IX

Superspace and the Nature of Quantum Geometrodynamics

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ALLOWABLE HISTORY SELECTED OUT OF ARENA OF DYNAMICS BY CONSTRUCTIVE INTERFERENCE

Particle dynamics takes place in the arena of spacetime. Geometrodynamics takes place in the arena of superspace. One needs only to mount to this point of view to have the whole content of Einstein's theory spread out before his eyes, as in the outlook from a mountain peak, and as well quantum geometrodynamics as classical geometrodynamics. What is the nature of the landscape that we see from this height? What structures can we hope to build upon this landscape? And what kinds of mysteries are hidden in the mists beyond?

In looking at the terrain of dynamics, happily everyone by now is long accustomed to the Hamilton–Jacobi theory. It belongs entirely to the world of classical physics. Yet it carries one in an instant into the world of the quantum of action.

When better has one ever seen the transition from quantum to classical than in the theory of a particle in motion? Call the potential $V = V(x)$. Call the energy of the particle E. Then there is not the slightest hope of discussing the movement of the particle in space and time. Complementarity forbids! The wave function is spread out all over space. That one sees in no way more easily than through the semiclassical approximation for the probability amplitude function,

$$\psi_E(x, t) = \begin{pmatrix}\text{SLOWLY VARYING} \\ \text{AMPLITUDE FUNCTION}\end{pmatrix} \exp\left(\frac{i}{\hbar}\right) S_E(x, t) \tag{1}$$

It is of no help in localizing the probability distribution that the Hamilton–Jacobi function S has in many applications a value large in comparison with the quantum of angular momentum $\hbar = 1.02 \times 10^{-27}$ g-cm^2/sec. It is of no help that this "dynamical phase"—to give S another name—obeys the simple Hamilton–Jacobi law of propagation,

$$\begin{aligned}\frac{\partial S}{\partial t} &= H\left(\frac{\partial S}{\partial x}, x\right) \\ &= \left(\frac{1}{2m}\right)\left(\frac{\partial S}{\partial x}\right)^2 + V(x)\end{aligned} \tag{2}$$

And finally, it is of no help that the solution of this equation for a particle of energy E is extraordinarily simple,

$$S(x, t) = -Et + \int^x \{2m[E - V(x)]\}^{1/2}\, dx + \delta_E \tag{3}$$

The probability is still spread all over everywhere! There is not the slightest trace of anything like a localized world line, $x = x(t)$!

How old the idea of building wave packets out of monofrequency waves and how easy! The probability amplitude is now a superposition of terms, qualitatively of the form

$$\psi(x, t) = \psi_E(x, t) + \psi_{E+\Delta E}(x, t) + \cdots \tag{4}$$

Destructive interference takes place almost everywhere. The wave packet is concentrated in the region of constructive interference. There the phases of the various waves agree; thus

$$S_E(x, t) = S_{E+\Delta E}(x, t) \tag{5}$$

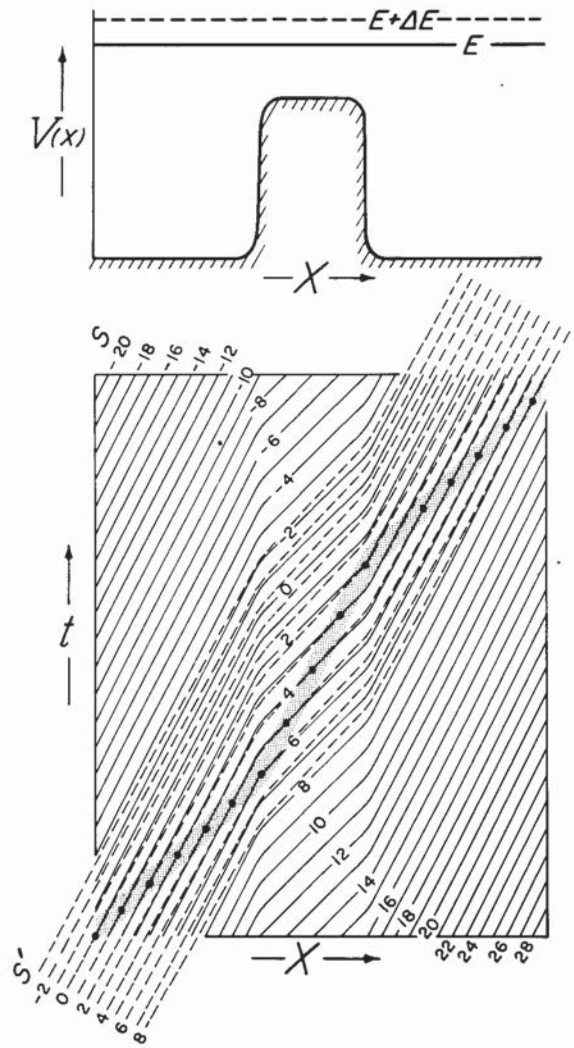

FIGURE 1. "Motion" and "world line" of a particle appear in quantum mechanics as the consequence of interference between wave trains that extend over all space. Above, potential energy as a function of distance for a model problem.

Below, smooth lines numbered −20, −18, ..., 28 are wave crests of probability amplitude function $\psi_E(x, t) \sim$ (slowly varying amplitude factor) $\exp{(i/\hbar)S(x, t)}$ for energy E. Dashed lines, same for energy $E + \Delta E$. Shaded area, region of constructive interference ("wave packet"). Black dots mark locus of classical world line ($S_{E+\Delta E} = S_E$).

At last a world line! And how easy to find the Newtonian motion from this condition of constructive interference:

$$0 = S_{E+\Delta E} - S_E$$
$$\Downarrow$$
$$0 = -t\,\Delta E + \int^x \Delta p_E(x)\,dx + (\delta_{E+\Delta E} - \delta_E)$$
$$\Downarrow$$
$$t = \int^x \frac{dx}{v_E(x)} + t_0 \qquad (\text{NEWTON}) \tag{6}$$

Here $v_E(x)$ denotes the velocity at the location x,

$$\frac{\Delta[2m(E-V)]^{1/2}}{\Delta E} = \frac{\Delta p_E}{\Delta E} \to \frac{\partial(\text{momentum})}{\partial E} = \frac{1}{v_E(x)} = \begin{pmatrix}\text{TIME TO COVER}\\ \text{A UNIT DISTANCE}\end{pmatrix} \tag{7}$$

and the quantity t_0 is an abbreviation for

$$\frac{\delta_{E+\Delta E} - \delta_E}{\Delta E} \to \frac{d\delta_E}{dE} \equiv t_0 \tag{8}$$

Marvelously, not one trace of the quantum of action appears in the final solution for the motion. Yet the quantum principle supplies the whole rationale and motivation for talking about "constructive interference." The quantum comes in only when one recognizes the finite spread of the wave packet (Fig. 1). Then the idea of a world line has to be renounced. A whole range of histories contribute to the propagation of the particle from start to finish. This is the way the real world of quantum physics operates!

Three Dimensions, Not Four

Similarly in geometrodynamics: Here the dynamic object is not spacetime. It is space. The geometrical configuration of space changes with time. But it is space, three-dimensional space, that does the changing. No surprise! In particle dynamics the dynamical object is not x *and* t, but only x. How to tell this to our friends in the world of mathematics? For so long they have heard us say that it was in default of the fourth dimension that Riemann could not have discovered general relativity. First there had to come special relativity and spacetime and the fourth dimension. Otherwise how could one have had any possibility to connect gravitation with the curvature of spacetime (Fig. 2)? This understood, how can physicists change their minds and "take back" one dimension? The answer is simple. A decade and more of work by Dirac, Bergmann, Schild, Pirani, Anderson, Higgs, Arnowitt, Deser, Misner, DeWitt, and others has taught us through many a hard knock that Einstein's geometrodynamics deals with the dynamics of geometry: of 3-geometry, not 4-geometry [1, 2].

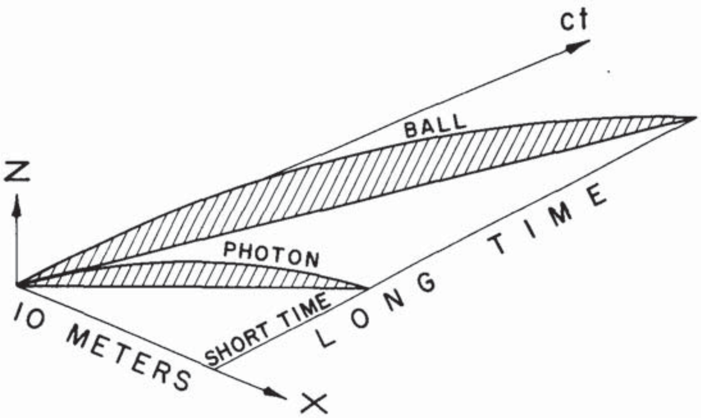

FIGURE 2. The track of the ball and the track of the photon through space (x, z plane) have very different curvatures, but in space time (x, z, ct space) the curvatures are comparable.

What is a 3-geometry? To simplify the question, rephrase it in an everyday context: What is a 2-geometry? Nothing illustrates a 2-geometry more clearly than an automobile fender. In whatever way coordinates are painted on its surface, in whatever way the points of that surface are named or renamed, the fender keeps the same 2-geometry. Similarly for a 3-geometry. In mathematical terms, a ${}^{(3)}\mathcal{G}$ is not a positive definite 3×3 metric; instead, it is an *equivalence class* of such metrics that are transformable, one into another, by diffeomorphisms.

Mere baggage is the right term not only for the points of the space, but also for the coordinates employed to label these points, and for the metric that tells the distance from each point to all its near neighbors. Behind all of that paraphernalia lies the real idea, the concept of 3-geometry, as solid and substantial as the 2-geometry of the fender. Down with "points"; up with "geometry"!

SUPERSPACE

One climbs up to the concept of geometry only to find a new height beyond—superspace [3]. Superspace is the manifold, a single point of which stands for an entire 3-geometry (Fig. 3). A 3-geometry stands at the halfway mark between point and superspace: risen though it is above the concept of point by abstractification, any one geometry of space counts as only a single point of superspace.

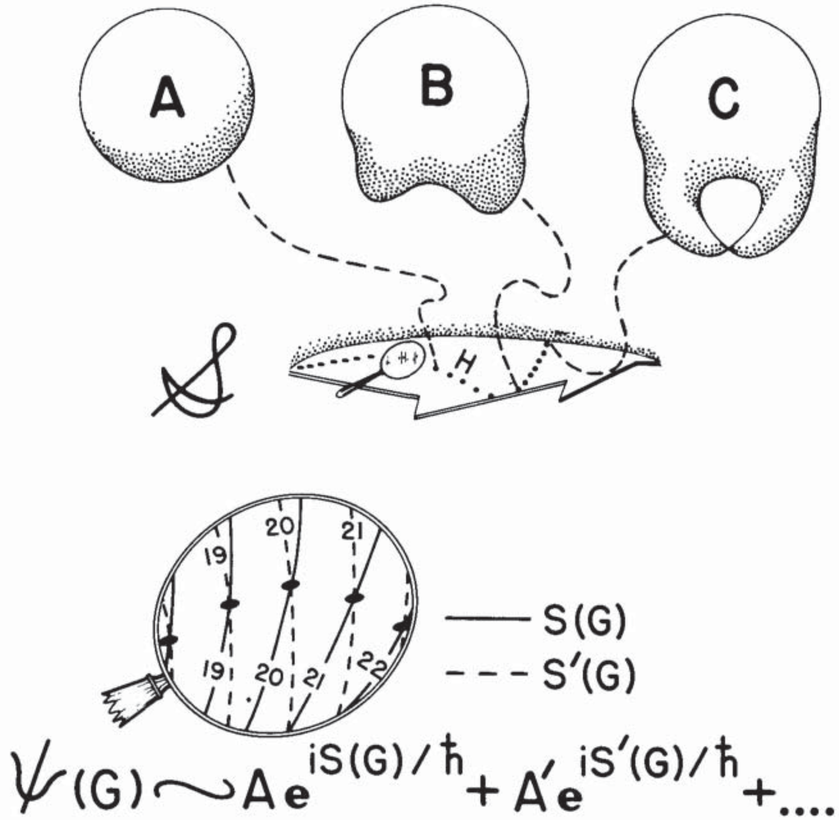

FIGURE 3. Superspace $\mathscr{S}$ is the manifold each of whose "points" A, B, C, ... is an abbreviation for one 3-geometry. A submanifold H of $\mathscr{S}$ is the "classical history of the geometry of space" when space has been started off under some particular set of dynamical initial conditions. In other words, H consists of all those spacelike 3-geometries that can be obtained as spacelike sections through one particular 4-geometry (that satisfies Einstein's classical field equations). The 3-geometries of the classical history H may alternatively but equivalently be distinguished from other 3-geometries by the fact that they satisfy the classical "condition of constructive interference" $S({}^{(3)}\mathscr{G}) = S'({}^{(3)}\mathscr{G}) = S''({}^{(3)}\mathscr{G}) = \cdots$ ("coincidence of wave crests" in lower magnified view of a region of superspace). In quantum theory the wave packet is not localized with unlimited sharpness. The probability amplitude ψ has significant values for 3-geometries at some small "distance" in superspace on each "side" of the classical history H; hence the "quantum fluctuations in the geometry of space" (depicted symbolically in more detail in Fig. 5).

Superspace is the arena for geometrodynamics, just as Lorentz–Minkowski space-time is the arena for particle dynamics (Table 1). The momentary configuration of the particle is an event, a single point in space-time. The momentary configuration of space is a 3-geometry, a single point in superspace.

TABLE 1 *Geometrodynamics Compared with Particle Dynamics*

Quality	Particle	Geometrodynamics
Dynamical entity	Particle	Space
Descriptors of momentary configuration	x, t (" event")	${}^{(3)}\mathscr{G}$ (" 3-geometry ")
History	$x = x(t)$	${}^{(4)}\mathscr{G}$ (" 4-geometry ")
History is a stockpile of configurations?	Yes. Every point on world line gives a momentary configuration of particle	Yes. Every spacelike slice through ${}^{(4)}\mathscr{G}$ gives a momentary configuration of space
Dynamic arena	Spacetime (totality of all points x, t)	Superspace (totality of all ${}^{(3)}\mathscr{G}$'s)

Is superspace a proper manifold? Its construction is simple enough: call a 3-geometry a " point," and put all such points together. Or why limit attention to 3-geometries with positive definite metric? Build "extended superspace"; call every 3-geometry a "point" even if its signature is not $+++$, and put all these "points" together. Does the resulting mathematical object possess a reasonable topology? Is it a true manifold? No! In a proper manifold each point has a neighborhood homeomorphic to an open set in a Banach space, and two distinct points have disjoint neighborhoods. Not so here. In an important investigation Michael Stern has shown [4] that for each point of extended superspace there exists another point, distinct from it, which nevertheless cannot be isolated from it by open sets. The topology is not Hausdorf. Extended superspace is not a manifold. It can hardly be regarded as an acceptable arena for dynamics.

Superspace, in contrast, Stern has shown, does have Hausdorf topology and does constitute a manifold. According to these mathematical considerations, *superspace is the proper arena for geometrodynamics.*

Physical considerations point to the same conclusion. To insist that a 3-geometry shall have positive definite metric is to guarantee that no light ray can traverse this 3-geometry. No physical effect can propagate from one point of the ${}^{(3)}\mathscr{G}$ to another. A physical quantity local to the one point and a physical quantity local to the other have zero reciprocal coupling. They commute. Such quantities lend themselves to simultaneous specification over the entire 3-geometry. No simpler example exists of a "complete observation" in quantum geometrodynamics. Closely related is the initial value problem of classical geometrodynamics.

"Observation?" Does not observation imply an observer? If there is an observer, does he not respond to every event on his past light cone? And consequently does not that past light cone with its $++0$ metric supply the appropriate geometry on which to specify physical conditions? No, and for two reasons. First, to know the state of the geometry on a past light cone, however completely, is still to be powerless to predict from Einstein's field equations the future of the geometry. Predict the 4-geometry within the past light cone? Yes [5]. Outside—to the future? No. Into the domain of the future, influences flow from afar without ever once impinging upon the light cone. "Demidynamics"—prediction into the past—one can do; full dynamics, no. The fault is in the choice of initial value hypersurface. The cone creates an unsymmetrical divide between past and future. How different from a spacelike initial value hypersurface! The initial value data on it allow a prediction of the complete geometrodynamical history, past and future.

Second, the original argument was mistaken that one should consider "a light cone converging upon an observer." The "observer" of dynamical theory is not and cannot be a single detector of events either in special relativity or in general relativity. Instead he "collects the printout" [6] from a multitude of detectors dotted densely about. Each of his detectors is sensitive only for an instant. These instants of sensitivity bear a spacelike relationship each to the other [7]. Collectively they define a spacelike hypersurface, a "simultaneity," a common moment of a rudimentary "time" variable that perhaps is not and certainly need not be specified any further. In its moment of sensitivity each detector responds to the appropriate influence: to particle proximity in particle physics, to local field strength in electrodynamics, and to local geometry in geometrodynamics. Whichever the dynamic entity, the measurements of it do not and cannot fully serve dynamic theory unless they span a spacelike hypersurface.

Distill this discussion! Where specify dynamically complete initial value data? On a spacelike 3-geometry, yes; on a null 3-geometry, no. What "points" belong to a topologically acceptable arena for geometrodynamics? Spacelike 3-geometries, yes; null 3-geometries, no. What a remarkable correspondence between dynamics and topology!

Wave Packet in Superspace and Its Propagation

So much for the dynamical entity, space; and so much for the arena in which the dynamics takes place, superspace; now for the dynamics itself. Not one trace of dynamics does one see when he examines the typical probability amplitude function, $\psi = \psi({}^{(3)}\mathscr{G})$, for it is spread all over superspace. No surprise! Already in classical theory the Hamilton–Jacobi function, $S = S({}^{(3)}\mathscr{G})$, is spread out over the manifold. Moreover, this "dynamical phase function" of classical geometrodynamics gives at once in the semiclassical approximation the actual phase of ψ, according to the formula

$$\psi({}^{(3)}\mathscr{G}) = \begin{pmatrix}\text{SLOWLY VARYING} \\ \text{AMPLITUDE FUNCTION}\end{pmatrix} \exp\left(\frac{i}{\hbar}\right) S({}^{(3)}\mathscr{G}) \tag{9}$$

—indication enough that ψ and S are both unlocalized! Dynamics first clearly becomes recognizable when sufficiently many such spread-out probability amplitude functions are superposed to build up a localized wave packet [8, 9]:

$$\psi = c_1\psi_1 + c_2\psi_2 + \cdots \tag{10}$$

Constructive interference occurs where the phases of the several individual waves agree [10]:

$$S_1({}^{(3)}\mathscr{G}) = S_2({}^{(3)}\mathscr{G}) = \cdots \tag{11}$$

The ${}^{(3)}\mathscr{G}$*'s compatible with these conditions of constructive interference constitute the classical geometrodynamical history of space* (Fig. 3): Marvelously, every 3-geometry that satisfies the conditions of constructive interference (11) can be obtained as a spacelike slice through a certain 4-geometry. More marvelously, this ${}^{(4)}\mathscr{G}$ satisfies the ten field equations of Einstein. In other words, *in addition to the principle of constructive interference one needs only the single equation of Hamilton and Jacobi for the "dynamical phase"* $S({}^{(3)}\mathscr{G})$ *to obtain all of classical geometrodynamics. The proof of this important point has been announced by Gerlach* [11].

