problematic, to say the least. Quantization of the dynamical picture based on the constraints is essentially equivalent to restricting the states of the resulting quantum systems to a "shell", which is determined by constraints that are classical in origin. An attempt to undertake a similar action in quantum mechanics or quantum field theory would be quite disastrous under all but very carefully selected conditions.

One way to resolve this dilemma would be to weaken the requirement of covariance, essentially discarding it in dynamical considerations and recovering it by imposing symmetries on solutions only to the extent and in the sense that is allowed by dynamics. General covariance in the traditional sense should be recovered in the classical limit. This requirement should determine, at least partially, what constitutes the classical limit of quantum geometrodynamics.

This can be achieved if York's analysis of gravitational degrees of freedom[8] is taken into account and actively implemented. The Hamilton–Jacobi equation takes its traditional form, familiar from classical mechanics or electrodynamics. The resulting description remains equivalent to the one commonly

accepted in classical geometrodynamics. However, quantization based on this new Hamilton–Jacobi equation provides an appropriate interpretation of the conceptual problems of time, making them quite natural statements concerning the properties of gravity quantization. It also seems to avoid the technical problems of time, such as the Hilbert space and spectral analysis problems, as it produces a Schrödinger equation for the state evolution, and the Hamiltonian does not include the square root operation. The procedure has been described elsewhere in the context of homogeneous quantum cosmologies[6, 7].

In this setting time can be introduced as a slicing parametrization on a spacetime manifold, and tied to the metric structure, without contradiction. The metric interpretation of time is coupled with geometrodynamical evolution, while the true meaning of time becomes completely determined only after the geometrodynamical evolution problem has been solved. In a sense, the quantum geometrodynamical configuration and time emerge together, and the meaning of the clock readings is influenced by the quantum gravitational system[7].

As noted above, the geometrodynamical approach differs from the standard one in its treatment of the constraints. This difference is rather subtle on the classical level and does not result in different predictions. On the quantum level, however, it becomes fundamental.

In previous work we have considered different aspects of geometrodynamical quantization in particular cases of homogeneous quantum cosmologies. Here, we develop, for the first time, the geometrodynamical quantization formalism in a general setting and show that it retains all of the essential features previously illustrated in the context of homogeneous cosmologies.

To achieve transparency of the discussion and to provide an appropriate platform for future applications, we start by reviewing the issue in classical geometrodynamics. We begin from the Lagrangian formulation, and then describe the transition to the Hamilton and Hamilton–Jacobi equations.

2 Constraints on the Configuration Space of 3–Metrics.

We start from the 3+1 Lagrangian expression for the action, obtained from the standard Hilbert action by expressing the 4–metric in terms of lapse N ,

shift N_i , and 3-metric g_{ij} . After eliminating total time derivatives and total divergences[4], the action becomes

$$I_c = \frac{1}{16\pi} \int \left[R - (\text{Tr}\mathbf{K})^2 + \text{Tr}(\mathbf{K}^2) \right] N \sqrt{g} d^3x dt = \int \mathcal{L} d^3x dt \quad (5)$$

where

$$\mathcal{L} = \frac{1}{16\pi} \left[R - (\text{Tr}\mathbf{K})^2 + \text{Tr}(\mathbf{K}^2) \right] N \sqrt{g}. \quad (6)$$

Following standard convention[4] we drop the factor of $1/16\pi$ from the gravitational action in the remainder of this paper. In vacuum the equations are equivalent, and the factor can be trivially included when matter is present.

In the equations above \mathbf{K} is used as a shorthand notation for the extrinsic curvature K_{ij} , defined as

$$K_{ij} = \frac{1}{2N} \left[N_{i|j} + N_{j|i} - \dot{g}_{ij} \right], \quad (7)$$

R is the 3-curvature associated with the 3-metric g_{ij} , g is the determinant of this 3-metric, and $\dot{g}_{ij} = \frac{\partial g_{ij}}{\partial t}$.

Components of the 3-metric g_{ij} are treated as dynamical variables or (functional) coordinates of the geometrodynamical configuration space. Following the standard prescription for transitioning from the Lagrangian description of dynamics to the Hamiltonian one, we introduce momenta π^{ij} conjugate to the dynamical variables g_{ij} , with

$$\pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{g}_{ij}}. \quad (8)$$

Computing the right hand side of this expression involves determining the derivatives of $K = \text{Tr}(\mathbf{K}) = g^{ij} K_{ij}$ and of $\text{Tr}(\mathbf{K}^2) = g^{im} g^{jk} K_{mj} K_{ki}$ with respect to the \dot{g}_{nl} using

$$\frac{\partial K_{ij}}{\partial \dot{g}_{km}} = -\frac{1}{2N} \delta^k_i \delta^m_j. \quad (9)$$

This computation yields

$$\frac{\partial \text{Tr}\mathbf{K}}{\partial \dot{g}_{nl}} = -\frac{1}{2N} g^{ij} \delta^n_i \delta^l_j = -\frac{1}{2N} g^{nl} \quad (10)$$

$$\begin{aligned}
\frac{\partial(\text{Tr}K^2)}{\partial\dot{g}_{nl}} &= -\frac{1}{2N} \left[g^{im}g^{jk}\delta^n{}_m\delta^l{}_j K_{ki} + g^{im}g^{jk}K_{mj}\delta^n{}_k\delta^l{}_j \right] \\
&= -\frac{1}{2N} \left[g^{in}g^{lk}K_{ki} + g^{jn}g^{lm}K_{mj} \right] = -\frac{1}{N} K^{nl} \quad (11)
\end{aligned}$$

which results in the expression for the momenta π^{ij} defined by (8),

$$\pi^{ij} = \sqrt{g} \left(K g^{ij} - K^{ij} \right). \quad (12)$$

In what follows we use notations $\Pi = \pi^{ij}$ and $\Gamma = \text{Tr}\Pi$. The last equation implies

$$\Gamma = \sqrt{g} (3K - K) = 2\sqrt{g}K \quad (13)$$

$$K^{ij} = \frac{1}{\sqrt{g}} \left(\frac{1}{2}\Gamma g^{ij} - \pi^{ij} \right) \quad (14)$$

$$\text{Tr}K^2 = \frac{1}{g} \left(\text{Tr}\Pi^2 - \frac{\Gamma^2}{4} \right) \quad (15)$$

which allows one to express \mathcal{L} as a function of g_{ij} and π^{ij} only, giving

$$\mathcal{L} = N \left[g^{\frac{1}{2}}R - g^{-\frac{1}{2}} \left(\frac{\Gamma^2}{2} - \text{Tr}\Pi^2 \right) \right]. \quad (16)$$

The standard definition of the Hamiltonian

$$\mathcal{H}_{WDW} = \pi^{ij}\dot{g}_{ij} - \mathcal{L} \quad (\text{Mod total divergence}) \quad (17)$$

can be transformed to express \mathcal{H}_{WDW} in terms of g_{ij} and the conjugate momenta. In order to achieve this, we use

$$\dot{g}_{ij} = N_{i|j} + N_{j|i} - 2NK_{ij} \quad (18)$$

which allows us to write the first term of (17) as

$$\begin{aligned}
\pi^{ij}\dot{g}_{ij} &= (N_{i|j} + N_{j|i})\pi^{ij} - 2N\pi^{ij}K_{ij} = 2N_{i|j}\pi^{ij} - 2N\pi^{ij}K_{ij} \\
&= 2(N_i\pi^{ij})_{|j} - 2N_i(\pi^{ij})_{|j} - 2N\pi^{ij}K_{ij}. \quad (19)
\end{aligned}$$

The first term in this expression is the total covariant divergence of a vector density. It can be expressed as the total divergence of a vector field,

$$(N_i\pi^{ij})_{|j} = \sqrt{g} \left[N_i(Kg^{ij} - K^{ij}) \right]_{|j} = \left[\sqrt{g} N_i(Kg^{ij} - K^{ij}) \right]_{,j} \quad (20)$$

and removed from the final expression for the Hamiltonian. The third term, after substituting K_{ij} as given by (14), reduces to

$$2N\pi^{ij}K_{ij} = 2g^{-\frac{1}{2}}N \left(\frac{\Gamma^2}{2} - \text{Tr}\Pi^2 \right), \quad (21)$$

and the expression for \mathcal{H}_{WDW} takes the form

$$\mathcal{H}_{WDW} = N_i \left(-2\pi^{ij} \right)_{|j} + N \left[g^{-\frac{1}{2}} \left(\text{Tr}\Pi^2 - \frac{\Gamma^2}{2} \right) - g^{\frac{1}{2}}R \right]. \quad (22)$$

It is common practice to write the Hamiltonian \mathcal{H}_{WDW} in the form

$$\mathcal{H}_{WDW} = N \mathcal{H} + N_i \mathcal{H}^i \quad (23)$$

where

$$\mathcal{H} = g^{-\frac{1}{2}} \left(\text{Tr}\Pi^2 - \frac{\Gamma^2}{2} \right) - g^{\frac{1}{2}}R \quad (24)$$

is called the superhamiltonian, and

$$\mathcal{H}^i = -2 \left(\pi^{ij} \right)_{|j} \quad (25)$$

are the supermomenta. The action in Hamiltonian form can thus be written as

$$\begin{aligned} I_c &= \int \left(\pi^{ij} \dot{g}_{ij} - \mathcal{H}_{WDW} \right) d^3x dt \\ &= \int \left(\pi^{ij} \dot{g}_{ij} - N \mathcal{H} - N_i \mathcal{H}^i \right) d^3x dt. \end{aligned} \quad (26)$$

Variations of this action with respect to π^{ij} and g_{ij} produces the Hamilton equations

$$\dot{g}_{ij} = \frac{\partial \mathcal{H}_{WDW}}{\partial \pi^{ij}} \quad (27)$$

$$\dot{\pi}^{ij} = -\frac{\partial \mathcal{H}_{WDW}}{\partial g_{ij}}, \quad (28)$$

while variations of the shift and lapse yield the superhamiltonian and supermomenta constraint equations

$$\mathcal{H}(g_{ij}, \pi^{ij}) = 0 \quad (29)$$

$$\mathcal{H}^i(g_{ij}, \pi^{ij}) = 0, \quad (30)$$

which will be discussed in more detail after we develop convenient notations and formal machinery to handle the relevant questions. It is important that these equations (27) can be obtained by inverting the kinematic relations (12); therefore, they are independent of the Hamiltonian dynamics.

