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Emergent Spacetime and Cosmic Inflation I

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ABSTRACT

We propose a background-independent formulation of cosmic inflation. The inflation in this picture corresponds to a dynamical process to generate space and time while the conventional inflation is simply an (exponential) expansion of a preexisting spacetime owing to the vacuum energy carried by an inflaton field. We observe that the cosmic inflation is triggered by the condensate of Planck energy into vacuum responsible for the dynamical emergence of spacetime and must be a single event according to the exclusion principle of noncommutative spacetime caused by the Planck energy condensate in vacuum. The emergent spacetime picture admits a background-independent formulation so that the inflation can be described by a conformal Hamiltonian system characterized by an exponential phase space expansion without introducing any inflaton field as well as an *ad hoc* inflation potential. This implies that the emergent spacetime may incapacitate all the rationales to introduce the multiverse hypothesis. In Part I we will focus on the physical foundation of cosmic inflation from the emergent spacetime picture to highlight the main idea. Its mathematical exposition will be addressed in Part II.

Keywords: Emergent spacetime, Cosmic inflation, Quantum gravity

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1 Introduction

History is a mirror to the future. If we do not learn from the mistakes of history, we are doomed to repeat them.[1] In the middle of the 19th century, Maxwell's equations for electromagnetic phenomena predicted the existence of an absolute speed, $c = 2.998 \times 10^8$ m/sec, which apparently contradicted the Galilean relativity, a cornerstone on which the Newtonian model of space and time rested. Since most physicists, by then, had developed deep trust in the Newtonian model, they concluded that Maxwell's equations can only hold in a specific reference frame, called the ether. However, by doing so, they reverted back to the Aristotelian view that Nature specifies an absolute rest frame. It was Einstein to realize the true implication of this quandary: It was asking us to abolish Newton's absolute time as well as absolute space. The ether was removed by the Einstein's special relativity by radically modifying the concept of space and time in the Newtonian dynamics. Time lost its absolute standing and the notion of absolute simultaneity was physically untenable. Only the four-dimensional spacetime has an absolute meaning. The new paradigm of spacetime has completely changed the Newtonian world with dramatic consequences.

The physics of the last century had devoted to the study of two pillars: general relativity and quantum field theory. And the two cornerstones of modern physics can be merged into beautiful equations, the so-called Einstein equations given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}, \tag{1.1}$$

where the right-hand side is the energy-momentum tensor whose contents are described by (quantum) field theories. Although the revolutionary theories of relativity and quantum mechanics have utterly changed the way we think about Nature and the Universe, new open problems have emerged which have not been resolved yet within the paradigm of the 20th century physics. For example, a short list of them is the cosmological constant problem, the hierarchy problem, dark energy, dark matter, cosmic inflation and quantum gravity. In particular, recent developments in cosmology, particle physics and string theory have led to a radical proposal that there could be an ensemble of universes that might be completely disconnected from ours [1]. Of course, it would be perverse to claim that nothing exists beyond the horizon of our observable universe. The observable universe is one causal patch of a much larger unobservable universe. However, a painful direction is to use the string landscape or multiverse to explain some notorious problems in theoretical physics based on the anthropic argument [2]. "And it's pretty unsatisfactory to use the multiverse hypothesis to explain only things we don't understand."[2] Taking history as a mirror, this situation is very reminiscent of the hypothetical luminiferous ether in the late 19th century. Looking forward to the future, we may need another turn of the spacetime picture to defend the integrity of physics.

[1]George Santayana (1863-1952).

[2]Graham Ross in *Quanta magazine* "At multiverse impasse, a new theory of scale" (August 18, 2014) and *Wired.com* "Radical new theory could kill the multiverse hypothesis."

In physical cosmology, cosmic inflation is the exponential expansion of space in the early universe. Suppose that spacetime evolution is determined by a single scale factor $a(t)$ and its Hubble expansion rate $H \equiv \frac{\dot{a}}{a}$ according to the cosmological principle and driven by the dynamics of a scalar field ϕ, called the inflaton [3, 4]. Then the Einstein equation (1.1) reduces to the Friedmann equation

$$H^2 = \frac{8\pi G_N}{3}\Big(\frac{1}{2}\dot{\phi}^2 + V(\phi)\Big). \tag{1.2}$$

The evolution equation of the inflaton in the Friedmann universe is described by

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\delta V}{\delta \phi} = 0. \tag{1.3}$$

The Friedmann equation (1.2) tells us that in the early universe with $V(\phi) \approx V_0$ and $\dot{\phi} \approx 0$, there was an inflationary epoch of the exponential expansion of space, i.e., $a(t) \propto e^{Ht}$ where $H = \sqrt{\frac{8\pi G_N V_0}{3}}$ is called the inflationary Hubble constant. In order to successfully fit to data, one finds [3, 4]

$$V_0 \geq (2 \times 10^{16}\mathrm{GeV})^4 \approx (10^{-2} M_P)^4 \tag{1.4}$$

where $M_P = 1/\sqrt{8\pi G_N}$ is the Planck mass.

Let us contemplate the inflationary scenario with a critical eye. According to this scenario [3, 4], inflation is described by the exponential expansion of the universe in a supercooled false vacuum state that is a metastable state without any fields or particles but with a large energy density. It should be emphasized that the inflation scenario so far has been formulated in the context of effective field theory coupled to general relativity. Thus, in this scenario, the existence of space and time is *a priori* assumed from the beginning although the evolution of spacetime is determined by Eq. (1.1). In other words, the inflationary scenario does not describe any generation (or creation) of spacetime but simply characterizes an expansion of a preexisting spacetime. It never addresses the (dynamical) origin of spacetime. However, there has to be a definite beginning to an inflationary universe [5]. This means that the inflation is incomplete to describe the very beginning of our universe and some new physics is needed to probe the past boundary of the inflating regions. One possibility is that there must have been some sort of quantum creation event as a beginning of the universe [6].

The Friedmann equation (1.2) shows that the cosmic inflation is triggered by the potential energy carried by an inflaton whose energy scale is near the Planck energy over which quantum gravity effects become strong and effective field theory description may be broken down. Although an inflating false vacuum is metastable, essentially all models of inflation lead to eternal inflation to the future since expansion rate is much greater than decay rate [3]. Once inflation starts, it never stops. If one identifies the slowly varying inflaton field $\phi(t)$ with a particle trajectory $x(t) = \phi(t)$ and $\dot{\phi}(t)$ with its velocity $v(t) = \dot{x}(t)$, the evolution equation (1.3) tells us that the frictional force, $3Hv(t)$, caused by the inflating spacetime is (almost) balanced with an external force $F(x) = -\frac{dV}{dx}$, i.e.,

$$\dot{x}(t) \approx \frac{F(x)}{3H}, \tag{1.5}$$

because $\ddot{x} \approx 0$ during inflation. This implies that the cosmic inflation as a dynamical system corresponds to a non-Hamiltonian system.[3]

Recent developments in string theory have revealed a remarkable and radical new picture about gravity. For example, the AdS/CFT correspondence illustrates a surprising picture that $U(N)$ gauge theory in lower dimensions defines a nonperturbative formulation of quantum gravity in higher dimensions [7]. In particular, the AdS/CFT duality shows a typical example of emergent gravity and emergent space because gravity in higher dimensions is defined by a gravityless field theory in lower dimensions. Now we have many examples from string theory in which spacetime is not fundamental but only emerges as a large distance, classical approximation [8]. Therefore, the rule of the game in quantum gravity is that space and time are an emergent concept. Since the emergent spacetime, we believe, is a significant new paradigm for quantum gravity, we want to apply the emergent spacetime picture to cosmic inflation. We will propose a background-independent formulation of the cosmic inflation.[4] This means that we do not assume the prior existence of spacetime but define a spacetime structure as a solution of an underlying background-independent theory such as matrix models. The inflation in this picture corresponds to a dynamical process to generate space and time which is very different from the standard inflation simply describing an (exponential) expansion of a preexisting spacetime. It turns out that spacetime is emergent from the Planck energy condensate in vacuum that generates an extremely large *Universe*. Our observable patch within cosmic horizon is a very tiny part $\sim 10^{-60}$ of the entire spacetime. Originally the multiverse hypothesis has been motivated by an attempt to explain the anthropic fine-tuning such as the cosmological constant problem [9] and boosted by the chaotic and eternal inflation scenarios [3, 4] and the string landscape derived from the Kaluza-Klein compactification of string theory [10, 11], which are all based on the traditional spacetime picture. Since emergent spacetime is radically different from any previous physical theories, all of which describe what happens in a given spacetime, the multiverse picture must be reexamined from the standpoint of emergent spacetime. The cosmic inflation from the emergent spacetime picture will certainly open a new prospect that may cripple all the rationales to introduce the multiverse hypothesis.

Since the concept of the multiverse raises deep conceptual issues even to require to change our view of science itself [2], it should be important to ponder on the real status of the multiverse whether it is simply a mirage developed from an incomplete physics like the ether in the late 19th century or it is of vital importance even in more complete theories. The main purpose of this paper is to illuminate how the emergent spacetime picture brings about radical changes of physics, especially, regarding to physical cosmology. In particular, a background-independent theory such as matrix models provides a

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[3]Nonetheless, the friction term does not lead to dissipative energy production. This fact can be seen by observing that Eq. (1.3) can be derived from the first law of thermodynamics, $dE + pdV = Vd\rho + (\rho + p)dV = 0$, where $\rho + p = \dot{\phi}^2$ and $\dot{\rho} = \left(\ddot{\phi} + \frac{\delta V}{\delta \phi}\right)\dot{\phi}$.

[4]Here we refer to a background-independent theory in which any spacetime structure is not *a priori* assumed but defined by the theory.

concrete realization of the idea of emergent spacetime which has a sufficiently elegant and explanatory power to defend the integrity of physics against the multiverse hypothesis. The emergent spacetime is a completely new paradigm so that the multiverse debate in physics circles has to seriously take it into account.

This is the first installment of a series of papers whose aim is to propose the cosmic inflation from emergent spacetime picture. In Part I we will focus on the physical motivation and argumentation to highlight the main idea, deferring the mathematical exposition to Part II. The Part II is intended to be self-contained as much as possible and mathematical backgrounds underlying our arguments will also be briefly reviewed in two Appendices. The Part I is organized as follows.

In Sec. 2, we explain the physical picture depicted in Figs. 1 and 2, whose mathematical exposition will be addressed in Part II. The background-independent formulation of emergent gravity crucially relies on the fact that noncommutative (NC) space arises as a solution of a large N matrix model in the Coulomb branch and this vacuum on the Coulomb branch admits a separable Hilbert space as quantum mechanics [12]. The gravitational metric is derived from a nontrivial inner automorphism of the NC algebra $\mathcal{A}_\theta$, in which the NC nature is essential to realize the emergent gravity [13, 14, 15, 16]. See also closely related works [17, 18, 19, 20]. An important point is that the matrix model does not presuppose any spacetime background on which fundamental processes develop. Rather the background-independent theory provides a mechanism of spacetime generation such that any spacetime structure including the flat spacetime arises as a solution of the theory itself [15].

In Sec. 3, we observe that the NC spacetime is caused by the Planck energy condensate responsible for the generation of spacetime and results in an extremely large spacetime. We demonstrate why the emergent gravity clearly resolves the notorious cosmological constant problem [13, 14]. A principal reason is that the huge vacuum energy being a perplexing cosmological constant in general relativity was simply used to generate flat spacetime and thus does not gravitate. The emergent gravity is in stark contrast to general relativity since it does not allow the coupling of the cosmological constant [21]. We note that the Planck energy condensate into vacuum must be a dynamical process and show that the cosmic inflation arises as a solution of a time-dependent matrix model, describing the dynamical process of the vacuum condensate. It turns out that the cosmic inflation corresponds to the dynamical mechanism for the instantaneous condensation of vacuum energy to enormously spread out spacetime. It is remarkable to see that the inflation can be described by time-dependent matrices only without introducing any inflaton field as well as an *ad hoc* inflation potential. Our work is not the first to address physical cosmology using matrix models. There have been interesting earlier attempts [22]. In particular, the cosmic inflation was addressed in very interesting works [23] using the Monte Carlo analysis of the type IIB matrix model in Lorentzian signature and it was found that three out of nine spatial directions start to expand at some critical time after which exactly (3+1)-dimensions dynamically become macroscopic.

In Sec. 4, we discuss why the emergent spacetime picture may incapacitate all the rationales to introduce the multiverse hypothesis. Since the emergent spacetime picture is radically different from the conventional picture in general relativity so that they are exclusive and irreconcilable each other, we reconsider main sources to introduce the multiverse hypothesis from the standpoint of emergent spacetime: (A) cosmological constant problem, (B) chaotic and eternal inflation scenarios, (C) string landscape. We argue [24] that the emergent spacetime certainly opens a new perspective that may cripple all the rationales to introduce the multiverse hypothesis.

2 Emergent spacetime from large N duality

String theory is defined by replacing particles (point-like objects) with strings (one-dimensional objects). In order to do this, we need to introduce a *new* constant α' whose physical dimension is $(\mathrm{length})^2$. It is well-known that the new constant α' introduces a new duality depicted by $R \to R' = \frac{\alpha'}{R}$. This is known as the T-duality in string theory [25], but it is not possible in particle theories $(\alpha' = 0)$. It is important to notice that a new physical constant such as $\hbar$ and α' introduces a deformation of some structure in a physical theory [13, 14]. For instance, the Planck constant $\hbar$ in quantum mechanics carries the physical dimension $[\hbar] = (\mathrm{length}) \times (\mathrm{momentum})$ and so it deforms the algebraic structure of particle phase space from commutative to NC space, i.e.,

$$xp - px = 0 \qquad \Rightarrow \qquad xp - px = i\hbar. \tag{2.1}$$

An educated reasoning motivated by the fact that $[\alpha'] = (\mathrm{length}) \times (\mathrm{length})$ leads to a natural speculation that α' brings about the deformation of the algebraic structure of spacetime itself such that

$$xy - yx = 0 \qquad \Rightarrow \qquad xy - yx = i\alpha'. \tag{2.2}$$

From the deformation theory point of view, replacing particles with strings is equivalent to the transition from commutative space to NC space. This may be supported by the fact that the NC space (2.2) defines only a minimal area whereas the concept of point is doomed as if $\hbar$ in quantum mechanics introduces a minimal area in the NC phase space (2.1). The minimal surface in the NC space (2.2) acts as a basic building block of string theory and behaves like the smallest units of spacetime blob. Remarkably the deformation (2.2) provides us an important clue for a background-independent formulation of string theory as will be discussed in Part II.