The Hamilton–Jacobi equation itself was first given explicitly in the literature by Peres [12] on the foundation of earlier work by himself and others on the Hamiltonian formulation of geometrodynamics [13]:

$$g^{-1}(g_{ik}g_{jl} - \tfrac{1}{2}g_{ij}g_{kl})\left(\frac{\delta S}{\delta g_{ij}}\right)\left(\frac{\delta S}{\delta g_{kl}}\right) + {}^{(3)}R = 0 \tag{12}$$

Here the g_{ij} are the coefficients in the metric on ${}^{(3)}\mathscr{G}$ and g is the determinant of these metric coefficients. The quantity ${}^{(3)}R$ is the local value of the scalar curvature invariant of the geometry intrinsic to ${}^{(3)}\mathscr{G}$. The Hamilton–Jacobi function depends only upon the 3-geometry, and not upon how that 3-geometry is expressed in terms of metric coefficients in a particular coordinate patch. However, S is treated in (12) as if it depended upon the metric coefficients individually, $S = S(g_{11}, g_{12}, \ldots, g_{33})$. With this understanding, $\delta S/\delta g_{ij}$ denotes the functional derivative of S with respect to alterations in the function $g_{ij}(x, y, z)$. The fact that coordinates in the end have nothing to do with the matter can be stressed by rewriting (12) symbolically in the form

$$\boxed{\left(\frac{\nabla S}{\delta^{(3)}\mathscr{G}}\right)^2 + {}^{(3)}R = 0} \tag{13}$$

This "Einstein–Hamilton–Jacobi equation" contains all of classical geometrodynamics in regions where no "real" sources of mass-energy are present. This one equation carries the entire content of Einstein's ten field equations. Consequently, this equation has been checked and is subject to further check, in the same sense and to the same degree that the predictions of Einstein's field equations have been verified, and are subject to further verification. In the simplest version of particle physics a world line has a simple meaning: It is a stockpile consisting of all those points (x, t) that satisfy the condition of constructive interference. There are ∞^1 of these points on the world line, if we use an obvious though loose way of counting. Dynamics marks out these points and gives preference to them over all the ∞^2 points in the arena of particle dynamics. In geometrodynamics a 4-geometry has a similar significance. It is a stockpile consisting of all those 3-geometries that satisfy the condition of constructive interference. There are ∞^{∞^3} of these 3-geometries, obtainable by making a spacelike slice through the 4-geometry in one or another way:

$$t = t(x, y, z) \tag{14}$$

thus,

(a) ∞^3 points (x, y, z)
(b) ∞^1 choices for t at each of these points

hence

(c) ∞^{∞^3} choices of 3-geometry altogether (15)

Classical geometrodynamics marks out these ${}^{(3)}\mathscr{G}$'s and gives preference to them over the infinitely more numerous totality of ${}^{(3)}\mathscr{G}$'s to be found in the entire arena of geometrodynamics. That arena, superspace, contains ${}^{(3)}\mathscr{G}$'s to the number $(\infty^3)^{\infty^3}$, calculated most simply as follows:

(a) ∞^3 points; and, in a coordinate system that makes the metric diagonal,
(b) 3 diagonal components of the metric specifiable per space point;

and hence

(c) ∞^3 choices of the metric per space point;

and therefore

(d) $(\infty^3)^{\infty^3}$ choices of ${}^{(3)}\mathscr{G}$ altogether (16)

From this totality of conceivable 3-geometries one has to pick out and exhibit all the dynamically allowed 3-geometries before he has told in all fullness how space evolves with time. No new lesson! One has long known that time in general relativity is a many-fingered entity. The hypersurface drawn through spacetime to give one ${}^{(3)}\mathcal{G}$ can be pushed forward in time a little here or a little there or a little somewhere else to give one or another or another new ${}^{(3)}\mathcal{G}$. "*Time*" conceived in these terms *means* nothing more or less than *the location of the* ${}^{(3)}\mathcal{G}$ *in the* ${}^{(4)}\mathcal{G}$. In this sense "3-geometry is a carrier of information about time" [14].

"Spacetime," a Concept of Limited Validity

The child's toy can be removed from its box only to reveal another box and—that taken away—another box, and so on, until eventually there are dozens of boxes scattered over the floor. Or conversely the boxes can be put back together, nested one inside the other, to reconstitute the original package. The packaging of ${}^{(3)}\mathcal{G}$'s into a ${}^{(4)}\mathcal{G}$ is much more sophisticated. Nature provides no monotonic ordering of the ${}^{(3)}\mathcal{G}$'s. Two of the dynamically allowed ${}^{(3)}\mathcal{G}$'s taken at random will often cross each other one or more times. When one shakes the ${}^{(4)}\mathcal{G}$ apart, he therefore gets enormously more ${}^{(3)}\mathcal{G}$'s "spread out over the floor" than he might otherwise have imagined. Conversely, when one puts back together all of the ${}^{(3)}\mathcal{G}$'s allowed by the condition of constructive interference, he gets a structure with a rigidity that he might not otherwise have foreseen. This rigidity arises from the infinitely rich interleaving and intercrossing of clear-cut well-defined ${}^{(3)}\mathcal{G}$'s one with another. In summary, (1) the ${}^{(3)}\mathcal{G}$'s allowed by (11) are the basic building blocks; (2) their interconnections give ${}^{(4)}\mathcal{G}$ its existence, its dimensionality [15], and its "magic structure"; and (3) in this structure every ${}^{(3)}\mathcal{G}$ has a rigidly fixed location of its own.

How different from the textbook concept of spacetime! There the geometry of spacetime is conceived as constructed out of elementary objects, or points, known as "events." Here, by contrast, the primary concept is 3-geometry, and the event is secondary: (1) The event lies at the "intersection" of such and such ${}^{(3)}\mathcal{G}$'s. (2) Its timelike relation to some other ${}^{(3)}\mathcal{G}$ is determined by the structure of the ${}^{(4)}\mathcal{G}$, which in turn derives from the intercrossings of all the other ${}^{(3)}\mathcal{G}$'s.

Whether one starts with ${}^{(3)}\mathcal{G}$'s as primary and regards the "event" as a derived concept, or vice versa, might make little difference if one were to remain in the domain of classical geometrodynamics. It makes all the difference when one turns to quantum geometrodynamics.

There is no such thing as a 4-geometry in quantum geometrodynamics, and for a simple reason. No probability amplitude function $\psi({}^{(3)}\mathcal{G})$ can propagate through superspace as an indefinitely sharp wave packet. It spreads (Fig. 3). It has a finite probability amplitude in a domain of superspace of finite measure. This domain encompasses a set of $^{(3)}\mathcal{G}$'s far too numerous to accommodate in any one $^{(4)}\mathcal{G}$. One can express this situation in various terms. One can say that propagation takes place in superspace, not by following any one classical history of space, not by following any one $^{(4)}\mathcal{G}$, but by summation of contributions from an infinite variety of such histories. This extension of Feynman's concept of "sum over histories" has received special attention from Misner [16]. In whatever way one states the matter, however, the facts are clear. The $^{(3)}\mathcal{G}$'s that occur with significant probability amplitude do not fit and cannot be fitted into any single $^{(4)}\mathcal{G}$. That "magic structure" of classical geometrodynamics simply does not exist. Without that building plan to organize the $^{(3)}\mathcal{G}$'s of significance into a definite relationship, one to another, even the "time ordering of events" is a notion devoid of all meaning.

These considerations reveal that the concepts of spacetime and time itself are not primary but secondary ideas in the structure of physical theory. These concepts are valid in the classical approximation. However, they have neither meaning nor application under circumstances when quantum-geometrodynamical effects become important. Then one has to forgo that view of nature in which every event, past, present, or future, occupies its preordained position in a grand catalog called "spacetime." There is no spacetime, there is no time, there is no before, there is no after. The question what happens "next" is without meaning.

The Planck Length and Gravitational Collapse

Under everyday circumstances these unexpected consequences of the quantum principle never come into evidence. The characteristic dimension of quantum geometrodynamics is the Planck length [17], $(\hbar G/c^3)^{1/2} = 1.6 \times 10^{-33}$ cm. By comparison the normally relevant scale of any geometry of interest is stupendous. Negligible on the scale of the geometry is the quantum-mechanical spread of the wave packet in superspace. Consequently the dynamical evolution of the geometry can be treated in the context of classical geometrodynamics. Thus the geometries that occur with significant probability amplitude can be idealized to a good approximation as if confined to a region in superspace of zero thickness. The $^{(3)}\mathcal{G}$'s of this limited set are sufficiently small in number to fit together into a single $^{(4)}\mathcal{G}$. They are sufficiently large in number to reproduce every conceivable spacelike slice of that $^{(4)}\mathcal{G}$. In this approximation it makes good sense to speak of "the classical geometrodynamical history of space."

If the dynamics of geometry thus normally lends itself to classical analysis, there are two contexts where it does not. One is the final stage of gravitational collapse. The other is analysis of the microscopic quantum-mechanical fluctuations in the geometry of space and their consequences for physics generally.

Of all the applications of quantum geometrodynamics, none would seem more immediate than gravitational collapse [18]. Here according to classical general relativity the dimensions of the collapsing system in a finite proper time are driven down to indefinitely small values. The phenomenon is not limited to the space occupied by matter. It occurs also in the space surrounding the matter. In a finite proper time the calculated curvature rises to infinity. At this point classical theory becomes incapable of further prediction. In actuality, classical considerations go wrong before this point. A prediction that is infinity is not a prediction. The wave packet in superspace does not and cannot follow the classical history when the geometry becomes smaller in scale than the quantum-mechanical spread of the wave packet. Not a new phenomenon! Throughout physics one sees examples of a wave impinging upon a region of interaction of dimensions small compared to a wavelength. The outcome is scattering or diffraction. One speaks of a probability for this, that, or the other outcome of the interaction. The photon or phonon or other entity entering from one direction emerges in another. The concept of a deterministic world line may serve adequately during the phase of approach to the zone of interaction, and during the phase of regression. It is completely out of place during the phase of scattering. So here. The concept of a deterministic history of geometry, a well-defined ${}^{(4)}\mathscr{G}$, makes sense in the early phase of gravitational collapse, but has absolutely no application in the decisive phase. There "space-time" is nonexistent, "events" and the "time ordering of events" are without meaning, and the question "what happens after the final phase of gravitational collapse" is a mistaken way of speaking.

The correct way of speaking deals with the propagation of the probability amplitude $\psi({}^{(3)}\mathscr{G})$ in superspace. The semiclassical treatment of the propagation (Eqs. (9), (13)) is appropriate in most of the domain of superspace of interest for gravitational collapse. Not so in the decisive region. There, as in elementary problems of scattering, the mathematical analysis has to go straight back to the full and accurate wave equation for its foundation. What is the appropriate question to ask of the mathematics? One has learned how to formulate the right question, in the case of scattering, through long experience with the physics. However, if one had not had that physical experience, he would have learned out of the mathematical formalism itself to speak about incoming plane waves and outgoing spherical waves, and scattering amplitudes, and about all the other relevant concepts. Similarly in quantum geometrodynamics, where one has so much less experience. The mathematical formalism itself must serve as the final arbiter on how to pose the central question about gravitational collapse as well as how to answer it!

Quantum Fluctuations in Geometry of Space

"Quantum fluctuations in the geometry of space" is the other pressing field of application of quantum geometrodynamics. What a strange combination of words! Fluctuations are well known. The term "quantum fluctuations" carries a deeper meaning. It stands for a movement that can never be frozen out, however low the temperature. Such fluctuations are universal. In the hydrogen molecule both the separation of the two atoms and their relative momentum continually flucuate. Fixity of both would violate the uncertainty principle. In the frozen vaccuum of quantum electrodynamics the electric and magnetic fields both fluctuate. Were both of these dynamically conjugate field variables to vanish, the uncertainty principle would likewise fail. The same is true of quantum geometrodynamics. There the conjugate variables are the "intrinsic curvature" of three-dimensional space and the "extrinsic curvature," telling how this space is bent relative to the geometry of any 4-dimensional space-time that might envelop it. For both dynamic quantities to be stilled would equally contradict Heisenberg's uncertainty relation. Thus all space at the quantum scale of distances is the seat of the liveliest geometrodynamics, as it is also everywhere the scene of the most violent small-scale fluctuations in the electromagnetic field.

No prediction of quantum electrodynamics has been more impressively verified in the whole post-World War II era than these vacuum fluctuations in the electric field. Their perturbing influence on the motion of the electron (Fig. 4) accounts for the major component in the Lamb shift in the energy levels of the hydrogen atom [19].

In putting numbers to the fluctuations in geometry one is guided by the example of electromagnetism and, at a still earlier stage, by the example of the harmonic oscillator. For the oscillator the expression for the energy is

$$E = \left\langle \begin{pmatrix}\text{kinetic}\\ \text{energy}\end{pmatrix} + \begin{pmatrix}\text{potential}\\ \text{energy}\end{pmatrix} \right\rangle = \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 x^2$$

$$= \int \psi^*(x)\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \tfrac{1}{2}m\omega^2 x^2\right]\psi(x)\,dx \tag{17}$$

For a probability amplitude function $\psi(x)$ of Gaussian form and range a,

$$\psi(x) = \pi^{-1/4}a^{-1/2}\exp\left(-\frac{x^2}{2a^2}\right) \tag{18}$$

the expectation value of the energy is

$$E = \frac{\hbar^2}{4ma^2} + \tfrac{1}{4}m\omega^2 a^2 \tag{19}$$

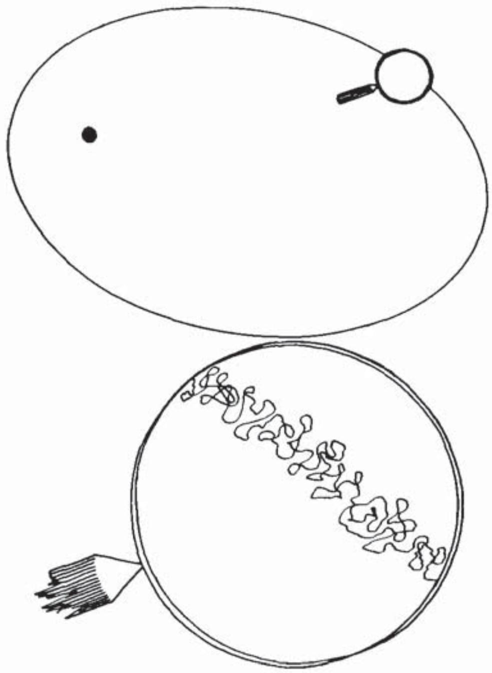

FIGURE 4. Symbolic representation of motion of electron in hydrogen atom as affected by fluctuations in electric field in vacuum ("vacuum" or "ground state" or "zero-point" fluctuations). The electric field associated with the fluctuation, $E_x(t) = \int E_x(\omega)e^{-i\omega t}\, d\omega$, brings about in the most elementary approximation the displacement $\Delta x = \int (e/m\omega^2)E_x(\omega)e^{-i\omega t}\, d\omega$. The average vanishes but the root mean square $\langle(\Delta x)^2\rangle$ does not. In consequence the electron feels an effective atomic potential altered from the expected value $V(x, y, z)$ by the amount

$$\Delta V(x, y, z) = \tfrac{1}{2}\langle(\Delta x)^2\rangle \nabla^2 V(x, y, z)$$

The average of this perturbation over the unperturbed motion accounts for the major part of the observed Lamb–Rutherford shift $\Delta E = \langle \Delta V(x, y, z)\rangle$ in the energy level. Conversely, the observation of the expected shift makes the reality of the vacuum fluctuations inescapably evident.

If quantum effects were absent, the first term would disappear. The minimum energy would be obtained by putting the oscillator at rest at the origin. However, in the real world of quantum physics such a sharp localization in position ($a = 0$) would make the effective wavelength zero and the momentum and kinetic energy arbitrarily large (divergent first term in Eq. (19)). The minimum energy, $E = \frac{1}{2}\hbar\omega$ ("half quantum" or "zero-point energy" or "fluctuation energy") is obtained for a range $a = (\hbar/m\omega)^{1/2}$ ("range of zero-point oscillations"). In other words, the oscillator in its ground state

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\left(\frac{m\omega}{2\hbar}\right)x^2\right] \tag{20}$$

can be said to "resonate" between locations in space ranging over a region of extent $\sim(\hbar/m\omega)^{1/2}$.

The electromagnetic field can be treated as an infinite collection of independent "field oscillators," with amplitudes $\xi_1, \xi_2, \ldots$. When the Maxwell field is in its state of lowest energy, the probability amplitude for the first oscillator to have amplitude ξ_1, and simultaneously the second oscillator to have amplitude ξ_2, the third ξ_3, and so on, is the product of functions of the form (20), one for each oscillator. When the scale of amplitudes for each oscillator is suitably normalized, the resulting infinite product takes the form

$$\psi(\xi_1, \xi_2, \ldots) = N \exp[-(\xi_1{}^2 + \xi_2{}^2 + \cdots)] \tag{21}$$

This expression gives the probability amplitude ψ for a configuration $\mathbf{B}(x, y, z)$ of the magnetic field that is described by the Fourier coefficients $\xi_1, \xi_2, \ldots$. One can forgo any mention of these Fourier coefficients if he so desires, however, and rewrite (21) directly in terms of the magnetic field configuration itself [20]:

$$\psi(\mathbf{B}(x, y, z)) = \mathcal{N} \exp\left(-\iint \frac{\mathbf{B}(x_1)\cdot\mathbf{B}(x_2)}{16\pi^3\hbar c r_{12}^2}\right) d^3x_1\, d^3x_2 \tag{22}$$

No longer does one speak of "the" magnetic field; he talks instead of the probability of this, that, or the other configuration of the magnetic field and this even under circumstances, as here, where the electromagnetic field is in its ground state.

It is reasonable enough under these circumstances that the configuration of greatest probability is $\mathbf{B}(x, y, z) = 0$. Consider for comparison a configuration where the magnetic field is again everywhere zero except in a region of dimension L. There let the field, subject as always to the condition $\operatorname{div} \mathbf{B} = 0$, be of the order of magnitude ΔB. The probability amplitude for this configuration will be reduced relative to the nil configuration by a factor $\exp(-I)$. Here the quantity I in the exponent is of the order $(\Delta B)^2 L^4/\hbar c$. Configurations for which I is large compared to 1 occur with negligible probability. Configurations for which I is small compared to 1 occur with practically the same probability as the nil configuration. In this sense, one can say that the fluctuations in the magnetic field in a region of extension L are of the order of magnitude

$$\Delta B \sim \frac{(\hbar c)^{1/2}}{L^2} \tag{23}$$

In other words, the field "resonates" between one configuration and another with the range of configurations of significance given by (23). Moreover, the smaller is the region of space under consideration, the larger are the field magnitudes that occur with appreciable probability.

Still another familiar way of speaking about electromagnetic field fluctuations gives additional insight relevant to geometrodynamics. One considers a measuring device responsive in comparable measure to the magnetic field at all points in a region of dimension L. One asks for the effect on this device of electromagnetic disturbances of various wavelengths. A disturbance of wavelength short compared to L will cause forces to act one way in some parts of the detector and will give rise to nearly compensating forces in other parts of it. In contrast, a disturbance of a long wavelength λ produces forces everywhere in the same direction, but of a magnitude too low to have much effect. Thus the field, estimated from the equation

$$\begin{pmatrix}\text{ENERGY OF ELECTROMAGNETIC}\\ \text{WAVE OF WAVELENGTH } \lambda \text{ IN A}\\ \text{DOMAIN OF VOLUME } \lambda^3\end{pmatrix} \sim \begin{pmatrix}\text{ENERGY OF ONE QUANTUM}\\ \text{OF WAVELENGTH } \lambda\end{pmatrix}$$

or

$$B^2\lambda^3 \sim \frac{\hbar c}{\lambda}$$

or

$$B \sim \frac{(\hbar c)^{1/2}}{\lambda^2} \tag{24}$$

is very small if λ is large compared to the domain size L. The biggest effect is caused by a disturbance of wavelength λ comparable to L itself. This line of reasoning leads directly from (24) to the standard fluctuation formula (23).