3 Transformation of Variables.

It is often convenient to replace g_{ij} with another set of variables (functions) which form the configuration space. For instance, in general analysis of the initial data problem or gravitational degrees of freedom the variables are split into the true dynamical variables, the scale factor and the gravitomagnetic vector (three components) that cannot be identified with components of the 3-metric g_{ij} . A similar parametrization is used in setting up the problems of homogeneous cosmologies[9]. In general, the g_{ij} are expressed as

$$g_{ij} = g_{ij}(q_A) \quad (31)$$

or

$$g_{ij} = g_{ij}(q_A, x^i) \quad (32)$$

where $q_A = q_A(x^i, t)$ are assumed to be independent, and $A = 1, \dots, n_q$ with $n_q \leq 6$. In the generic case $n_q = 6$ and all components of shift are present in the spacetime metric. If this is not the case, some symmetries have been used to fix the form of the metric, which typically leads to the loss of covariance. Some of the supermomenta constraints might be lost or not be independent. It is hard to list and consider all possibilities, but as a rule these degenerate cases do not cause essential troubles in any particular case. In what follows we will be interested mostly in the generic, non-degenerate case, although most of the conclusions will also be true for degenerate cases.

The transition from the Lagrangian to the Hamiltonian action begins with the Lagrangian (6). The extrinsic curvature is given by (7), except now \dot{g}_{ij} is given by

$$\dot{g}_{ij} = \frac{\partial g_{ij}}{\partial q_A} \dot{q}_A = M^A{}_{ij} \dot{q}_A \quad (33)$$

which results in the expression for K_{ij}

$$K_{ij} = \frac{1}{2N} \left[N_{i|j} + N_{j|i} - M^A{}_{ij} \dot{q}_A \right] \quad (34)$$

and

$$\frac{\partial K_{ij}}{\partial \dot{q}_A} = -\frac{1}{2N} M^A_{ij}, \quad (35)$$

where M^A_{ij} is a 3×3 symmetric matrix for each value of A . Alternatively, one can narrow it down to six components of g_{ij} such that $i \leq j$ (right upper triangle) and consider pairs (ij) as collective indices, in which case this matrix becomes the Jacobian of transformation between q_A and $g_{ij}, i \leq j$. The assumption that the variables q_A are independent implies that the rank of this matrix is equal to the number of q_A (less or equal to six). In the generic cases this rank is equal to six, which means that the system of equations (33) can be solved, providing expressions for \dot{q}_A in terms of \dot{g}_{ij} .

Equations (35) allow us to calculate the derivatives of

$$\text{Tr}\mathbf{K} = g^{ij} K_{ij} \quad (36)$$

$$\text{Tr}(\mathbf{K}^2) = g^{im} g^{jk} K_{mj} K_{ki} \quad (37)$$

with respect to \dot{q}_A , which yields

$$\frac{\partial \text{Tr}\mathbf{K}}{\partial \dot{q}_A} = -\frac{1}{2N} g^{ij} M^A_{ij} \quad (38)$$

$$\begin{aligned} \frac{\partial (\text{Tr}(\mathbf{K}^2))}{\partial \dot{q}_A} &= g^{im} g^{jk} \left(-\frac{1}{2N} M^A_{mj} \right) K_{ki} + g^{im} g^{jk} K_{mj} \left(-\frac{1}{2N} M^A_{ki} \right) \\ &= -\frac{1}{2N} \left[K^{jm} M^A_{mj} + K^{ik} M^A_{ki} \right] \\ &= -\frac{1}{N} K^{ij} M^A_{ij}. \end{aligned} \quad (39)$$

The momentum p^A conjugate to the variable q_A is given by

$$\begin{aligned} p^A &= \frac{\partial \mathcal{L}}{\partial \dot{q}_A} = \sqrt{g} \left[K g^{ij} M^A_{ij} - K^{ij} M^A_{ij} \right] \\ &= \sqrt{g} \left[K g^{ij} - K^{ij} \right] M^A_{ij} \end{aligned} \quad (40)$$

and comparison of this expression with (12) yields the very useful relations

$$p^A = \pi^{ij} M^A_{ij}. \quad (41)$$

The Lagrangian \mathcal{L} (6), with \mathbf{K} given by (34), becomes a function of the new variables

$$\mathcal{L} = \mathcal{L}(q_A, \dot{q}_A, N, N_i), \quad (42)$$

and the transition to the Hamiltonian action can proceed in the standard way, expressing the Hamiltonian

$$\widetilde{\mathcal{H}} = p^A \dot{q}_A - \mathcal{L}, \quad (43)$$

in terms of canonical variables q_A, p^A . This procedure presumably results in an expression for the Hamiltonian of the same type as above:

$$\widetilde{\mathcal{H}} = N \mathcal{H}(q_A, p^A) + N_i \mathcal{H}^i(q_A, p^A). \quad (44)$$

This is obvious in the generic case, when the rank of $M^A{}_{ij}$ is maximal (six), since (44) can then be obtained by inverting (31) and (41) and substituting the expressions for $\pi^{ij} = \pi^{ij}(q_A, p^A)$, together with (31), into (23)–(25). The resulting Hamiltonian $\widetilde{\mathcal{H}}$ depends on the new variables q_A, p^A , and inverting (31), followed by the substitution of $q_A = q_A(g_{ij})$ and (41) into this new Hamiltonian, yields

$$\widetilde{\mathcal{H}}(q_A(g_{ij}), p^A(g_{ij}, \pi^{ij}), N, N_i) = \mathcal{H}_{WDW}(g_{ij}, \pi^{ij}, N, N_i), \quad (45)$$

which implies

$$\dot{g}_{ij} = M^A{}_{ij} \dot{q}_A = \frac{\partial \mathcal{H}_{WDW}}{\partial \pi^{ij}} = \frac{\partial p^A}{\partial \pi^{ij}} \frac{\partial \widetilde{\mathcal{H}}}{\partial p^A} = M^A{}_{ij} \frac{\partial \widetilde{\mathcal{H}}}{\partial p^A}. \quad (46)$$

Thus

$$M^A{}_{ij} \left(\dot{q}_A - \frac{\partial \widetilde{\mathcal{H}}}{\partial p^A} \right) = 0, \quad (47)$$

and this system of equations has the unique solution

$$\dot{q}_A = \frac{\partial \widetilde{\mathcal{H}}}{\partial p^A}. \quad (48)$$

Just as in the previous section, these expressions for \dot{q}_A are equivalent to the definitions of momenta p^A given by (40).

This logic fails when the number n_q of variables q_A is less than six. Nevertheless, the basic structure remains the same. In particular, the Hamiltonian $\widetilde{\mathcal{H}}(q_A, p^A, N, N_i)$ can be expressed by (44) as a combination of the superhamiltonian and supermomenta (neither of which depend on shift and lapse). Note, however, that this expression cannot be obtained by the simple inversions described above. Just as in the non-degenerate case, the Hamilton equations (48) remain equivalent to the definitions of momenta (40).

This procedure is more complex, and requires lengthier computations, but we believe that it leads to a greater understanding of the structure of the equations, and thus we present it here. As an additional benefit, this procedure provides explicit expressions for \dot{q}_A and p^A in terms of each other, as well as explicit expressions for the superhamiltonian and supermomenta.