It turns out [15] that the NC space (2.2) denoted by $\mathbb{R}^2_{\alpha'}$ is much more radical and mysterious than we thought. In order to understand NC spacetime correctly, we need to deactivate the thought patterns that we have installed in our brains and taken for granted for so many years. The reason is the following. As we have learned from quantum mechanics, the NC phase space (2.1) introduces the wave-particle duality. Indeed the NC space (2.2) also brings about a radical change of physics since the NC nature of spacetime is responsible for a new kind of duality, known as the gauge-gravity duality. The underlying mathematical principle is the well-known duality between geometry and algebra. A primary cause of the radical change of physics in quantum mechanics is that the NC phase space (2.1) introduces a *complex* vector space called the Hilbert space [26]. This is also true for the NC space (2.2) since its mathematical structure is essentially the same as quantum mechanics. Similarly to quantum mechanics, the NC space $\mathbb{R}^2_{\alpha'}$ also admits a nontrivial inner automorphism. For example, for an arbitrary NC field $f(x,y)$, we have the relation given by

$$f(x+a,y) = U(a)^\dagger f(x,y) U(a), \qquad f(x,y+b) = U(b)^\dagger f(x,y) U(b) \tag{2.3}$$

where $U(a) = \exp(-\frac{iay}{\alpha'})$ and $U(b) = \exp(\frac{ibx}{\alpha'})$. Thus a striking feature of the NC space is that every points are unitarily equivalent because translations in $\mathbb{R}^2_{\alpha'}$ are simply a unitary transformation acting on the Hilbert space $\mathcal{H}$. This means that the concept of space is doomed and the classical space is replaced by a state in the Hilbert space $\mathcal{H}$. This fact leads to an important picture that classical spacetime is somehow a derived concept and a NC algebra and its Hilbert space play a more fundamental role. In other words, NC spacetime necessarily implies emergent spacetime if spacetime at microscopic scales should be viewed as NC and any dynamical variable defined on $\mathbb{R}^2_{\alpha'}$ becomes an operator acting on the Hilbert space $\mathcal{H}$. In particular, any NC field can be regarded as a linear operator acting on the Hilbert space. Note that the NC space (2.2) is equivalent to the Heisenberg algebra of harmonic oscillator, i.e. $[a, a^\dagger] = 1$, if the annihilation operator is defined by $a = \frac{1}{\sqrt{2\alpha'}}(x+iy)$. Thus the Hilbert space for $\mathbb{R}^2_{\alpha'}$ is the Fock space and has a countable basis. Therefore the representation of NC fields on the Hilbert space $\mathcal{H}$ is given by $N \times N$ matrices where $N = \dim(\mathcal{H}) \to \infty$. Consequently, the NC space (2.2) leads to an interesting equivalence between a lower-dimensional large N gauge theory and a higher-dimensional NC $U(1)$ gauge theory [15].

To be specific, let us consider a $2n$-dimensional NC space denoted by $\mathbb{R}^{2n}_\theta$ whose coordinate generators obey the commutation relation

$$[y^a, y^b] = i\theta^{ab}, \qquad a,b = 1, \cdots, 2n, \tag{2.4}$$

where $(\theta)^{ab} = \alpha'(\mathbf{1}_n \otimes i\sigma^2)$ is a $2n \times 2n$ constant symplectic matrix and $l_s \equiv \sqrt{\alpha'}$ is a typical length scale set by the vacuum. The NC space $\mathbb{R}^2_{\alpha'}$ corresponds to the $n=1$ case. Let us denote the NC $\star$-algebra generated by $\mathbb{R}^{2n}_\theta$ by $\mathcal{A}_\theta$. Similarly to the $n=1$ case, the NC space (2.4) is equivalent to the Heisenberg algebra of n-dimensional harmonic oscillator. Hence the underlying Hilbert space on which $\mathcal{A}_\theta$ acts is given by the Fock space defined by

$$\mathcal{H} = \{ |\vec{n}\rangle \equiv |n_1, \cdots, n_n\rangle |\, n_i \in \mathbb{Z}_{\geq 0},\ i = 1, \cdots, n\}, \tag{2.5}$$

which is orthonormal, i.e., $\langle \vec{n}|\vec{m}\rangle = \delta_{\vec{n},\vec{m}}$ and complete, i.e., $\sum_{\vec{n}=0}^{\infty} |\vec{n}\rangle\langle\vec{n}| = \mathbf{1}_{\mathcal{H}}$, as is well-known from quantum mechanics. Since the Fock space (2.5) has a countable basis, it is convenient to introduce a one-dimensional basis using the "Cantor diagonal method" to put the n-dimensional non-negative integer lattice in $\mathcal{H}$ into one-to-one correspondence with the natural numbers:

$$\mathbb{Z}^n_{\geq 0} \leftrightarrow \mathbb{N} : |\vec{n}\rangle \leftrightarrow |n\rangle,\ n = 1, \cdots, N \to \infty. \tag{2.6}$$

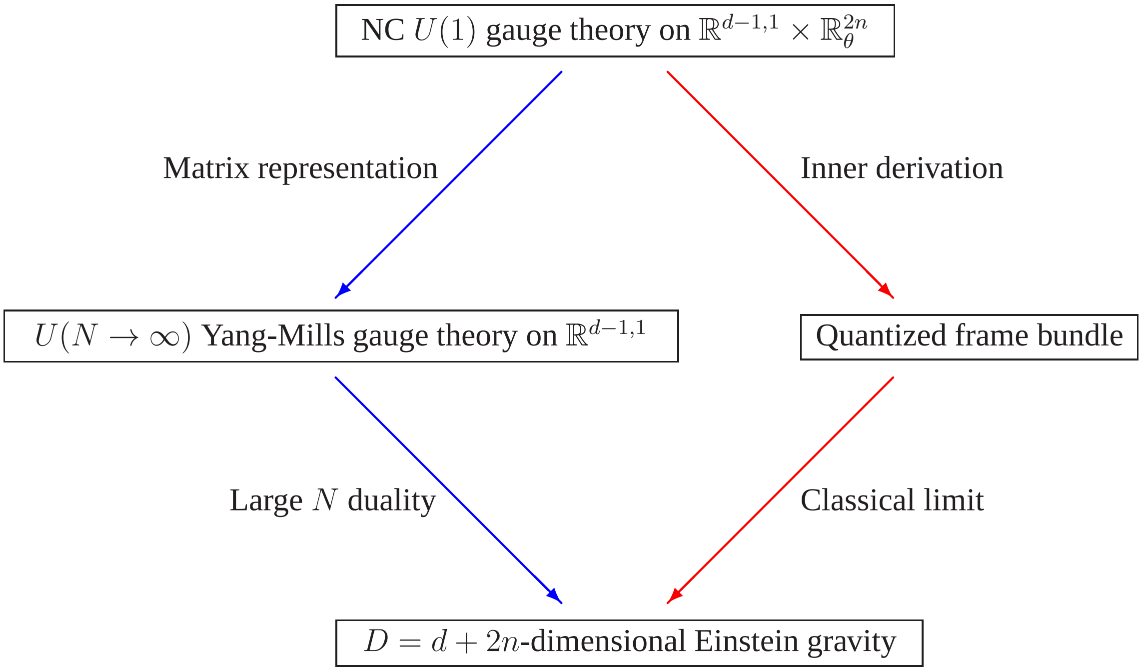

Figure 1: Flowchart for emergent gravity

In this one-dimensional basis, the completeness relation of the Fock space (2.5) is now given by $\sum_{n=1}^\infty |n\rangle\langle n| = \mathbf{1}_\mathcal{H}$. Since NC fields in $\mathcal{A}_\theta$ are linear operators acting on the Fock space $\mathcal{H}$, the representation of the NC fields in $\mathcal{A}_\theta$ is given by $N \times N$ matrices in $\mathrm{End}(\mathcal{H}) \equiv \mathcal{A}_N$ where $N = \dim(\mathcal{H}) \to \infty$. Here we have denoted the set of $N \times N$ matrices in $\mathrm{End}(\mathcal{H})$ by $\mathcal{A}_N$. In the one-dimensional basis (2.6), the trace over $\mathcal{A}_\theta$ can also be transformed into the trace over $N \times N$ matrices in $\mathcal{A}_N$, i.e.,

$$\int_M \frac{d^{2n}y}{(2\pi)^n |\mathrm{Pf}\theta|} = \mathrm{Tr}_\mathcal{H} = \mathrm{Tr}_N. \tag{2.7}$$

Using the matrix representation, one can show [13, 27, 28, 29] that the $D = (d+2n)$-dimensional NC $U(1)$ gauge theory on $\mathbb{R}^{d-1,1} \times \mathbb{R}^{2n}_\theta$ is exactly mapped to the d-dimensional $U(N \to \infty)$ Yang-Mills theory on $\mathbb{R}^{d-1,1}$:

$$S = -\frac{1}{G^2_{YM}} \int d^D Y \frac{1}{4} (\widehat{F}_{AB} - B_{AB})^2 \tag{2.8}$$

$$= -\frac{1}{g^2_{YM}} \int d^d x \mathrm{Tr}\Big(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi_a D^\mu \phi_a - \frac{1}{4} [\phi_a, \phi_b]^2 \Big) \tag{2.9}$$

where $G^2_{YM} = (2\pi)^n |\mathrm{Pf}\theta| g^2_{YM}$ and

$$B_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab} \end{pmatrix}.$$

We emphasize that the equivalence between the D-dimensional NC $U(1)$ gauge theory (2.8) and d-dimensional $U(N \to \infty)$ Yang-Mill theory (2.9) is not a dimensional reduction but an exact mathematical identity. A remarkable point is that the large N gauge theories described by the action (2.9) arise as a nonperturbative formulation of string/M theories [30]. For instance, we get the IKKT matrix model for $d = 0$ [31], the BFSS matrix quantum mechanics for $d = 1$ [33] and the matrix string theory for $d = 2$ [34]. The most interesting case arises for $d = 4$ and $n = 3$ which suggests an engrossing duality [12] that the 10-dimensional NC $U(1)$ gauge theory on $\mathbb{R}^{3,1} \times \mathbb{R}^6_\theta$ is equivalent to the bosonic action of 4-dimensional $\mathcal{N} = 4$ supersymmetric $U(N)$ Yang-Mills theory, which is the large N gauge theory of the AdS/CFT duality [7]. According to the large N duality or gauge-gravity duality, the resulting large N gauge theory must be dual to a higher dimensional gravity or string theory as summarized in Fig. 1. Hence it should not be surprising that the NC $U(1)$ gauge theory should describe a theory of gravity (or a string theory) in the same dimensions. In spite of the apparent relationship depicted in Fig. 1, this important possibility unfortunately has been largely ignored until recently.

The blue arrows on the right-hand side of Fig. 1 show how to derive D-dimensional Einstein gravity from NC $U(1)$ gauge theory on $\mathbb{R}^{d-1,1} \times \mathbb{R}^{2n}_\theta$, which should be expected if we accept the conjectural large N duality. However we can use the emergent gravity from NC $U(1)$ gauge theory to verify the conjectural large N duality by realizing the equivalence between the actions (2.8) and (2.9) in a reverse way. It is based on the observation [12, 15] that there are two different kinds of vacua in Coulomb branch if we consider the $N \to \infty$ limit and the NC space (2.4) arises as a vacuum solution of the d-dimensional $U(N \to \infty)$ Yang-Mills theory (2.9) in the Coulomb branch. See Fig. 2. The conventional choice of vacuum in the Coulomb branch of $U(N)$ Yang-Mills theory is given by

$$[\phi_a, \phi_b]|_{\text{vac}} = 0 \qquad \Rightarrow \qquad \langle \phi_a \rangle_{\text{vac}} = \text{diag}\big((\alpha_a)_1, (\alpha_a)_2, \cdots, (\alpha_a)_N\big) \tag{2.10}$$

for $a = 1, \cdots, 2n$. In this case the $U(N)$ gauge symmetry is broken to $U(1)^N$. If we consider the $N \to \infty$ limit, the large N limit opens a new phase of the Coulomb branch given by

$$[\phi_a, \phi_b]|_{\text{vac}} = -iB_{ab} \qquad \Rightarrow \qquad \langle \phi_a \rangle_{\text{vac}} = p_a \equiv B_{ab} y^b \tag{2.11}$$

where $B_{ab} = (\theta^{-1})_{ab}$ and the vacuum moduli y^a satisfy the Moyal-Heisenberg algebra (2.4). This vacuum will be called the NC Coulomb branch. Note that the Moyal-Heisenberg vacuum (2.11) saves the NC nature of matrices while the conventional vacuum (2.10) dismisses the property.