Fluctuations Superposed on Classical Background

Nothing says that the electromagnetic field has to be in its ground state. It can be excited by a distant wireless antenna so that locally it is oscillating up and down at 10^6 cycles/sec in the range $B = \pm 3.3 \times 10^{-8}$ gauss, for example (accompanying electric fields 1 millivolt/meter). By comparison with this deterministic classical history of the local magnetic field,

$$B_z = 3.3 \times 10^{-8} \text{ gauss} \cos 10^6 t \tag{25}$$

the quantum-mechanical fluctuation field of the same frequency (Eq. (24)) is quite negligible:

$$\begin{aligned} B &\sim \frac{6 \times 10^{-9}(\text{erg cm})^{1/2}}{(3 \times 10^4 \text{ cm})^2} \\ &\sim 10^{-17} \text{ gauss} \end{aligned} \tag{26}$$

Even the total effect of all the independent quantum fluctuations in the magnetic field is small as sensed by a detector of dimension $L \sim 1$ cm; thus, from Eq. (23),

$$\Delta B \sim \frac{6 \times 10^{-9}(\text{erg cm})^{1/2}}{(1\ \text{cm})^2}$$

$$\sim 6 \times 10^{-9}\ \text{gauss} \tag{27}$$

However, when attention is fixed on the field in a still smaller domain, say $L \sim 0.1$ cm or less, then the quantum fluctuations in the magnetic field dominate over the deterministic classical field of (25):

$$\Delta B > 6 \times 10^{-7}\ \text{gauss} \tag{28}$$

So much for the coexistence of classical fields and quantum fluctuations in electrodynamics!

Similar considerations apply in geometrodynamics [21]. Quantum fluctuations in the geometry are superposed on and coexist with the large-scale slowly varying curvature predicted by classical deterministic general relativity. Thus, in a region of dimension L, where in a local Lorentz frame the normal values of the metric coefficients will be $-1, 1, 1, 1$, there will occur fluctuations in these coefficients of the order

$$\Delta g \sim \frac{L^*}{L} \tag{29}$$

fluctuations in the first derivatives of the g_{ik}'s of the order

$$\Delta \Gamma \sim \frac{\Delta g}{L} \sim \frac{L^*}{L^2} \tag{30}$$

and fluctuations in the curvature of space of the order

$$\Delta R \sim \frac{\Delta g}{L^2} \sim \frac{L^*}{L^3} \tag{31}$$

Here

$$L^* = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.6 \times 10^{-33}\ \text{cm} \tag{32}$$

is the so-called [22] Planck length. It is appropriate to look at orders of magnitude. The curvature of space within and near the earth, according to classical Einstein theory, is of the order

$$R \sim \left(\frac{G}{c^2}\right)\rho \sim (0.7 \times 10^{-28}\ \mathrm{cm/g})(5\ \mathrm{g/cm^3})$$
$$\sim 4 \times 10^{-28}\ \mathrm{cm}^{-2} \tag{33}$$

This quantity has a very direct physical significance. It measures the "tide-producing component of the gravitational field" as sensed, for example, in a freely falling elevator or in a space ship in free orbit around the earth [23]. By comparison the quantum fluctuations in the curvature of space are only

$$\Delta R \sim 10^{-33}\ \mathrm{cm}^{-2} \tag{34}$$

even in a domain of observation as small as 1 cm in extent. Thus the quantum fluctuations in the geometry of space are completely negligible under everyday circumstances.

Even in atomic and nuclear physics the fluctuations in the metric,

$$\Delta g \sim \frac{10^{-33}\ \mathrm{cm}}{10^{-8}\ \mathrm{cm}} \sim 10^{-25}$$

and

$$\Delta g \sim \frac{10^{-33}\ \mathrm{cm}}{10^{-13}\ \mathrm{cm}} \sim 10^{-20} \tag{35}$$

are so small that it is completely in order to idealize the physics as taking place in a flat Lorentzian spacetime manifold.

The quantum fluctuations in the geometry are nevertheless inescapable, if we are to believe the quantum principle and Einstein's theory. They coexist with the geometrodynamical development predicted by classical general relativity. The fluctuations widen the narrow swathe cut through superspace by the classical history of the geometry. In other words, the geometry is not deterministic, even though it looks so at the everyday scale of observation. Instead, at a submicroscopic scale it "resonates" between one configuration and another and another. This terminology means no more and no less than the following: (1) Each configuration ${}^{(3)}\mathscr{G}$ has its own probability amplitude $\psi = \psi({}^{(3)}\mathscr{G})$. (2) These probability amplitudes have comparable magnitudes for a whole range of 3-geometries included within the limits (29) on either side of the classical swathe through superspace. (3) This range of 3-geometries is far too variegated on the submicroscopic scale to fit into any one 4-geometry, or any one classical geometrodynamical history. (4) Only when one overlooks these small-scale fluctuations ($\sim 10^{-33}$ cm) and examines the larger-scale features of the 3-geometries do they appear to fit into a single space-time manifold, such as comports with the classical field equations.

Extrapolate Geometrodynamics to the Planck Scale of Distances?

Is it not preposterous to apply existing theory in a realm of dimensions smaller than nuclear sizes by twenty powers of ten? What a fantastic extrapolation! Yet it is the tradition of theoretical physics to adopt what might be called a "strong bargaining posture." It is not the custom to give up any long-established principle without pushing it to the limit and finding out where, if anywhere, it goes wrong. A direct contradiction between a prediction and an observation, or between two points of principle, is ordinarily necessary if one is to have any solid ground for change, or even any indication where to make a change! The physicist does not have the habit of giving up something unless he gets something better in return.

To pursue systematically the consequences of quantum geometrodynamics is recommended not merely by the absence of any contradiction and by the absence of any more comprehensive theory. It is also made attractive by an example out of the past. Who in the 1850's, measuring the attraction between electric charges and testing the Coulomb law at distances from meters to millimeters, could have predicted that it would be proved valid in 1911 to 10^{-12} cm, in 1933 to 10^{-13} cm, and in 1963 to 10^{-14} cm? The fantastic extrapolatory power of basic physical theory always seems a miracle [24]!

Electrodynamics contains no natural length. Neither do general relativity (G, c) or the quantum principle ($\hbar$) individually. The union of geometrodynamics and the quantum does: $L^* = (\hbar G/c^3)^{1/2}$. The importance of what is essentially this length was stressed by Planck as early as 1899 [25]. He had taken up the study of blackbody radiation not least because it is universal: independent of the shape and size of the container, independent of the properties of its walls, and independent of the complexities of atomic, molecular, and solid state physics. In keeping with this search for the universal, he asked for standards of length, mass, and time that are independent of such special circumstances as the size of the planet we happen to inhabit, its period of rotation, and the density of the fluid that covers it. In excluding reference to special substances he found it natural also to exclude reference to special particles: both the electron, with its then known mass and charge, and all heavier entities—objects that even today pose unsolved structural problems. Excluding all else, he was left with the speed of light, the Newtonian constant of gravitation, and the constant newly discovered from the analysis of blackbody radiation as the quantities he was willing to accept as truly fundamental. Out of these three quantities there is but one way to construct a length. Planck's length, introduced to science before either special or general relativity, first acquired an understandable role in the context of quantum geometrodynamics as measure of the fluctuations in the geometry of space.

Accept seriously a length as small as 10^{-33} cm? Try to assess in any detail at all the physics that goes on at a scale of distances shorter by twenty powers of ten than the 10^{-13} cm of elementary particle physics? What could be more preposterous? Only three numbers are still more preposterous than 10^{20}: the factor of 10^{40} that distinguishes electric forces from gravitational forces; the 10^{40} from elementary particle dimensions to the estimated radius of the universe at the phase of maximum expansion; and the 10^{80} that furnishes an order of magnitude estimate as good as any that one knows for the number of particles in the universe. Eddington [26], Dirac [27], Jordan [28], Dicke [29], and Hayakawa [30] argue that it is unreasonable to think of such enormous numbers as having independent roles in physics. The correspondence between these numbers cannot be purely accidental, they stress [31]; there could hardly be a "regularity of the large numbers" if there were not a deep connection between cosmology, general relativity, and elementary particle physics. But where to begin in looking for this connection? Begin with asking why there are so and so many particles in the universe? Hardly. Physics can elucidate laws of motion, but it has proved powerless to explain initial conditions. Begin with asking why the universe has such and such dimensions? Again a matter of initial conditions, outside of the present scope of physics. Begin with trying to explain the charge structure of elementary particles, or the characteristic dimension of 10^{-13} cm? More hopeful, perhaps, but still beyond present power. No, of all the quantities coupled by the large numbers, one alone has a clear status within existing theory: the Planck length. Where else than here can one begin?

It could seem plausible to stop considering physics a little below 10^{-13} cm because accelerator budgets stop a little above $100 million a year. What good is it, one sometimes asks, to analyze what goes on if one has no way to observe it? Happily, experience has taught views less biased by mankind's temporary limitations. Wait until one had mastered the work hardening of metals before one took up the microscope and saw dislocations? Wait until one had explained dislocations before one started the study of atoms? Not so. The route to understanding did not go down the ladder, 1 cm $\rightarrow 10^{-4}$ cm $\rightarrow 10^{-8}$ cm, but up: 10^{-8} cm $\rightarrow 10^{-4}$ cm $\rightarrow$ 1 cm. One had to understand something about atoms before he could explain dislocations, and something about dislocations before he could uncover the rationale of work hardening. Is it possible that one similarly must have some perspective on what happens at 10^{-33} cm before one can find the rationale of particles and 10^{-13} cm [32]? Right or wrong [33], quantum geometrodynamics alone has any suggestions to offer on this point. What then does it say?

Every new perspective offered by this long established theory radiates out from the central prediction: *geometry fluctuates violently at small distances*. This concept opens new views on the nature of electric charge, on the nature of the vacuum, and on the nature of particles.

ELECTRICITY AS LINES OF FORCE TRAPPED IN THE TOPOLOGY OF SPACE

To arrive at a new vision of electricity it is enough to question an old view of topology: "Space is Euclidean in character at small distances." The view is reasonable enough for everyday purposes. Equally reasonable is the conception of the surface of the ocean as endowed with Euclidean topology—reasonable to one flying miles above it. To one in a small boat the opposite impression is inescapable. He sees the breaking waves and the foam. He knows that the surface is multiply connected at the scale of millimeters and centimeters. If the ocean is violent, the geometry of space on the Planck scale of distances is even more violent. Nowhere is there any region of calm. Moreover, if the equations of hydrodynamics are nonlinear, so are the equations of geometrodynamics. What a contrast to the linearity of electrodynamics! There the predicted fluctuations in potential

$$\Delta A \sim \frac{(\hbar c)^{1/2}}{L} \tag{36}$$

and in field

$$\Delta F \sim \frac{(\hbar c)^{1/2}}{L^2} \tag{37}$$

preserve always the same character, regardless of the smallness of the distances L to which one goes in his probing. There is no natural magnitude to mark off large fluctuations as different in nature from small ones. The contrary is the case for fluctuations in the metric, governed by the formula

$$\Delta g \sim \frac{L^*}{L} \tag{38}$$

Values of Δg comparable to unity and larger indicate changes in geometry so drastic that the word "curved space" is hardly adequate to describe them. "Changes in topology" seems a more reasonable description.

It is not so natural in mathematics as in physics to consider a transformation that alters one topology to another. An oscillating drop of water undergoes fission. The topology changes. A point marks the place of separation of the two masses of liquid. That point lacks the full neighborhood of points that characterizes a normal point. Such a critical point is ruled out from any proper manifold by the very definition of the term "manifold" in mathematics. Before the division, the surface of the drop constituted a manifold. After the division, it is again a manifold, consisting of two disparate pieces. At the instant of division it is not a manifold. But little attention does the drop pay to this distinction. It divides, despite all definitions. No more reason does one see in the definition of "manifold" against *space* changing *its* topology.

No principle is at hand that would give one topology perpetual preference over all others. On the contrary, the field equations of relativity are purely local in character. They make no statements at all about global topology, as Einstein himself emphasized more than once. Moreover, the whole character of physics speaks for the theme that "everything that can happen will happen." An alpha particle penetrates through a region classically forbidden to it; the side group on a chain molecule undergoes "hindered rotation"; and the umbrella structure of an ammonia molecule turns inside out despite the apparent contradiction to the law of conservation of energy. It is difficult to resist the conclusion that likewise the topology of space can change and does change.

If these general considerations are relevant, and if fluctuations alter the topology of space as well as its curvature [34], the consequences are decisive for the nature of the physics that goes on at small distances, and even for the nature of superspace itself. Superspace has to be broadened from the totality of positive definite 3-geometries built on one topology to the totality of positive definite 3-geometries built on the totality of all topologies. It has new implications to say that the probability amplitude $\psi({}^{(3)}\mathcal{G})$ is appreciable for a swathe of points in superspace, with a finite spread about the deterministic history of classical geometrodynamics (Fig. 5): Geometry in the small fluctuates not only from one microscopic pattern of curvature to another, but much more, from one microscopic topology to another. Moreover, those structures that are everywhere full of submicroscopic "handles" or "wormholes" are overwhelmingly more numerous than 3-geometries of simpler topology. In other words, space "resonates" between one foamlike structure and another [35]. The space of quantum geometrodynamics can be compared to a carpet of foam spread over a slowly undulating landscape. The undulations symbolize deterministic classical geometrodynamics. The continual microscopic changes in the carpet of foam as new bubbles appear and old ones disappear symbolize the quantum fluctuations in the geometry. The fluctuations change the microscopic connectivity of space itself. No longer is one entitled to take it for granted that space is Euclidean in the small.

Nowhere in physics does the structure of geometry in the small play a larger part than in electricity. Electric lines of force converge onto a region of space and none come out of it. Something strange must go on in that region. Either Maxwell's equations break down, or the region is filled with a special substance, an electric jelly, a magic fluid beyond further explanation. From the one picture or the other there has never been an escape. The reason is simple. The region is tacitly assumed to have Euclidean topology. Give up this assumption [36], but hold to Maxwell's field equations for empty space. Then the conclusion changes. The region in question must contain the mouth of at least one "handle" or "wormhole" (Fig. 6). Electric lines of force converge upon this mouth only to emerge from the other mouth, located somewhere else in space. One comes in this way to a new picture of electricity: *A classical geometrodynamical electric charge is a set of lines of force trapped in the topology of space.*

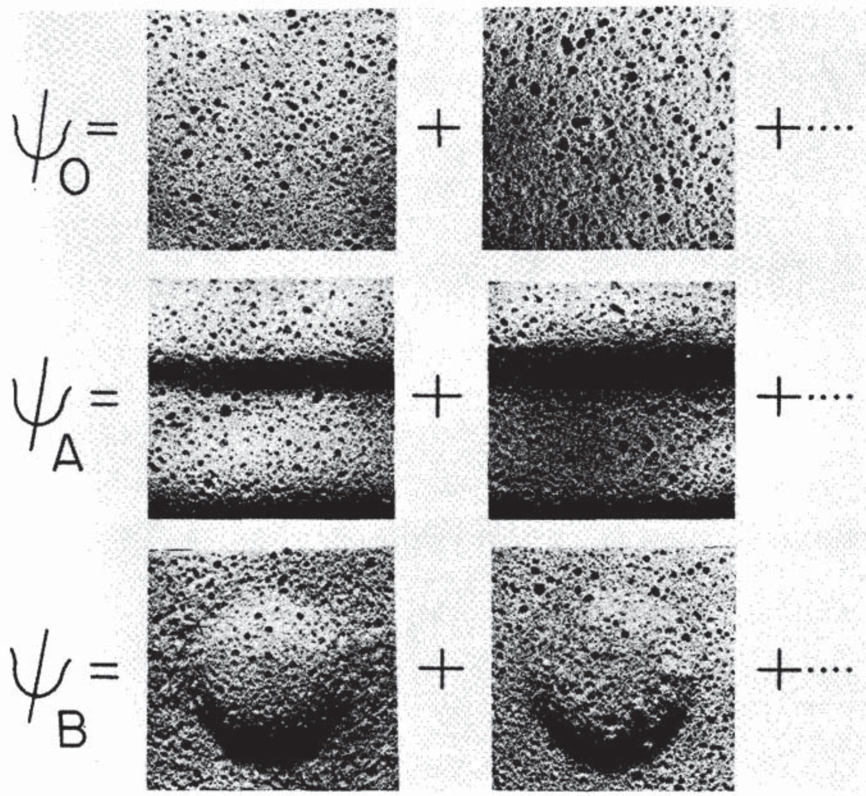

FIGURE 5. Symbolic representations of three alternative probability functions $\psi({}^{(3)}\mathscr{G})$. Above, ψ_0, normal fluctuations alone; middle, ψ_A, macroscopic classical gravitational wave plus superposed fluctuations; below, ψ_B, localized excitation plus superposed fluctuations. The complex number ψ gives the probability amplitude for the occurrence of the 3-geometry ${}^{(3)}\mathscr{G}$. Those 3-geometries that contribute most to the totalized probability are highly multiply connected at the scale of the Planck length ("foamlike structure of space in the small").

Someone can be imagined who first studies topology, next takes to heart Maxwell's equations for empty space, and then for the first time sees an electric charge. He takes it as experimental evidence that space must be multiply connected in the small (Table 2). Nothing prevents us from adopting the same point of view. On this view, *the occurrence of electric charges in nature is the single most impressive piece of evidence available today for the reality of the fluctuations that quantum theory predicts in the geometry of space, and suggests in the topology of space, at the Planck scale of distances.*

The "wormholes" predicted by quantum geometrodynamics are a property of all space, are submicroscopic, and they and the fluxes through

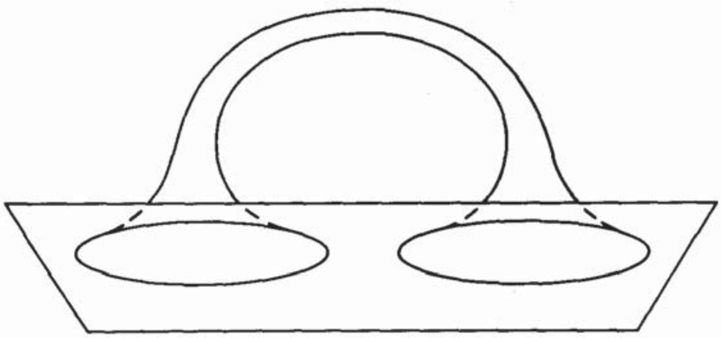

FIGURE 6. Classical geometrodynamical concept of charge as "electric lines of force trapped in the topology of space." The "wormhole" connects two regions in one otherwise nearly Eucildean space, not two different Euclidean spaces. The distance between the two mouths of the wormhole (1) via the nearly Euclidean space and (2) via the route through the wormhole are permitted by Einstein's field equations to be quite different, even in order of magnitude, contrary to what one might assume from the figure. There the geometry (one dimension suppressed!) is depicted for simplicity as if embedded in flat Euclidean 3-space. The third dimension (distance "off" of the geometry) is to be considered as unattainable as the stratosphere is unattainable to an ant crawling on the surface of the earth. An observer endowed with an instrument of inadequate resolving power sees one wormhole mouth as a positive charge, the other as a negative charge. To be contrasted with this classical picture of single indentifiable wormholes in the quantum-mechanical picture (Fig. 4) of a submicroscopic foamlike wormhole structure constantly fluctuating throughout all space.

them arise spontaneously, through quantum fluctuations. Nothing prevents one from considering also a single wormhole, of macroscopic dimensions, created *ab initio*, with a prescribed flux threading through it, and evolving deterministically in time in accordance with the classical field equations. However, this classical electric charge has not the slightest direct connection with the charges of the real world of quantum physics and requires no consideration here.

