On relativistic gravitation

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The existence of a gravitational analog of the magnetic field is demonstrated very easily by considering a special case.

In a recent article[1] we show that the origin of the torque on a magnetic dipole (as formed by a revolving point charge) in the magnetic field of a moving line charge needs different explanations when viewed in the rest frame of the line charge, depending on the dipole orientation. When the point charge is moving in a plane parallel to the line, Lorentz contraction or relativity of simultaneity has to be invoked. When it is moving in a plane perpendicular to the line, momentum conservation has to be used.

This interesting interplay between dynamics and geometry applies equally well to the gravitational interaction; an immediate consequence of this is the existence of a gravitational analog of the magnetic field, as well as the corresponding "magnetic" force on a moving mass. This aspect of relativistic gravitation has not received the attention it deserves, and is missing from standard texts.

Consider, as in Fig. 1, a point mass m moving perpendicularly towards a long line of rest mass density λ. In this frame, S', the line is at rest, and the instantaneous velocity of m is $\mathbf{v}' = (0,v',0)$. If we take the gravitational interaction to be governed by rest mass,[2] the force on m is

$$F'_x = 0, \quad F'_y = \frac{2Gm\lambda}{r}, \quad F'_z = 0, \tag{1}$$

where r is the distance from m to the line and m is the rest mass of the particle.

In order to transform F' to a frame, S, in which the velocity of the line is $(0,0,u)$, we use 4-forces[3] (see Fig. 2)

$$f = \gamma(\mathbf{v})\{\mathbf{F};(\mathbf{F}\cdot\mathbf{v})/c^2\},$$

where $\mathbf{F}$ is the 3-force on a particle, $\mathbf{v}$ is its velocity, and $\gamma(\mathbf{v}) = \gamma = (1 - v^2/c^2)^{-1/2}$. [We shall also abbreviate $\gamma(\mathbf{v}')$ by γ' and $\gamma(U)$ by Γ]. In our case,

$$f' = \gamma'(0, F'_y, 0; F'_y v'/c^2)$$

and so, since $f_y = f'_y$ (Lorentz transformation of a 4-vector),

$$F_y = (\gamma'/\gamma) F'_y = F'_y/\Gamma. \tag{2}$$

This last step follows from the easily derived result, $\gamma = \gamma'\Gamma$. Also, $f_x = f'_x$ gives $F_x = 0$, and finally, the Lorentz transformation gives

$$f_z = \Gamma(f'_z + f'_0 U) = F'_y \gamma' \Gamma \frac{v'U}{c^2} \tag{3}$$

or

$$F_z = \frac{f_z}{\gamma} = F'_y \frac{v'U}{c^2} = \frac{2Gm\lambda}{r} \Gamma \frac{v_y U}{c^2}. \tag{4}$$

Thus, in addition to the radial force on the particle due to the gravitational attraction by the rod, there is a force on the moving particle, parallel to the line.

That such a force is necessary is easily understood: The mass is accelerating toward the line, and so, since work is being done on it, its z component of momentum is increasing even though its z component of velocity is constant. Whether we interpret this as due to an increase in inertial mass γm or as an increase in $m(\partial z/\partial \tau)$ need not detain us here; the fact is that a force on m in the z direction is required to keep its velocity in that direction constant. That this force is the gravitational analog of the magnetic force is seen as follows.

In the S frame, the line mass density increases by a factor of Γ due to "length contraction."[1] (The rest masses of the line's constituent "atoms," which govern the interaction, remain the same, but they get closer together.) Thus the analog of the electric force on m is

$$F^e_y = \frac{2Gm\Gamma\lambda}{r} = \frac{F'_y}{\Gamma}\Gamma^2. \tag{5}$$

Therefore, there must be another force on m in the y direction, which from Eq. (2) is

$$F^m_y = \frac{F'_y}{\Gamma} - F^e_y = \frac{F'_y}{\Gamma}(1 - \Gamma^2)$$

$$= -\frac{F'_y}{\Gamma}\Gamma^2 \frac{U^2}{c^2} = -\frac{2Gm\lambda}{r}\Gamma \frac{v_z U}{c^2}. \tag{6}$$

The analog of the magnetic force on m is thus, from Eqs. (4) and (6),

$$\mathbf{F}^m = \frac{G}{c^2} \cdot \frac{2m\Gamma\lambda U}{r}(0,v_z,v_y).$$

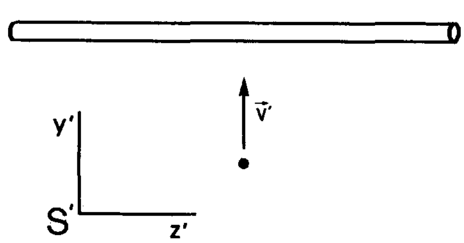

Fig. 1. Rest frame, S', of the line charge.