Suppose that fluctuations around the vacuum (2.11) take the form

$$D_\mu = \partial_\mu - i\widehat{A}_\mu(x, y), \qquad \phi_a = p_a + \widehat{A}_a(x, y). \tag{2.12}$$

We denote the NC $\star$-algebra on $\mathbb{R}^{d-1,1} \times \mathbb{R}^{2n}_\theta$ by $\mathcal{A}^d_\theta \equiv \mathcal{A}_\theta\big(C^\infty(\mathbb{R}^{d-1,1})\big) = C^\infty(\mathbb{R}^{d-1,1}) \otimes \mathcal{A}_\theta$. The adjoint scalar fields in Eq. (2.12) now obey the deformed algebra given by

$$[\phi_a, \phi_b] = -i(B_{ab} - \widehat{F}_{ab}) \in \mathcal{A}^d_\theta, \tag{2.13}$$

where

$$\widehat{F}_{ab} = \partial_a \widehat{A}_b - \partial_b \widehat{A}_a - i[\widehat{A}_a, \widehat{A}_b] \tag{2.14}$$

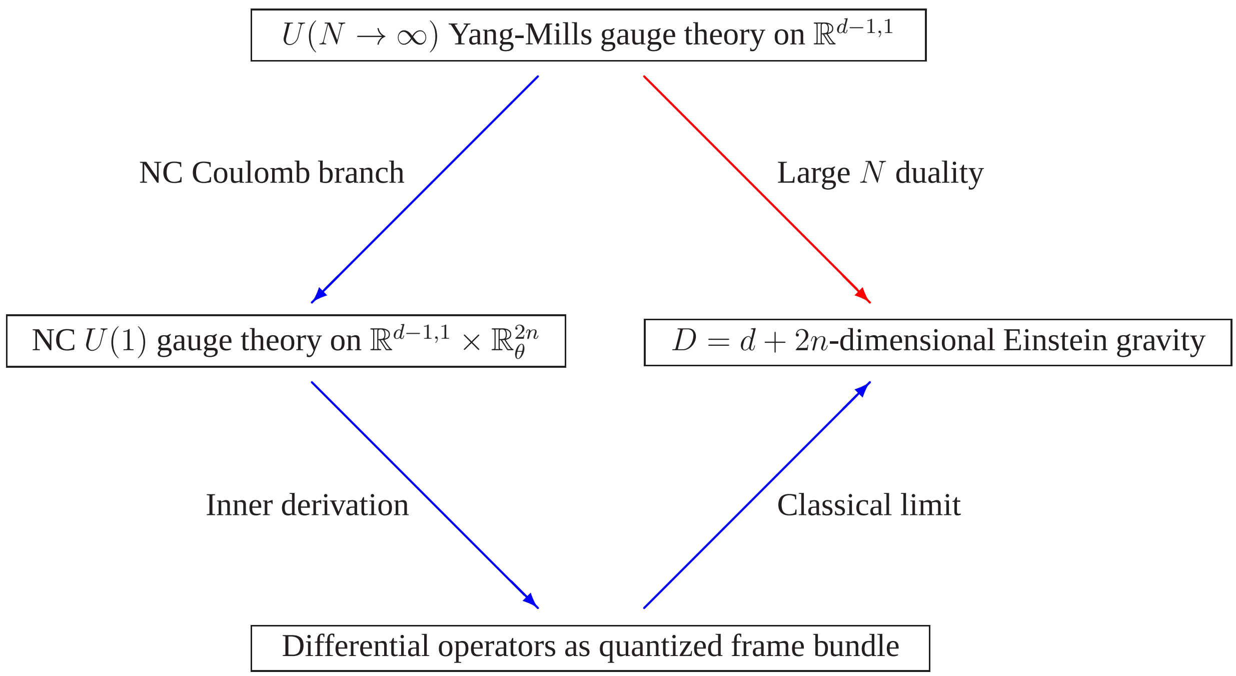

Figure 2: Flowchart for large N duality

with the definition $\partial_a \equiv \mathrm{ad}_{p_a} = -i[p_a, \cdot]$. Plugging the fluctuations in Eq. (2.12) into the d-dimensional $U(N \to \infty)$ Yang-Mills theory (2.9), we finally get the $D = (d+2n)$-dimensional NC $U(1)$ gauge theory. Thus we arrive at the reversed version of the equivalence [12, 15]:

$$
\begin{aligned}
S &= -\frac{1}{g_{YM}^2} \int d^d x \mathrm{Tr}\Big(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu \phi_a D^\mu \phi_a - \frac{1}{4}[\phi_a, \phi_b]^2\Big) \\
&= -\frac{1}{G_{YM}^2} \int d^D Y \frac{1}{4}(\widehat{F}_{AB} - B_{AB})^2, \qquad (2.15)
\end{aligned}
$$

where $\widehat{A}_A(x,y) = (\widehat{A}_\mu, \widehat{A}_a)(x,y)$ are $D = (d+2n)$-dimensional NC $U(1)$ gauge fields. It might be remarked that the NC space (2.11) is a consistent vacuum solution of the action (2.9) and the crux to realize the equivalence (2.15). If the conventional commutative vacuum (2.10) were chosen, we would have failed to realize the equivalence (2.15). Indeed it turns out [12] that the NC Coulomb branch is crucial to realize the emergent gravity from matrix models or large N gauge theories as depicted in Fig. 2.

Some remarks are in order. The relationship between a lower-dimensional large N gauge theory and a higher-dimensional NC $U(1)$ gauge theory in Figs. 1 and 2 is an exact mathematical identity. The identity in Fig. 1 is derived from the fact that the NC space (2.4) admits a separable Hilbert space and NC $U(1)$ gauge fields become operators acting on the Hilbert space. The identity in Fig. 2 is based on the fundamental fact that the NC space (2.4) is a consistent vacuum solution of a large N gauge theory in the Coulomb branch and more general solutions are generated by all possible (on-shell) deformations of the vacuum. This means that there exists an isomorphic map from the NC $U(1)$

gauge theory to the Einstein gravity which completes the large N duality. To be precise, consider the inverse metric in Einstein gravity given by

$$\Big(\frac{\partial}{\partial s}\Big)^2 = E_A \otimes E_A = g^{MN}(X)\partial_M \otimes \partial_N, \tag{2.16}$$

where $E_A = E_A^M(X)\partial_M$ are orthonormal frames on the tangent bundle $T\mathcal{M}$ of a D-dimensional spacetime manifold $\mathcal{M}$. The large N (or gauge-gravity) duality in Figs. 1 and 2 can be achieved by realizing the vector fields $E_A = E_A^M(X)\partial_M \in \Gamma(T\mathcal{M})$ in terms of NC $U(1)$ gauge fields.

A decisive clue is coming from the fact that the NC $\star$-algebra $\mathcal{A}_\theta$ generated by the Moyal-Heisenberg algebra (2.4) always admits a nontrivial inner automorphism $\mathfrak{I}$ as was already illustrated in Eq. (2.3) for the $n = 1$ case. In general, for any dynamical variable $\widehat{\Phi}(x,y) \in \mathcal{A}_\theta^d$, one has the relation

$$\widehat{\Phi}(x, y+d) = U(d)^\dagger \widehat{\Phi}(x,y) U(d), \qquad U(d) = e^{ip_a d^a} \in \mathfrak{I}. \tag{2.17}$$

In the presence of NC $U(1)$ gauge fields $\widehat{A}_A(x,y) = (\widehat{A}_\mu, \widehat{A}_a)(x,y)$ which appear in the form of background-independent variables $\phi_A(x,y) \equiv (iD_\mu, \phi_a)(x,y)$, one can covariantize the inner automorphism with $U(d) = e^{i\phi_A d^A} \in \mathfrak{I}$ by introducing open Wilson lines [35]. See section 3.2 in [13] for more details. The infinitesimal generators of $\mathfrak{I}$ form an inner derivation defined by the adjoint operation

$$\mathcal{A}_\theta^d \to \mathfrak{D}^d : f \mapsto \mathrm{ad}_f = -i[f, \ \cdot] \tag{2.18}$$

for any $f \in \mathcal{A}_\theta^d$. The module of derivations $\mathfrak{D}^d$ is a direct sum of the submodules of horizontal and inner derivations [36]:

$$\mathfrak{D}^d = \mathrm{Hor}(\mathcal{A}_\theta^d) \oplus \mathfrak{D}(\mathcal{A}_\theta^d), \tag{2.19}$$

where horizontal derivation is locally generated by a vector field

$$k^\mu(x,y)\frac{\partial}{\partial x^\mu} \in \mathrm{Hor}(\mathcal{A}_\theta^d). \tag{2.20}$$

Definitely the derivation $\mathfrak{D}^d$ is a Lie algebra homomorphism, i.e.,

$$\mathrm{ad}_{[f,g]} = i[\mathrm{ad}_f, \mathrm{ad}_g] \tag{2.21}$$

for $f, g \in \mathcal{A}_\theta^d$ and their commutator $[f,g] \in \mathcal{A}_\theta^d$. In particular, we are interested in the derivation algebra generated by the dynamical variables in Eq. (2.12). It is defined by

$$\widehat{V}_A = \{\mathrm{ad}_{\phi_A} = -i[\phi_A, \ \cdot]|\phi_A(x,y) = (iD_\mu, \phi_a)(x,y) \in \mathcal{A}_\theta^d\} \in \mathfrak{D}^d. \tag{2.22}$$

In a large-distance limit, i.e. $|\theta| \to 0$, one can expand the NC vector fields $\widehat{V}_A$ in Eq. (2.22) using the explicit form of the Moyal $\star$-product. The result takes the form

$$\widehat{V}_A = V_A^M(x,y)\frac{\partial}{\partial X^M} + \sum_{p=2}^{\infty} V_A^{a_1\cdots a_p}(x,y)\frac{\partial}{\partial y^{a_1}}\cdots\frac{\partial}{\partial y^{a_p}} \in \mathfrak{D}^d, \tag{2.23}$$

where $X^M = (x^\mu, y^a)$ are local coordinates on a D-dimensional emergent *Lorentzian* manifold $\mathcal{M}$ and $V_A^\mu = \delta_A^\mu$. Thus the Taylor expansion of NC vector fields in $\mathfrak{D}^d$ generates an infinite tower of the so-called polyvector fields [15]. Note that the leading term gives rise to the ordinary vector fields that will be identified with a frame basis associated to the tangent bundle $T\mathcal{M}$ of an emergent manifold $\mathcal{M}$. It is important to perceive that the realization of emergent geometry through the derivation algebra in Eq. (2.22) is intrinsically local. Therefore it is necessary to consider patching or gluing together the local constructions to form a set of global quantities. We will assume that local coordinate patches have been consistently glued together to yield global (poly)vector fields. See Refs. [37] for a global construction of NC $\star$-algebras and Ref. [15] for the globalization of emergent geometry. It will also be recapitulated in Part II. Let us truncate the above polyvector fields to ordinary vector fields given by

$$\mathfrak{X}(\mathcal{M}) = \Big\{ V_A = V_A^M(x,y)\frac{\partial}{\partial X^M} | A, M = 0, 1, \cdots, D-1 \Big\}. \tag{2.24}$$

The orthonormal vielbeins on $T\mathcal{M}$ are then defined by the relation [38]

$$V_A = \lambda E_A \in \Gamma(T\mathcal{M}) \tag{2.25}$$

or on $T^*\mathcal{M}$

$$v^A = \lambda^{-1} e^A \in \Gamma(T^*\mathcal{M}). \tag{2.26}$$

The conformal factor $\lambda \in C^\infty(\mathcal{M})$ is determined by the volume-preserving condition

$$\mathcal{L}_{V_A}\nu_t = \big(\nabla \cdot V_A + (2-d-2n)V_A \ln\lambda\big)\nu_t = 0, \qquad \forall A = 0, 1, \cdots, D-1, \tag{2.27}$$

where the invariant volume form on $\mathcal{M}$ is given by

$$\begin{aligned} \nu_t \equiv d^d x \wedge \nu &= \lambda^2 d^d x \wedge v^1 \wedge \cdots \wedge v^{2n} \\ &= \lambda^{2-d-2n}\nu_g \end{aligned} \tag{2.28}$$

and $\nu_g = e^0 \wedge \cdots \wedge e^{D-1}$ is the D-dimensional Riemannian volume form.

Define the structure equations of vector fields $V_A = \lambda E_A \in \Gamma(T\mathcal{M})$ by

$$[V_A, V_B] = -g_{AB}{}^C V_C. \tag{2.29}$$

Then the volume-preserving condition (2.27) can equivalently be written as [13, 14]

$$g_{BA}{}^B = V_A \ln\lambda^2. \tag{2.30}$$

In the end, the Lorentzian metric on a D-dimensional spacetime manifold $\mathcal{M}$ is given by [13, 14, 15]

$$\begin{aligned} ds^2 &= \mathcal{G}_{MN}(X) dX^M \otimes dX^N = e^A \otimes e^A \\ &= \lambda^2 v^A \otimes v^A = \lambda^2\big(\eta_{\mu\nu}dx^\mu dx^\nu + v_b^a v_c^a (dy^b - \mathbf{A}^b)(dy^c - \mathbf{A}^c)\big) \end{aligned} \tag{2.31}$$

where $\mathbf{A}^b := A^b_\mu(x,y)dx^\mu$. The above metric completely determines a D-dimensional Lorentzian spacetime emergent from the NC $U(1)$ gauge fields described by the action (2.15). Therefore the NC field theory representation of the d-dimensional large N gauge theory in the NC Coulomb branch provides a powerful machinery to identify gravitational variables dual to large N matrices.