TABLE 2 *The Concepts of Electric Charge in Quantum and Classical Geometrodynamics, Compared and Contrasted*

Concept	Classical	Quantum
This type of electric charge interpretable as electric lines of force trapped in the topology of space?	Yes	Yes
Any "real" charge considered to be present?	No	No
Nature of topological trap?	Wormhole in 3-geometry	Wormhole in 3-geometry
Where are wormholes located?	Connecting opposite charges	Throughout all space
Dimension of wormhole	Enormous compared to 10^{-33} cm	Comparable to 10^{-33} cm
Status of wormhole?	Classical deterministic development with time; macroscopic size; this size determined by initial conditions; no observational evidence that any such macroscopic wormhole ever occurred, although by definition it is one of the solutions of Einstein's field equations	"Fluctuation"; consequence of fact geometry is not deterministic; that is, a consequence of the nonzero probability that exists for quite diverse 3-geometries; 3-geometries with wormholes almost everywhere (scale of 10^{-33} cm) are the most numerous ("foamlike 3-geometry"); wormholes do not have to be initiated; they occur naturally ("zero-point disturbance of the vacuum") and cannot be avoided

TABLE 2—(*Continued*)

Concept	Classical	Quantum
Wormhole pinches off?	Yes. Throat of radius a undergoes gravitational collapse in time of order a/c (10^{-10} sec for $a = 3$ cm)	Question strictly speaking is undefined; no meaning to deterministic small-scale geometrodynamics in context of quantum geometrodynamics; but in the loose way of speaking that is so useful in certain parts of quantum field theory one can say that "virtual wormholes" are continually being created and annihilated as "space resonates between one foamlike 3-geometry and another"
Charge as identified by flux through wormhole?	Does not change with time; a constant of the motion; not quantized	Flux through any one wormhole, like shape and dimensions of wormhole, subject to quantum-mechanical fluctuations; order of magnitude of this "fluctuation charge," $(\hbar c)^{1/2} \sim 12e$; not "constant" and not "quantized"
Relation of charge on an elementary particle to this kind of geometrodynamical charge?	Not the slightest direct connection with classical geometrodynamical electric charge	Not the slightest direct connection with the charge on an elementary "fluctuation wormhole." Particle viewed as a quantum state of collective excitation of the entire geometrical continuum ("geometrodynamical exciton"). Charge associated with this exciton to be calculated when one understands how to do the calculation!

THE ENERGY OF THE VACUUM

If one insight out of fluctuations in geometry has to do with the nature of electricity, a second has to do with the energy of the vacuum. Already in quantum electrodynamics one has long known from both observation and theory that when one examines a region of the vacuum of dimension L (1) the fluctuation energy is found to be of the order $\hbar c/L$, and (2) the effective density of energy of the fluctuation is found to be of the order $\hbar c/L^4$. Not known out of electrodynamics is any natural lower limit for the L values that come into consideration. In quantum geometrodynamics formulas of the same type hold, but there is now a natural cutoff: the Planck length. It is unreasonable to apply linear theory when L is of the order of $L^* = (\hbar G/c^3)^{1/2} = 1.6 \times 10^{-33}$ or less. In other words, there is a certain sense in which one can say: (1) Elementary fluctuations measure in energy up to $\hbar c/L^*$. The magnitude of this characteristic mass-energy is $\sim 10^{-5}$ g or $\sim 10^{28}$ eV; that is, about twenty powers of ten greater than the mass of an elementary particle and nine powers of ten greater than the energy of the most energetic cosmic ray that has ever been found. (2) These fluctuations take place throughout all space. (3) The density of the electromagnetic energy associated with these fluctuations [37] is of the characteristic order of magnitude $\hbar c/L^4 = c^5/\hbar G^2 \sim 10^{95}$ g/cm^3, stupendous in comparison with the 10^{14} g/cm^3 of nuclear matter. (4) The effective density of the "gravitational" or geometrodynamical wave energy [38] associated with these fluctuations is of the same order of magnitude.

Every observation shows that the net density of energy in space is negligible by comparison with these huge figures. The enormous positive energy must be compensated in some way. How the compensation comes about was a major concern of Niels Bohr over the years. A new approach to the problem shows up when one considers the fluctuations at the Planck scale of distances. Two such fluctuations, each of mass-energy $\sim(\hbar c/G)^{1/2} \sim 10^{-5}$ g, interacting gravitationally at the distance L^*, have a coupling energy that is negative,

$$E_{\text{grav}} = -\frac{Gm_1m_2}{r_{12}} \sim -\frac{G\left[\left(\frac{\hbar c}{G}\right)^{1/2}\right]^2}{L^*} \sim -\frac{\hbar c}{L^*} \tag{39}$$

and of the order -10^{-5} g. This coupling between neighboring "fluctuons" thus has such a sign and such a magnitude as to be appropriate for compensating the energies of the individual fluctuons. It would seem surprising if this mechanism did not play a dominant part in bringing about compensation of vacuum energies.

Despite all the mysteries that enshroud the compensation process, one conclusion stands out clear: *Individually the components of the vacuum energy are enormous, and collectively they compensate.*

A PARTICLE AS A GEOMETRODYNAMICAL EXCITON

Quantum fluctuations in geometry offer not only a new picture of electricity and a new view of the violence of the vacuum, but also a third vista: the concept of a particle as a quantum state of excitation of the geometry of space.

A bit of nuclear matter with its density of $\sim 10^{14}$ g/cm^3 is completely unimportant compared to the calculated density of $\sim 10^{95}$ g/cm^3 of the fluctuation energy of the vacuum. A particle means less to the physics of the vacuum than a cloud (10^{-6} g/cm^3) means to the physics of the sky (10^{-3} g/cm^3). No single fact points more powerfully than this to the conclusion that a "particle" is not the right starting point for the description of nature.

From the standpoint of geometrodynamics the primordial entity is not one particle, nor an intercoupled family of particle fields, but the geometry of empty space itself. On this view a particle is not itself a 10^{-33}-cm fluctuation in the geometry; instead, it is a fantastically weak alteration in the pattern of these fluctuations, extending over a zone containing very many such 10^{-33} regions. In brief, a particle is a quantum state of excitation of the geometry; it is a *geometrodynamical exciton.* In mathematical terms, the vacuum is described by one probability amplitude $\psi_0 = \psi_0({}^{(3)}\mathscr{G})$; and states where one or more particles are present are described by other functionals $\psi = \psi({}^{(3)}\mathscr{G})$.

On this interpretation elementary particle physics ranks as a new and beautiful kind of chemistry. First came the chemistry of atoms and molecules, marvelous in its complexities and also in its regularities, all built on one single simple dynamical entity, the electron. Then came "nuclear chemistry," with nuclear shapes and energies and reaction rates all going back for their explanation to the dynamics of another elementary dynamical entity, the nucleon. Today we deal in effect with a chemistry of the elementary particles themselves. Wonderful advances in the subject classify the particles into families, tie their masses together into mathematical regularities, and systematize their rates of transformation—without, however, revealing the identity of the elementary dynamical entity beneath it all. That entity, on the present view, is geometry itself.

When in the first half of the nineteenth century Berzelius proposed that chemical forces are a manifestation of electrical forces [39], he excited investigations by many workers that eventually discredited his hypothesis. The homopolar bond: how could the observed affinity be reconciled with the known repulsion between like electric charges? Homopolar forces, ionic forces, van der Waals forces, valence forces: how could all this variety of magnitudes and particularities possibly be compatible with electrical forces, pure and simple? The tide against the electrical interpretation of chemical forces only turned with the discovery of the electron by J. J. Thomson in 1897. The dynamical entity once identified, the unraveling of the mystery eventually had to follow. Still it was not easy for the imagination to grasp what organizing power the quantum principle possesses. In encounters in the mid 1920's more than one physicist told his colleague from the laboratory across the way, "Your chemistry is now passé. All that jumble can now be explained in terms of electrons and quantum numbers." In more than one case the then justified reply came back, "What makes you think your circular and elliptic orbits have anything to do with chemistry? Have you ever heard of the valence angles of ammonia or the tetrahedral bonds of carbon? Don't ever forget that electrical forces are electrical forces and chemical forces are chemical forces." The Coulomb law had to be supplemented by the concept of probability amplitude before Heitler and London could explain valence forces. Today no one doubts that the Schroedinger equation accounts in principle for all of chemistry. Yet no surer way could be found to stop the advance of chemistry than to require everyone to calculate the wave function of his new compound before making it. Not the contemplation of 600-dimensional configuration space, but the analysis of the regularities between molecule and molecule, proves to be the fruitful way to make progress. It can hardly be otherwise when the energy of binding is the very small difference between the very much larger total energies of the associated and dissociated states.

In "elementary particle chemistry" the art of analyzing regularities is already highly advanced [40], thanks not least to applications of group theory even more far-reaching than the applications made in the chemistry of molecules. On the other hand, the possibility to derive particle masses from first principles would seem even more remote, from the standpoint of geometrodynamics, than the possibility to calculate the binding energy of a complex molecule from first principles. The energy with which one hopes to end up, 10^{-27} g to 10^{-24} g, is smaller by twenty powers of ten than the characteristic energy of the theory with which one starts. Still it would seem unwise to discount in advance the ingenuity of available methods to calculate reliably small effects against enormously larger backgrounds, as for example in the case of superconductivity [41] ($\sim 10^{-4}$ eV versus ~ 10 eV).

How can the geometrodynamical interpretation of particles (Table 3) be tested? Sooner to be expected than quantitative calculations are qualitative predictions and conceptual developments. Of such developments none gives more incentive than gravitational collapse [18] to believe in a tie between particles and geometry; and none gives more encouragement to believe in the relevance of the Planck length than the concept of charge as electric lines of force trapped in the topology of space.

TABLE 3 *Quantum–Geometrodynamical Interpretation of Particles and Forces*

Ultimate dynamical object	Not an electron or any other kind of particle, but geometry itself
Geometry of space	Not unique and classical, but everywhere resonating at the scale of the Planck length between configurations of varied submicroscopic curvature and varied topology
Topology of space	Those geometries that occur with appreciable probability amplitude are highly multiply connected throughout all space ("foamlike structure")
Particle	Not a foreign and physical entity moving about within the geometry of space, but a quantum state of excitation of that geometry itself; as unimportant for the physics of the vacuum as a cloud is unimportant for the physics of the sky. Not a localized ripple in the geometry, not a submicroscopic "wormhole" in the geometry of space at the Planck scale of distances, but an excitonlike change in the phase relations in the probability amplitude for a very large number of such wormholes
Charge	Not a place where Maxwell's equations fail, not a mysterious "foreign and physical" jelly introduced into geometry from outside, but "lines of electric force trapped in the topology of space" (Table 2 and Problem 2)
Spin	Not a dynamical object added to geometry but the nonclassical two-valuedness associated with geometry itself—because distinct probability amplitudes attach to a multiply connected 3-geometry endowed with alternative triad fields or "spin structure" (Problem 2)
Force	Strong forces, weak forces, and intermediate forces no more distinct in their character than van der Waals forces, ionic forces, and valence forces; not themselves primordial, but the residual effect of percentage-wise negligible changes in the enormous density of the energy of the zero-point fluctuations taking place in the geometry of space at small distances

Widening vistas open out for further investigation [42]. (1) What can one give in the way of simple principles to shortcircuit all the usual derivations of Einstein's field equations and pass in one leap from postulates to the Hamilton–Jacobi equation itself? (2) What deeper insights can one win into the structure of superspace? And (3) at what new point in geometrodynamics does one draw that old line between dynamic law and initial conditions that one sees throughout all of physics?

These issues are the foothills. The mountain looms above them: Is a particle a state of excitation of the geometry of space?

Einstein, above his work and writing, held a long-term vision: There is nothing in the world except curved empty space [43]. Geometry bent one way here describes gravitation. Rippled another way somewhere else it manifests all the qualities of an electromagnetic wave. Excited at still another place, the magic material that is space shows itself as a particle. There is nothing that is foreign and "physical" immersed in space. Everything that is, is constructed out of geometry. This is the dream. Is this dream coming to life?

Note: This article is the revised and extended version of reports made at the Académie Internationale de Philosophie des Sciences, Oberwolfach/Freiburg im Breisgau, July, 1966; the International School of Nonlinear Mathematics and Physics, Munich, July, 1966; the Society of Engineering Science, Raleigh, North Carolina, October, 1966; the Colloque Internationale sur Fluides et Champ Gravitationel en Relativité Générale, Paris, June, 1967; and the Battelle Rencontres in Mathematics and Physics, Seattle, July, 1967. It appears in the proceedings of the second, third, and fifth organizations in English and in the proceedings of the fourth in French. Appreciation is expressed to the organizers of these conferences for their hospitality and to many colleagues for discussions and advice, among them especially Y. Choquet, Bryce DeWitt, Cecile DeWitt, L. Ehrenpreis, U. Gerlach, H. Leutwyler, C. W. Misner, and M. Stern.

PROBLEM 1. "DERIVATION" OF "EINSTEIN–HAMILTON–JACOBI EQUATION"

If one did not know the Einstein–Hamilton–Jacobi (EHJ) equation (Eq. (12) or (13)), how might one hope to derive it straight off from plausible first principles, without ever going through the formulation of the Einstein field equations themselves? To find one such direct derivation of the EHJ equation would not seem rash to hope for when one already knows five ways to derive the field equations in their traditional form: (1) Einstein's original derivation based upon the correspondence with Newtonian gravitational theory; (2) Weyl's derivation based upon enumeration of all covariant differential operations on the metric tensor that are linear in the second derivatives and that contain no higher derivatives; (3) Hilbert's derivation from a variational principle; (4) Cartan's capture [44] of the geometrical content of the field equations [45]; and (5) the derivation by Gupta, Thirring, and Feynman starting with the theory of a field of spin two and mass zero in flat space.

The central starting point in the proposed derivation would necessarily seem to be "imbeddability." On the basis of experience one can reasonably ask that the desired Hamilton–Jacobi equation, of course combined as always with the conditions of constructive interference, should pick out ${}^{(3)}\mathscr{G}$'s that will fit together into a ${}^{(4)}\mathscr{G}$.

In what way would one violate the "condition of imbeddability" if, for example, one left the differential operator unchanged in (13) but replaced the term ${}^{(3)}R$ by the square or by some other function of this curvature scalar? Without directly answering this question, one can say that the special form of the equation is governed in the most direct possible way by the four-dimensional character of space-time. When geometrodynamics is put into simplest terms, it comes out as the statement

(CURVATURE) = (DENSITY OF MASS-ENERGY)

In the present considerations one looks apart from situations where there is any "real" mass-energy present. To write the equation as if there were such a term on the right is, however, a reminder that one is speaking about a tensorial quantity, dependent therefore not only upon one's choice of point, but also upon the choice of direction, or unit 4-vector, at that point. This circumstance illuminates the meaning of the curvature term on the left [44, 45]. It has to do with curvature of the 4-geometry in a tangent plane normal to the 4-vector in question. Good! But we are considering conditions, not merely at one point, but at a threefold infinity of points, that form a spacelike 3-geometry. This 3-geometry is not necessarily "free of extrinsic curvature" (vanishing "tensor K_{ij} of extrinsic curvature" or vanishing "second fundamental form") at the point under study. If it is, excellent! Then the desired curvature is given directly by the scalar curvature invariant ${}^{(3)}R$ of the geometry intrinsic to ${}^{(3)}\mathscr{G}$ at the point in question. However, any nonvanishing "extrinsic curvature of the 3-geometry relative to the enveloping 4-geometry" makes an additional contribution to the scalar curvature intrinsic to ${}^{(3)}\mathscr{G}$. One has to correct for this contribution before one secures a proper measure,

$$(\mathrm{Tr}\,\mathbf{K})^2 - \mathrm{Tr}\,\mathbf{K}^2 + {}^{(3)}R$$

of the curvature of the 4-geometry itself in the tangent plane in question (Gauss–Codazzi formula). Why this special bilinear expression in the tensor K_{ij} of extrinsic curvature and not some other one [46]? The rationale shows most directly when one considers the special case where the 4-geometry itself is flat. In this case the "correction terms" in the expression for the desired curvature must exactly compensate ${}^{(3)}R$. Let the 3-geometry be imbedded in the 4-geometry with "principal radii of curvature" at the point in question equal to ρ_1, ρ_2, and ρ_3. Then the scalar curvature invariant of the intrinsic geometry is ${}^{(3)}R = -2/\rho_2\rho_3 - 2/\rho_3\rho_1 - 2/\rho_1\rho_2$. The difference in sign here compared to the familiar ${}^{(3)}R = 6/a^2$ for a 3-sphere of radius a arises from the fact that the "radius of curvature" is being measured in a 4-geometry of signature $- + + +$. The tensor of extrinsic curvature is

$$\mathbf{K} = \begin{Vmatrix} \frac{1}{\rho_1} & 0 & 0 \\ 0 & \frac{1}{\rho_2} & 0 \\ 0 & 0 & \frac{1}{\rho_3} \end{Vmatrix}$$

It is desired to build up out of this tensor an expression that (1) is bilinear in the reciprocals of the ρ_i; (2) contains no terms in $1/\rho_1^2$, etc.; and (3) compensates ${}^{(3)}R$. These three requirements lead uniquely to the stated formula. The considerations outlined here by no means come close to providing a derivation of the Einstein–Hamilton–Jacobi equation, but they perhaps suggest some of the factors that may play a natural part in such a derivation.

Subscript on Relation of Hamilton–Jacobi Method to Conventional Analytic Solutions of Field Equations

To look toward the Hamilton–Jacobi equation as an illuminating way to found general relativity is not to look away from the field equations as a way to solve problems in general relativity. The Hamilton–Jacobi equation is divorced from the equations of motion no more in geometrodynamics than in particle mechanics. Even in the most elementary problem the connection between the two methods is inescapable. Yes, one can solve Lagrange's equation

$$m\frac{d^2x}{dt^2} + \frac{\partial V}{\partial x} = 0$$

by direct numerical integration. Yes, one can translate the problem into Hamilton–Jacobi formalism, write down the equation

$$-\frac{\partial S}{\partial t} = \left(\frac{1}{2m}\right)\left(\frac{\partial S}{\partial x}\right)^2 + V(x, t)$$

and integrate *it* by the most elementary numerical methods (replacement of (x, t) continuum by lattice space, $t = m\tau$, $x = n\delta$, with m and n integers; replacement of partial differential equation by difference equation). However, the most penetrating method to integrate the partial differential equation has long been known to go straight back to the Lagrange equation itself for its start. Not everywhere does one at first evaluate $S(x, t)$, but only along a classical world line or history H,

$$x = x_H(t)$$
$$\dot{x} = \frac{dx_H(t)}{dt}$$

thus

$$s(t) = S(x_H(t), t) = \int_{t_0}^{t} [\tfrac{1}{2} m\dot{x}_H{}^2 - V(x_H(t), t)]\, dt$$

or, in a more general problem, with Lagrange function L,

$$s(t) = S(x_H(t), t) = \int_{t_0}^{t} L(\dot{x}_H, x_H, t)\, dt$$

From a knowledge of S along the world line one goes to a knowledge of S in a narrow band on either side of the world line by using the relations

$$\begin{pmatrix}\text{RATE OF CHANGE}\\ \text{OF ACTION } S\\ \text{WITH POSITION } x\end{pmatrix} = (\text{MOMENTUM}) = \frac{\partial L}{\partial \dot{x}}\,(\text{GENERAL}) = m\dot{x}_H(t)\,(\text{HERE})$$

and

$$-\begin{pmatrix}\text{RATE OF CHANGE OF}\\ \text{ACTION } S \text{ WITH}\\ \text{TIME } t\end{pmatrix} = (\text{ENERGY}) = \dot{x}\,\frac{\partial L}{\partial \dot{x}} - L\,(\text{GENERAL})$$
$$= \tfrac{1}{2} m\dot{x}_H{}^2 + V(x_H(t), t)\,(\text{HERE})$$

Thus at the point

$$x = x_H(t^*) + \delta x$$
$$t = t^* + \delta t$$

a little way off the world line, the Hamilton–Jacobi function ("classical phase"; $\hbar$ times phase of quantum-mechanical wave function) is

$$S(x, t) = s(t^*) + \left(\frac{\partial S}{\partial x}\right)\delta x + \left(\frac{\partial S}{\partial t}\right)\delta t$$

An identical procedure gives $S_{\text{new}}(x, t)$, a new solution with infinitesimally different initial conditions, throughout a band extending on either side of an infinitesimally different world line, $S_{\text{new}}(x, t)$. The "condition of constructive interference" between the new "wave" and the old "wave,"

$$S(x, t) = S_{\text{new}}(x, t)$$

gives back at once the original solution of the equation of motion—and all the history H that goes with that solution. So too in geometrodynamics!