It is convenient to rewrite equation (40) for p^A in the form

$$p^A = \sqrt{g} \left(K g^{ij} M^A_{ij} - K^{ij} M^A_{ij} \right). \quad (49)$$

As before, we have

$$K_{lm} = \frac{1}{2N} \left[N_{l|m} + N_{m|l} - M^B_{lm} \dot{q}_B \right] \quad (50)$$

which yields

$$K = g^{lm} K_{lm} = \frac{1}{2N} \left[g^{lm} (N_{l|m} + N_{m|l}) - g^{lm} M^B_{lm} \dot{q}_B \right] \quad (51)$$

$$K^{ij} = g^{il} g^{jm} K_{lm} = \frac{1}{2N} \left[g^{il} g^{jm} (N_{l|m} + N_{m|l}) - g^{il} g^{jm} M^B_{lm} \dot{q}_B \right], \quad (52)$$

and substituting the last two expressions into (49) leads to

$$p^A = \frac{\sqrt{g}}{2N} \left[-(g^{il} g^{jm} - g^{ij} g^{lm}) M^A_{ij} (N_{l|m} + N_{l|m}) \right. \\ \left. + (g^{il} g^{jm} - g^{ij} g^{lm}) M^A_{ij} M^B_{lm} \dot{q}_B \right]. \quad (53)$$

This can be written in a more compact form if we introduce the notations

$$G^{ijlm} = g^{il} g^{jm} - g^{ij} g^{lm} \quad (54)$$

or its symmetrized version

$$G^{ijlm} = \frac{1}{2} g^{il} g^{jm} + \frac{1}{2} g^{jl} g^{im} - g^{ij} g^{lm} \quad (55)$$

and

$$Q^{AB} = G^{ijlm} M^A_{ij} M^B_{lm}. \quad (56)$$

With these notations, the momenta becomes

$$p^A = \frac{\sqrt{g}}{2N} \left[-G^{ijlm} M^A_{ij} (N_{l|m} + N_{l|m}) + Q^{AB} \dot{q}_B \right]. \quad (57)$$

We have assumed that the q_A are independent. This implies that the square matrix Q^{AB} (with dimension $n_q \times n_q$) has rank n_q and, consequently, is invertible. In what follows we use the same letters G and Q with lower indices for the elements of the inverse matrices,

$$(G_{ijlm}) = (G^{ijlm})^{-1} \quad (58)$$

$$(Q_{AB}) = (Q^{AB})^{-1}, \quad (59)$$

so that

$$G_{ijkn} G^{knlm} = \delta_i^l \delta_m^j \quad (60)$$

and

$$Q_{AB} Q^{BC} = \delta_A^C. \quad (61)$$

These matrices have the quite obvious symmetry properties

$$G^{ijlm} = G^{jilm} = G^{ijml} = G^{lmij} \quad (62)$$

$$G_{ijlm} = G_{jilm} = G_{ijml} = G_{lmij} \quad (63)$$

as well as

$$Q^{AB} = Q^{BA}; \quad Q_{AB} = Q_{BA}. \quad (64)$$

There are two more useful relations that follow from simple observations. By definition

$$Q_{BC} Q^{CA} = Q_{BC} M^C_{ij} G^{ijlm} M^A_{lm} = \delta_B^A \quad (65)$$

which implies

$$M^B_{kn} Q_{BC} M^C_{ij} G^{ijlm} M^A_{lm} = \delta_B^A M^B_{kn} = M^A_{kn} = \delta_k^l \delta_n^m M^A_{lm}. \quad (66)$$

If $\{q_A\}_{A=1}^{n_q}$ are a complete set of variables (g_{ik} depend only on q_A and do not have any other arguments) then this results in the two relations

$$M^B_{kn} Q_{BC} M^C_{ij} G^{ijlm} = \delta_k^l \delta_n^m \quad (67)$$

$$M^B_{kn} Q_{BC} M^C_{ij} = G_{knij} \quad (68)$$

that will be used, together with symmetries, in computations below.

The expression (57) for the momenta can be considered as a system of linear equations for \dot{q}_B . Its solution

$$\dot{q}_B = Q_{BA} G^{ijlm} M^A_{ij} (N_{lm} + N_{ml}) + 2Ng^{-\frac{1}{2}} Q_{BA} p^A \quad (69)$$

expresses \dot{q}_B in terms of the momenta p^A .

We begin to calculate the Hamiltonian $\widetilde{\mathcal{H}}$ by expressing the extrinsic curvature K_{lm} , $\text{Tr}(\mathbf{K})$ and $\text{Tr}(\mathbf{K}^2)$ in terms of momenta. The last term in square brackets in equation (50) for K_{lm} can be written in terms of momenta by substituting \dot{q}_B given by (69):

$$\begin{aligned} M^B{}_{lm} \dot{q}_B &= \underbrace{M^B{}_{lm} q_{BA} G^{ijlm} M^A{}_{ij}}_{\delta_l^k \delta_m^n} (N_{k|n} + N_{n|k}) + 2Ng^{-\frac{1}{2}} Q_{BA} p^A M^B{}_{lm} \\ &= N_{l|m} + N_{m|l} + 2Ng^{-\frac{1}{2}} Q_{BA} p^A M^B{}_{lm}. \end{aligned} \quad (70)$$

This results in

$$K_{lm} = -g^{-\frac{1}{2}} Q_{BA} p^A M^B{}_{lm} \quad (71)$$

$$\text{Tr}\mathbf{K} = -g^{-\frac{1}{2}} Q_{BA} p^A g^{lm} M^B{}_{lm} \quad (72)$$

$$K^{ij} = -g^{-\frac{1}{2}} Q_{BA} p^A g^{il} g^{jm} M^B{}_{lm} \quad (73)$$

$$K_{ij} = -g^{-\frac{1}{2}} Q_{CD} p^D M^C{}_{ij} \quad (74)$$

which, together with $\text{Tr}(\mathbf{K}^2) = K^{ij} K_{ij}$, allows us to express the Lagrangian \mathcal{L} in terms of momenta:

$$\begin{aligned} \mathcal{L} &= \left[R - (\text{Tr}\mathbf{K})^2 + \text{Tr}(\mathbf{K}^2) \right] \sqrt{g} N \\ &= N \left[g^{\frac{1}{2}} R + g^{-\frac{1}{2}} Q_{BA} Q_{CD} \underbrace{G^{ijlm} M^C{}_{ij} M^B{}_{lm}}_{Q^{CB}} p^A p^D \right] \\ &= N \left[g^{\frac{1}{2}} R + g^{-\frac{1}{2}} Q_{AB} p^A p^B \right]. \end{aligned} \quad (75)$$

In addition, (69) implies

$$p^A \dot{q}_A = 2Ng^{-\frac{1}{2}} Q_{AB} p^A p^B + 2N_{i|j} Q_{AB} p^A M^B{}_{lm} G^{lmij} \quad (76)$$

which results in the expression for the Hamiltonian

$$\begin{aligned} \widetilde{\mathcal{H}} &= p^A \dot{q}_A - \mathcal{L} \text{ (Mod total divergence)} \\ &= N \left[g^{-\frac{1}{2}} Q_{AB} p^A p^B - g^{\frac{1}{2}} R \right] + 2N_{i|j} Q_{AB} p^A M^B{}_{lm} G^{lmij}. \end{aligned} \quad (77)$$

The last term in this expression can be written as

$$\begin{aligned} 2N_{i|j} Q_{AB} p^A M^B{}_{lm} G^{lmij} &= \left(2N_i Q_{AB} p^A M^B{}_{lm} G^{lmij} \right)_{|j} \\ &\quad - N_i \left(2Q_{AB} p^A M^B{}_{lm} G^{lmij} \right)_{|j} \end{aligned} \quad (78)$$

The covariant divergence of a vector density is a total divergence and can be thrown out. This reduces the Hamiltonian to

$$\widetilde{\mathcal{H}} = N \left[g^{-\frac{1}{2}} Q_{AB} p^A p^B - g^{\frac{1}{2}} R \right] + N_i \left(-2 Q_{AB} p^A M^B{}_{lm} G^{lmij} \right)_{|j} \quad (79)$$

which is usually written as

$$\widetilde{\mathcal{H}} = N \mathcal{H}(q_A, p^A) + N_i \mathcal{H}^i(q_A, p^A) \quad (80)$$

where

$$\mathcal{H}(q_A, p^A) = g^{-\frac{1}{2}} Q_{AB} p^A p^B - g^{\frac{1}{2}} R \quad (81)$$

is the superhamiltonian, and

$$\mathcal{H}^i(q_A, p^A) = \left(-2 Q_{AB} p^A M^B{}_{lm} G^{lmij} \right)_{|j} \quad (82)$$

are supermomenta in the superspace of q_A . It is trivial to verify, by computing appropriate derivatives of (79), that the Hamilton equation

$$\dot{q}_A = \frac{\partial \widetilde{\mathcal{H}}}{\partial p^A} \quad (83)$$

simply reproduce the results obtained by inverting the definition of the momenta.

The action can now be written in Hamiltonian form as

$$\begin{aligned} I_c &= \int \left(p^A \dot{q}_A - \widetilde{\mathcal{H}}(q_A, p^A, N, N_i) \right) d^3x dt \\ &= \int \left(p^A \dot{q}_A - N \mathcal{H}(q_A, p^A) - N_i \mathcal{H}^i(q_A, p^A) \right) d^3x dt. \end{aligned} \quad (84)$$

Variation of this action with respect to p^A and q_A reproduce the Hamilton equations

$$\dot{q}_A = \frac{\partial \widetilde{\mathcal{H}}}{\partial p^A} \quad (85)$$

and

$$\dot{p}^A = -\frac{\partial \widetilde{\mathcal{H}}}{\partial q_A}, \quad (86)$$

while variations of shift and lapse yield the superhamiltonian and supermomenta constraint equations

$$\mathcal{H}(q_A, p^A) = 0 \quad (87)$$

$$\mathcal{H}^i(q_A, p^A) = 0 \tag{88}$$

that will be discussed in more detail after we develop convenient notations and formal machinery to handle relevant questions. At this point we only wish to stress once more that equations (85) can be obtained by inverting the kinematic relations (57) and, thus, can be treated as independent of the Hamiltonian dynamics.