The prescription (2.25) implies that the metric $g_V = v^A \otimes v^A$ determined by the gauge theory basis V_A is in the same conformally equivalent class with the Einstein metric $g_E = e^A \otimes e^A$ for the orthonormal frame E_A and thus the Weyl tensors are the same for both metrics. Hence this prescription is particularly useful for Ricci-flat manifolds [38]. However, for other cases such as conformally flat manifolds, the curvature tensors, i.e. Ricci tensors, determined by the metrics g_V and g_E are in general not the same. For the latter case, there exists a more natural prescription given by

$$(V_\mu, V_a) = (E_\mu, \lambda E_a) \in \Gamma(T\mathcal{M}), \tag{2.32}$$

where an arbitrary positive function λ is still determined by solving Eq. (2.30). But the volume-preserving condition is replaced by

$$\mathcal{L}_{V_A}\nu_t = \big(\nabla \cdot V_A + (2-2n)V_A \ln\lambda\big)\nu_t = 0, \qquad \forall A = 0,1,\cdots,D-1, \tag{2.33}$$

because $\nu_t = \lambda^{2-2n}\nu_g$ is the invariant volume form in this case. With this prescription, the emergent metric is now given by

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + \lambda^2 v^a_b v^a_c (dy^b - \mathbf{A}^b)(dy^c - \mathbf{A}^c). \tag{2.34}$$

It is straightforward to see that the condition (2.33) reads as

$$\partial_\mu \rho + \partial_a(\rho A^a_\mu) = 0 \quad \& \quad \partial_b(\rho V^b_a) = 0, \tag{2.35}$$

where $\rho = \lambda^2 \det v^a_b$. Thus the new prescription can be implemented as before if there exists a solution $\lambda(x,y)$ obeying Eq. (2.35). In particular, it provides a more convenient basis for a product manifold. For example, if NC $U(1)$ gauge fields show a factorized dependence given by $\widehat{A}_A(x,y) = \big(\widehat{A}_\mu(x), \widehat{A}_a(y)\big)$, we expect that such gauge fields will generate a product manifold of the form $\mathbb{R}^{d-1,1} \times \mathcal{M}_{2n}$. This is the case for Eq. (2.32) since $\lambda = \lambda(y)$ and $\mathbf{A}^a = 0$ in this case, while Eq. (2.25) gives rise to a warped product metric. Later we will take the prescription (2.32) to describe the cosmic inflation in a comoving frame in which the inflationary metric takes the form

$$ds^2 = -dt^2 + a(t)^2 d\mathbf{y} \cdot d\mathbf{y}. \tag{2.36}$$

We have implicitly assumed that the dynamical variables in Eq. (2.22) satisfy the equations of motion derived from the action (2.15). This means that the fluctuations in Eq. (2.12) must arise as a solution of NC $U(1)$ gauge theory defined by the action (2.15). Using the relation between $\mathcal{A}^d_\theta$ and $\mathfrak{D}^d$, it is in principle possible to translate the equations of motion for NC gauge fields in the algebra $\mathcal{A}^d_\theta$ into some geometrical equations for polyvector fields in the derivation $\mathfrak{D}^d$ whose commutative limit corresponds to gravitational field equations for the metric (2.31) or (2.34). This translation for the $d = 0$ case is relatively simple in lower dimensions as was done in [13, 14] for $D = 2, 3, 4$ dimensions. Recently we also identified the Einstein's equation for six-dimensional NC $U(1)$ gauge fields obeying the Hermitian Yang-Mills equations [39]. However the problem for general NC $U(1)$ gauge fields in higher dimensions may be nontrivial even in the classical limit. If we include higher spin fields in polyvector fields defined by Eq. (2.23), the problem will be much more complicated. Nevertheless it should be important to determine the precise form of gravitational equations and their derivative corrections because the higher-order terms in Eq. (2.23) are interpreted as quantum corrections according to the emergent quantum gravity picture [14, 15]. We hope to address this problem in the near future.

In conclusion, the general large N duality depicted in Fig. 2 can be explained via the duality chain

$$\mathcal{A}_N^d \quad \Longrightarrow \quad \mathcal{A}_\theta^d \quad \Longrightarrow \quad \mathfrak{D}^d, \tag{2.37}$$

where $\mathcal{A}_N^d \equiv \mathcal{A}_N\big(C^\infty(\mathbb{R}^{d-1,1})\big) = C^\infty(\mathbb{R}^{d-1,1}) \otimes \mathcal{A}_N$. The dynamical variables in d-dimensional Yang-Mills gauge theory in Fig. 2 take values in $\mathcal{A}_N^d$ while those in $D = (d + 2n)$-dimensional NC $U(1)$ gauge theory take values in $\mathcal{A}_\theta^d$. These two NC algebras $\mathcal{A}_N^d$ and $\mathcal{A}_\theta^d$ are related to each other by considering the NC Coulomb branch for the algebra $\mathcal{A}_N^d$.

3 Cosmic inflation from time-dependent matrices

From now on we will focus on the matrix quantum mechanics (MQM), i.e., the $d = 1$ case in Eq. (2.15), to address the background-independent formulation of cosmic inflation. The underlying action in this case is given by

$$\begin{aligned} S &= \frac{1}{g^2} \int dt \, \mathrm{Tr}\Big(\frac{1}{2}(D_0\phi_a)^2 + \frac{1}{4}[\phi_a, \phi_b]^2\Big) \\ &= \frac{1}{4g^2} \int dt \, \eta^{AC}\eta^{BD} \mathrm{Tr}[\phi_A, \phi_B][\phi_C, \phi_D], \end{aligned} \tag{3.1}$$

where $\phi_0 \equiv iD_0 = i\frac{\partial}{\partial t} + A_0(t)$, $\phi_A(t) = (\phi_0, \phi_a)(t)$ and $\eta^{AB} = \mathrm{diag}(-1, 1, \cdots, 1)$, $A, B = 0, 1, \cdots, 2n$. With the notation of the symbol η^{AB}, it is easy to see that the matrix action (3.1) has a global automorphism given by

$$\phi_A \to \phi'_A = \Lambda_A{}^B \phi_B + c_A \tag{3.2}$$

if $\Lambda_A{}^B$ is a rotation in $SO(2n, 1)$ and c_A are constants proportional to the identity matrix. It will be shown later that the global symmetry (3.2) is responsible for the Poincaré symmetry of flat spacetime emergent from a vacuum in the Coulomb branch of MQM and so will be called the Poincaré automorphism. We remark that the time t in the action (3.1) is not a dynamical variable but a parameter. The concept of emergent time will be defined in Part II by considering a one-parameter family of deformations of zero-dimensional matrices which is parameterized by the coordinate t. Then the one-parameter family of deformations can be regarded as the time evolution of a dynamical system. A close analogy with quantum mechanics implies that the concept of emergent time is related to the time evolution of the dynamical system. In this context, the one-dimensional matrix model (3.1) can be interpreted as a Hamiltonian system of a zero-dimensional (e.g., IKKT) matrix model [15].

The equations of motion for the matrix action (3.1) are given by

$$D_0^2\phi_a + [\phi_b, [\phi_a, \phi_b]] = 0, \tag{3.3}$$

which must be supplemented with the Gauss constraint

$$[\phi_a, D_0\phi_a] = 0. \tag{3.4}$$

In order to achieve the NC field theory representation for the action (2.15), we have considered the NC Coulomb branch defined by

$$\langle\phi_A\rangle_{\rm vac} = p_A = \Big(i\frac{\partial}{\partial t} + \mathcal{E}, p_a\Big), \tag{3.5}$$

where $\mathcal{E} \equiv \langle A_0(t)\rangle_{\rm vac}$ is a constant vacuum energy density and the vacuum moduli p_a satisfy the commutation relation (2.11). We emphasize that the NC Coulomb branch (3.5) is a consistent *vacuum* solution of MQM since it satisfies the equations of motion (3.3) as well as the Gauss constraint (3.4). Since $\mathcal{E}$ is proportional to the identity matrix, it plays no role in the temporal covariant derivative D_0 and so it can be dropped without loss of generality. The notation (3.5) makes a merit of the emphasis that the temporal differential operator in ϕ_0 must be regarded as a timelike background on an equal footing with the spatial vacuum moduli p_a. Let us consider all possible deformations of the vacuum (3.5) and parameterize them as Eq. (2.12). Plugging the fluctuations into the action (3.1) leads to the identity

$$\begin{aligned} S &= \frac{1}{g^2}\int dt\,\mathrm{Tr}\Big(\frac{1}{2}(D_0\phi_a)^2 + \frac{1}{4}[\phi_a,\phi_b]^2\Big) \\ &= -\frac{1}{4G_{YM}^2}\int d^{2n+1}y\,\eta^{AC}\eta^{BD}(\widehat{F}_{AB} - B_{AB})(\widehat{F}_{CD} - B_{CD}), \end{aligned} \tag{3.6}$$

where $G_{YM}^2(2\pi)^n|\mathrm{Pf}\theta|g^2$ is the $(2n+1)$-dimensional gauge coupling constant.

Let us contemplate how we have obtained the $(2n+1)$-dimensional emergent spacetime $\mathcal{M}$ described by the Lorentzian metric (2.34). At the outset, we have considered a background-independent theory in which any existence of spacetime is not assumed but defined by the theory itself. Of course, the background-independent theory does not mean that the physics is independent of the background. Background independence here means that, although a physical phenomenon occurs in a particular background with a specific initial condition, an underlying theory itself describing such a physical event should presuppose neither any kind of spacetime nor material backgrounds. Therefore the background itself should arise from a vacuum solution of the underlying theory. In particular, the background-independent theory has to make no distinction between geometry and matter since it has no predetermined spacetime. We have defined a most primitive vacuum such that it generates a simple spacetime structure. General and more complicated spacetime structures are obtained by deforming the primitive vacuum in all possible ways. These deformations correspond to physical processes that happen upon a particular (spacetime) background. Hence they are regarded as a dynamical system. Motivated by a close analogy with quantum mechanics, we argue in Part II that the deformations of spacetime structure supported on a vacuum solution must be understood as the time evolution of the dynamical system. As a consequence, the fundamental action (3.1) describes a dynamical system, from which an emergent $(2n+1)$-dimensional Lorentzian spacetime $\mathcal{M}$ with the metric (2.34) is derived.

The large N duality in Fig. 2 says that the gravitational variables such as vielbeins in general relativity arise from the commutative limit of NC $U(1)$ gauge fields via the map (2.37). Then one may ask where flat Minkowski spacetime comes from. Let us look at the metric (2.34) to identify the origin of the flat Minkowski spacetime. Definitely the Lorentzian manifold $\mathcal{M}$ becomes the Minkowski spacetime when all fluctuations die out, i.e., $v^a_b \to \delta^a_b, \ \mathbf{A}^a \to 0$. Therefore the vacuum geometry for the metric (2.34) was originated from the vacuum configuration (3.5) in which $V^{(0)}_A \equiv \langle V_A \rangle_{\rm vac} = \delta^M_A \frac{\partial}{\partial X^M}$, so $\lambda^2 \to 1$ according to Eq. (2.30). In other words, the $(2n+1)$-dimensional flat Minkowski spacetime is emergent from the vacuum condensate (3.5) since the corresponding vielbeins and the metric are given by $E^{(0)}_A = \widehat{V}^{(0)}_A = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y^a}\right)$ and $ds^2 = -dt^2 + d\mathbf{y}\cdot d\mathbf{y}$ [13, 14]. We have to emphasize that the vacuum algebra responsible for the emergence of the Minkowski spacetime is the Moyal-Heisenberg algebra (2.11). But the NC Coulomb vacuum induces a nontrivial vacuum energy density caused by the condensate (2.11). We can calculate it using the action (3.6):

$$\rho_{\rm vac} = \frac{1}{4G^2_{YM}} |B_{ab}|^2. \tag{3.7}$$

A striking fact is that the vacuum (2.11) responsible for the generation of flat spacetime is not empty. Rather the flat spacetime had been originated from the uniform vacuum energy (3.7) known as the cosmological constant in general relativity. This is a tangible difference from Einstein gravity, in which $T_{\mu\nu} = 0$ in flat spacetime as one can see from Eq. (1.1). Consequently, the emergent gravity reveals a remarkable picture that a uniform vacuum energy such as Eq. (3.7) does not gravitate. As a result, the emergent gravity does not contain the coupling of cosmological constant like $\int d^{2n+1}x\sqrt{-\mathcal{G}}\Lambda$, so it presents a striking contrast to general relativity. This important conclusion may be strengthened by applying the Lie algebra homomorphism (2.21) to the commutators in Eq. (2.13), which reads as

$$-i\mathrm{ad}_{[\phi_a,\phi_b]} \equiv \widehat{V}_{\widehat{F}_{ab}-B_{ab}} = \widehat{V}_{\widehat{F}_{ab}} = [\widehat{V}_a, \widehat{V}_b] \in \mathfrak{D}^1 \tag{3.8}$$

for a constant field strength B_{ab}. To stress clearly, the gravitational fields emergent from NC $U(1)$ gauge fields must be insensitive to the constant vacuum energy such as Eq. (3.7). In the end, the emergent gravity clearly dismisses the notorious cosmological constant problem [13, 14, 21].

In order to estimate the dynamical energy scale for the vacuum condensate (3.5), note that the Newton constant G_N according to emergent gravity picture has to be determined by field theory parameters only such as the gauge coupling constant G_{YM} and $\theta = B^{-1}$ defining the NC $U(1)$ gauge theory (3.6). A simple dimensional analysis leads to the result [13, 14]

$$\frac{G_N \hbar^2}{c^2} \sim G_{YM}^2 |\theta|, \tag{3.9}$$

where $|\theta| := |\mathrm{Pf}\theta|^{\frac{1}{n}}$. To be specific, when considering the four-dimensional case in which $M_P = (8\pi G_N)^{-1/2} \sim 10^{18}$ GeV and $G_{YM}^2 \sim \frac{1}{137}$, the vacuum energy (3.7) due to the condensate (2.11) is at a moderate estimate given by

$$\rho_{\text{vac}} = \frac{1}{4G_{YM}^2} |B_{ab}|^2 \sim G_{YM}^2 M_P^4 \sim 10^{-2} M_P^4. \tag{3.10}$$

Amusingly emergent gravity discloses that the perverse vacuum energy $\rho_{\text{vac}} \sim M_P^4$ was actually the origin of flat spacetime. It is worthwhile to remark that the Planck mass M_P naturally sets a dynamical scale for the emergence of gravity and spacetime if quantum gravity should be formulated in a background-independent way so that the spacetime geometry emerges from a vacuum configuration of some fundamental ingredients in the underlying theory. Therefore it may be not a surprising result but rather an inevitable consequence that the Planck energy density (3.10) in vacuum was the genetic origin of spacetime.