One can use a known solution of the Einstein field equations to find the Hamilton–Jacobi function $S({}^{(3)}\mathscr{G})$ throughout a certain narrow swathe through superspace. For this purpose one will find it easiest to start with one of the well-known analytic solutions of Einstein's field equations. Let the 4-geometry be written in the form

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

where the metric coefficients $g_{\alpha\beta}$ are certain known functions of the four coordinates x^μ. The 4-geometry expressed in this way summarizes the dynamical history H of the 3-geometry of space. It also defines a submanifold—call it H—of the superspace $\mathscr{S}$. How now to determine the values taken on by the Hamilton–Jacobi function throughout the submanifold? First define the equivalent of the starting time t_0 in the one-particle problem. This equivalent is not a single number t_0, but in general a different number $t_0(x, y, z)$ for each point in 3-space; or better stated, it is a spacelike initial value hypersurface σ_0 slicing through the given ${}^{(4)}\mathscr{G}$. Next, give the equivalent of the point $x_H(t_0)$, t_0 along the classical history. It is the "momentary state of the geometry of space" on the hypersurface σ_0; that is, it is the 3-geometry ${}^{(3)}\mathscr{G}_0$ defined by the metric

$$ds^2(\sigma_0) = \left[g_{00}\left(\frac{\partial x^0}{\partial x^m}\right)\left(\frac{\partial x^0}{\partial x^n}\right) + 2g_{0m}\left(\frac{\partial x^0}{\partial x^n}\right) + g_{mn}\right] dx^m\, dx^n$$

Similarly with the equivalent of the running time t and the state $x_H(t)$ of the particle at this time. Thus, give a spacelike hypersurface σ by giving $t = t(x, y, z)$; and calculate the corresponding metric $ds^2(\sigma)$, which defines a 3-geometry ${}^{(3)}\mathscr{G}$ "on the classical history H." The value of the classical phase function S on this classical history is given by a fourfold integral. This integral is extended over the region of spacetime bounded by the two hypersurfaces. It has the form

$$s(\sigma) = \int_{\sigma_0}^{\sigma} \mathscr{L}(x^\alpha)\, d^4x$$

Here the Lagrange density $\mathscr{L}$ is given, for example, by Arnowitt, Deser, and Misner ([2]; see also GMD). Consider now a "point" ${}^{(3)}\mathscr{G}$ in superspace a little ways removed from some "point" ${}^{(3)}\mathscr{G}^*$ that lies on the classical history H. What is the value of S for this new 3-geometry? The difference between the two 3-geometries is expressed most conveniently for the present expository purpose in terms of the difference of the metric coefficients at corresponding points:

$$\delta g_{mn}(x, y, z) = g_{mn}(x, y, z; \sigma) - g_{mn}(x, y, z; \sigma^*)$$

This difference in 3-geometries has to be multiplied by the value of the conjugate geometrodynamical momentum $\pi^{mn}(x, y, x)$ at the point ${}^{(3)}\mathscr{G}^*$ on the classical history H (details of definition and calculation of π^{mn} given, for example, in GMD) and integrated to give the change in the Hamilton-Jacobi function. Thus, throughout a thin swathe in superspace one can write

$$S({}^{(3)}\mathscr{G}) = s(\sigma^*) + \int \pi^{mn}\delta g_{mn}\, d^3x$$

PROBLEM 2. STRUCTURE OF SUPERSPACE

Spacetime is the arena of particle dynamics. Superspace is the arena of geometrodynamics. How different our knowledge of these two arenas! In particle dynamics the structure of spacetime is taken to be given as from on high: an everywhere flat ideal Minkowski-Lorentz manifold. In geometrodynamics the structure of superspace is to be considered as defined entirely internally; that is to say, by the very form of the "Einstein–Schroedinger equation" itself. We write this equation symbolically as

$$-\frac{\nabla^2\psi}{(\delta^{(3)}\mathscr{G})^2} + {}^{(3)}R\psi = 0$$

To ask about the structure of this equation is therefore nothing more or less than to ask, what is the structure of superspace?

It is conceivable that a given geometry may be more appropriately identified with a whole class of points in superspace than with a single point, when that 3-geometry has less than maximal symmetry. In this way, Professor Stephen Smale kindly points out, one may keep from introducing "conical singularities" in superspace. To avoid such singularities would seem to be essential if superspace is to rise from a mere topological manifold to the status of a differentiable manifold.

Quantum theory once known, classical theory follows easily and uniquely, by way of the correspondence principle,

$$\psi \simeq \begin{pmatrix}\text{SLOWLY VARYING}\\ \text{AMPLITUDE FACTOR}\end{pmatrix} \exp\left(\frac{iS}{\hbar}\right)$$

However, classical theory alone being known, it is ordinarily difficult or impossible uniquely to determine the form of the quantum wave equation [47]. No system better illustrates this point than a particle moving in a prescribed external electromagnetic field. To know the Hamilton–Jacobi equation alone

$$g_{\alpha\beta}\left(\frac{\partial S}{\partial x^\alpha} + \frac{eA_\alpha}{c}\right)\left(\frac{\partial S}{\partial x^\beta} + \frac{eA_\beta}{c}\right) + m^2c^2 = 0$$

is to have no way to decide between the Schroedinger–Klein–Gordon equation, the Dirac equation, or one or another wave equation of higher spin; they all reduce to the same Hamilton–Jacobi equation in the appropriate semi-classical limit. Similarly in geometrodynamics. From the well-defined "Einstein–Hamilton–Jacobi" equation (12, 13)

$$\left(\frac{\nabla S}{\delta^{(3)}\mathscr{G}}\right)^2 + {}^{(3)}R = 0$$

no one knows a satisfying way to go to a unique "Einstein–Schroedinger" equation [48].

In the example of the particle most of the ambiguity about the form of the wave equation is resolved as soon as observation or other evidence reveals the spin [49]—or more directly—how many possible orientations there are for any spin degree of freedom over and above the three degrees of freedom that tell location in space. In other words, in addition to the Hamilton–Jacobi equation, one must know the "structure of configuration space" in order to end up with a well-defined wave equation. Hence our question here: What is the structure of superspace?

Unravel the structure of superspace? Hardly all in one jump, and hardly today! Instead a step by step penetration of the issues is more to be anticipated, if the history of electromagnetism or atomic structure or other branches of physics is any guide. At least three levels of analysis are to be perceived. First, one already knows the structure relevant to classical geometrodynamics: (a) Superspace has Hausdorf topology (Stern [4]). (b) Superspace is endowed with an indefinite metric, defined by the expression [50]

$$\left(\frac{1}{2g}\right)(g_{ik}\,g_{jl} + g_{il}\,g_{jk} - g_{ij}\,g_{kl})$$

that occurs in the Hamilton–Jacobi equation (Eq. (12)). Second, one searches for those features in the structure of superspace that come into play when space changes its topology. Third, one hopes in the longer term to uncover those deeper properties of superspace that give ordinary space its dimensionality, its metric structure, and its ability to propagate electromagnetic fields and neutrinos with the speed of light. It is appropriate to discuss the structure of superspace a little further at all three of these levels.

Level 1. Classical Geometrodynamics; Topology Does Not Change

In classical geometrodynamics space does not change its topology. Space may be "preparing" to change its topology, but it cannot actually make the change within the context of classical theory [51]. It can at most signal its "intention" to change topology by developing somewhere a curvature that increases without limit ("gravitational collapse" [18]). To go further with the analysis of the collapse phenomenon and treat changes in topology forces one to go outside the framework of classical theory. So long as one stays inside classical theory, he must restrict his attention in one dynamical problem to one topology. Which topologies are acceptable?

Of all topologies none is more familiar than that associated in Einstein's theory with the geometry in and around a dilute center of attraction such as the sun. Curved at small distances, the geometry becomes asymptotically flat at large distances. The topology, as distinguished from the geometry, is Euclidean: E_3 or $R \times R \times R$. In a broader context, however, Einstein thought of the space around one star as part of the space engulfing all stars [52], and of this total space as closed and endowed with the topology of the 3-sphere, S_3. No one has stated more strongly than he the arguments for considering space to be closed [53].

One acquires a new reason to consider space to be closed when he considers how hard it is to define "an open and asymptotically flat space" in the context of quantum geometrodynamics: Those ${}^{(3)}\mathscr{G}$'s that occur with overwhelming probability are everywhere endowed with all kinds of ripples and other geometrical structures at the scale of the Planck length. There is no direction that one can take and there is no distance that one can travel that will erase this structure. Under these circumstances it is difficult to attribute any well-defined meaning whatsoever to the term "asymptotically flat." On the other hand, one knows more than one example of a space that is open and *not* asymptotically flat, and that becomes wilder and wilder at great distances [54]. Not having any means to distinguish one open space from another as "good," one would seem justified at this stage in the development of the subject to exclude from attention all open spaces. This approach is the more attractive in that it appears possible [55] to distinguish between one type of closed 3-geometry and another by straightforward classificatory integers similar to the Betti numbers. In contrast, the concept of "asymptotically flat"—even in contexts where it is relevant—is much more complex to formulate [56].

Out of a manifold with the topology E_3 it is possible to cut out a block with the shape of a cube and obtain the topology of the 3-torus, $S_1 \times S_1 \times S_1$. On a manifold with this topology the initial value problem of classical geometrodynamics presents difficulties in certain cases, according to unpublished considerations of Brill and Avez. It is conceivable that these difficulties indicate that the topology $S_1 \times S_1 \times S_1$ is not acceptable. In that event one might almost say that this topology has inherited a "defective gene" from its parent topology E_3.

Another "black gene" that one can perhaps reasonably exclude is non-orientability. We assume in effect that "transport" of a right-handed glove shall never bring it back to its starting point left handed.

Compatible with these very tentative principles for selecting "acceptable topologies" are the 3-sphere (S_3); the 3-sphere with addition of one handle or "wormhole" ($S_2 \times S_1 = W_1$); and the 3-sphere with n wormholes (W_n). There may or may not be further acceptable topologies not included in this list [55].

Fixed Topology Excludes GMD Account of Particles and Fields

Taking the most familiar of these "acceptable topologies," S_3, as case example, one can discuss and treat quantitatively at the classical level a rich variety of physical processes, including gravitational radiation [38]; gravitational geons [20]; the planetary motion, collisions, and breakup of geons [20]; the Taub model of an expanding and recontracting universe [57]; and more complex model universes [38]. Even so, one is limited in the physics that he can include in any of these models. One has no place for particles. Nor can one treat classically the final stages of gravitational collapse. These limitations seem unrelated to each other. In classical theory they are unrelated. In that view, the geometry of space remains forever tied to a unique topology. Not so in the purely quantum-geometrodynamical model of physics. There (1) a particle is pictured in terms of space resonating from one topology to another [58]; and there (2) the final stages of gravitational collapse are viewed as a coupling of macroscopic motion and microscopic topology ("waterfall and foam"). In excluding from consideration all changes in topology, classical theory, on these views, also excludes any account of the rationale of particles. Particles have to be introduced as foreign and physical entities and space has to be viewed as arena, not as structural material. In this limited conceptual framework the particles and other fields are counted as having degrees of freedom over and above those of the geometry. The electromagnetic field, in particular, is viewed as an entity additional to geometry.

Attempts have been made to regard electromagnetism as an aspect of geometry. This enterprise limits itself to that classical framework of ideas where one looks apart from submicroscopic fluctuations in the geometry and the topology of space. One attempt has had some minor success. The second-order equations of Maxwell and the second-order equations of Einstein have been combined into one set of equations of the fourth order that make no reference to any but geometric magnitudes [59]. The Maxwell field is recognized by the "footprints" it leaves on the geometry of space. In a certain sense those "footprints" *are* the electromagnetic field; hence the name, "already unified theory" of gravitation and electromagnetism. Faithfully though this theory reproduces in other respects the dynamic content of the equations of Maxwell and Einstein, it turns out not to be adapted to treat the initial value problem. Geometrical measurements alone on an initial spacelike hypersurface do not always suffice completely to determine the future time evolution of the geometry [60]. In other words, one cannot uphold in "already unified field theory" a strict division of dynamics into "equations of motion" and "initial value data for these equations." Yet on that division one had learned to insist, not least by reason of hard-won lessons from the Hamilton–Jacobi theory and from quantum theory. Therefore the electromagnetic field, like a particle, can hardly be treated as anything but a foreign or "physical" entity immersed in space so long as one looks away from the microscopic geometry of space.

Formalism of Field When Treated as "Foreign and Physical"

In this nongeometric or "arena" approach to physics one considers the degrees of freedom of the "foreign and physical" entity as additional to those of the geometry. In the most elementary example, the case where only one such entity is contemplated, the pure source-free electromagnetic field, one writes the Hamilton–Jacobi function or the Schroedinger probability amplitude, as the case may be, in the form

$$S = S(g_{ik}(x, y, z), A_m(x, y, z))$$

or

$$\psi = \psi(g_{ik}, A_m)$$

Similarly when more fields are involved. Even at this level of analysis the geometrical approach has a contribution to make. The Hamilton–Jacobi function, ostensibly dependent upon the individual components of the metric and of the electromagnetic vector potential, actually has to be understood to depend only on the 3-geometry ${}^{(3)}\mathscr{G}$ and on the 2-form $\mathbf{B} = \mathbf{dA}$. Consequently it cannot change S to "push the rubber sheet on which the coordinates are painted" over space in such a way that the point P, which formerly had the coordinates x^i, now acquires the coordinates $x^i - \xi^i$. Similarly the point $P + dP$, which formerly had the coordinates $x^i + dx^i$, now acquires the coordinates

$$x^i + dx^i - \xi^i - \left(\frac{\partial \xi^i}{\partial x^j}\right) dx^j$$

The separation between the two points of course remains unchanged by this alteration in coordinates:

$$\begin{aligned} ds^2 &= g_{ij}(x^s)\, dx^i\, dx^j \\ &= g_{ij}^{\text{new}}(x^s - \xi^s)(dx^i - \xi^i_{,m}\, dx^m)(dx^j - \xi^j_{,n}\, dx^n) \end{aligned}$$

To terms of the first order in the displacement ξ^s the alteration in the metric coefficients is

$$\delta g_{ij} = g_{ij}^{\text{new}} - g_{ij} = \left(\frac{\partial g_{ij}}{\partial x^s}\right)\xi^s + \xi_{i,j} + \xi_{j,i} = \xi_{i\,|\,j} + \xi_{j\,|\,i}$$

where the subscript $|\,j$ denotes covariant differentiation, in the space of the 3-geometry, with respect to the jth coordinate. In a similar way one finds the alteration in the electromagnetic vector potential,

$$\delta A_i = A_i^{\text{new}} - A_i = A_{i,s}\xi^s + A_j \xi^j_{,i}$$

These changes in the metric coefficients, taking place throughout all space, ostensibly produce a change in the Hamilton–Jacobi function determined by the functional derivatives of S; thus,

$$\delta S = \int \left[\left(\frac{\delta S}{\delta g_{ij}}\right)\delta g_{ij} + \left(\frac{\delta S}{\delta A_i}\right)\delta A_i\right] d^3x$$

However, the mere shift in coordinates cannot produce any real physical change. Consequently the quantity δS must vanish, independent of the choice of the coordinate displacements ξ^i. The consequences of this coordinate invariance are easily traced out. The functional derivatives that come into consideration have well-defined physical interpretations. Thus

$$\pi^{ij} = \frac{\delta S}{\delta g_{ij}}$$

is the geometrodynamical momentum [2, 3] conjugate to the geometrodynamical field coordinate g_{ij}. It is closely connected with the "extrinsic curvature" K^{ij} of the 3-geometry with respect to the yet-to-be constructed 4-manifold. The other derivative,

$$\mathfrak{E}^i = \frac{\delta S}{\delta A_i}$$

the electrodynamic momentum conjugate to the vector potential A_i, is nothing other than the electric field. This quantity satisfies the divergence condition [61]

$$\mathfrak{E}^i_{,i} = 0$$

The condition that S be invariant with respect to coordinate changes takes the form

$$0 = \delta S = \int [\pi^{ij}(\xi_{i\,|\,j} + \xi_{j\,|\,i}) + \mathfrak{E}^i(A_{i,j}\xi^j + A_j \xi^j_{,i})]\, d^3x$$

Integrating by parts, readjusting the positions of indices as appropriate, and making use of the divergence relation, one finds

$$0 = \int [-2\pi^{\,|\,s}_{js} + \mathfrak{E}^i(A_{i,j} - A_{j,i})]\xi^j\, d^3x$$

This must vanish for arbitrary choices of the field of coordinate displacements ξ^j. Consequently the quantity in square brackets must vanish. One finds in this way those three of Einstein's field equations that link the curvature of space with the Poynting density of flow of electromagnetic field energy [62]:

$$2\pi_{js}^{|s} = (\mathfrak{E} \times \mathbf{B})_j$$

How remarkable that one gets so much—including the Poynting vector itself—from the elementary condition that S should depend, not upon the components g_{ij} and A_i individually, but only upon the coordinate-independent geometrical quantities that are "dressed up" in these components [63]!

Express electrodynamics in geometric language, yes. See the footprints of the electromagnetic field on the geometry of space, yes. Conceive of classical geometrodynamical electric charge as lines of force trapped in the topology of space, yes. But explain geometrically the "necessity" for electromagnetism and other "physical" entities, no. Not within the context of classical geometrodynamics and its never-changing topology.

Level 2. Space Resonating between 3-Geometries of Varied Topology

A new world opens out for analysis in quantum geometrodynamics. The central new concept is space resonating between one foamlike structure and another. For this multiple connectedness of space at submicroscopic distances no single feature of nature speaks more powerfully than electric charge. Yet at least as impressive as charge is the prevalence of spin $\frac{1}{2}$ throughout the world of elementary particle physics. "It is impossible to accept any description of elementary particles that does not have a place for spin $\frac{1}{2}$." This quotation from the book *Geometrodynamics* [64] of 1962 goes on to say, "What then has any purely geometrical description to offer in explanation of spin $\frac{1}{2}$ in general? More particularly and more importantly, what possible place is there in quantum geometrodynamics for the neutrino—the only entity of half integral spin which is a pure field in its own right, in the sense that it has zero rest mass and moves with the speed of light? No clear or satisfactory answer is known to this question today. Unless and until an answer is forthcoming, *pure quantum geometrodynamics must be judged deficient as a basis for elementary particle physics*." Happily the concept of spin manifold [65] has subsequently come to light, not least through the work of John Milnor. This concept suggests a new and interesting *interpretation of a spinor field* within the context of the resonating microtopology of quantum geometrodynamics, *as the nonclassical two-valuedness that attaches to the probability amplitude for otherwise identical 3-geometries endowed with alternative "spin structures."*

The Orientation Entanglement Relation or "Version"

Spin and topology: what is the connection? Take a cube (Fig. 7). To its upper northeast corner attach one end of a long elastic string. Run the other end to the upper northeast corner of the room and fasten it there. With seven other elastic strings attach the other corners of the cube to the corresponding corners of the room. Now select any axis running through the center of the cube and rotate the figure about that axis through 360°. The cube resumes its original configuration. Not so the strings. They are in a

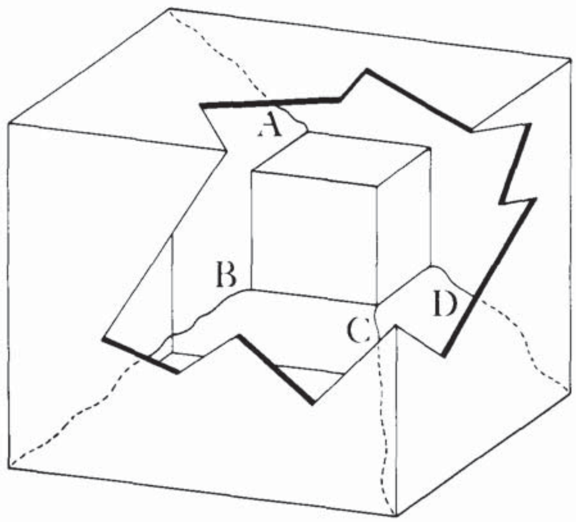

FIGURE 7. The elastic strings attached to the central cube keep account of its orientation entanglement relation with its surroundings.

tangle. Moreover, rejecting cuts, one has no way to untangle the strings. Consequently it needs more than orientation to tell the relation between the cube and its surroundings. The necessary bookkeeping is provided by a spinor,

$$s = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

Under a rotation through the angle θ about an axis making angles α, β, γ with the x, y, and z axes this spinor is transformed to the new spinor $s' = \mathbf{qs}$. Here the quaternion or "versor" **q** has a value indicated in various ways in various well-known systems of nomenclature:

$$\begin{aligned}\mathbf{q} &= \cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta(\mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma)\\ &\qquad (\text{with } \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \text{ etc.})\\ &= \cos\tfrac{1}{2}\theta - i\sin\tfrac{1}{2}\theta(\sigma_x\cos\alpha + \sigma_y\cos\beta + \sigma_z\cos\gamma)\\ &\qquad (\text{with } i = (-1)^{1/2} \text{ and } \sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z, \text{ etc.})\\ &= \left\| \begin{matrix} (\cos\tfrac{1}{2}\theta + i\sin\tfrac{1}{2}\theta\cos\gamma) & \sin\tfrac{1}{2}\theta(\cos\beta - i\cos\alpha)\\ \sin\tfrac{1}{2}\theta(-\cos\beta - i\cos\alpha) & (\cos\tfrac{1}{2}\theta - i\sin\tfrac{1}{2}\theta\cos\gamma)\end{matrix}\right\|\end{aligned}$$

The important point is the change of sign under a 360° rotation: $\mathbf{q}(360°)\cdot s = -s$. The 360° rotation alters what one may most appropriately call the "orientation entanglement relation" between the cube and its surroundings—or, more briefly, the "version" of the cube. The spinor keeps account of this orientation entanglement relation. Two successive rotations by 360° restore the cube to its original orientation entanglement relation with its surroundings. The strings at first appear to be tangled up with twice the twist they had before. Nevertheless, they can now be untangled completely, as one confirms by direct trial or by elementary reasoning [66, 67].