4 Geometrodynamical Superspace.

York's analysis of the geometrodynamical degrees of freedom suggests that the appropriate configuration space for geometrodynamics is not the superspace of 3-metrics (or 3-geometries), but rather the space of conformal 3-geometries. We describe here the ideas of such dynamics in a generalized form for a case when the 3-metric components g_{ik} are given as functions of n_q other variables q_A , $A = 1, \dots, n_q \leq 6$,

$$g_{ij} = g_{ij}(q_A). \tag{89}$$

The functions q_A are assumed to be independent and form a complete set. Following York's analysis, we split the set of variables $\{q_A\}_{A=1}^{n_q}$ into a subset $\{\beta_I\}_{I=1}^{n_d}$ ($n_d \leq 2$) of the true dynamic variables and a subset $\{\alpha_\mu\}_{\mu=0}^{n_e}$ ($n_e \leq 3$) of the so-called embedding variables, with the identifications facilitating a comparison with the equations of the previous section:

$$\beta_I = q_I \tag{90}$$

$$\alpha_\mu = q_{n_d+\mu+1} \equiv q_\mu. \tag{91}$$

This allows us to freely switch notations to better suit the context. It is clear that

$$n_q = n_d + n_e + 1. \tag{92}$$

We wish to reformulate geometrodynamics on the configuration space (geometrodynamical superspace) of the true dynamical degrees of freedom $q_I = \beta_I$ (conformal superspace as opposed to the superspace of 3-metrics of the previous two sections).

The Lagrangian \mathcal{L} , equation (6) with \mathbf{K} given by (34), remains a function of the same variables as in the previous section. It can be written more

appropriately as

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(q_A, \dot{q}_A, N, N_i) \\ &= \mathcal{L}(q_I, \dot{q}_I, q_\mu, \dot{q}_\mu, N, N_i) = \mathcal{L}(\beta_I, \dot{\beta}_I, \alpha_\mu, \dot{\alpha}_\mu, N, N_i),\end{aligned}\quad (93)$$

where only $q_I, (\dot{q})_I$ are related to the new configuration space. The rest of the arguments of \mathcal{L} are functional parameters, which make the Lagrangian \mathcal{L} explicitly time dependent (although the time dependence is introduced through functions $q_\mu, \dot{q}_\mu, N, N_i$). Only the momenta p^I conjugate to q_I

$$p^I = \frac{\partial \mathcal{L}}{\partial \dot{q}_I} \quad (94)$$

retain their dynamical meaning (and eventually become arguments of the Hamiltonian). The similar quantities

$$p^\mu = \frac{\partial \mathcal{L}}{\partial \dot{q}_\mu} \quad (95)$$

can be introduced and used in developing the theory, but cannot be treated as momenta.

The transition to a Hamiltonian formulation, and the analysis of the constraints, can be performed almost the same as in section 3. It is especially simple in the case when the matrix Q^{AB} , given by (56), is such that $Q^{\mu I} = Q^{I\mu} = 0$. That is, Q^{AB} has the block structure

$$(Q^{AB}) = \begin{pmatrix} Q^{IJ} & 0 \\ 0 & Q^{\mu\nu} \end{pmatrix} \quad (96)$$

in which case the inverse matrix Q_{AB} has the structure

$$(Q_{AB}) = \begin{pmatrix} Q_{IJ} & 0 \\ 0 & Q_{\mu\nu} \end{pmatrix} \quad (97)$$

where Q_{IJ} is the inverse matrix of Q^{IJ} and $Q_{\mu\nu}$ is the inverse matrix of $Q^{\mu\nu}$. This case is rather common in applications (for instance, it includes all diagonal homogeneous cosmologies). The true meaning is determined by the expressions for p^I and p^μ , replacing (57) with

$$p^I = \frac{\sqrt{g}}{2N} \left[-G^{ijlm} M^I{}_{ij} (N_{l|m} + N_{l|m}) + Q^{IJ} \dot{q}_J \right] \quad (98)$$

and

$$p^\mu = \frac{\sqrt{g}}{2N} \left[-G^{ijlm} M^\mu{}_{ij} (N_{l|m} + N_{l|m}) + Q^{\mu\nu} \dot{q}_\nu \right]. \quad (99)$$

It is important to notice that the p^μ depend only on time derivatives of q_μ , and do not involve time derivatives of the true dynamical variables.

All the computations in the previous section can be repeated literally, up until the Lagrangian is expressed in terms of momenta, equation (75). In view of (98) and (99), the final expression for \mathcal{L} takes the form

$$\begin{aligned} \mathcal{L} &= N \left[g^{\frac{1}{2}} R + g^{-\frac{1}{2}} Q_{AB} p^A p^B \right] \\ &= N \left[g^{\frac{1}{2}} R + g^{-\frac{1}{2}} \left(Q_{IJ} p^I p^J + Q_{\mu\nu} p^\mu p^\nu \right) \right], \end{aligned} \quad (100)$$

which is essentially the same as in the previous section, except now the p^μ are not momenta but merely functions

$$p^\mu = p^\mu(q_A, \dot{q}_\mu, N, N_i) \quad (101)$$

given by (99). This makes the Lagrangian \mathcal{L} a function of the new arguments

$$\mathcal{L} = \mathcal{L}(q_A, p^I, \dot{q}_\mu, N, N_i) = \mathcal{L}(q_I, q_\mu, p^I, \dot{q}_\mu, N, N_i). \quad (102)$$

The Hamiltonian \mathcal{H}_{DYN} on the dynamical superspace

$$\mathcal{H}_{DYN} = p^I \dot{q}_I - \mathcal{L}(q_A, p^I, \dot{q}_\mu, N, N_i), \quad (103)$$

is distinctly different from the Hamiltonian $\widetilde{\mathcal{H}}$ on the superspace of 3-metrics described in the previous section. A useful form of this Hamiltonian can be obtained by writing it as

$$\mathcal{H}_{DYN} = \underbrace{p^A \dot{q}_A - \mathcal{L}(q_A, p^I, \dot{q}_\mu, N, N_i)}_{\widetilde{\mathcal{H}}(q_A, p^I, \dot{q}_\mu, N, N_i)} - p^\mu \dot{q}_\mu, \quad (104)$$

where the first two terms form $\widetilde{\mathcal{H}}$ of the previous section, given by equations (80)–(82), with expressions (99) substituted for p^μ . More precisely,

$$\mathcal{H}_{DYN} = \widetilde{\mathcal{H}}(q_A, p^I, \dot{q}_\mu, N, N_i) - p^\mu \dot{q}_\mu \quad (105)$$

where

$$\widetilde{\mathcal{H}}(q_A, p^I, \dot{q}_\mu, N, N_i) = N \mathcal{H}(q_A, p^I, \dot{q}_\mu, N, N_i) + N_i \mathcal{H}^i(q_A, p^I, \dot{q}_\mu, N, N_i) \quad (106)$$

with

$$\begin{aligned}\mathcal{H}(q_A, p^I, \dot{q}_\mu, N, N_i) &= g^{-\frac{1}{2}} Q_{AB} p^A p^B - g^{\frac{1}{2}} R \\ &= g^{-\frac{1}{2}} \left(Q_{IJ} p^I p^J + Q_{\mu\nu} p^\mu p^\nu \right) - g^{\frac{1}{2}} R\end{aligned}\quad (107)$$

and

$$\begin{aligned}\mathcal{H}^i(q_A, p^I, \dot{q}_\mu, N, N_i) &= \left(-2 Q_{AB} p^A M^B{}_{lm} G^{lmij} \right)_{|j} \\ &= \left[-2 \left(Q_{IJ} p^I M^J{}_{lm} + Q_{\mu\nu} p^\mu M^\nu{}_{lm} \right) G^{lmij} \right]_{|j}\end{aligned}\quad (108)$$

The Hamiltonian action on the geometrodynamical superspace,

$$\begin{aligned}I &= \int [p^I \dot{q}_I - \mathcal{H}_{DYN}] d^3x dt \\ &= \int [p^I \dot{q}_I - (\widetilde{\mathcal{H}}(q_A, p^I, \dot{q}_\mu, N, N_i) - p^\mu \dot{q}_\mu)] d^3x dt\end{aligned}\quad (109)$$

can be used to derive n_d pairs of Hamilton equations

$$\begin{aligned}\dot{q}_I &= \frac{\partial \mathcal{H}_{DYN}}{\partial p^I} \\ \dot{p}^I &= \frac{\partial \mathcal{H}_{DYN}}{\partial q_I}\end{aligned}\quad (110)$$

by varying p^I and q_I .

Variations of the action with respect to N and N_i yield n_c constraints, with $n_c \leq 4$. In view of equations (105) – (108), the functional dependence of \mathcal{H}_{DYN} on N, N_i can be expressed as

$$\begin{aligned}\mathcal{H}_{DYN}(q_A, p^I, p^\mu(N, N_i, \dot{q}_\mu), N, N_i) \\ = \underbrace{N \mathcal{H}(q_A, p^I, p^\mu(N, N_i, \dot{q}_\mu)) + N_i \mathcal{H}^i(q_A, p^I, p^\mu(N, N_i, \dot{q}_\mu))}_{\widetilde{\mathcal{H}}(q_A, p^I, p^\mu(N, N_i, \dot{q}_\mu), N, N_i)} \\ - p^\mu(N, N_i, \dot{q}_\mu) \dot{q}_\mu.\end{aligned}\quad (111)$$

Variation of N produces the constraint

$$\frac{\partial \mathcal{H}_{DYN}}{\partial N} + \frac{\partial \mathcal{H}_{DYN}}{\partial p^\mu} \frac{\partial p^\mu}{\partial N} = 0,\quad (112)$$

and equation (111) implies

$$\frac{\partial \mathcal{H}_{DYN}}{\partial N} = \mathcal{H}(q_A, p^I, p^\mu(N, N_i, \dot{q}_\mu)), \quad (113)$$

so, together with the trivial non-dynamic expression (83) of the previous section,

$$\frac{\partial \mathcal{H}_{DYN}}{\partial p^\mu} = \frac{\partial \widetilde{\mathcal{H}}}{\partial p^\mu} - \dot{q}_\mu = \dot{q}_\mu - \dot{q}_\mu = 0, \quad (114)$$

which reduces (112) to the superhamiltonian constraint

$$\mathcal{H}(q_A, p^I, p^\mu(N, N_i, \dot{q}_\mu)) = 0. \quad (115)$$

Likewise, variation of N_i leads to the supermomenta constraints

$$\mathcal{H}^i(q_A, p^I, p^\mu(N, N_i, \dot{q}_\mu)) = 0. \quad (116)$$

These superhamiltonian and supermomenta constraints on the geometrodynamical superspace are obtained from the constraints on the superspace of 3-geometries by a simple substitution of p^μ as given by (99).