We observed before that the MQM admits a global automorphism given by Eq. (3.2). Let us see what is the consequence of the Poincaré automorphism (3.2) on the emergent spacetime geometry. The Poincaré automorphism leads to the transformation $V_A^{(0)} \to V_A^{\prime(0)} = \Lambda_A{}^B V_B^{(0)}$. However, this transformation does not change λ^2 because $\det \Lambda = 1$. The geometry for the transformed vacuum p'_A is determined by the metric (2.34) that is again the flat Minkowski spacetime $\mathbb{R}^{2n,1}$. Therefore, we see that the vacuum configuration responsible for the generation of flat spacetime is not unique but degenerate up to the Poincaré automorphism.[5] After all, the global Poincaré symmetry of the Minkowski spacetime is emergent from the Poincaré automorphism (3.2) of MQM.

Note that the Planck energy condensate in vacuum resulted in an extremely extended spacetime as the metric (2.34) clearly indicates. However, since we have started with a background-independent theory in which any spacetime structure has not been assumed in advance, the spacetime was not existent at the beginning but simply emergent from the vacuum condensate (3.5). Therefore the Planck energy condensation into vacuum must be regarded as a dynamical process. Since the dynamical scale for the vacuum condensate is about of the Planck energy, the time scale for the condensation will be roughly of the Planck time $t_P \sim 10^{-44}$ sec. Inflation scenario asserts that our Universe at the beginning had undergone an explosive inflation era lasted roughly $\sim 10^{-33}$ seconds. Thus it is natural to consider the cosmic inflation as a dynamical process for the instantaneous condensation of vacuum energy $\rho_{\text{vac}} \sim M_P^4$ to enormously spread out spacetime [21]. Now we will explore how the cosmic

[5] Note that the vacuum solution (3.5) is further degenerated under the scaling $p_a \to p'_a = \beta p_a$ or $y^a \to y'^a = \beta^{-1} y^a$ as far as $\beta \in \mathbb{R} \setminus \{0\}$ is a nonzero constant. We will use this freedom to normalize the initial length scale such that $|y^a(t=0)| = L_P$ or $l_s = \sqrt{\alpha'}$.

inflation is triggered by the condensate of Planck energy in vacuum responsible for the dynamical emergence of spacetime.

First let us understand intuitively Eqs. (1.2) and (1.3) to get some dear insight from the old wisdom. Suppose that a test particle with mass m is placed in the condensate (3.10). Consider a ball of radius $r(t)$ and the test particle placed on its surface. According to the Gauss's law, the particle will be subject to the gravitational potential energy $V(r) = -\frac{G_N M(r) m}{r}$ caused by the condensate (3.10), where $M(r) = \frac{4\pi r(t)^3 \rho_{\rm vac}}{3}$ is the total mass inside the ball.[6] In order to preserve the total energy E of the particle, the ball has to expand so that the kinetic energy $K(r) = \frac{1}{2} m \dot{r}(t)^2$ generated by the expansion compensates the negative potential energy. That is, the energy conservation implies the following relation

$$H^2 = \frac{8\pi G_N \rho_{\rm vac}}{3} - \frac{k}{r(t)^2}, \tag{3.11}$$

where $H = \frac{\dot{r}(t)}{r(t)}$ is the expansion rate and $k \equiv -\frac{2E}{m}$. By comparing the above equation with the Friedmann equation (1.2) after the identification $r(t) = Ra(t)$, we see that Eq. (3.11) corresponds to $\rho_{\rm vac} = V(\phi) \approx V_0$ and $\dot{\phi} \approx 0$ with $k = 0$. At the outset we actually assumed the spatially flat universe, $k = 0$, for the Friedmann equation (1.2). In our approach with a background-independent theory, the condition $k = 0$ is automatic since the very beginning should be absolutely nothing! This conclusion is consistent with the metric (2.34) which describes a final state of cosmic inflation. Hence we may moderately claim that the background-independent theory for cosmic inflation predicts a spatially flat universe, in which the constant k must be exactly zero.

From the above simple argument, we see that the size of the ball exponentially expands, i.e.,

$$a(t) = a_0 e^{Ht} \tag{3.12}$$

where

$$H = \sqrt{\frac{8\pi G_N \rho_{\rm vac}}{3}} \tag{3.13}$$

is a constant. Let us introduce fluctuations around the inflating solution (3.12) by considering $\rho_{\rm vac} \to \rho_{\rm vac} + \delta\rho$ and $\dot{\phi} \neq 0$, where $\delta\rho$ is the mechanical energy due to the fluctuations of the inflaton $\phi(t)$. Then the evolution equation (3.11) is replaced by

$$H^2 = \frac{8\pi G_N}{3} (\rho_{\rm vac} + \delta\rho), \tag{3.14}$$

and the dynamics of the inflaton is described by Eq. (1.3). As we already remarked in Eq. (1.5), the dynamics of the inflaton must be described by a non-Hamiltonian system, whose mathematical basis

[6]It might be remarked that this experiment is a simple twist of the well-known solution of Gauss's law for gravity inside the earth, in which the minus sign in the gravitational potential energy presupposes a repulsive force rather than the usual attractive force. Moreover the repulsive force is given by $\mathbf{F} = k_g \mathbf{r} = -\nabla V(r)$ where $k_g = \frac{4\pi G_N m \rho_{\rm vac}}{3}$ and $V(r) = -\frac{G_N M(r) m}{2r}$ is the gravitational potential energy in Newtonian gravity. The change of sign and the factor 2 enhancement are due to the general relativity effect since $\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho_{\rm vac} + 3p) = -\frac{4\pi G_N}{3}(-2\rho_{\rm vac})$.

will be reviewed in Part II. Therefore, in order to describe the inflationary universe in the context of emergent gravity, we need to extend the module $\mathfrak{D}^1$ of differential operators in Eq. (2.22) so that the exponential behavior (3.12) is derived from it. In classical limit, such vector fields are known as conformal vector fields whose flow preserves a symplectic form up to a constant, so they appear in the conformal Hamiltonian dynamics such as simple mechanical systems with friction [40, 41].

As we have advocated the vitality of the background-independent formulation of emergent spacetime, it is desirable to realize the inflationary universe as a solution of the matrix model (3.1). Now we will show that the cosmic inflation arises as such a time-dependent solution describing the dynamical process of Planck energy condensate into vacuum without introducing any inflaton field as well as an *ad hoc* inflation potential. It is not difficult to show that the dynamical process for the vacuum condensate is described by the time-dependent vacuum configuration given by

$$\langle \phi_a(t) \rangle_{\rm vac} = p_a(t) = e^{\frac{\kappa t}{2}} p_a, \qquad \langle \widehat{A}_0(t) \rangle_{\rm vac} = \widehat{a}_0(t,y), \tag{3.15}$$

where the temporal gauge field is given by an open Wilson line [35]

$$\widehat{a}_0(t,y) = \frac{\kappa}{2} \int_0^1 d\sigma \frac{dy^a(\sigma)}{d\sigma} p_a(\sigma) \tag{3.16}$$

along a path parameterized by the curve $y^a(\sigma) = y_0^a + \zeta^a(\sigma)$ where $\zeta^a(\sigma) = \theta^{ab} k_b \sigma$ with $0 \leq \sigma \leq 1$ and $y^a(\sigma = 0) \equiv y_0^a$ and $y^a(\sigma = 1) \equiv y^a$. The constant κ will be identified with the inflationary Hubble constant H. First note that the second term in Eq. (3.3) identically vanishes for the background (3.15). Therefore it is necessary to impose the condition

$$D_0 \phi_a = e^{\frac{\kappa t}{2}} \Big(\frac{\kappa}{2} p_a - i[\widehat{A}_0, p_a] \Big) = 0 \tag{3.17}$$

to satisfy both (3.3) and (3.4). In terms of the NC $\star$-algebra $\mathcal{A}^1_\theta$, Eq. (3.17) reads as

$$\frac{\partial \widehat{a}_0(t,y)}{\partial y^a} = \frac{\kappa}{2} p_a. \tag{3.18}$$

Using the formula

$$\frac{\partial}{\partial y^a} \int_0^1 d\sigma \frac{dy^b(\sigma)}{d\sigma} K\big(y(\sigma)\big) = \delta_a^b K(y) \tag{3.19}$$

for some differentiable function $K(y)$, one can easily check that the temporal gauge field in Eq. (3.16) satisfies Eq. (3.18).

Before calculating the metric (2.34) for the inflating background (3.15), we want to discuss some physical significance of the nonlocal term (3.16). First we point out that the temporal gauge field (3.16) corresponds to a background Hamiltonian density in the comoving frame. (See footnote 7 for a different choice of coordinate frame.) We will see soon that the gravitational metric including the effect of the nonlocal term (3.16) is still local as it should be. It was already noticed in [42] that nonlocal observables in emergent gravity are in general necessary to describe some gravitational metric that is nonetheless local. Moreover the appearance of such nonlocal terms should not be surprising in NC gauge theories, in which there exist no local gauge invariant observables. Indeed it was shown in [35] that nonlocal observables are the NC generalization of gauge invariant operators in NC gauge theories.

Now let us determine the metric (2.34) for the inflating background (3.15). The $(2n+1)$-dimensional vector fields defined by Eq. (2.22) take the following form

$$V_0(t) = \frac{\partial}{\partial t} - \frac{\kappa}{2} y^a \frac{\partial}{\partial y^a}, \qquad V_a(t) = e^{\frac{\kappa t}{2}} \frac{\partial}{\partial y^a}. \tag{3.20}$$

It may be stressed that the result (3.20) is exact, i.e., higher-order derivative terms in Eq. (2.23) identically vanish. Note that the vector fields take the local form again as the result of applying the formula (3.19) and the open Wilson line (3.16) leads to a conformal vector field $Z \equiv \frac{1}{2} y^a \frac{\partial}{\partial y^a}$ known as the Liouville vector field [40, 41]. Then the dual orthogonal one-forms are given by

$$v^0(t) = dt, \qquad v^a(t) = e^{-\frac{\kappa t}{2}} (dy^a + \mathbf{a}^a) = e^{-\kappa t} dy^a_t \tag{3.21}$$

where

$$\mathbf{a}^a = \frac{\kappa}{2} y^a dt, \qquad y^a_t \equiv e^{\frac{\kappa t}{2}} y^a. \tag{3.22}$$

One can see that the vector fields in Eq. (3.20) satisfy $[V_0, V_a] = \kappa V_a$ and thus

$$g_{AB}{}^C = \begin{cases} g_{0a}{}^b = -g_{a0}{}^b = \kappa \delta^b_a, & a, b = 1, \cdots, 2n; \\ 0, & \text{otherwise.} \end{cases} \tag{3.23}$$

From this result, we get $\lambda = e^{n\kappa t}$ since $g_{BA}{}^B = V_A \ln \lambda^2$ [14]. One can see that the volume-preserving condition (2.35) is definitely satisfied since $\rho = e^{n\kappa t}$ and $A^a_0 = -\frac{\kappa}{2} y^a$. In the end, the time-dependent metric for the inflating background (3.15) is given by

$$ds^2 = -dt^2 + e^{2Ht} d\mathbf{y}_t \cdot d\mathbf{y}_t, \tag{3.24}$$

where we have identified the inflationary Hubble constant $H \equiv (n-1)\kappa$. We emphasize that the temporal gauge field (3.16) is crucial to satisfy Eqs. (3.3) and (3.4). Note that the metric (3.24) is conformally flat, i.e., the corresponding Weyl tensors identically vanish and so describes a homogeneous and isotropic inflationary universe known as the Friedmann-Robertson-Walker metric in physical cosmology.

We showed that the cosmic inflation arises as a time-dependent solution of a background-independent theory describing the dynamical process of Planck energy condensate in vacuum without introducing any inflaton field as well as an *ad hoc* inflation potential. Let us generalize the cosmic inflation by also including arbitrary fluctuations around the inflationary background (3.15). Such a general inflationary universe in $(2n+1)$-dimensional Lorentzian spacetime can be realized by considering a time-dependent NC algebra given by[7]

$$ {}^t\mathcal{A}^1_\theta \equiv \Big\{ \widehat{\phi}_0(t,y) = i\frac{\partial}{\partial t} + \widehat{A}_0(t,y), \quad \widehat{\phi}_a(t,y) = e^{\frac{\kappa t}{2}}\big(p_a + \widehat{A}_a(t,y)\big)\Big\}. \tag{3.25} $$

We denote the corresponding time-dependent matrix algebra by ${}^t\mathcal{A}^1_N$ which consists of a time-dependent solution of the action (3.1). Then the general Lorentzian metric describing a $(2n+1)$-dimensional inflationary universe can be obtained by the following duality chain:

$$ {}^t\mathcal{A}^1_N \quad \Longrightarrow \quad {}^t\mathcal{A}^1_\theta \quad \Longrightarrow \quad {}^t\mathfrak{D}^1. \tag{3.26} $$

The module ${}^t\mathfrak{D}^1$ of derivations of the NC algebra ${}^t\mathcal{A}^1_\theta$ is given by

$$ {}^t\mathfrak{D}^1 = \Big\{ \widehat{V}_A(t) = (\widehat{V}_0, \widehat{V}_a)(t) | \widehat{V}_0(t) = \frac{\partial}{\partial t} + \mathrm{ad}_{\widehat{A}_0(t,y)}, \quad \widehat{V}_a(t) = e^{\frac{\kappa t}{2}}\Big(\frac{\partial}{\partial y^a} + \mathrm{ad}_{\widehat{A}_a(t,y)}\Big)\Big\}, \tag{3.27} $$

where the adjoint operations are defined by Eq. (2.22). In the classical limit of the module (3.27), we get a general inflationary universe described by

$$ ds^2 = -dt^2 + e^{2Ht}(1+\delta\lambda)^2 v^a_b v^a_c (dy^b_t - \mathbf{A}^b)(dy^c_t - \mathbf{A}^c), \tag{3.28} $$

where $v^a_b := v^a_b(t,y)$, $\delta\lambda := \delta\lambda(t,y)$ and $\mathbf{A}^b := \delta a^b_0(t,y)dt$. If all fluctuations are turned off for which $v^a_b = \delta^a_b$ and $\delta\lambda = \mathbf{A}^b = 0$, we recover the inflation metric (3.24).