Arbitrarily pick out one way of placing the cube, call it the "standard orientation entanglement relation" between the cube and its surroundings, or the standard "version" of the cube, and associate with it the spinor $\begin{pmatrix}0\\1\end{pmatrix}$. Proceed similarly at other points of space, taking care only that any changes in the standard version from point to point shall take place smoothly.

A special situation develops when an orientable 3-geometry [68] is endowed with a handle or wormhole. The cube can be transported in imagination from A to B "through the surrounding nearby space" or "through the wormhole." With one cube possessing a certain pattern of colored faces follow one route and with another identically colored cube follow the other route. It makes physical sense to ask if the two cubes have at B identical orientation entanglement relations to their surroundings [67]. If they do not, alter the definition of the standard orientation within the wormhole by a rotation. Let this rotation increase continuously from zero at the mouth of the handle near A to 360° at the mouth near B. At each point outside the wormhole a triad of axes a, b, c defines the direction of the axes of the cube when the cube is located at that point in its standard orientation. These triads are not affected by the alterations made inside the wormhole. They vary in direction as smoothly as ever from point to point. At the two mouths of the wormhole they join on as smoothly to the new field of triads inside as they did to the old field of triads. Now, however, one has at last achieved an everywhere continuous field of standard orientation entanglement relations. The associated spinor field is likewise continuous, having everywhere the standard value $\begin{pmatrix}0\\1\end{pmatrix}$, whereas before it underwent somewhere a discontinuous change from this value to its negative. One can restate in mathematical language what has been accomplished. First, one has laid down a "spin structure" upon the manifold. This structure is defined by the field of triads, or by "the class of such fields that are equivalent under homotopy." Second, one has laid down a "spinor field" upon the manifold. This spinor field is defined with respect to the given "spin structure." Other spinor fields can of course be laid down, with components also varying continuously from place to place.

Alternative Spin Structures in a Multiply Connected Space

Start again with another closed orientable 3-manifold endowed with the original topology and metric. One might hope to obtain an acceptable spin structure on this new manifold by taking over to it the identical field of triads that serves for the original manifold. It may be that this supposition is correct about the particular new manifold that one happens to have picked out. In this event identical colored cubes "taken from A to B by inequivalent routes" will preserve their orientation entanglement relation, one to the other. However, the supposition can equally well be incorrect. If so, the consequences are direct. The two cubes, carried by different routes from A to B, always in alignment with the canonical field of triads, will end up at B inequivalent to the extent of a 360° rotation. Moreover, this inequivalence in the orientation entanglement relation with the surroundings can be detected in principle by direct physical measurement [67]. In other words, the spin structure of the original manifold does not apply to the new manifold. To serve in the new manifold, the field of triads has to be modified to the extent of a 360° rotation within the wormhole or by an equivalent change. The difference between the new manifold and the original manifold expresses itself in these terms: the two manifolds have the same topology and metric, but they have inequivalent spin structures. *The difference between the two manifolds is not only mathematical. It is physical.*

The Multisheeted Character of Superspace

One does not classify the closed orientable 3-manifold of physics completely when he gives its topology, its differential structure, and its metric. He must tell in addition which spin structure it has. The spin structure, like the metric, lends itself in principle to observation. Consequently it is not enough for the purposes of quantum geometrodynamics to give the probability amplitude for a "3-geometry" as the term 3-geometry was previously understood. One must introduce a new two-valued descriptor, w_k, for each ($k = 1, 2, \ldots, n$) of the n wormholes of the manifold, to distinguish the two inequivalent ways to lay down a spin structure in the interior of that wormhole. The new, enlarged concept of a 3-geometry ${}^{(3)}\mathscr{G}$ adjoins these descriptors to the continuous infinity of parameters which alone served previously to distinguish one 3-geometry, ${}^{(3)}\mathscr{G}^{\text{old}}$, from another; thus,

$$^{(3)}\mathscr{G} = ({}^{(3)}\mathscr{G}^{\text{old}}; w_1, w_2, \ldots, w_n)$$

and

$$\psi({}^{(3)}\mathscr{G}) = \psi({}^{(3)}\mathscr{G}^{\text{old}}; w_1, w_2, \ldots, w_n)$$

In other words, *superspace acquires a multisheeted character* (Fig. 8) with 2^n distinct sheets in that region of superspace where the 3-geometry is endowed with n wormholes.

For the two values to be assigned to the descriptor w_n it is natural to pick $+1$ and -1. However, there is nothing anomalous about the one spin structure as compared to the other. No canonical way has ever been proposed to give preference to one as compared to the other. Therefore it is a matter of arbitrary choice to which spin structure to assign the descriptor $+1$, and to which, -1.

The words "spin structure" can mislead. They suggest that there is something special about "laying down a spinor field" upon the manifold. They conjure up visions of laying down upon the 3-geometry other kinds of fields that transform according to groups other than the spinor group $SU(2)$; for example, $SU(n)$ or $SL(n)$. However, the relevant point in the whole analysis is not the field of spinors, but the field of triads and their orientation entanglement relations. There is nothing in the concept of spin structure that one could not have conveyed, with less chance of being misunderstood, by using the phrase "triad structure." Moreover, there is not the slightest indication that there is any other structure of a closed orientable 3-manifold that remains to be brought to light. Consequently we take ${}^{(3)}\mathscr{G}$ in its new and enlarged sense to be the full indicator of the configuration of space, and as containing the full set of variables upon which ψ depends. In other words, we take the new ${}^{(3)}\mathscr{G}$ to comprise a set of commuting observables, *complete* in the sense of quantum mechanics, and therefore suitable for analysis of the probability amplitude ψ.

We do not add a spinor field to geometry. Quite the contrary. We take a spinor field away from geometry. A 3-geometry, augmented by a spin structure (as might for example be indicated by the descriptor $(w_1, w_2, \ldots, w_5) = (+1, +1, +1, -1, +1)$) is a possible habitation for a spinor field—but we have thrown out the inhabitant [69].

Spin $\frac{1}{2}$, if it occurs naturally in the context of quantum geometrodynamics, can hardly occur in any other sense than that in which Pauli spoke of

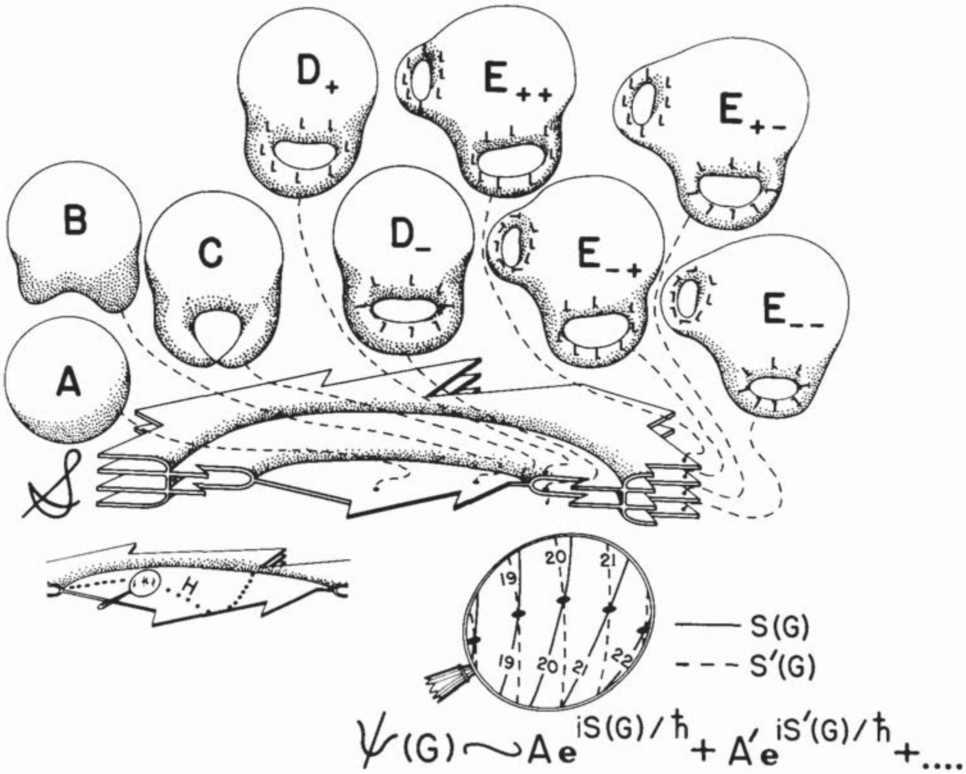

FIGURE 8. The multisheeted character of superspace.

spin from the very beginning, as a "nonclassical two-valuedness." Moreover, a nonclassical two-valuedness is already inescapable in the formalism. There are separate probability amplitudes for a 3-geometry with descriptor $w_k = +1$ and for an otherwise identical 3-geometry with descriptor $w_k = -1$. Does this circumstance imply that quantum geometrodynamics supplies all the machinery one needs to describe fields of spin $\frac{1}{2}$ in general and the neutrino field in particular? That is the proposal. That is the only way that has ever turned up within the framework of Einstein's general relativity and Planck's quantum principle to account for spin. Is this the right path? It is difficult to name any question more decisive than this in one's assessment of "everything as geometry."

Electromagnetism as a Statistical Aspect of Geometry? Other Questions

It would be tempting to stop with this major issue if it did not open the door to so many tributary questions. (1) When a new handle develops and the number of descriptors rises by one, what boundary condition in superspace connects the probability amplitude ψ for 3-geometries of the original topology with the probability amplitudes ψ_+ and ψ_- for the two spin structures of the new topology? (2) One may wish to describe for each newly developing wormhole something like "the fractional amplitude going from ψ into ψ_+." However, with the fantastic number of wormholes that typically come into consideration ($\sim 10^{99}/\text{cm}^3$) any such individual bookkeeping would seem for many purposes to be out of the question. It is equally impractical to keep account of the orientation of each of the $\sim 10^{23}$ spins in a ferromagnet. It is much more appropriate to speak of the "density of magnetization" and of perturbations in that density carried by "magnons" [71]. What are the analogous quantities and concepts in geometrodynamics? What statistical approach is best suited to keep account of the many "matching ratios" ψ_+/ψ associated with all the nascent wormholes in the geometry? (3) One speaks of "magnetization," knowing well that the term "magnetization" has not the slightest real meaning at subatomic distances. Is the term "electromagnetic field" equally without submicroscopic physical significance? In other words, among the statistical parameters most appropriate for keeping account of the 10^{99} "matching ratios" per cubic centimeter, *is there one set of statistical parameters that one can identify with the electromagnetic field?*

The Example of Two-Geometries

No one can ask about the physics of changes in the topology of 3-space without at least a look at the mathematics of changes in the topology of 2-space. The superspace built on all 3-geometries is like no mathematical object so much as the superspace "built on all 2-geometries." No one did so much to bring this mathematical object to light as Riemann, the same Bernhard Riemann who taught that the curvature of space is a branch of physics, and who provided the mathematical machinery to describe not only curvature (the Riemann curvature tensor, $R_{\alpha\beta\gamma\delta}$) but also topology (the Betti numbers R_n).

In his 1857 paper Riemann noted that all algebraic 2-geometries endowed with the topology of a 2-sphere are equivalent to each other under conformal transformation (multiplication of all three metric coefficients by a common position-dependent factor λ). In other words the equivalence class of conformally equivalent 2-geometries of the given topology (S_2; or "genus $g=0$") consists of a single object. It therefore constitutes a single "point" in what is not really a superspace itself, as we have been using that term, but a "reduced superspace": reduced in the sense that the "$(\infty^2)^\infty$" degrees of freedom in λ have here been "strained out" of superspace.

Two-geometries with the topology of the torus (T_2; or one wormhole W_1; or "genus $g=1$"), Riemann showed, are not all equivalent to one another under conformal transformation. Instead, when conformally equivalent 2-geometries of the topology T_2 are identified, the family of objects that results is a complex continuum of dimension 1 (two real dimensions) [72]. Two-geometries endowed with a larger number of wormholes (topology W_g; genus $g \geqslant 2$), after extraction of the conformal degrees of freedom, reduce to a family of objects described by $3g - 3$ complex parameters ($6g - 6$ real parameters). "Reduced superspace," built on the totality of conformally equivalent closed orientable 2-geometries of all topologies, thus appears to consist of a series of disjoint spaces, the first of dimension 0, the second of dimension 1, the next of dimension 3, the next of dimension 6, and so on. However, great developments in the analysis have taken place since the days of Riemann through the efforts of many investigators [73]. Today, thanks not least to the works of Lipman Bers, one knows how to define one single infinite-dimensional reduced superspace in which all these apparently disparate parts fit smoothly together [74]. What a model for the mathematical treatment of the superspace of general relativity! Yet, at the risk of seeming overdemanding, one has to ask for more. The superspace of physics is not to have any "conformal factor strained out of it"; rather, it is if anything to be enlarged, so as to include all the descriptors w_k of the spin structure. How fit all the pieces of *this* superspace smoothly together? Challenging problem, at the very heart of quantum geometrodynamics!

Other Aspects of Superspace

Why all this emphasis on the structure of superspace? Why not simply spell out explicitly the form of the "Einstein–Schroedinger equation"? For the more elementary problem of a particle moving in flat 3-space it was straightforward for Schroedinger to derive his wave equation. Simple considerations of invariance with respect to translation and rotation show that ∇^2 is the only simple differential operator that can come into play. That clear, the principle of correspondence with classical physics gives all the rest. One can hope that equally compelling considerations will fix the detailed mathematical form of the expression that we write down so far only symbolically,

$$\frac{\nabla^2 \psi}{(\delta^{(3)}\mathscr{G})^2}$$

However, a precise formulation of such considerations would seem to be out of reach until one has a knowledge of the tranformations of superspace comparable to one's knowledge of the transformations of 3-space. Hence the emphasis on the structure of superspace.

A problem of such depth can hardly be examined from too many points of view. Six more aspects of superspace seem worthy of mention. One can summarize them under the names "metric," "residual causality," "initial value," "conjugate momentum," "collapse," and "pregeometry." (1) There could hardly be a more helpful guide to the structure of superspace than the metric that obtains in it,

$$\left(\frac{1}{2g}\right)(g_{ik}\,g_{jl} + g_{il}\,g_{jk} - g_{ij}\,g_{kl})$$

Reference is made to [2] for the most illuminating discussion of this metric given to date. (2) This metric has a "light cone" associated with it. This light cone makes propagation proceed anisotropically in superspace. This anisotropy differs in character, however, from place to place according as ${}^{(3)}R$ is positive or negative. This anisotropy imposes a kind of "residual causality" upon superspace. Charles W. Misner has pointed out in a conversation that one cannot forget this residual causality when one says that the customary ideas of "before" and "after" lose their meaning at the scale of the Planck length. Perhaps one can say more: If the principle of causality has been of service in analyzing the structure of flat spacetime, it can hardly fail to help in studying the structure of superspace.

Tangent Vectors on Superspace and the Classical Initial Value Problem

(3) In the classical mechanics of a particle one is accustomed to specifying freely x_0 and $(dx/dt)_0$. These initial conditions determine the whole future history of the particle. What are the analogous freely disposable initial value data of classical geometrodynamics? One is tempted to say: conceive of a continuous one-parameter family of 3-geometries, specified for example in one coordinate patch by the 6 metric coefficients $g_{ik}(x, y, z; \lambda)$; and use this one-parameter family to define data analogous to x_0 and $(dx/dt)_0$ in the one-particle problem; thus, ${}^{(3)}\mathscr{G}_0$ stands for the class of metrics equivalent to $g_{ik}(x, y, z; 0)$; and $(d^{(3)}\mathscr{G}/d\lambda)_0$ is the "tangent vector in superspace" defined by $[\partial g_{ik}(x, y, z, \lambda)/\partial\lambda]_{\lambda=0}$, modulo the group of coordinate transformations.

It is easy for a mere coordinate shift to mock up the appearance of a change in geometry. Let the coordinates be shifted so that the point P, formerly characterized by the coordinates x^i, is now characterized by $x^i - \lambda\xi^i$, with the vector field ξ^i a continuous function of position. The metric g_{ik} is altered by this shift to

$$g_{ik} + \lambda(\xi_{i\,|\,k} + \xi_{k\,|\,i})$$

The derivative $(d^{(3)}\mathscr{G}/d\lambda)_0$ is $\xi_{i\,|\,k} + \xi_{k\,|\,i}$, modulo the group of coordinate transformations. But this quantity, by reason of its very origin, is obviously annullable by a coordinate transformation. Consequently there is in this case no real change in the geometry. In other words, one cannot admit any otherwise reasonable-looking field of values for $(\partial g_{ik}/\partial\lambda)_0$ without running the risk of deception. One wants what has been called in [2] (GMD & IFS) a 3-geometry (of "acceptable" topology) and another "nearby" 3-geometry in order—one trusts (central hypothesis of the subject!)—to be able to determine the entire past and future of the space, and thereby a complete 4-geometry. But if one has been "deceived" in describing what is ostensibly a second and "nearby" 3-geometry, he may merely be repeating all over again the previously given 3-geometry. In that event he has only half the amount of initial value data needed to predict the dynamics. One can restate the situation in the following terms in the context of the 4-geometry (regarded temporarily as known!). A spacelike slice is made through the 4-geometry. That gives the one 3-geometry demanded as one of the essential ingredients of the initial value data. However, that one slice is not adequate to distinguish the given 4-geometry from any number of other, different 4-geometries that admit as slice the same 3-geometry. To complete the selection of the given 4-geometry from these alternative 4-geometries, erect vectors λn^α at each of the points of the 3-geometry, with n^α a continuous function of position. Their tips define a new hypersurface, the coordinates in which are connected continuously with the coordinates in the original hypersurface. Evaluate the metric coefficients $g_{ik}(x, y, z, \lambda)$ on this hypersurface. This hypersurface can be said to have been "pushed forward" with respect to the original hypersurface. But has it? Yes, if the normal component of λn^α nowhere vanishes. However, Hans Ohanian and Elliot Belasco have emphasized in unpublished remarks that it may happen that there are whole regions of the hypersurface where the normal component of λn^α vanishes. In that case one has not really pushed the hypersurface ahead at all in ${}^{(4)}\mathscr{G}$. In this event the second component of the initial value data, the derivatives $(\partial g_{ik}/\partial\lambda)_0$, will simply be inadequate for the purposes of the elliptic initial value equations [75]. This situation will be signaled by the fact that $(\partial g_{ik}/\partial\lambda)_0$ can be written in the form $\xi_{i\,|\,k} + \xi_{k\,|\,i}$. In other regions the 3-geometry *will* have been pushed forward in time. This situation will be signaled, except in special circumstances (time-symmetric initial value problem; change in g_{ik} proportional to λ^2 rather than λ; situation covered by appropriate care in the formulation), by the fact that $(\partial g_{ik}/\partial\lambda)_0$ (or $(\partial g_{ik}/\partial\lambda^2)_0$ in special cases like the time-symmetric initial value problem) is *not* representable in the form $\xi_{i\,|\,k} + \xi_{k\,|\,i}$.