In the general case the matrix Q^{AB} does not have a block structure. Instead, it can be written as

$$(Q^{AB}) = \begin{pmatrix} Q^{IJ} & Q^{I\nu} \\ Q^{\mu J} & Q^{\mu\nu} \end{pmatrix}, \quad (117)$$

in which case the inverse matrix \tilde{Q}_{AB} has the structure

$$(\tilde{Q}_{AB}) = \begin{pmatrix} \tilde{Q}_{IJ} & \tilde{Q}_{I\nu} \\ \tilde{Q}_{\mu J} & \tilde{Q}_{\mu\nu} \end{pmatrix}. \quad (118)$$

It is clear that $\tilde{Q}_{IJ} \neq Q_{IJ}$ and that \tilde{Q}_{IJ} is not the inverse matrix of the sub-matrix Q^{IJ} . Likewise, $\tilde{Q}_{\mu\nu} \neq Q_{\mu\nu}$ and $\tilde{Q}_{\mu\nu}$ is not the inverse matrix of the sub-matrix $Q^{\mu\nu}$.

Expressions for p^I and p^μ take the form (compare with (98) and (99)):

$$p^I = \frac{\sqrt{g}}{2N} \left[-G^{ijlm} M^I{}_{ij} (N_{l|m} + N_{l|m}) + Q^{IJ} \dot{q}_J + Q^{I\nu} \dot{q}_\nu \right] \quad (119)$$

and

$$p^\mu = \frac{\sqrt{g}}{2N} \left[-G^{ijlm} M^\mu{}_{ij} (N_{l|m} + N_{l|m}) + Q^{\mu J} \dot{q}_J + Q^{\mu\nu} \dot{q}_\nu \right]. \quad (120)$$

An important change is that now the p^μ depend not only on time derivatives of q_μ but also on time derivatives of the true dynamical variables.

Expressing the Lagrangian \mathcal{L} in terms of momenta yields

$$\mathcal{L} = N \left[g^{\frac{1}{2}} R + g^{-\frac{1}{2}} \left(\tilde{Q}_{IJ} p^I p^J + \tilde{Q}_{I\nu} p^I p^\nu + \tilde{Q}_{\mu J} p^\mu p^J + \tilde{Q}_{\mu\nu} p^\mu p^\nu \right) \right] \quad (121)$$

which is again essentially the same as in the previous section, except now p^μ are functions

$$p^\mu = p^\mu(q_A, \dot{q}_I, \dot{q}_\mu, N, N_i) \quad (122)$$

given by (120). This in turn makes the Lagrangian \mathcal{L} a function of the new arguments

$$\mathcal{L} = \mathcal{L}(q_A, p^I, p^\mu(q_A, \dot{q}_I, \dot{q}_\mu, N, N_i), N, N_i). \quad (123)$$

The dependence of the Lagrangian on \dot{q}_I in the general case seems to break the argument developed above in the case when Q^{AB} has block structure. In fact, this is not the case, and the problem can be easily remedied.

As before, we assume that the true dynamical variables q_I are independent, which implies that the sub-matrix Q^{IJ} in (119) is invertible (we keep the notation Q_{IJ} for the elements of the matrix inverse to the sub-matrix Q^{IJ}). This means that (119) can be considered as a system of linear equations with respect to \dot{q}_I . Its solution

$$\dot{q}_I = Q_{IJ} G^{ijklm} M^J_{ij} (N_{l|m} - N_{l|m}) - Q_{IJ} Q^{J\nu} \dot{q}_\nu + 2Ng^{-\frac{1}{2}} Q_{IJ} p^J \quad (124)$$

expresses \dot{q}_I as functions

$$\dot{q}_I = \dot{q}_I(q_A, p^I, \dot{q}_\mu, N, N_i). \quad (125)$$

Their substitution in (122) transforms p^μ to

$$p^\mu = p^\mu(q_A, \dot{q}_I(q_A, p^I, \dot{q}_\mu, N, N_i), \dot{q}_\mu, N, N_i) = p^\mu(q_A, p^I, \dot{q}_\mu, N, N_i) \quad (126)$$

and \mathcal{L} into the form

$$\begin{aligned} \mathcal{L} &= \mathcal{L}(q_A, p^I, p^\mu(q_A, \dot{q}_I(q_A, p^I, \dot{q}_\mu, N, N_i), \dot{q}_\mu, N, N_i), N, N_i) = \\ &= \mathcal{L}(q_A, p^I, \dot{q}_\mu, N, N_i). \end{aligned} \quad (127)$$

With this in mind, we can follow the same steps as before in developing the Hamiltonian formalism. We introduce the geometrodynamical Hamiltonian as

$$\mathcal{H}_{DYN} = p^I \dot{q}_I - \mathcal{L} \quad (128)$$

where \mathcal{L} is given by (121). This allows us to recover the same expressions for \mathcal{H}_{DYN}

$$\mathcal{H}_{DYN} = p^A \dot{q}_A - \mathcal{L} - p^\mu \dot{q}_\mu = \widetilde{\mathcal{H}} - p^\mu \dot{q}_\mu, \quad (129)$$

with

$$\widetilde{\mathcal{H}} = N \mathcal{H} + N_i \mathcal{H}^i \quad (130)$$

where $\widetilde{\mathcal{H}}$, \mathcal{H} , \mathcal{H}^i are given by the same expressions as in the previous section, except that p^μ is now given by (120) and \dot{q}_I is given by (124) (also, Q_{AB} from the previous section should be replaced by \widetilde{Q}_{AB}).

Tracing the chain of arguments developed above results in the action

$$\begin{aligned} I &= \int [p^I \dot{q}_I - \mathcal{H}_{DYN}] d^3x dt \\ &= \int [p^I \dot{q}_I - (\widetilde{\mathcal{H}} - p^\mu \dot{q}_\mu)] d^3x dt, \end{aligned} \quad (131)$$

with \mathcal{H}_{DYN} and $\widetilde{\mathcal{H}}$ arguments best described by equations

$$\begin{aligned} \mathcal{H}_{DYN} &= \mathcal{H}_{DYN}(q_A, p^I, p^\mu(q_A, \dot{q}_I, \dot{q}_\mu, N, N_i), N, N_i) \\ \widetilde{\mathcal{H}} &= \widetilde{\mathcal{H}}(q_A, p^I, p^\mu(q_A, \dot{q}_I, \dot{q}_\mu, N, N_i), N, N_i) \\ \dot{q}_I &= \dot{q}_I(q_A, p^I, \dot{q}_\mu, N, N_i). \end{aligned} \quad (132)$$

Following the same line of arguments as before leads to the Hamilton equations (110) and to the n_c constraint equations

$$\mathcal{H}(q_A, p^I, p^\mu(N, N_i, \dot{q}_I(q_A, p^I, \dot{q}_\mu, N, N_i), \dot{q}_\mu)) = 0 \quad (133)$$

$$\mathcal{H}^i(q_A, p^I, p^\mu(N, N_i, \dot{q}_I(q_A, p^I, \dot{q}_\mu, N, N_i), \dot{q}_\mu)) = 0. \quad (134)$$

These superhamiltonian and supermomenta constraints on the geometrodynamical superspace in the general case are obtained from the constraints on the superspace of 3-geometries by a simple substitution of p^μ as given by (120) and \dot{q}_I as given by (124).

5 The Hamilton–Jacobi Equation.

A detailed description of the Hamilton–Jacobi equation on the configuration space of 3-metrics g_{ik} can be found in the literature, including monographs,

together with a detailed explanation of how the concept of a functional (bubble) derivative takes an active part in writing the equation. We therefore provide only a very brief discussion of this subject here.