To appreciate the physical picture of the vacuum configuration (3.15), recall that a NC space such as $\mathbb{R}^2_{\alpha'}$ cannot occupy a single point of the plane but rather lies in a region of the plane. Thus there must be a basic length scale, below which the notion of space (and time) does not make sense. Let us fix such a typical length scale at $t=0$ as $|y^a(t=0)| \sim L_P$ or $l_s = \sqrt{\alpha'}$ using the scaling freedom noted in footnote 5. It should be reasonable to identify L_P with the Planck length. Since $y^a(t=0)$ are operators acting on a Hilbert space, this means that the inflationary vacuum (3.15) creates a spacetime of the Planck size. After the creation, the universe evolves to the inflation epoch as a solution of time-dependent matrix model unlike the traditional inflationary models that describe just the exponential expansion of a preexisting spacetime. This picture is similar to the birth of inflationary universes in Ref. [6] in which the universe is spontaneously created by quantum tunneling from nothing into a de Sitter space. Here by nothing we mean a state without any classical spacetime. According to the standard inflation scenario, the universe expanded by at least a factor of e^{60} during the inflation. In order to know the duration of the inflation exactly, we need to understand the precise mechanism of reheating, which unfortunately goes beyond our ability at present. Since the radius of the universe

[7] One may wonder why the time direction is not inflating. This is due to our choice of a coordinate frame to describe the dynamical system. The time evolution operator $\widehat{\phi}_0(t,y)$ is defined in the so-called comoving frame. In general, one can choose an arbitrary frame in which the time evolution is described by $k(t,y)\frac{\partial}{\partial t} \in \mathrm{Hor}(\mathcal{A}^1_\theta)$, i.e., the $d=1$ case of Eq. (2.20). A particularly interesting frame is the conformal coordinates with which the metric is given by $ds^2 = a(\eta)^2(-d\eta^2 + d\mathbf{x}\cdot d\mathbf{x})$ where $a(\eta) = -\frac{1}{H\eta}$ and $-\infty < \eta < 0$. The conformal coordinates can be easily transformed to the comoving coordinates by $a(\eta)d\eta = dt$.

at the beginning of inflation is about L_P, 60 e-foldings at $t = t_{\text{end}} = 10^{-36} \sim 10^{-33}$ sec mean that $Ht_{\text{end}} \gtrsim 60$ and the size of universe at the end of inflation amounts to $|y^a(t = t_{\text{end}})| = e^{Ht_{\text{end}}}|y^a(t = 0)| \gtrsim e^{60}L_P$. Since 1 eV $= (6.6 \times 10^{-16}\text{sec})^{-1}$, this informs us of the energy scale of the inflationary Hubble constant given by $H \gtrsim 10^{11} \sim 10^{14}$GeV [3, 4].

4 Discussion

It is well-known [43, 44] that NC field theories arise as a low-energy effective theory in string theory, in particular, on D-branes upon turning on a constant B-field. A remarkable aspect of the NC field theory is that it can be mapped to a large N matrix model as depicted in Fig. 1. The relation between NC gauge theories and matrix models is quite general since any Lie algebra or Moyal-type NC space such as (2.4) always admits a separable Hilbert space and NC gauge fields become operators acting on the Hilbert space [29]. The matrix representation of NC gauge fields implies that they can be embedded into a background-independent formulation in terms of a matrix model. Here we refer to a background-independent theory in which any spacetime structure is not *a priori* assumed but defined by the theory. The background-independent variables are identified as the degrees of freedom of the underlying matrix model. The relation with the matrix model gives a physical interpretation of the background independence for the NC gauge theories by the observation [12, 28] that the NC space (2.4) is a consistent vacuum solution of a large N gauge theory in the Coulomb branch. The matrices are the original dynamical variables of the matrix model which are manifestly background-independent and the NC gauge fields are now derived from fluctuations in the NC Coulomb branch as depicted in Fig. 2. These matrix models can be embedded into string theories or M-theory. For example, the $d = 0$ $(n = 5)$ and $d = 2$ $(n = 4)$ cases in the matrix action (2.15) are precisely the IKKT matrix model [31] and the matrix string theory [32, 34], respectively. However its relation to the BFSS matrix model [33] is not straightforward since the matrix model (2.15) contains only even number of adjoint scalar fields while the BFSS matrix model requires 9. Nevertheless, the DLCQ M-theory compactified on an odd-dimensional torus $\mathbb{T}^p$ can be described by the matrix action (2.15) with $d = p+1$ and $n = \frac{9-p}{2}$ because it is known [30] that the former is described by the $(p+1)$-dimensional $U(N)$ supersymmetric Yang-Mills theory on a dual torus $(\mathbb{T}^p)^*$. Although it remains open to realize the original BFSS matrix model as the Hilbert space representation of a NC $U(1)$ gauge theory, it is a separate issue from the background-independent formulation of an emergent inflationary spacetime. The latter arises from a time-dependent solution to a one-dimensional matrix quantum mechanics which does not presuppose any spacetime background.

In string theory, there are two exclusive spacetime pictures based on the Kaluza-Klein (KK) theory vs. emergent gravity although they are conceptually in deep discord with each other. On the one hand, the KK gravity is defined in higher dimensions as a more superordinate theory and gauge theories in lower dimensions are derived from the KK theory via compactification. Since the KK theory is just the Einstein gravity in higher dimensions, the prior existence of spacetime is *a priori* assumed. On the other hand, in emergent gravity picture, gravity in higher dimensions is not a fundamental force but a collective phenomenon emergent from more fundamental ingredients defined in lower dimensions. In emergent gravity approach, the existence of spacetime is not *a priori* assumed but the spacetime structure is defined by the theory itself. This picture leads to the concept of emergent spacetime. In some sense, emergent gravity is the inverse of KK paradigm, schematically summarized by

$$(1 \otimes 1)_S \rightleftarrows 2 \oplus 0 \tag{4.1}$$

where $\rightarrow$ means the emergent gravity picture while $\leftarrow$ indicates the KK picture.

Recent developments in string theory have revealed growing evidences for the emergent gravity and emergent spacetime. The AdS/CFT correspondence and matrix models are typical examples supporting the emergence of gravity and spacetime [7]. Since the emergent spacetime is a new fundamental paradigm for quantum gravity and radically different from any previous physical theories, all of which describe what happens in a given spacetime, it is required to seriously reexamine all the rationales to introduce the multiverse hypothesis from the perspective of emergent spacetime. However, we do not intend to make an objection to the existence of more diverse subregions in the *Universe*. The Universe is rather likely much larger than we previously thought. Actually the emergent spacetime picture implies that our observable patch within cosmic horizon is a very tiny part $\sim 10^{-60}$ of the entire spacetime, as we will discuss soon. Instead we will pose the issue whether the existence of more diverse subregions besides ours means that the laws of physics are ambiguous or all these subregions follow the same laws of physics and the physical laws of our causal patch in the Universe can be understood as accurately as possible without reference to the existence of other subregions.

First let us summarize the main (not exhausting) sources of the multiverse idea [1]:

A. Cosmological constant problem.

B. Chaotic and eternal inflation scenarios.

C. String landscape.

First of all, we have to point out that these are all based on the traditional spacetime picture. The cosmological constant problem (A) is the problem in all traditional gravity theories such as Einstein gravity and modified gravities. So far any such a theory has not succeeded to resolve the problem A. The inflation scenarios (B) are also based on the traditional gravity theory coupled to an effective field theory for inflaton(s). Thus, in these scenarios, the prior existence of spacetime is simply assumed. The string landscape (C) also arises from the conventional KK compactification of string theory although the string theory is liberal enough to allow two exclusive spacetime pictures, as we already remarked above. Since superstring theories can consistently be defined only in ten-dimensions, extra six-dimensional internal spaces need to be compactified to explain our four-dimensional world. Moreover it is important to determine the shape and topology of an internal space to make contact with a low-energy phenomenology in four-dimensions because the internal geometry of string theory determines a detailed structure of the multiplets for elementary particles and gauge fields via the KK compactification. The string landscape (C) means that the huge variety of compactified internal geometries exist, typically, in the range of 10^{500} and almost the same number of four-dimensional worlds with different low-energy phenomenologies accordingly survive [10, 11].

We have to stress again that the emergent spacetime picture is radically different from the conventional picture in general relativity so that they are exclusive and irreconcilable each other. Therefore, if the emergent spacetime picture is correct to explain our Universe, we have to give up the traditional spacetime picture and KK paradigm. For this reason, we will reconsider all the rationales (A,B,C) from the standpoint of emergent spacetime and the background independentness.

We already justified at the beginning of Sec. 3 why emergent gravity definitely dismisses the cosmological constant problem (A). See also Refs. [13, 14, 21] for more extensive discussion of this issue. There is no cosmological constant problem in emergent gravity approach founded on the emergent spacetime. The foremost reason is that the huge vacuum energy (3.7) or (3.10) that is a cosmological constant in general relativity was simply used to generate the flat spacetime and thus it does not gravitate any more. The emergent gravity does not allow the coupling of the cosmological constant thanks to the general property (3.8), which is a tangible difference from general relativity. Consequently there is no demanding reason to rely on the anthropic fine-tuning to explain the tiny value of current dark energy. We will also discuss later what dark energy is from the emergent gravity picture following the observation in Refs. [13, 14, 21].

The multiverse picture arises in inflationary cosmology (B) as follows [3, 4]. In theories of inflationary model, even though false vacua are decaying, the rate of exponential expansion is always much faster than the rate of exponential decay. Once inflation starts, the total volume of the false vacuum continues to grow exponentially with time. The chaotic inflation is also eternal, in which large quantum fluctuations during inflation can significantly increase the value of the energy density in some parts of the universe. These regions expand at a greater rate than their parent domains, and quantum fluctuations inside them lead to production of new inflationary domains which expand even faster. Jumps of the inflaton field due to quantum fluctuations lead to a process of eternal self-production of inflationary universe. In most inflationary models, once inflation happens, it produces not just one universe, but an infinite number of universes.

Now an important question is whether the emergent spacetime picture can also lead to the eternal inflation. The answer is certainly no. The reason is the following. We showed that the inflationary vacuum (3.15) arises as a solution of the (BFSS-like) matrix model (3.1). In order to define the matrix model (3.1), however, we have not introduced any spacetime structure. The vacuum (3.15) corresponds to the creation of spacetime unlike the traditional inflationary models that describe just the exponential expansion of a preexisting spacetime. Moreover, the inflationary vacuum (3.15) describes a dynamical process of the Planck energy condensate responsible for the emergence of spacetime. In general relativity the Minkowski spacetime with metric $g_{\mu\nu} = \eta_{\mu\nu}$ must be a completely empty space because the Einstein equation (1.1) requires $T_{\mu\nu} = 0$. However, in emergent gravity, it is not an empty space but full of the Planck energy as Eq. (3.10) clearly indicates. An important point is that the Planck energy condensate results in a highly coherent vacuum called the NC space. As the NC phase space in quantum mechanics necessarily brings about the Heisenberg's uncertainty relation, $\Delta x \Delta p \geq \frac{\hbar}{2}$, the NC space (2.4) also leads to the spacetime uncertainty relation. Therefore any further accumulation of energy over the vacuum (3.15) must be subject to the exclusion principle known as the UV/IR mixing [45]. Consequently, it is not possible to further accumulate the Planck energy density $\delta\rho \sim M_P^4$ over the inflationary vacuum (3.15). This means that it is impossible to superpose a new inflating subregion over the inflationary vacuum. In other words, the cosmic inflation triggered by the Planck energy condensate into vacuum must be a single event [21]. In the end we have a beautiful picture: The NC spacetime is necessary for the emergence of spacetime and the exclusion principle of NC spacetime guarantees the stability of spacetime. In conclusion, the emergent spacetime does not allow the pocket universes appearing in the eternal inflation.

The above argument suggests an intriguing picture for the dark energy too. Suppose that the inflation ended. This means that the inflationary vacuum (3.15) in nonequilibrium makes a (first-order) phase transition to the vacuum (2.11) in equilibrium in some way. We do not know how to do it. We will discuss a possible scenario in Part II. Since the vacuum (2.11) satisfies the NC commutation relation, any local fluctuations over the vacuum (2.11) must also be subject to the spacetime uncertainty relation or UV/IR mixing. This implies that any UV fluctuations are paired with corresponding IR fluctuations. For example, the most typical UV fluctuations are characterized by the Planck mass M_P and these will be paired with the most typical IR fluctuations with the largest possible wavelength denoted by $L_H = M_H^{-1}$. This means that these UV/IR fluctuations are extended up to the scale L_H which may be identified with the current size of cosmic horizon. By a simple dimensional analysis one can estimate the energy density of these fluctuations:

$$\delta\rho \sim M_P^2 M_H^2 = \frac{1}{L_P^2 L_H^2}. \tag{4.2}$$

It may be emphasized that, if the microscopic spacetime is NC, then the UV/IR mixing is inevitable and the extended (nonlocal) energy (4.2) is necessarily induced [21]. If we identify L_H with the cosmic horizon of our observable universe, $L_H \sim 1.3 \times 10^{26}$ m, $\delta\rho$ is roughly equal to the current dark energy, i.e.,

$$\delta\rho = M_{DE}^4 \sim (10^{-3}\mathrm{eV})^4. \tag{4.3}$$

Thus the emergent gravity predicts the existence of dark energy whose scale is characterized by the size of our visible universe. Since the characteristic scale of entire spacetime is set by the Planck mass M_P only, this implies that our observable universe is one causal patch out of much larger unobservable patches. According to the cosmic uroborus [2], we estimate the total number of causal patches in our Universe to be $M_P/M_H = M_P^2/M_{DE}^2 \sim 10^{60}$.