It is natural to try to summarize the whole situation in the following form. *Give a point in superspace and give a "fully developed direction" at this point in superspace. Then (hypothesis!) this information is sufficient, together with Einstein's equations, uniquely to determine the entire 4-geometry.* Here the term "point in superspace" implies, as earlier, the demand that the 3-geometry in question have acceptable topology. The term "fully developed direction" implies that there is no point on the 3-geometry where the quantity

$(\partial g_{ik}/\partial\lambda)_0$ (or if it vanishes, the quantity $(\partial g_{ik}/\partial\lambda^2)_0$) can be expressed in the form $\xi_{i|k} + \xi_{k|i}$. In brief, does superspace provide a new approach to the classical initial value problem? And in turn, does that initial value problem throw new light on the concept of "direction" in superspace?

(4) Superspace is built on the concept of 3-geometry; but dynamically conjugate to 3-geometry is the geometrodynamical momentum, with components π_{ij}. Out of these objects, with all the varied topologies that *they* can have, one can build a "conjugate superspace." What are *its* properties?

(5) No crisis stands out more insistently in all of physics than gravitational collapse. No topic connects so immediately the world of the very large and the very small. What insights can one gain from the concept of superspace into the cause and consequences of gravitational collapse?

(6) How far can one go in analyzing the properties of superspace without getting into the problems of "pregeometry"?

Level 3. Pregeometry

Weyl remarks [76] "... a more detailed scrutiny of a surface might disclose that, what we had considered an elementary piece, in reality has tiny handles attached to it which change the connectivity character of the piece, and that a microscope of even greater magnification would reveal ever new topological complications of this type, *ad infinitum*." Under such circumstances it would seem difficult to uphold the concept of dimensionality at the smallest distances. General arguments [77] emphasize the same point. Moreover, if electromagnetism and other fields have to do with the quantum-mechanical resonance of space between one topology and another, why should not the concept of metric itself be likewise a derived concept, going back for its foundation to topological or pretopological—and at any rate to pregeometric—ideas ("distance between A and B" being defined in the last analysis, for example, by "the ramification of the connections between A and B")? One cannot even mention these topics without recalling the universal sway of the quantum principle, and without stressing the "order of creation" as one thinks of it from physical evidence: Not first geometry and then the quantum principle, but first the quantum principle and then geometry!

It is enough to raise these issues, with all their depth, to see into what difficulties one can get with quantum geometrodynamics if one tries to think of it as an "ultimate" theory. However, physics has never depended for its progress on having an ultimate theory. There is no reason to think that the situation is different today. While one can raise ultimate issues of all kinds, there is no reason to believe that they all have to settled now! Nor that one *can* resolve them now! The subject presents an ever widening list of issues that have lively physical interest and lend themselves to well-known methods of analysis [78].

PROBLEM 3. INITIAL CONDITIONS

The classical initial value problem has already been discussed. What can one say about the corresponding problem in quantum geometrodynamics? In other words, how much information must one give about $\psi({}^{(3)}\mathscr{G})$ on an appropriate submanifold of superspace in order to be able to predict this probability amplitude everywhere in superspace? And what is the character of this submanifold? In this connection one recalls that the "Einstein–Schroedinger wave equation" is of the second order. This second-order character raises a question of principle: In order to be able to calculate ψ everywhere, must one know on a hypersurface of superspace not only ψ but also its normal derivative? No, Leutwyler suggests in a most interesting paper [79]. He points out in the context of a simplified model that the natural features of superspace itself impose certain natural boundary conditions. They reduce the effective order of the equation from second to first.

Wider questions of principle are also posed by the very structure of quantum geometrodynamics. The arena of the dynamics is not space, but superspace. At first this development seems preposterous. How can one speak sensibly of any physical predictions when the outcome depends on what is taking place in unreachable regions of superspace? Nothing could seem more at variance with the spirit of science as dealing only with the knowable. However, a closer look shows that one has broken not at all with the traditional spirit of dynamics, but only with the details. In classical dynamics a clean distinction has always been maintained between (1) the equations of motion, which one can hope to know and understand, and (2) the origin of the initial conditions for those equations of motion—which is beyond one's power to investigate [80]. Quantum geometrodynamics maintains a similar cut between the knowable and the unknowable, but the cut comes in a new place [81]. Nothing seems to exclude the possibility ultimately to know (1) the detailed form of the Einstein–Schroedinger equation, and the concomitant structure of superspace; but as for (2) the source of the initial conditions on ψ, that would seem as far as ever beyond one's power ever to know. Happily, in neither classical dynamics nor quantum geometrodynamics does one have to know all initial conditions to make useful predictions! On the contrary, as Wigner has so often stressed [82], the role of physics is to predict the *correlations* between observations.

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R. P. Feynman, mimeographed letter to V. F. Weisskopf dated 4 January to 11 February 1961; *Acta Physica Polonica* **24**, 697 (1963); *Lectures on Gravitation* (notes mimeographed by F. B. Morinigo and W. G. Wagner, California Institute of Technology, 1963); report in *Proceedings of the 1962 Warsaw Conference on the Theory of Gravitation* (PWN-Editions Scientifiques de Pologne, Warszawa, 1964); S. N. Gupta, report in *Recent Developments in General Relativity*, Pergamon, New York, 1962; S. Mandelstam, *Proc. Roy. Soc. (London)* **A270**, 346 (1962) and *Annals of Physics* **19**, 25 (1962); J. L. Anderson in *Proceedings of the 1962 Eastern Theoretical Conference* (M. E. Rose, ed.), Gordon and Breach, New York, 1963, p. 387; I. B. Khriplovich, "Gravitation and Finite Renormalization in Quantum Electrodynamics" (mimeographed report, Siberian Section Academy of Science, U.S.S.R., Novosibirsk, 1965); H. Leutwyler, *Phys. Rev.* **134**, B1155 (1964); B. S. DeWitt, "Dynamical Theory of Groups and Fields," in *Relativity Groups and Topology* (C. DeWitt and B. DeWitt, eds.), Gordon and Breach, New York, 1964; S. Weinberg, *Phys. Rev.* **135**, B1049 (1964), **138**, B988 (1965), and **140**, B516 (1965); M. A. Markov, *Progr. Theor. Phys., Yukawa Suppl.*, 1965, p. 85.

2. P. W. Higgs, *Phys. Rev. Letters* **1**, 373 (1958) and **3**, 66 (1959); R. Arnowitt, S. Deser, and C. W. Misner, a series of papers summarized in "The Dynamics of General Relativity" in *Gravitation: An Introduction to Current Research* (L. Witten, ed.), Wiley, New York, 1962; A. Peres, *Nuovo Cimento* **26**, 53 (1962); R. F. Baierlein, D. H. Sharp, and J. A. Wheeler, *Phys. Rev.* **126**, 1864 (1962); cf. also the Princeton A. B. Senior Thesis of D. H. Sharp, May 1960 (unpublished); J. A. Wheeler, *Geometrodynamics*, Academic Press, New York, 1962, cited hereafter as GMD, and "Geometrodynamics and the Issue of the Final State," cited hereafter as GMD & IFS, a chapter in *Relativity, Groups and Topology* (C. DeWitt and B. DeWitt, eds.), Gordon and Breach, New York, 1964; B. DeWitt, *Phys. Rev.* **160**, 1113 (1967), **162**, 1195 (1967) and **162**, 1239 (1967), together cited hereafter as QTG.

3. J. A. Wheeler, GMD & IFS; B. DeWitt, QTG.

4. Michael D. Stern, *Investigations of the Topology of Superspace*, Princeton A. B. Senior Thesis, May 1967 (unpublished) and *Proc. Natl. Acad. Sci. U.S.A.* (submitted for publication).

5. R. Penrose, *An Analysis of the Structure of Spacetime*, Adams Prize Essay (mimeographed for limited distribution, Princeton University, Princeton, New Jersey, December 1966), gives a beautiful procedure to prescribe on the past light cone exactly enough geometrical information to determine out of Einstein's field equations the complete 4-geometry everywhere within the past light cone. The treatment is given only in the analytic case (in which case the difference between "inside" and "outside" the light cone does not make itself felt) but from general

considerations one must expect in the nonanalytic case that the data in question determine the 4-geometry only *within* the past light cone. When the light cone points into the future instead of the past, similar considerations of course apply, obtained by the interchange of the words "past" and "future" in what is said in the text. Problems arise with such formulations of the initial value problem "on the light cone" when the propagation proceeds far in a space of variable curvature. Then the light cone develops more than one sheet. Compare the several claps of thunder often heard from a single localized explosion!

6. For a discussion of the "observer" as a "collector of printout" see for example E. F. Taylor and J. A. Wheeler, *Spacetime Physics*, W. H. Freeman, San Francisco, 1966.

7. For the idea of a general spacelike hypersurface as the manifold on which the magnitudes of quantum field theory are to be measured, see especially S. Tomonaga, *Progr. Theor. Phys.* **1**, 34 (1946) and J. Schwinger, *Phys. Rev.* **74**, 1449 (1948).

8. J. A. Wheeler, GMD & IFS.

9. B. DeWitt, QTG.

10. This way of writing the conditions for constructive interference is symbolic only. In actuality the Hamilton–Jacobi function S depends not only upon the ${}^{(3)}\mathcal{G}$, but upon an infinity of parameters that distinguish one solution of the Hamilton–Jacobi equation from another. Thus, in a problem with one degree of freedom we write $S = S_0(x, E) + \delta(E)$ and in a problem with n degrees of freedom $S = S_0(x_1, \ldots, x_n; \alpha_1, \ldots, \alpha_n) + \delta(\alpha_1, \ldots, \alpha_n)$. In geometrodynamics there are two degrees of freedom per space point, the magnitudes associated with which may be designated by α and β. Thus the infinitude of freely disposable parameters may be indicated by two freely disposable functions, $\alpha(u, v, w)$ and $\beta(u, v, w)$. The ∞^3 points are given by the ∞^3 possible choices of u, v, and w. The u, v, w manifold may be, but is not required to be, the same as the manifold x, y, z of points in the 3-geometry ("alternative choices of parametrization of Hamilton–Jacobi function"). In any case we write S as a functional of α and β; thus, $S = S_0({}^{(3)}\mathcal{G};\ \alpha(u, v, w), \beta(u, v, w)) + \delta(\alpha(u, v, w), \beta(u, v, w))$. Then the conditions of constructive interference become statements about functional derivatives; thus,

$$\frac{\delta S}{\delta \alpha} = 0$$

and

$$\frac{\delta S}{\delta \beta} = 0$$

This is an explicit form of the symbolic Eq. (11) of the text. Other ways of writing the equations of constructive interference also exist.

11. Ulrich Gerlach, *Bull. Am. Phys. Soc.* for the Washington meeting of April, 1966, paper DE7, p. 340.

12. A. Peres, *Nuovo Cimento* **26**, 53 (1962). Here the unit of length is $(16\pi)^{1/2}L^* \quad (16\pi\hbar G/c^3)^{1/2}$

13. See [1] and [2].

14. R. F. Baierlein, D. H. Sharp, and J. A. Wheeler, *Phys. Rev.* **126**, 1864 (1962).

15. The term "dimensionality" can be translated as the requirement of "imbeddability" of all the $^{(3)}\mathscr{G}$'s in a $^{(4)}\mathscr{G}$, a requirement that would seem the natural starting point for a derivation of the Einstein–Hamilton–Jacobi equation straight from first principles (see Problem 1).

16. C. W. Misner, *Rev. Mod. Phys.* **29**, 497 (1957); see also H. Leutwyler, *Phys. Rev.* **134**, B1155 (1964) and B. S. DeWitt, QTG.

17. M. Planck, *Sitzungsber. Preussische Akad. Wiss. Berlin*, *Math.-Phys. Klasse*, 1899, p. 440; J. A. Wheeler, GMD.

18. For a review of the subject of gravitational collapse, see for example B. K. Harrison, K. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse*, Univ. of Chicago Press, 1965; also A. G. Doroschkevich, Ya. B. Zel'dovich, and I. D. Novikov, *J. Exptl. Theor. Phys.* **49**, 170 (1965), English translation in *Soviet Physics JETP* **22**, 122 (1966) and Ya. B. Zel'dovich and I. D. Novikov, *Usp. Fiz. Nauk* **84**, 377 (1964) and **86**, 447 (1965), English translations in *Soviet Physics Usp.* **7**, 763 (1965) and **8**, 522 (1965).

19. For a presentation of the quantum electrodynamical calculation of the major part of the Lamb shift of hydrogen from this point of view, see T. A. Welton, *Phys. Rev.* **74**, 1157 (1948) and F. J. Dyson, *Advanced Quantum Mechanics*, Cornell Univ., Ithaca, 1954 (mimeographed), p. 54.

20. J. A. Wheeler, GMD.

21. J. A. Wheeler, GMD and GMD & IFS; B. S. DeWitt, QTG and "The Quantization of Geometry" in *Gravitation: An Introduction to Current Research* (L. Witten, ed.), Wiley, New York, 1962, p. 342 ff.

22. In GMD.

23. For an elementary discussion of the identity between the tide-producing component of the gravitational force and the Riemann curvature, see for example E. F. Taylor and J. A. Wheeler, *Spacetime Physics*, Freeman, San Francisco, 1966.

24. E. P. Wigner, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," in his book *Symmetries and Reflections*, Indiana Univ. Press, Bloomington, 1967; reprinted from *Comm. Pure App. Math.* **13**, No. 1 (February, 1960).

25. Reference 17.

26. A. S. Eddington, *Relativity Theory of Protons and Electrons*, Cambridge Univ. Press, 1936, and *Fundamental Theory*, Cambridge Univ. Press, 1946; also *Proc. Camb. Phil. Soc.* **27**, 15 (1931).

27. P. A. M. Dirac, *Nature* **139**, 323 (1937), *Proc. Roy. Soc.* (*London*) **A165**, 199 (1938).

28. P. Jordan, *Schwerkraft und Weltall*, Vieweg und Sohn, Braunschweig, 1955, and *Z. Physik* **157**, 112 (1959).

29. R. H. Dicke, *Science* **129**, 3349 (1959) and *The Theoretical Significance of Experimental Relativity*, Gordon and Breach, New York, 1964, p. 72.

30. S. Hayakawa, *Progr. Theor. Phys.* **33**, 538 (1965) and *Progr. Theor. Phys. Suppl.* p. 532 (1965).

31. It is permissible to take at full force the argument of Eddington, Dirac, Jordan, Dicke, and Hayakawa, that a physical correlation exists between 10^{20}, 10^{40} and 10^{80}, without accepting the suggestion sometimes made in the same context, that the physical constants may "change with time." There is increasing observational evidence against such changes, and no incontrovertible evidence for them has ever been found. Among the relevant observations one can cite as examples R. H. Dicke, *Nature* **183**, 170 (1959) and *Nature* **192**, 440 (1961); also R. H. Dicke and P. J. E. Peebles, *J. Geophys. Res.* **67**, 10 and 4063 (1962) and *Phys. Rev.* **128**, 5 and 2006 (1962), showing no detectable change with time in the relative rates of selected processes of radioactive decay. To search for changes with time in the reciprocal fine structure constant, $\alpha^{-1} = \hbar c/e^2 = 137.03$, is simple in principle. One has only to compare the wavelength of the 21-cm line of hydrogen (red shifted because it was given out by a rapidly receding galaxy, far away and long ago) with the wavelength of a line in the optical spectrum (which has undergone the same red shift). The ratio R of the two wavelengths is α^{-1} multiplied by a known function of the atomic number of the source (an integer) and of the relevant quantum numbers (also integers):

$$R = \alpha^{-1} \text{ times function of integers}$$

The value of R for a source 1.4×10^9 light-years away (recession rate $\beta = v/c = 0.1$) should differ by several percent from the value of R for a laboratory source if any of the suggestions are correct that physical constants might change in proportion to the time (or some significant power of the time) measured from the start of the expansion of the universe. The writer is indebted to the kindness of Professor R. Minkowski of Berkeley for the following (July 28, 1967) summary of the observational situation: (1) Hopes have been dashed to observe the 21-cm line in the spectrum of galaxies anywhere near as far away as would correspond to a recession velocity of $\beta = 0.1$. (2) Observations have been made on 30 nearer objects by Dieter, Epstein, Lilley, and Roberts, *Astrophys. J.* **67**, 270 (1962) as supplemented by Roberts, *Astrophys. J.* **142**, 148 (1965). The red shift is the same for the 21-cm line and for the optical lines within the limits of error of the observations. It is difficult to evaluate the accuracy because individual motions amount to as much as 20–30% of the average recession velocity of $v = 1600$ km/sec ($\beta = 0.005$). It is probably safe to say that any change in α^{-1} must be less than a few percent to be compatible with the observations. However, a change anyway smaller than this limit would be expected on almost any of the varied theories of the change of α^{-1} with time, since the time lapse in this case is only one two-hundredth of the Hubble time. (3) Instead of comparing the wavelength of the 21-cm line with the wavelength of an optical transition, one can measure the fine structure separation of a related pair of lines in the optical spectrum itself. The fractional splitting $\Delta\lambda/\lambda$ should be independent of the red shift of the source if α^{-1} is constant. From the observations of R. Minkowski, *Astrophys. J.* **123**, 373 (1956), on Cygnus A ($v = 16830$ km/sec, $\beta = 0.056$: W. Baade and R. Minkowski, *Astrophys. J.* **119**, 206 (1954)), it is again probably safe to say that any change in the fine structure constant must be less than a few percent. Bahcall, Sargent, and Schmidt, *Ap. J. Lett*, **149**, 11 (1967) give $|\Delta\alpha/\alpha| < 0.05$ for $z = 2$.

32. The quark, so useful in doing bookkeeping on the beautiful regularities of elementary particle physics (summarized, for example, in M. Gell-Mann and Y. Ne'eman, *The Eightfold Way*, Benjamin, New York, 1964 and F. Dyson, *Symmetry Groups in Nuclear and Particle Physics*, Benjamin, New York, 1966) has sometimes been taken much more seriously, as if it were an actual "primordial building block" of matter. That view may or may not be correct. If it is, and if one still continues to take quantum geometrodynamics as the only available indicator of what goes on at very small distances, then it would still seem reasonable to expect that one must have some perspective on what happens at 10^{-33} cm before one can find the rationale of quarks and particles. That there is no such thing as a quark in the literal sense is, however, a point of view accepted by many investigators, and stressed especially by Heisenberg and Dürr: W. Heisenberg, *Introduction to the Unified Field Theory of Elementary Particles*, Wiley, New York, 1966 and H. P. Dürr, "On the non-linear spinor theory of elementary particles," *Acta Physica Austriaca*, Suppl. III, 1966. Dürr has made the same point even more vividly (kind personal communication of June, 1967) by considering in effect what one would conclude out of the first several dozen atomic energy levels of an atom such as, for example, carbon or iron if one had (1) good measurements of the energies and transition probabilities, and (2) today's aptitude for searching for symmetries, but (3) not the slightest idea of the actual internal machinery of an atom. He shows how groups of high symmetry will make their appearance. His discussion leads one to ask whether the innocent investigator will not conclude that the atom is made out of quarks!