Following standard approaches we introduce the Hamilton principal functional,

$$S = S[t, g_{ik}] \quad (135)$$

as the extremal value of the action (26), evaluated between the 3-slices given by (t', g'_{ik}) and (t, g_{ik}) . The primed slice is assumed to be fixed:

$$S[t, g_{ik}] = I_{extremum} = \int_{(t', g'_{ik})}^{(t, g_{ik})} \left(\pi^{ij} \dot{g}_{ij} - \mathcal{H}_{WDW} \right) d^3x dt \quad (136)$$

The integral on the right hand side of this expression is extremized with both ends fixed. If the upper limit is released, the integral becomes a functional of t and g_{ik} given by (136). The Hamilton–Jacobi equation for this functional is obtained by variation of the upper limit. Sometimes this variation is preceded by imposing the constraints[4]. However, such a move weakens arguments following the variational procedure. Instead one can vary the integral before imposing the constraints, as it is done in mechanics. It is easy to see that the variation $(\delta t, \delta g_{ik})$ on the final hypersurface produces variation in S given by

$$\delta S = \int \left[\pi^{ik} \delta g_{ik} - \mathcal{H}_{WDW}(g_{ik}, \pi^{ik}) \delta t \right] d^3x. \quad (137)$$

The standard expression for this variation in terms of functional derivatives of S is

$$\delta S = \int \left[\frac{\delta S}{\delta t} \delta t + \frac{\delta S}{\delta g_{ik}} \delta g_{ik} \right] d^3x, \quad (138)$$

and comparison of the two equations yields the expression for momenta

$$\pi^{ik} = \frac{\delta S}{\delta g_{ik}} \quad (139)$$

and one more equation

$$\frac{\delta S}{\delta t} = -\mathcal{H}_{WDW}(g_{ik}, \pi^{ik}). \quad (140)$$

Together, these result in a functional differential equation for S ,

$$-\frac{\delta S}{\delta t} = \mathcal{H}_{WDW} \left(g_{ik}, \frac{\delta S}{\delta g_{ik}} \right) \quad (141)$$

that should be considered the Hamilton–Jacobi equation. However, in Hamiltonian geometrodynamics on the superspace of 3–metrics, the constraints (29) and (30) change the nature of this equation in such a way that, although being an important statement concerning the nature of time, it no longer performs the same functions as the Hamilton–Jacobi equation in mechanics, at least in the standard treatment of the subject. The standard approach is to first impose the constraints, which take the form

$$\mathcal{H}\left(g_{ij}, \frac{\delta S}{\delta g_{ik}}\right) = 0 \quad (142)$$

$$\mathcal{H}^i\left(g_{ij}, \frac{\delta S}{\delta g_{ik}}\right) = 0. \quad (143)$$

Equation (23) for \mathcal{H}_{WDW} then allows us to write the Hamilton–Jacobi equation (141) in the form

$$-\frac{\delta S}{\delta t} = N \mathcal{H}\left(g_{ik}, \frac{\delta S}{\delta g_{ik}}\right) + N_i \mathcal{H}^i\left(g_{ik}, \frac{\delta S}{\delta g_{ik}}\right) \quad (144)$$

which, together with (142) and (143) yields

$$\frac{\delta S}{\delta t} = 0. \quad (145)$$

From the dynamical point of view the partial functional derivative of S with respect to t is computed with only g_{ik} fixed. Equation (145) might seem strange, since (136) implies that S also depends on N and N_i , which might in turn depend on t . This does not create problems because, as is easy to see,

$$\frac{\delta S}{\delta N} = \frac{\delta S}{\delta N_i} = 0. \quad (146)$$

This means that equation (145) only states that S does not depend explicitly on time and that any dependence on t can emerge only through the components of g_{ik} . In other words, information about time is carried by the 3–metric of the slice, which, after proper refinements provides the basis for Baierlane, Sharp and Wheeler’s concept of intrinsic time. It also implies that the functional $S[t, g_{ik}]$ is a functional of the slice 3–metric only

$$S = S[g_{ik}]. \quad (147)$$

In any case, equation (140), although reminiscent of the Hamilton–Jacobi equation of mechanics, does not perform functions that are expected from the Hamilton–Jacobi equation. This statement requires more refined considerations, based on the observation that equations (143) can be written as (cf. (25))

$$\left(\frac{\delta S}{\delta g_{ij}} \right)_{|j} = 0, \quad (148)$$

which can be interpreted[4] as invariance of S with respect to the choice of coordinates (diffeomorphism invariance), and can be expressed by the statement that S is not even the functional of the 3–metric, but only of its diffeomorphically invariant part, called the 3–geometry ${}^{(3)}\mathcal{G}$

$$S = S[{}^{(3)}\mathcal{G}]. \quad (149)$$

This reduces the left hand side of equation (144), together with the second term on the right hand side of the same equation, to the kinematic statement expressed by (149). The only remaining part that can possibly have dynamic content can be written as

$$\mathcal{H} \left({}^{(3)}\mathcal{G}, \frac{\delta S}{\delta {}^{(3)}\mathcal{G}} \right) = 0. \quad (150)$$

This equation is identified as the Hamilton–Jacobi equation on the superspace of 3–geometries. It can be shown to encode all the dynamic content of the theory, but the equation is purely symbolic. In practice, this equation can be solved by picking for $S[g_{ik}]$ a form satisfying (148), and subsequently adjusting the functional parameters of this solution to satisfy (142). Alternatively, we could in principle solve equation (140) or (144) and then adjust parameters of the solution to satisfy (142) and (143). The first step of this procedure produces solutions of (140) both on and off the constraint shell, which involves providing information (including but not limited to information concerning shift and lapse) that gets eliminated when the solutions are restricted to the shell. This procedure is equivalent to the first one, but is not as practical in classical geometrodynamics. We will return to it later in the context of quantum geometrodynamics.

The transformation of variables described in section 3, and given by equations (31), result from replacing the variables g_{ik} with the new variables q_A . This modifies the equations but leaves intact the content of the theory and

its interpretation in the generic case. Slight modifications are required in non-generic cases, when the dimension of the configuration superspace is reduced.

On the superspace of q_A , the Hamilton principal functional

$$S = S[t, q_A] \quad (151)$$

is

$$S[t, q_A] = I_{extremum} = \int_{(t', q'_A)}^{(t, q_A)} \left(p^A \dot{q}_A - \widetilde{\mathcal{H}}(N, N_i, q_A, p^A) \right) d^3x dt \quad (152)$$

where $\widetilde{\mathcal{H}}$ is given by equations (79) – (82). The variational procedure on the superspace of q_A is similar to the one used on the superspace of g_{ik} , and yields the expression for the momenta

$$p^A = \frac{\delta S}{\delta q_A} \quad (153)$$

and the equation

$$\frac{\delta S}{\delta t} = -\widetilde{\mathcal{H}}(N, N_i, q_A, p^A). \quad (154)$$

Together these result in the functional differential equation for S ,

$$-\frac{\delta S}{\delta t} = \widetilde{\mathcal{H}} \left(N, N_I, q_A, \frac{\delta S}{\delta q_A} \right). \quad (155)$$

Following the logic of standard Hamiltonian dynamics this is considered to be the Hamilton–Jacobi equation. In view of (80), this equation can be written as

$$-\frac{\delta S}{\delta t} = N \mathcal{H} \left(q_A, \frac{\delta S}{\delta q_A} \right) + N_i \mathcal{H}^i \left(q_A, \frac{\delta S}{\delta q_A} \right). \quad (156)$$

The same line of reasoning as before leads to the constraints

$$\mathcal{H} \left(q_A, \frac{\delta S}{\delta q_A} \right) = 0 \quad (157)$$

$$\mathcal{H}^i \left(q_A, \frac{\delta S}{\delta q_A} \right) = 0, \quad (158)$$

which can also be expressed as

$$\frac{\delta S}{\delta N} = \frac{\delta S}{\delta N_i} = 0. \quad (159)$$

The three equations (156)–(158) imply, as before,

$$\frac{\delta S}{\delta t} = 0. \quad (160)$$

The last two equations can be summarized by the statement that $S[t, q_A]$ is, in fact, a functional of q_A only:

$$S = S[q_A]. \quad (161)$$

Additional considerations, similar to those leading to equations (148) – (150), are based on the observation that equations (158) can be written as (cf. (82))

$$\left(-2 Q_{AB} \frac{\delta S}{\delta q_A} M^B{}_{lm} G^{lmij} \right)_{|j} = 0 \quad (162)$$

which can again be interpreted, in the generic case, as invariance of S with respect to the choice of coordinates (diffeomorphism invariance). This can be expressed by the statement that S is not even a functional of q_A , but only of the diffeomorphically invariant information carried by q_A , the 3-geometry ${}^{(3)}\mathcal{G}$ (cf. equation (149)). This reduces the left hand side of equation (156), together with the second term on the right hand side of the same equation, to the kinematic statement expressed by (149). The only remaining part that can possibly have dynamic content can be expressed by (150), or in practice, by equation (157), which might be identified as the Hamilton–Jacobi equation on the configuration superspace of q_A . Using (81), we can also write

$$g^{-\frac{1}{2}} Q_{AB} \frac{\delta S}{\delta q_A} \frac{\delta S}{\delta q_B} - g^{\frac{1}{2}} R = 0. \quad (163)$$

All the comments made above concerning the solution of the Hamilton–Jacobi equation on the superspace of g_{ik} can be repeated with no essential change in the generic case of the superspace of q_A .

The non-generic, degenerate, cases require more attention. A straightforward application of variational principles in these cases might not lead to the desired result because of restrictions imposed on variations of the action.

These arise from the fixed form of expressions for g_{ik} , as well as restrictions on variations of the shift and lapse (often referred to as gauge conditions).