The gauge/gravity duality such as the AdS/CFT correspondence has clarified how a higher dimensional gravity can emerge from a lower dimensional gauge theory. A mysterious point is that the emergence of gravity requires the emergence of spacetime too. If spacetime is emergent, everything supported on the spacetime should be emergent too for an internal consistency of the theory. In particular, matters cannot exist without spacetime and thus must be emergent together with the spacetime. Eventually, the background-independent theory has to make no distinction between geometry and matter [15]. This is the reason why the emergent spacetime picture cannot coexist peacefully with the KK paradigm. As we pointed out before, the string landscape has been derived from the KK compactification of string theory. Therefore, if the emergent spacetime picture is correct, we need to carefully reexamine the string landscape (C) from that point of view. The emergent spacetime picture may endow the string landscape with a completely new interpretation since reversing the arrow in (4.1) accompanies a radical change of physics. For example, a geometry is now derived from a gauge theory while previously the gauge theory was derived from the geometry.

The KK compactification of string theory advocates that the Standard Model in four dimensions is determined by a six-dimensional internal geometry, e.g., a Calabi-Yau manifold. Thus different internal geometries mean different physical laws in four dimensions, so different universes governed by the different Standard Models. However, the emergent gravity reverses the arrow in (4.1). Rather internal geometries are determined by microscopic configurations of gauge fields and matter fields in four dimensions. As a consequence, different internal geometries mean different microscopic configurations of four-dimensional particles and nonperturbative objects such as solitons and instantons. This picture may be more strengthened by the fact [39] that Calabi-Yau manifolds are emergent from six-dimensional NC $U(1)$ instantons and thus the origin of Calabi-Yau manifolds is actually a gauge theory. If the microscopic configuration changes by interactions, then the corresponding change of the internal geometry will also be induced by the interactions. If so, the huge variety of internal geometries may correspond to the ensemble of microscopic configurations in four dimensions and 10^{500} would be the Avogadro number for the microscopic ensemble. Recall that NC geometry begins from the rough correspondence–contravariant functor–between the category of topological spaces and the category of commutative algebras over $\mathbb{C}$ and then changes the commutative algebras by NC algebras to define corresponding NC spaces. In this correspondence, different internal geometries correspond to choosing different NC algebras. We have observed that the latter allows a background-independent formulation which does not require a background geometry and a large amount (possibly infinitely many) of spacetime geometries can be described by generic deformations of a vacuum algebra in a master theory. Hence a background-independent quantum gravity seems to bring a new perspective that cripples all the rationales to introduce the multiverse hypothesis.

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Emergent Spacetime and Cosmic Inflation II

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ABSTRACT

In Part I, we have proposed a background-independent formulation of cosmic inflation. It was shown that the inflationary universe arises as a time-dependent solution of a background-independent theory such as matrix models without introducing any inflaton field as well as an *ad hoc* inflation potential. The emergent spacetime picture admits a background-independent formulation so that the inflation is responsible for the dynamical emergence of spacetime described by a conformal Hamiltonian system. In this sequel, we explore the mathematical foundation for the background-independent formulation of cosmic inflation and generalize the emergent spacetime picture to matrix string theory.

Keywords: Emergent spacetime, Cosmic inflation, Quantum gravity

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1 Summary of Part I

In Part I [1], we have shown that the emergent gravity from noncommutative (NC) $U(1)$ gauge theory is basically the large N duality and it can be applied to cosmic inflation. It has been based on the observation that the $N \to \infty$ limit of $U(N)$ Yang-Mills theory opens a new phase of the so-called NC Coulomb branch given by

$$[\phi_a, \phi_b]|_{\mathrm{vac}} = -iB_{ab} \qquad \Rightarrow \qquad \langle \phi_a \rangle_{\mathrm{vac}} = p_a \equiv B_{ab} y^b \tag{1.1}$$

where $B_{ab} = (\theta^{-1})_{ab}$ and the vacuum moduli y^a satisfy the Moyal-Heisenberg algebra

$$[y^a, y^b] = i\theta^{ab}, \qquad a, b = 1, \cdots, 2n. \tag{1.2}$$

A fundamental fact is that the NC space (1.2) denoted by $\mathbb{R}^{2n}_\theta$ is a consistent vacuum solution of a large N gauge theory in the Coulomb branch and more general solutions are generated by all possible (on-shell) deformations of the vacuum (1.1). To be specific, suppose that the deformations take the form

$$D_\mu = \partial_\mu - i\widehat{A}_\mu(x, y), \qquad \phi_a = p_a + \widehat{A}_a(x, y). \tag{1.3}$$

The adjoint scalar fields in Eq. (1.3) now obey the deformed algebra given by

$$[\phi_a, \phi_b] = -i(B_{ab} - \widehat{F}_{ab}) \in \mathcal{A}^d_\theta, \tag{1.4}$$

where

$$\widehat{F}_{ab} = \partial_a \widehat{A}_b - \partial_b \widehat{A}_a - i[\widehat{A}_a, \widehat{A}_b] \tag{1.5}$$

with the definition $\partial_a \equiv \mathrm{ad}_{p_a} = -i[p_a, \cdot]$. Plugging the fluctuations in Eq. (1.3) into the d-dimensional $U(N \to \infty)$ Yang-Mills theory, we get a remarkable identity [2, 3] given by

$$\begin{aligned} S &= -\frac{1}{g^2_{YM}} \int d^d x \mathrm{Tr}\Big(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi_a D^\mu \phi_a - \frac{1}{4}[\phi_a, \phi_b]^2\Big) \\ &= -\frac{1}{G^2_{YM}} \int d^D Y \frac{1}{4}(\widehat{F}_{AB} - B_{AB})^2, \end{aligned} \tag{1.6}$$

where $\widehat{A}_A(x, y) = (\widehat{A}_\mu, \widehat{A}_a)(x, y)$ are $D = (d + 2n)$-dimensional NC $U(1)$ gauge fields. We emphasize that the NC Coulomb branch (1.1) is crucial to realize the emergent gravity from matrix models or large N gauge theories. We summarize the emergent gravity picture from a large N gauge theory with the flowchart depicted in Fig. 1.

In order to complete the large N duality in Fig. 1, it is necessary to know how to map the NC $U(1)$ gauge theory to the Einstein gravity. Although the answer has already been known thanks to the works [3, 4, 5], we will give here a self-contained exposition to clarify the issues regarding to physical cosmology addressed in Part I. We observed in Part I [1] that the cosmic inflation arises as a time-dependent solution of matrix quantum mechanics (MQM), i.e. the $d = 1$ case in Eq.

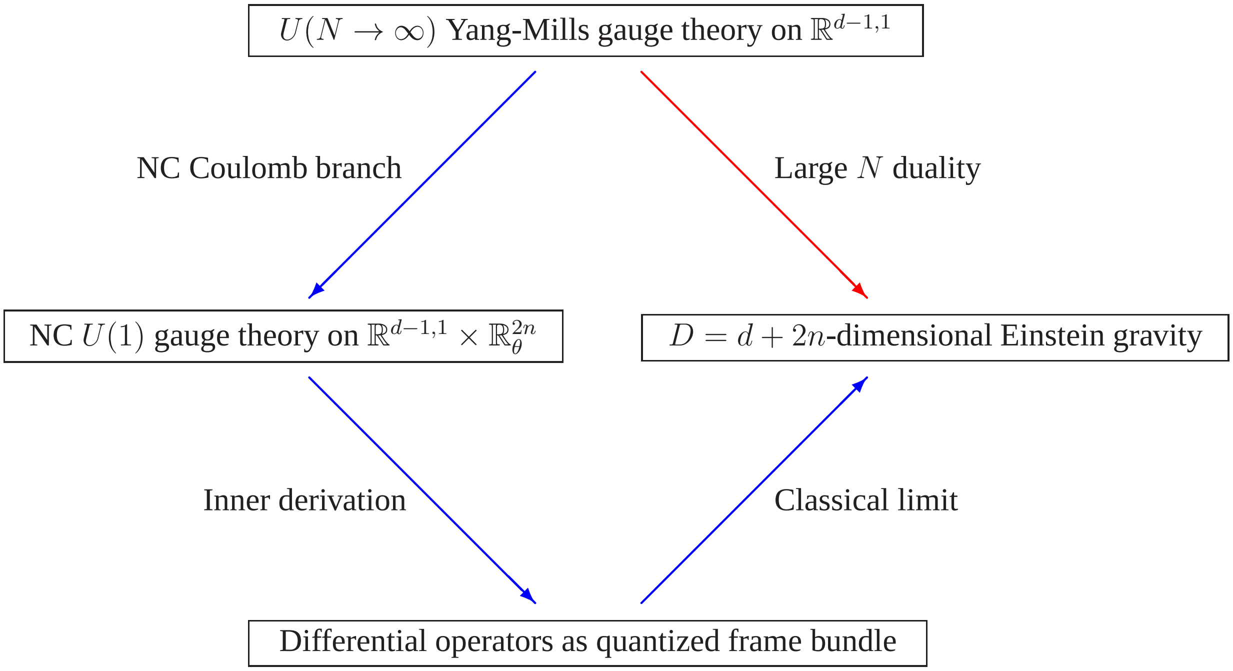

Figure 1: Flowchart for large N duality

(1.6), without introducing any inflaton field as well as an *ad hoc* inflation potential. In particular, the emergent spacetime picture admits a background-independent formulation of the cosmic inflation as the dynamical generation of spacetime. We have shown that the time-dependent vacuum configuration given by

$$\langle \phi_a(t) \rangle_{\text{vac}} = p_a(t) = e^{\frac{\kappa t}{2}} p_a, \qquad \langle A_0(t,y) \rangle_{\text{vac}} = \widehat{a}_0(t,y), \tag{1.7}$$

satisfies the equations of motion for the MQM, where κ is related to the inflationary Hubble constant $H = (n-1)\kappa$ and

$$\widehat{a}_0(t,y) = \frac{\kappa}{2} \int_0^1 d\sigma \frac{dy^a(\sigma)}{d\sigma} p_a(\sigma) \tag{1.8}$$

is an open Wilson line [6] along a path parameterized by the curve $y^a(\sigma) = y_0^a + \zeta^a(\sigma)$. The inflating background (1.7) determines the time-dependent metric given by

$$ds^2 = -dt^2 + e^{2Ht} d\mathbf{y}_t \cdot d\mathbf{y}_t, \tag{1.9}$$

where $y_t^a \equiv e^{\frac{\kappa t}{2}} y^a$. We emphasize that the temporal gauge field (1.8) is crucial to satisfy the equations of motion and generates a conformal vector field for the exponential behavior in Eq. (1.9) [1]. Note that the metric (1.9) is conformally flat, i.e., the corresponding Weyl tensors identically vanish and so describes a homogeneous and isotropic inflationary universe known as the Friedmann-Robertson-Walker metric in physical cosmology.

We can further consider standard cosmological perturbations by including arbitrary fluctuations around the inflationary background (1.7). Such a general inflationary universe in $(2n+1)$-dimensional

Lorentzian spacetime can be realized by considering a time-dependent NC algebra given by

$$ {}^t\mathcal{A}_\theta^1 \equiv \Big\{ \widehat{\phi}_0(t,y) = i\frac{\partial}{\partial t} + \widehat{A}_0(t,y), \quad \widehat{\phi}_a(t,y) = e^{\frac{\kappa t}{2}}\big(p_a + \widehat{A}_a(t,y)\big) \Big\}. \tag{1.10} $$

The module ${}^t\mathfrak{D}^1$ of derivations of the NC algebra ${}^t\mathcal{A}_\theta^1$ is given by

$$ {}^t\mathfrak{D}^1 = \Big\{ \widehat{V}_A(t) = (\widehat{V}_0, \widehat{V}_a)(t) | \widehat{V}_0(t) = \frac{\partial}{\partial t} + \mathrm{ad}_{\widehat{A}_0}, \quad \widehat{V}_a(t) = e^{\frac{\kappa t}{2}}\Big(\frac{\partial}{\partial y^a} + \mathrm{ad}_{\widehat{A}_a}\Big) \Big\}, \tag{1.11} $$

where the adjoint operations are defined by the derivation of $\mathcal{A}_\theta^1$ when $\kappa = 0$. In the classical limit of the module (1.11), we get a general inflationary universe described by

$$ ds^2 = -dt^2 + e^{2Ht}(1+\delta\lambda)^2 v_b^a v_c^a (dy_t^b - \mathbf{A}^b)(dy_t^c - \mathbf{A}^c)\big), \tag{1.12} $$

where $v_b^a := v_b^a(t,y)$, $\delta\lambda := \delta\lambda(t,y)$ and $\mathbf{A}^b := \delta a_0^b(t,y)dt$. If all fluctuations are turned off for which $v_b^a = \delta_b^a$ and $\delta\lambda = \mathbf{A}^b = 0$, we recover the inflation metric (1.9).

Since the cosmic inflation is simply the dynamical generation of spacetime according to the emergent spacetime picture, a particularly important issue is to understand the origin of space and time in the context of physical cosmology. The emergence of space is relatively easy to understand compared to the notorious issue on the emergent time. In order to grasp the emergence of time in quantum gravity, we will get a valuable lesson by examining how the time evolution of a dynamical system is defined in quantum mechanics. We have a great virtue by the fact that the mathematical structure of NC spacetime is basically equivalent to the NC phase space in quantum mechanics. Motivated by the close analogy with quantum mechanics, we argue that the evolution of spacetime structure supported on a vacuum solution must be understood as a dynamical system defined by large N matrices. We show that the resulting dynamical system can be described by the MQM corresponding to the $d = 1$ case in Eq. (1.6).