33. The strongest statement easily available against taking general relativity seriously at small distances appears to be that made by Robert Oppenheimer in his article "On Albert Einstein" (*New York Review*, March 17, 1966, pp. 4, 5): "He also worked with a very ambitious program, to combine the understanding of electricity and gravitation in such a way as to explain what he regarded as the semblance—the illusion—of discreteness, of particles in nature. I think that it was clear then, and believe it to be obviously clear today, that the things that this theory worked with were too meager, left out too much that was known to physicists but had not been known much in Einstein's student days. Thus it looked like a hopelessly limited and historically rather accidentally conditioned approach."

34. For further discussion of the rationale of changes in topology see Problem 2.

35. GMD.

36. To take it as self-evident that space is Euclidean in character at small distances became impossible after Riemann. His Göttingen inaugural lecture of June 10, 1854 pointed out that space can be highly rippled at submicroscopic distances and yet look smooth to all ordinary means of observation: "Uber die Hypothesen welche der Geometrie zugrunde liegen" in his *Gesammelte Mathematische Werke* (H. Weber, ed.), 2nd ed., reprinted by Dover, New York, 1953, also in a translation in *Nature* **8**, 14 (1873), by W. K. Clifford. Clifford himself went further in his lecture before the Cambridge Philosophical Society February 21, 1870, "On the Space-Theory of Matter," reprinted in his *Mathematical Papers* (R. Tucker, ed.),

London, 1882, also in his *Lectures and Essays* (L. Stephen and F. Pollock, eds.), Vol. 1, London, 1879. He proposed to consider a particle as made up of nothing but curved empty space, differing from the surrounding space precisely in this localized curvature—and perhaps also in its connectivity or local topology. In *Was ist Materie* (Springer, Berlin, 1924, esp. pp. 57, 58) Hermann Weyl again pointed out that space may be multiply connected in the small, and consequently: "The argument that the charge of the electron must be spread over a finite region, because otherwise it would possess infinite inertial mass, has thus lost its force. One cannot at all say, here is charge, but only, this closed surface encloses charge." He went on to comment that the enormous value of the ratio of electric to gravitation forces "seems to indicate that the total number of electrons in the universe is important for the constitution of the individual electron." Albert Einstein and Nathan Rosen, *Phys. Rev.* **48**, 73 (1935), proposed the concept of two nearly Euclidean spaces, connected here and there by thin bridges or tubes, through which electric lines of force thread, to give the appearance of charges of variegated signs in the "upper" space and corresponding charges of the opposite sign in the "lower" space. J. A. Wheeler, *Phys. Rev.* **97**, 511 (1955), reprinted in GMD, proposed instead the concept of a tube or handle or "wormhole" reaching between two different localities in one and the same Euclidean space. It is an automatic consequence of this picture that the universe should contain equal amounts of positive and negative electricity. It is another consequence, proved by Misner in 1957 straight from Maxwell's equations for empty space, that the charge, or flux of lines through the wormhole, must stay constant with time. The proof, C. W. Misner and J. A. Wheeler, *Annals of Physics* **2**, 525 (1957), reprinted in GMD, holds no matter how tortuously the lines of force may be twisted, no matter how wanting in symmetry the geometry of the wormhole may be, and no matter how violently the field and the geometry may subsequently change with time. Misner also showed here the beautiful ties that exist between the Maxwell theory in a multiply connected empty space and the mathematics of differential forms and homology groups. In his analysis the field and the geometry were assumed to evolve deterministically in time, in accordance with the classical equations of electrodynamics and geometrodynamics. Reasons out of fluctuation theory to consider "wormholes" a property, not of particles, but of all space, were first given by J. A. Wheeler in *Annals of Physics* **2**, 604 (1957), expanded in GMD.

37. No one has pointed out a more direct tie between the energy of vacuum fluctuations and macroscopic physics than H. B. G. Casimir, *Proc. Nederland Akad. Wetenschappen, Amsterdam* **60**, 793 (1948), who predicted a force between two parallel metal plates. No attempt is made here to cite the extensive literature that verifies the existence and the predicted magnitude of this force. The same kind of fluctuations that are verified by this force at macroscopic distances are also checked at distances $\sim 10^{-12}$ cm by the Lamb shift, the most impressive single development in quantum electrodynamics in the post-World War II period (Fig. 4 and [19]).

38. D. R. Brill and J. B. Hartle, *Phys. Rev.* **135**, B271 (1964).

39. For a historical survey that treats chemistry and atomic physics as the two parts of a single development, see for example: W. G. Palmer, *A History of the Concept of Valency to 1930*, Cambridge Univ. Press, 1965, and especially J. J. Lagowski, *The Chemical Bond*, Houghton Mifflin, Boston, 1966.

40. On these regularities see for example the books cited in [32].

41. J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Phys. Rev.* **108**, 1175 (1957).

42. The three issues listed here are taken up in more detail in the Appendix.

43. A. Einstein, in P. A. Schilpp, ed., *Albert Einstein: Philosopher Scientist* (Library of Living Philosophers, Evanston, Illinois, 1949, p. 81) remarks: "If one had the field-equation of the total field, one would be compelled to demand that the particles themselves would *everywhere* be describable as singularity-free solutions of the completed field-equations. Only then would the general theory of relativity be a *complete* theory."

44. E. Cartan, *Leçons sur la géométrie des espaces de Riemann*, Gauthier-Villars, Paris, 2nd ed., 1959, Chapter 8.

45. J. A. Wheeler, Chapter 4 on Cartan's geometrical interpretation of Einstein's field equations in *Gravitation and Relativity* (H. Y. Chiu and W. F. Hoffmann, eds.), Benjamin, New York, 1964.

46. Arguments that the propagator appropriate for any particle of spin two and mass zero necessarily has a leading term of the form

$$k^{-2}(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\alpha\beta}g^{\mu\nu})$$

are given by S. Weinberg, *Phys. Rev.* **138**, B988 (1965); also in expanded form in his contribution to S. Deser and K. W. Ford, eds., Brandeis Summer Institute in Theoretical Physics, 1964, Vol. 2, *Lectures on Particle and Field Theory*, Prentice-Hall, Englewood Cliffs, New Jersey, 1965.

47. See for example W. Pauli in "Die allgemeinen Prinzipien der Wellenmechanik" in *Handbuch der Physik* (Geiger and Scheel, eds.), Springer, Berlin, 1933, Vol. 24, part 1; reprinted in revised form in the new *Handbuch der Physik* (S. Flügge, ed.), Springer, Berlin, 1958, Vol. 5, part 1.

48. See the discussion of the problem of factor ordering in B. DeWitt, QTG; also the references to earlier discussions of this issue cited by him there.

49. For a determination of the wave equation from (1) the principle of Lorentz covariance and (2) a selection of one or another set of spin quantum numbers see E. P. Wigner, *Ann. of Math.* **40**, 149 (1939) and V. Bargmann and E. P. Wigner, *Proc. Natl. Acad. Sci. U.S.A.* **34**, 211 (1946), both reprinted in the collection *Symmetry Groups in Nuclear and Particle Physics* (F. J. Dyson, ed., Benjamin, New York, 1966.

50. For an analysis of the metric of superspace see B. DeWitt, QTG [2], and other work cited by DeWitt; see also S. Weinberg [46], for the propagator in the linear theory of gravitation in the de Donder gauge, which mysteriously has the same form as the metric in superspace (but indices 0, 1, 2, 3 as compared to 1, 2, 3).

51. For the proof that the topology of space cannot change within the context of classical geometrodynamics, see R. P. Geroch, *J. Math. Phys.* **8**, 782 (1967).

52. For a model of a closed universe with the topology S_3 put together out of 720 identical pieces each endowed with the Schwarzschild geometry ("lattice universe") see R. W. Lindquist and J. A. Wheeler, *Rev. Mod. Phys.* **29**, 432 (1957) and further treatment in GMD & IFS, pp. 370–379.

53. A. Einstein, end of chapter dealing with Mach' principle in *The Meaning of Relativity*, Princeton Univ. Press, Princeton, New Jersey, 3rd ed., 1950.

54. See for example some of the solutions of Einstein's equations given by B. K. Harrison, *Phys. Rev.* **116**, 1285 (1959) and his fuller Princeton University Ph.D. thesis, *Exact Three-Variable Solutions of the Field Equations of General Relativity*, 1959 (unpublished).

55. Here it is assumed that the conjecture of H. Poincaré is correct, that every simply connected compact differentiable three-dimensional manifold is homeomorphic to the three-sphere. See C. D. Papakyriakopoulos, "The theory of differentiable manifolds since 1950," *Proc. Intern. Congr. Mathematicians, 1958* (Cambridge Univ. Press, 1960), pp. 433–440; also J. Milnor, *Topology from the Differentiable Viewpoint*, Univ. of Virginia Press, Charlottesville, 1965, and the bibliography cited by Milnor; also J. Derwent, "Handle decomposition of manifolds," *J. Math. Mech.* **15**, 329 (1966).

56. Particularly to be emphasized is the distinction between "asymptotically flat" as that concept is so often understood in the classical context of a 4-geometry, and the concept of flatness as it is applied to a 3-geometry in the context of Hamilton–Jacobi theory or quantum geometrodynamics. No example illustrates this distinction more clearly than the Schwarzschild geometry. There the rate of approach of the 4-geometry to flatness at infinity determines always a unique value for the mass of the center of attraction; but the analogous calculation for a spacelike 3-geometry slicing through the 4-geometry gives quite different values for the apparent mass, depending upon the choice of slice. Thus, in the 4-geometry

$$ds^2 = -(1 - 2m/r)dt^2 + (1 - 2m/r)^{-1}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

take at large distances the spacelike slice

$$t = t_0 + (8\alpha r)^{1/2}$$

so that

$$dt = \left(\frac{2\alpha}{r}\right)^{1/2} dr$$

On this slice one finds a 3-geometry in which the coefficient of dr^2, also at large distances, is

$$1 + \frac{2(m-\alpha)}{r}$$

The dependence of the "effective mass" $(m - \alpha)$ upon the choice of slice, through the parameter α, suggests some of the many dangers that seem to lurk in the concept of "asymptotic flatness" as applied to *three*-geometries.

57. In the Taub universe the effective mass-energy arises entirely from excitation of that mode of gravitational radiation which has the longest wavelength capable of fitting into this universe. For the metric of this model see A. Taub, *Ann. of Math.* **53**, 472 (1959) and C. W. Misner, *J. Math. Phys.* **4**, 924 (1963).

58. To say that a particle is "pictured in terms of space resonating from one topology to another" means more precisely that it is "pictured as a geometrodynam-

ical exciton—a state of excitation in which space resonates from one topology and geometry to another according to a probability amplitude function slightly different from, and orthogonal to, the probability amplitude function $\psi({}^{(3)}\mathcal{G})$ that describes the vacuum."

59. For a systematic development of "already unified field theory" see C. W. Misner and J. A. Wheeler, *Annals of Physics* **2**, 525 (1957) (reprinted in GMD) where also reference is made to the earlier work of G. Y. Rainich.

60. In "already unified field theory" the electromagnetic field tensor is expressed, in accordance with Einstein's field equations, in terms of the "Maxwell square root" of the Ricci curvature tensor and its dual

$$F_{\mu\nu} = (\text{“}R^{1/2}\text{”})_{\mu\nu} \cos\alpha + {}^*(\text{“}R^{1/2}\text{”})_{\mu\nu} \sin\alpha$$

The change of the "complexion" α of the electromagnetic field from place to place is fully determined by Maxwell's equations in places where there is a field. However, consider a spacelike initial value hypersurface. On this hypersurface consider two regions, I, II, endowed with field and separated by a region III free of field. Within each region individually the relative complexion is well determined, which is all that matters momentarily for the electrodynamics. However, the complexion of region II relative to region I can never be found from purely geometrical measurements limited to this initial spacelike hypersurface. Moreover, this relative complexion is all important for the dynamic development of the electromagnetic field at those later points in space-time that can be reached by disturbances both from I and from II. In this sense the initial value problem of already unified field theory does not lend itself to purely geometrical formulation. For more on this topic see the chapter by L. Witten in the book of which he is also the editor, *Gravitation: An Introduction to Current Research*, Wiley, New York, 1962.

61. The divergence condition is well known to follow from the invariance of the Hamilton–Jacobi function with respect to the gauge transformation, $A_i^{\text{new}} = A_i + \partial\lambda/\partial x^i$; thus,

$$\begin{aligned} 0 = \delta S &= \int \left(\frac{\delta S}{\delta A_i}\right) \delta A_i \, d^3x \\ &= \int \left(\mathfrak{E}^i \frac{\partial\lambda}{\partial x^i}\right) d^3x \\ &= -\int \mathfrak{E}^i{}_{,i} \lambda \, d^3x \end{aligned}$$

The vanishing of this expression for arbitrary λ gives the desired relation. It should be emphasized that the quantity $\mathfrak{E}^i$ as employed here is not a contravariant vector, but $(g)^{1/2}$ times a contravariant vector ("vector density"). Were the contravariant vector itself employed, the divergence relation would have to be expressed in terms of covariant derivatives rather than ordinary derivatives, complicating the derivation in the text.

62. Here the symbol $(\mathfrak{E} \times B)_j$ stands for the covariant vector density $\mathfrak{E}^i B_{ij}$.

63. As Hermann Weyl emphasized long ago, *Math. Z.* **23**, 271 (1925), "In den geometrischen und physikalischen Anwendungen zeigte sich stets, dass eine Grössart nicht allein durch Angabe der Tensorstufe, sondern durch Symmetrie bedingungen charakterisiert ist." In other words, every physical quantity is represented by an irreducible tensorial quantity; that is to say, by what S. S. Chern terms "a geometrical object." Weyl conceived of these geometrical entities as local. However, it is a natural extension of his line of thought to speak of a *functional* S or ψ that depends *globally* upon a 3-geometry, and upon a 2-form imbedded in that 3-geometry.

64. GMD [2], p. 88.

65. John Milnor, "A survey of cobordism theory," *L'enseignement mathématique* **8**, 16 (1962); "Spin structures on manifolds," *ibid.* **9**, 198 (1963); "On the Stiefel–Whitney numbers of complex manifolds and of spin manifolds," *Topology* **3**, 223 (1965); "Remarks concerning spin manifolds" in S. S. Cairns, ed., *Differential and Combinatorial Topology*, Princeton Univ. Press, Princeton, New Jersey, 1965, p. 55; Andre Lichnerowicz, *Compt. Rend. Acad. Sci. Paris* **252**, 3742 (1961), **253**, 940 (1961), and **253**, 983 (1961), and summary of these results in the third part of the chapter by Lichnerowicz, "Propagateurs, Commutateurs et Anticommutateurs en Relativité Générale," in C. and B. DeWitt, eds., *Relativity, Groups and Topology*, Gordon and Breach, New York, 1964; D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson, "Spin cobordism," *Bull. Am. Math. Soc.* **72**, 256 (1966); "*SU*-cobordism, *KO*-characteristic numbers and the Kervaire invariant," *Ann. of Math.* **83**, 54 (1966); W. C. Hsiang and B. J. Sanderson, "Twist-spinning spheres in spheres," *Illinois J. Math.* **9**, 651 (1965). Appreciation is expressed to John Milnor, Roger Penrose, and Robert Geroch for discussions clarifying the concept of spin structure.

66. Take a belt. Stretch it out flat and taut. Keeping the left-hand end A fixed in the left hand, twist the right-hand end B through 720°. Maintaining A and B all the time parallel to their present orientations, move B in a complete circle about A (releasing for an instant one's hold on B). The belt straightens out. Not so when there is only a 360° twist in it. The belt is relevant to the cube, the room, and the eight elastic strings. Before the cube is rotated at all, it can be pulled out through a window to some distance from the room. The eight elastic strings then take on the configuration of the belt. The distinction between 360° and 720° rotation for the belt applies equally to the "pseudo-belt" made up of the eight strings. Another way of seeing that a 720° rotation restores the orientation entanglement relation between the cube and its surroundings (picture of one cone rolling on another) is presented by R. Penrose and W. Rindler in a preprint of an appendix to a book that they have in preparation. Appreciation is expressed to Professor Penrose for the privilege of seeing this preprint.

67. The possibility has been suggested that one may eventually be able to detect what is here called the "orientation entanglement relation" between an object and its surroundings by measuring the contact potential between one metallic object (subject to rotation) and another (held fixed): Y. Aharonov and L. Susskind, *Phys. Rev.* **158**, 1237 (1967).

68. Cf. section on orientability, Problem 2, Level 1.

69. It is not new to abstractify geometry. Einstein's curved space-time was in the beginning nothing if it was not a home for geodesics. How else, one asked,

could he predict a planetary motion. Later Einstein, Grommer, Infeld, and Hoffman threw out the geodesics. The field equations themselves, they showed, predict the evolution of geometry with time, and hence the motion of concentrations of mass-energy.

70. For a history of the concept of spin, see W. Pauli's 1945 Nobel prize lecture, *Exclusion Principle and Quantum Mechanics*, Editions Grisson, Neuchatel, 1947, also the relevant discussion in M. Fierz and V. F. Weisskopf, *Theoretical Physics in the Twentieth Century: A Memorial Volume to Wolfgang Pauli*, Interscience, New York, 1960.

71. See for example C. Kittel, *Introduction to Solid-State Physics*, 3rd ed., Wiley, New York, 1966.

72. The torus can be converted into a single sheet by two cuts, and can then be conceived as laid out on the complex plane, with one corner at the origin. One adjacent corner can be identified arbitrarily with the number $1 + 0i$, by appropriate choice of scale ("conformal transformation"). The location of the other adjacent corner $\tau = \tau_1 + i\tau_2$ is then completely determined. Also completely determined is the concomitant 2-geometry, modulo the group of conformal transformations. The quantity τ can be identified with the complex parameter mentioned in the text.

73. For a survey, with many references to the literature, see H. E. Rauch, "A transcendental view of the space of algebraic Riemann surfaces," *Bull. Am. Math. Soc.* **71**, 1 (1965). Appreciation is expressed to Professor Leon Ehrenpreis for elucidation of the subject and for this and the following reference.

74. Lipman Bers, *On the Moduli of Riemann Surfaces*; lectures at the Forschunginstitut für Mathematik, Eidgenössische Technische Hochschule, Zürich, 1964. Notes by L. M. and R. J. Sibner (mimeographed).

75. For the initial value equations of classical geometrodynamics see G. Darmois, *Les équations de la gravitation einsteinienne*, Gauthier-Villars, Paris, 1927; K. Stellmacher, *Math. Ann.* **115**, 136 (1937); A. Lichnerowicz, *J. math. pure appl.* **23**, 37 (1944); *Helv. Phys. Acta Suppl.* **4**, 176 (1956); *Théories relativistes de la gravitation et de l'électromagnétisme*, Masson, Paris, 1955; Yvonne Fourés-Bruhat, *Acta Math.* **88**, 141 (1952); *J. Rational Mech. Anal.* **5**, 951 (1956); and the chapter by Y. Fourés (now Y. Choquet) in Louis Witten, ed., *Gravitation: An Introduction to Current Research*, Wiley, New York, 1962.

76. H. Weyl, *Philosophy of Mathematics and Natural Science* (original German in 1927; translation by O. Helmer), Princeton Univ. Press, Princeton, New Jersey, 1949, p. 91.

77. GMD & IFS, pp. 495–499.

78. An extensive list of problems open for further investigation is to be found in GMD & IFS.

79. H. Leutwyler in *Battelle Rencontres: 1967 Lectures in Mathematics and Physics*, this volume p. 309.

80. One is reminded in this connection of the statement of William James over a half a century ago, that "Actualities seem to float in a wider sea of possibilities

from out of which they were chosen; and *somewhere*, indeterminism says, such possibilities exist, and form a part of truth." Appreciation is expressed to Paul Van de Water for this quotation.

81. E. P. Wigner, *Symmetries and Reflections*, Indiana Univ. Press, Bloomington, Indiana, 1967; see J. M. Jauch, E. P. Wigner, and M. M. Yanase, *Nuovo Cimento* **48**, 144 (1967), also B. DeWitt, QTG; also H. Everett III, *Rev. Mod. Phys.* **29**, 454 (1957); J. A. Wheeler, *Rev. Mod. Phys.* **29**, 463 (1957) and GMD, p. 75.