The geometrodynamical superspace has been described in section 4. It can be thought of as the configuration space of the true dynamical variables q_I . The Hamilton principal functional on the geometrodynamical superspace

$$S = S[t, q_I] \quad (164)$$

is given by

$$S[t, q_I] = I_{extremum} = \int_{(t', q'_I)}^{(t, q_I)} (p^I \dot{q}_I - \mathcal{H}_{DYN}) d^3x dt \quad (165)$$

where

$$\mathcal{H}_{DYN} = \widetilde{\mathcal{H}} - p^\mu \dot{q}_\mu, \quad (166)$$

and as before,

$$\widetilde{\mathcal{H}} = N \mathcal{H} + N_i \mathcal{H}^i. \quad (167)$$

The arguments of \mathcal{H}_{DYN} and $\widetilde{\mathcal{H}}$ are given by (132), with functions \dot{q}_I and p^μ given by equations (125) and (126). In the discussion of the Hamilton–Jacobi equation below, it is useful to describe the arguments of \mathcal{H}_{DYN} , $\widetilde{\mathcal{H}}$, \mathcal{H} , and \mathcal{H}^i as follows

$$\begin{aligned} \mathcal{H}_{DYN} &= \mathcal{H}_{DYN} \left(q_A, p^I, p^\mu \left(q_A, \dot{q}_I(q_A, p^I, \dot{q}_\mu, N, N_i), \dot{q}_\mu, N, N_i \right), N, N_i \right) \\ \widetilde{\mathcal{H}} &= \widetilde{\mathcal{H}} \left(q_A, p^I, p^\mu \left(q_A, \dot{q}_I(q_A, p^I, \dot{q}_\mu, N, N_i), \dot{q}_\mu, N, N_i \right), N, N_i \right) \\ \mathcal{H} &= \mathcal{H} \left(q_A, p^I, p^\mu \left(q_A, \dot{q}_I(q_A, p^I, \dot{q}_\mu, N, N_i), \dot{q}_\mu, N, N_i \right), N, N_i \right) \\ \mathcal{H}^i &= \mathcal{H}^i \left(q_A, p^I, p^\mu \left(q_A, \dot{q}_I(q_A, p^I, \dot{q}_\mu, N, N_i), \dot{q}_\mu, N, N_i \right), N, N_i \right). \end{aligned} \quad (168)$$

The variational procedure on the geometrodynamical superspace of q_I , similar to the one used before (variation of the endpoints), yields an expression for the momenta conjugate to the true dynamic variables q_I ,

$$p^I = \frac{\delta S}{\delta q_I}, \quad (169)$$

and the equation

$$\frac{\delta S}{\delta t} = -\mathcal{H}_{DYN} \left(q_A, p^I, p^\mu \left(q_A, \dot{q}_I(q_A, p^I, \dot{q}_\mu, N, N_i), \dot{q}_\mu, N, N_i \right), N, N_i \right), \quad (170)$$

which together result in the functional differential equation for S

$$-\frac{\delta S}{\delta t} = \mathcal{H}_{DYN} \left(q_A, \frac{\delta S}{\delta q_I}, p^\mu \left(q_A, \dot{q}_I \left(q_A, \frac{\delta S}{\delta q_I}, \dot{q}_\mu, N, N_i \right), \dot{q}_\mu, N, N_i \right), N, N_i \right) \quad (171)$$

This is considered to be the Hamilton–Jacobi equation. As before, the constraints are enforced by requirements that can be expressed as the functional differential equations

$$\frac{\delta S}{\delta N} = \frac{\delta S}{\delta N_i} = 0, \quad (172)$$

or equivalently (cf. section 4)

$$\mathcal{H} \left(q_A, \frac{\delta S}{\delta q_I}, p^\mu \left(q_A, \dot{q}_I \left(q_A, \frac{\delta S}{\delta q_I}, \dot{q}_\mu, N, N_i \right), \dot{q}_\mu, N, N_i \right), N, N_i \right) = 0. \quad (173)$$

$$\mathcal{H}^i \left(q_A, \frac{\delta S}{\delta q_I}, p^\mu \left(q_A, \dot{q}_I \left(q_A, \frac{\delta S}{\delta q_I}, \dot{q}_\mu, N, N_i \right), \dot{q}_\mu, N, N_i \right), N, N_i \right) = 0 \quad (174)$$

On the geometrodynamical superspace we have, in general,

$$\frac{\delta S}{\delta t} \neq 0 \quad (175)$$

even on the constraints shell. The reason for this is that, unlike the variational derivative of S with respect to t of equation (160) computed at fixed q_A , the derivative of equation (175) is computed at the fixed true dynamic variables q_I only.

As before, there are two ways to solve the equations for S , similar to those described above for the configuration superspace of q_A . The analog of the traditional choice would be to solve first the system of equations (171) and (174), which will partially fix the form of S . After that, solve (173), which is considered the proper Hamilton–Jacobi equation. However, such a choice mixes the true dynamical variables with embedding parameters at both stages and loses all the advantages of the similar procedure on the superspace of all q_A . This goes against the structure of the theory and implies the loss of any similarity to the Hamilton–Jacobi theory in mechanics. The second approach is to solve (171) first, considering it the Hamilton–Jacobi equation, and then using (173) and (174) to adjust the functional parameters of the solution. The first stage of the procedure provides, in principle, solutions of the Hamilton–Jacobi equation both on and off shell,

and thus includes information that disappears at the second stage when the solution is forced onto the constraint shell. In practice, this procedure is more involved than it appears, but in the end is equivalent to the procedure on the superspace of q_A described above.

To summarize, the Hamilton–Jacobi theories on the superspace of 3–metrics (parametrized by g_{ik} or q_A) and on the geometrodynamical superspace are found to be equivalent. Any possible difference is erased by forcing solutions onto the constraint shell. Identification of the Hamilton–Jacobi equation is, to an extent, arbitrary. The total content of the Hamilton–Jacobi theory is expressed by three equations, be it on the superspace of 3–geometries (equations (156), (157), and (158)) or on the geometrodynamical superspace (equations (171), (173), and (174)). Identification of one of the equations as the Hamilton–Jacobi equation is related to the choice of solution strategy and does not influence the final solution. It is mostly a matter of convenience and, as such, is problem dependent. The situation changes when the theory is quantized because it is the Hamilton–Jacobi equation that is converted into the wave equation, while the other two equations merely supply additional information.

6 Canonical Quantization

Canonical quantization of any field theory is based on the Hamilton–Jacobi representation of the theory and consists of steps that are determined by this representation. Instead of the classical system determined by the functional S , a quantum system determined by the state functional Ψ is introduced. The arguments of Ψ are assumed to be the same as the arguments of S , and are determined by the choice of the configuration superspace and by the assignment of the equation to be treated as the Hamilton–Jacobi equation.

The classical Hamilton–Jacobi equation is then transformed into the wave equation of the quantum theory. The functional derivatives of S in the Hamilton–Jacobi equation are replaced by operators acting on Ψ as follows

$$-\frac{\delta S}{\delta t} \implies i\hbar \frac{\delta}{\delta t} \tag{176}$$

$$\frac{\delta S}{\delta g_{ik}} \implies \hat{\pi}^{ik} = \frac{\hbar}{i} \frac{\delta}{\delta g_{ik}} \tag{177}$$

$$\frac{\delta S}{\delta q_A} \implies \hat{p}^A = \frac{\hbar}{i} \frac{\delta}{\delta q_A} \tag{178}$$

$$\frac{\delta S}{\delta q_I} \implies \hat{p}^I = \frac{\hbar}{i} \frac{\delta}{\delta q_I}. \quad (179)$$

The rest of the expressions participating in the Hamilton–Jacobi equations are interpreted as c–numbers acting either on the density ψ of Ψ , defined by

$$\Psi = \int \psi d^3x, \quad (180)$$

or the densities produced by functional differentiation of Ψ . For instance, if A is an expression that does not contain momenta, the action of the operator \hat{A} associated to it will be given by

$$\hat{A} \Psi = A \psi \quad (181)$$

$$\hat{A} \frac{\delta \Psi}{\delta q_A} = A \frac{\delta \Psi}{\delta q_A} \quad (182)$$

$$\hat{A} \frac{\delta^2 \Psi}{\delta q_A \delta q_B} = A \frac{\delta^2 \Psi}{\delta q_A \delta q_B} \quad (183)$$

and so on.

On the configuration superspace of 3–metrics, with equation (142) identified as the Hamilton–Jacobi equation, this procedure produces the Hamilton operator (we ignore here technical problems such as factor ordering)

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}(g_{ij}, \hat{\pi}^{ij}) \quad (184)$$

and the wave equation

$$\hat{\mathcal{H}}(g_{ij}, \hat{\pi}^{ij}) \Psi = 0, \quad (185)$$

known as the Wheeler–DeWitt equation.

Similarly, the change of variables that introduces the superspace of q^A , and the assignment of (157) as the Hamilton–Jacobi equation, produces the Hamilton operator

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}(q_A, \hat{p}^A) \quad (186)$$

and the wave equation

$$\hat{\mathcal{H}}(q_A, \hat{p}^A) \Psi = 0. \quad (187)$$

A more detailed form of this (Wheeler–DeWitt) equation, based on (163), is

$$-\hbar^2 g^{-\frac{1}{2}} Q_{AB} \frac{\delta^2 \Psi}{\delta q_A \delta q_B} - g^{\frac{1}{2}} \hat{R} \Psi = 0, \quad (188)$$

which is a second order functional differential equation. The second functional derivatives do not participate in the classical Hamilton–Jacobi theory; simple descriptions of these operations can be found in the literature[10] (more sophisticated treatments are also easy to find). This equation does not contain the time derivative of Ψ . For it to describe the time evolution of Ψ , time must be inserted into the equation by assigning a suitable function of the 3–metric. The resulting equation cannot be interpreted as a Schrödinger equation. Its structure is more similar to that of the Klein–Gordon equation, and this is the cause of some of the problems of time in the quantum picture associated with this approach.