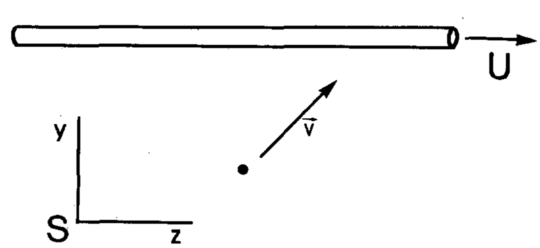

Fig. 2. The frame, S, in which the line charge has velocity U.

It is not difficult to show that this follows from the interpretation

$$\mathbf{F}^m = m\mathbf{v}\times\mathbf{h},$$

where

$$\mathbf{h} = \frac{G}{c^2}\cdot\frac{2\Gamma\lambda U}{r}\ \text{(out of paper, i.e., left-hand rule)}$$

is the gravitational analog of the magnetic field. One could follow through this program as in electromagnetic theory. There are however two differences, due to the absence of sign on the "gravitational charges" (like masses attract and there are no unlike masses). First, the gravito-magnetic interaction is always like the electromagnetic interaction between *unlike* charges. Second, as there is no way to compensate the gravito-static interaction between moving masses with the help of stationary "unlike" masses, there is no possibility of easy laboratory observability of **h** in the gravitational case since the "electric" analog, **g**, will always dominate by a factor of c^2. Nonetheless, on a cosmological scale, it is possible to imagine significant contribution of this interaction to, for example, angular momentum transfers between stellar or galactic systems.

[1]D. Bedford and P. Krumm, Am. J. Phys. (submitted).

[2]E. F. Taylor and J. A. Wheeler, *Spacetime Physics* (Freeman, San Francisco, 1966), p. 135.

[3]W. Rindler, *Essential Relativity* (Springer-Verlag, New York, 1977), 2nd ed., p. 88.

Speed dependence of the efficiency of heat engines

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We have studied the efficiency and power output of a simple model of an irreversible heat engine as a function of cyclic operating frequency. The model adopted is defined by a reversibly cycled working substance coupled to heat source and sink by thermally conducting walls. The maximum operating frequency corresponding to zero external power output is associated with the traversal of a limit cycle in the working substance. Closed form expressions for the maximum operating frequencies are derived for the special cases of isothermal and polytropic limit cycles.

The *quasistatic efficiency* of a heat engine yields a quantity which is independent of frequency or engine speed since it is assumed the cycle is traversed infinitely slowly. An engine which requires an infinite time to complete its cycle can produce no power. Although heat engines which run at nonzero speeds and produce power are obviously more interesting, progress in understanding their thermodynamics has come only comparatively recently. The basic reason for such slow progress lies in the fact that nonzero speed implies irreversible processes in the transfer of heat into the engine and in the working fluid itself. In the following article we concern ourselves only with the effects of irreversible heat transfer *into* the engine. The internal processes are assumed to be reversible and the working fluid is taken to be an ideal gas.

The point of this paper is to calculate the maximum angular frequency of operation for which power can be delivered by the engine. To calculate this quantity we exploit the fact that zero power output and zero engine efficiency occur together in the limit when the engine is operated such that the heat input to an ideal gas working fluid over a cycle equals the heat rejected. This limit corresponds to the traversal of some type of limit cycle in the space of thermodynamic coordinates. The limit cycle must be specified in order to define the type of engine operation for which the maximum operating frequency is to be calculated. In general, the lower operating frequency at which maximum power output occurs will depend on the type of operating cycle actually employed. The efficiency at maximum power output is of considerable practical interest and for a Carnot cycle a remarkably simple formula for it has been derived by Curzon and Ahlborn.[1] In this paper, however, we focus instead on characterizing the *maximum* operating frequency for selected types of operating cycles.

Figure 1 shows the efficiency and power output calculated for the model considered previously by Curzon and Ahlborn[1] and by Rubin.[2] The model consists of a Carnot cycle engine connected to the external heat source and sink by walls of conductance[3] κ. As a second example we consider a similarly connected engine with an elliptic cycle in which all quantities vary sinusoidally in time, such as considered by Richter and Ross.[4] Its cycle in the P–V plane is shown in Fig. 2 and the efficiency and power output are given in Fig. 3.

Despite the large differences in the basic cycles the results shown in Figs. 1 and 3 are similar qualitatively. This suggests a conjecture that the general expression for the efficiency of engines which transfer heat through conduct-