The wave equation constitutes only a part of the theory. In addition, commutation relations are imposed on the coordinates of the configuration superspace (g_{ij} or q_A) and their conjugate momenta ($\hat{\pi}^{ij}$ or \hat{p}^A). The state functional is a distribution over the superspace of configurations, which makes inserting time into the quantum picture, as described above, more troublesome than it initially appears.

The quantum version of the auxiliary relations (143) and (158) (super-momentum constraints) are obtained by forming the operators

$$\widehat{\mathcal{H}}^i = \widehat{\mathcal{H}}^i(g_{ij}, \hat{\pi}^{ij}) \quad (189)$$

or

$$\widehat{\mathcal{H}}^i = \widehat{\mathcal{H}}^i(q_A, \hat{p}^A) \quad (190)$$

and enforcing the constraints by operator equations that can be written, for both cases, as

$$\widehat{\mathcal{H}}^i \Psi = 0. \quad (191)$$

In the case of the configuration superspace of q_A this equation is obtained from (162), and can be written as (again, ignoring factor ordering issues)

$$\left(-2 \frac{\hbar}{i} Q_{AB} \frac{\delta \Psi}{\delta q_A} M^B{}_{lm} G^{lmij} \right)_{|j} = 0. \quad (192)$$

The tendency to enforce the supermomentum constraints as operator equations, after the superhamiltonian constraint has been interpreted as the wave equation and written as an operator equation, is quite understandable. After all, the classical versions of these equations are merely different components of an equation that ensures energy–momentum conservation. However, once

implemented, these equations generate the problems of time and prevent the introduction of any concept of time or of geometrodynamical evolution[5].

Comparing this quantum gravity picture with quantum electrodynamics, we find that the equations in question play a role similar to that of the Lorentz gauge condition (charge conservation). In quantum electrodynamics the Lorentz gauge is imposed as a statement concerning expectation values rather than as an operator equation[11]. A similar approach in quantum gravity does not seem to be viable in the picture based on the Wheeler–DeWitt equation.

In contrast, canonical quantization on the geometrodynamical superspace of q_I , with equation (171) identified as Hamilton–Jacobi equation, appears to produce a quantum gravitational picture that avoids the problems of time, and which is similar in spirit to quantum electrodynamics. The geometrodynamical quantum Hamiltonian operator is based on the expression for \mathcal{H}_{DYN} on the right hand side of equation (171),

$$\widehat{\mathcal{H}}_{DYN} = \widehat{\mathcal{H}}_{DYN} \left(q_A, \widehat{p}^I, p^\mu \left(q_A, \dot{q}_I \left(q_A, \widehat{p}^I, \dot{q}_\mu, N, N_i \right), \dot{q}_\mu, N, N_i \right), N, N_i \right), \quad (193)$$

where \widehat{p}^I is given by

$$\widehat{p}^I = \frac{\hbar}{i} \frac{\delta}{\delta q_I}. \quad (194)$$

The wave equation of the geometrodynamical quantum theory (Schrödinger equation),

$$-i\hbar \frac{\delta \Psi}{\delta t} = \widehat{\mathcal{H}}_{DYN} \Psi, \quad (195)$$

is obtained from (171) in the standard way.

The commutation relations are imposed only on the true dynamical variables q_I , and their conjugate momenta \widehat{p}^I . Embedding variables and their velocities are c -numbers, and as such generate only trivial commutation relations. Time, suitable for describing quantum geometrodynamical evolution, can easily be introduced through these embedding variables.

The auxiliary conditions (173) and (174) of the classical theory are replaced by quantum conditions, based on operators derived from the left hand side of these equations. This is achieved using a procedure similar to the one used to form $\widehat{\mathcal{H}}_{DYN}$:

$$\widehat{\mathcal{H}} = \widehat{\mathcal{H}} \left(q_A, \widehat{p}^I, p^\mu \left(q_A, \dot{q}_I \left(q_A, \widehat{p}^I, \dot{q}_\mu, N, N_i \right), \dot{q}_\mu, N, N_i \right), N, N_i \right) \quad (196)$$

$$\widehat{\mathcal{H}}^i = \widehat{\mathcal{H}}^i(q_A, \widehat{p}^I, p^\mu(q_A, \dot{q}_I(q_A, \widehat{p}^I, \dot{q}_\mu, N, N_i), \dot{q}_\mu, N, N_i), N, N_i). \quad (197)$$

It has been mentioned above that imposing the constraints in operator form leads to numerous difficulties, including time problems. A weaker way to impose the constraints, similar to that of quantum electrodynamics, is to impose the constraints on expectation values.

The first step of this procedure is to solve the Schrödinger equation (195), with appropriate initial and boundary conditions, assuming that embedding variables are represented by c-numbers which are unknown but assumed to exist. The resulting solution Ψ_s is a functional that can be represented as

$$\Psi_s(t, q_I) = \int \psi_s(t, q_I; x^i) d^3x. \quad (198)$$

The action of any observable \widehat{A} on this produces a function on slices. The expectation of this observable over the solution can be written symbolically as

$$A_s = \langle \Psi_s | \widehat{A} | \Psi_s \rangle \quad (199)$$

and computed following the prescription

$$A_s = \int \psi_s^* \widehat{A} \Psi_s \mathcal{D}q_I \mathcal{D}q_J, \quad (200)$$

which includes functional integration over the geometrodynamical configuration superspace but not over slices.

Quantum constraints on the level of expectations can then be formed as

$$\langle \Psi_s | \widehat{\mathcal{H}} | \Psi_s \rangle = 0 \quad (201)$$

$$\langle \Psi_s | \widehat{\mathcal{H}}^i | \Psi_s \rangle = 0, \quad (202)$$

with $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{H}}^i$ given by (196) and (197).

7 Discussion.

In this paper we have extended the geometrodynamical quantization approach from quantum cosmological models[6] to the generic gravitational field. We find no inconsistencies in this broader setting. Quantum geometrodynamical evolution is determined by the Schrödinger equation (195), together with the constraints (201) and (202) imposed on expectations. Lapse and shift should

be either given explicitly or determined by four additional conditions, which determine the interpretation of the time parameter t . If the classical version of the conditions includes the true dynamical variables and their conjugate momenta, their quantum version is imposed in the form of expectations, just like the constraints.

Solving any particular problem can be thought of as a three-step procedure. First, the Schrödinger equation (195) is solved, assuming that the embedding variables and their time derivatives are unique, although unknown, functions of time and the spatial coordinates. The same is assumed of lapse and shift, unless they are given explicitly. Solving the the Schrödinger equation implies that appropriate boundary and initial conditions for the state functional on the geometrodynamical configuration superspace are given. The resulting solution is a functional that depends on the embedding variables and their time derivatives, as well as shift and lapse. The expectations of the constraints (201) and (202) over the solution of the Schrödinger equation are then computed. This procedure produces four differential equations for the four embedding variables if the lapse and shift are given explicitly. Alternatively, one can simply couple the four constraint conditions with four functional conditions for the lapse and shift. These procedures determine the meaning of time. The last step is to solve these equations and substitute the solutions for the embedding variables, their time derivatives, and lapse and shift into the expression for the state functional.

This whole quantization procedure for the general gravitational field parallels the quantum cosmological examples considered elsewhere[6]. Considerable complications are caused by the algebraic complexity of the expressions for the geometrodynamical Hamiltonian and the constraints, as well as by the functional nature of equations. These complications do not, however, stop the solution procedure in principle, although they do introduce a rather complex coupled system. This complexity places demands on our ability to gain a proper understanding of the problem, especially in setting appropriate initial and boundary conditions on the configuration superspace.

In principle, however, the geometrodynamical quantization formalism in the general setting retains all of the essential features previously illustrated in the context of homogeneous cosmologies.

References

- [1] J. Butterfield and C. J. Isham, “Spacetime and the Philosophical Challenge of Quantum Gravity”, gr-qc/9903072, (1999).
- [2] K. Kuchař, “Canonical Quantum Gravity”, in R. J. Gleiser, C. N. Kosameh and O. M. Moreschi, editors, *General Relativity and Gravitation* (IOP Publishing, Bristol, 1993).
- [3] C. W. Misner, “Feynman Quantization of General Relativity,” *Rev. Mod. Phys.* **29**, 497–509 (1957).
- [4] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (W. H. Freeman and Co., San Francisco, 1970).
- [5] Kuchař, K. V., “Time and Interpretations of Quantum Gravity” in *Proc. 4th Canadian Conference on General Relativity and Relativistic Astrophysics* eds. Kunstatter G, Vincent D E, and Williams J G (World Scientific; Singapore, 1992).
- [6] A. Kheyfets and W. A. Miller, “Quantum Geometrodynamics: Quantum-Driven Many-Fingered Time,” *Phys. Rev.* **D51**, n. 2, 493-501
- [7] A. Kheyfets, D. E. Holz and W. A. Miller, “The Issue of Time Evolution in Quantum Gravity,” *Int. J. Mod. Phys.* **A11**, n. 16, 2977-3002, (1996).
- [8] J. W. York, “Role of Conformal Three-Geometry in the Dynamics of Gravitation,” *Phys. Rev. Lett.* **28**, 1082-1085 (1972).
- [9] M. P. Ryan and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton Univ. Press, Princeton, NJ, 1975) Ch. 11.4.
- [10] I. M. Gelfand and S. V. Fomin, *Calculus of Variations* (Dover Pubns, Inc., New York, NY, 2000)
- [11] W. Heitler, *The Quantum Theory of Radiation* (Dover Pubns, Inc., New York, NY, 3rd ed., 1984).