The Part II is organized as follows. In Sec. 2, we compactly review the background-independent formulation of emergent gravity and emergent spacetime in terms of matrix models [3, 4, 5, 7, 8, 9]. See also [10, 11, 12, 13]. The crux of the underlying argument is the realization that the NC space $\mathbb{R}_\theta^{2n}$ arises as a solution of a large N matrix model in the Coulomb branch and this vacuum admits a separable Hilbert space as quantum mechanics [3]. General solutions are generated by considering arbitrary deformations of a primitive vacuum such as $\mathbb{R}_\theta^{2n}$ obeying the Heisenberg algebra. These deformations can be arranged into a one-parameter family. Since any automorphism of the matrix algebra is inner, this means that they are described by the general inner automorphism of an underlying NC algebra $\mathcal{A}_\theta$. Thus these deformations are intrinsically dynamical. The (emergent) time is defined through the Hamiltonian description of the dynamical system like quantum mechanics. The emergent geometry is then simply derived from the nontrivial inner automorphism of the NC algebra $\mathcal{A}_\theta$, in which the NC nature is crucial to realize the emergent gravity [3, 8]. An important point is that the matrix model does not presuppose any spacetime background on which physical processes develop. Rather the matrix model provides a mechanism of spacetime generation such that every spacetime structure including the flat spacetime arises as a solution of the theory.

It is important to keep in mind that the inflationary scenario is at best an incomplete picture of the very early universe since it is known to be past incomplete [14]. This implies that we need to go beyond the inflationary cosmology if we really want to understand the very earliest moments of the universe. In Sec. 3, we observe that the vacuum configuration in the NC Coulomb branch is the Planck energy condensate responsible for the generation of spacetime and results in an extremely large spacetime. Because the Planck energy condensate into vacuum must be a dynamical process, we explore the dynamical mechanism for the instantaneous condensation of vacuum energy to enormously spread out spacetime. We show that the cosmic inflation as a dynamical system can be described by a locally conformal (co)symplectic manifold (see Appendix A for the definition) which is a generalized phase space of a time-dependent Hamiltonian system. Since the generalized symplectic manifold admits a rich variety of vector fields, in particular, Liouville vector fields that generate an exponential phase space expansion, the inflation can be described by the so-called conformal Hamiltonian system [15, 16] without introducing any inflaton field as well as an *ad hoc* inflation potential. It is remarkable to see that an inflationary vacuum describing the dynamical emergence of spacetime simply arises as a solution of time-dependent matrix model as far as a nonlocal temporal gauge field is introduced.

In Sec. 4, we emphasize that NC spacetime necessarily implies emergent spacetime if spacetime at microscopic scales should be viewed as NC. Although spacetime at the microscopic scale is intrinsically NC, we understand the NC spacetime through the quantization of a symplectic manifold. Since the most natural object to probe the symplectic geometry is a string rather than a particle [3] or a pseudoholomorphic curve which is a stringy generalization of a geodesic worldline in Riemannian geometry [17], we need a mathematically precise framework for describing strings in a background-independent way to make sense of the emergent spacetime proposal. We show that the pseudoholomorphic curve can be lifted to a NC spacetime by the matrix string theory [18, 19]. We argue that any NC spactime may be viewed as a second-quantized string for the background-independent formulation of quantum gravity, which is still elusive in the usual string theory. Hence we need to read old literatures with the new perspective.

In Sec. 5, we discuss a speculative mechanism for a graceful exit from inflation by some nonlinear damping through interactions between the inflating background and ubiquitous local fluctuations. We also discuss possible approaches to understand our real world $\mathbb{R}^{3,1}$ that is unfortunately beyond our current approach because $\mathbb{R}^{3,1}$ does not belong to the class of (almost) symplectic manifolds.

In the first appendix, we briefly review the mathematical foundation of locally conformal cosymplectic (LCC) manifolds that correspond to a natural phase space describing the cosmic inflation of our universe. In the second appendix, we give a brief exposition of harmonic oscillator with time-dependent mass to illustrate how a nonconservative dynamical system with friction can be formulated by a time-dependent Hamiltonian system which may be useful to understand the cosmic inflation as a dynamical system

2 Emergent spacetime from matrix model

Let us start with a zero-dimensional matrix model with a bunch of $N \times N$ Hermitian matrices, $\{\phi_a \in \mathcal{A}_N | a = 1, \cdots, 2n\}$, whose action is given by [20]

$$S = -\frac{1}{4} \sum_{a,b=1}^{2n} \mathrm{Tr}\, [\phi_a, \phi_b]^2. \tag{2.1}$$

In particular, we are interested in the matrix algebra $\mathcal{A}_N$ in the limit $N \to \infty$. We require that the matrix algebra $\mathcal{A}_N$ is associative, from which we get the Jacobi identity

$$[\phi_a, [\phi_b, \phi_c]] + [\phi_b, [\phi_c, \phi_a]] + [\phi_c, [\phi_a, \phi_b]] = 0. \tag{2.2}$$

We also assume the action principle, from which we yield the equations of motion:

$$\sum_{b=1}^{2n} [\phi_b, [\phi_a, \phi_b]] = 0. \tag{2.3}$$

We emphasize that we have not introduced any spacetime structure to define the action (2.1). It is enough to suppose the matrix algebra $\mathcal{A}_N$ consisted of a bunch of matrices which are subject to a few relationships given by Eqs. (2.2) and (2.3).

First suppose that the vacuum configuration of $\mathcal{A}_N$ is given by

$$\langle \phi_a \rangle_{\mathrm{vac}} = p_a \in \mathcal{A}_N, \tag{2.4}$$

which must be a solution of Eqs. (2.2) and (2.3). An obvious solution in the limit $N \to \infty$ is given by the Moyal-Heisenberg algebra[1]

$$[p_a, p_b] = -iB_{ab}, \tag{2.5}$$

where $(B_{ab}) = -L_P^{-2}(\mathbf{1}_n \otimes i\sigma^2)$ is a $2n \times 2n$ constant symplectic matrix and L_P is a typical length scale set by the vacuum. A general solution will be generated by considering all possible deformations of the Moyal-Heisenberg algebra (2.5). It is assumed to take the form

$$\phi_a = p_a + \widehat{A}_a \in \mathcal{A}_N, \tag{2.6}$$

obeying the deformed algebra given by

$$[\phi_a, \phi_b] = -i(B_{ab} - \widehat{F}_{ab}), \tag{2.7}$$

[1]The conventional choice of vacuum in Coulomb branch is given by $[\phi_a, \phi_b]|_{\mathrm{vac}} = 0$ and so $\langle \phi_a \rangle_{\mathrm{vac}} = \mathrm{diag}\big((\alpha_a)_1, (\alpha_a)_2, \cdots, (\alpha_a)_N\big)$. However, it turns out (see Section III.C in [5]) that, in order to describe a classical geometry from a background-independent theory, it is necessary to have a nontrivial vacuum defined by a coherent condensation obeying the algebra (2.5). For this reason, we will choose the Moyal-Heisenberg vacuum instead of the conventional vacuum. A similar reasoning was also advocated in footnote 2 in Ref. [2].

where

$$\widehat{F}_{ab} = \partial_a \widehat{A}_b - \partial_b \widehat{A}_a - i[\widehat{A}_a, \widehat{A}_b] \in \mathcal{A}_N \tag{2.8}$$

with the definition $\partial_a \equiv \mathrm{ad}_{p_a} = -i[p_a, \cdot]$. For the general matrix $\phi_a \in \mathcal{A}_N$ to be a solution of Eqs. (2.2) and (2.3), the set of matrices $\widehat{F}_{ab} \in \mathcal{A}_N$, called the field strengths of NC $U(1)$ gauge fields $\widehat{A}_a \in \mathcal{A}_N$, must obey the following equations

$$\widehat{D}_a \widehat{F}_{bc} + \widehat{D}_b \widehat{F}_{ca} + \widehat{D}_c \widehat{F}_{ab} = 0, \tag{2.9}$$

$$\sum_{b=1}^{2n} \widehat{D}_b \widehat{F}_{ab} = 0, \tag{2.10}$$

where

$$\widehat{D}_a \widehat{F}_{bc} \equiv \mathrm{ad}_{\phi_a} \widehat{F}_{bc} = -i[\phi_a, \widehat{F}_{bc}] = -[\phi_a, [\phi_b, \phi_c]]. \tag{2.11}$$

The algebra $\mathcal{A}_N$ admits a large amount of inner automorphism denoted by $\mathrm{Inn}(\mathcal{A}_N)$. Note that any automorphism of the matrix algebra $\mathcal{A}_N$ is inner. Suppose that $\mathcal{A}'_{\widetilde{N}} = \{\phi'_a | a = 1, \cdots, m\}$ is an another matrix algebra composed of m elements of $\widetilde{N} \times \widetilde{N}$ Hermitian matrices. We will identify two matrix algebras, i.e. $\mathcal{A}_N \cong \mathcal{A}'_{\widetilde{N}}$ if $m = 2n$ and $\widetilde{N} = N$ and there exists a unitary matrix $U_a \in \mathrm{Inn}(\mathcal{A}_N)$ such that $\phi'_a = U_a \phi_a U_a^{-1}$ for each $a = 1, \cdots, 2n$. It is important to recall that the NC algebra $\mathcal{A}_N$ generated by the vacuum operators p_a admits an infinite-dimensional separable Hilbert space

$$\mathcal{H} = \{|n\rangle | n = 1, \cdots, N \to \infty\}, \tag{2.12}$$

that is the Fock space of the Moyal-Heisenberg algebra (2.5). As is well-known from quantum mechanics [21], there is a one-to-one correspondence between the operators in $\mathrm{Hom}(V)$ and the set of $N \times N$ matrices over $\mathbb{C}$ where V is an N-dimensional complex vector space. In our case, $V = \mathcal{H}$ is a Hilbert space and $N = \dim(\mathcal{H}) \to \infty$. Thus the matrix algebra $\mathcal{A}_N$ can be realized as a Hilbert space representation of the NC $\star$-algebra

$$\mathcal{A}_\theta = \{\widehat{\phi}_a(y) \in \mathrm{Hom}(\mathcal{H}) | a = 1, \cdots, 2n\}, \tag{2.13}$$

which is generated by the set of coordinate generators in Eq. (1.2). The commutator (1.2) is related to the Moyal-Heisenberg algebra (2.5) by $\theta^{ab} = (B^{-1})^{ab}$ and $p_a = B_{ab} y^b$. To be specific, given a Hermitian operator $\widehat{\phi}_a(y) \in \mathcal{A}_\theta$, we have a matrix representation in $\mathcal{H}$ as follows:

$$\widehat{\phi}_a(y) = \sum_{n,m=1}^{\infty} |n\rangle \langle n | \widehat{\phi}_a(y) | m \rangle \langle m| = \sum_{n,m=1}^{\infty} (\phi_a)_{nm} |n\rangle\langle m| \tag{2.14}$$

using the completeness of $\mathcal{H}$, i.e. $\sum_{n=1}^{\infty} |n\rangle\langle n| = \mathbf{1}_{\mathcal{H}}$. The unitary representation of the operator algebra $\mathcal{A}_\theta$ can thus be understood as a linear transformation acting on an N-dimensional Hilbert space $\mathcal{H}_N$:

$$\mathcal{A}_\theta : \mathcal{H}_N \to \mathcal{H}_N. \tag{2.15}$$

That is, we have the identification

$$\mathcal{A}_N \cong \mathrm{End}(\mathcal{H}_N) \cong \mathcal{A}_\theta. \tag{2.16}$$

As a result, the inner automorphism $\mathrm{Inn}(\mathcal{A}_N)$ of the matrix algebra $\mathcal{A}_N$ is translated into that of the NC $\star$-algebra $\mathcal{A}_\theta$, denoted by $\mathrm{Inn}(\mathcal{A}_\theta)$. Its infinitesimal generators consist of an inner derivation $\mathfrak{D}$ defined by the map [3, 4, 5, 8]

$$\mathcal{A}_\theta \to \mathfrak{D} : \mathcal{O} \mapsto \mathrm{ad}_{\mathcal{O}} = -i[\mathcal{O}, \cdot]_\star \tag{2.17}$$

for any operator $\mathcal{O} \in \mathcal{A}_\theta$. Using the Jacobi identity of the NC $\star$-algebra $\mathcal{A}_\theta$, one can easily verify the Lie algebra homomorphism:

$$[\mathrm{ad}_{\mathcal{O}_1}, \mathrm{ad}_{\mathcal{O}_2}] = -i\mathrm{ad}_{[\mathcal{O}_1, \mathcal{O}_2]_\star} \tag{2.18}$$

for any $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{A}_\theta$. In particular, we are interested in the set of derivations determined by NC gauge fields in Eq. (2.13):

$$\{\widehat{V}_a \equiv \mathrm{ad}_{\widehat{\phi}_a} \in \mathfrak{D} | \widehat{\phi}_a(y) = p_a + \widehat{A}_a(y) \in \mathcal{A}_\theta,\ a = 1, \cdots, 2n\}. \tag{2.19}$$

In a large-distance limit, i.e. $|\theta| \to 0$, one can expand the NC vector fields $\widehat{V}_a$ using the explicit form of the Moyal $\star$-product. The result takes the form[2]

$$\widehat{V}_a = V_a^\mu(y)\frac{\partial}{\partial y^\mu} + \sum_{p=2}^{\infty} V_a^{\mu_1 \cdots \mu_p}(y)\frac{\partial}{\partial y^{\mu_1}} \cdots \frac{\partial}{\partial y^{\mu_p}} \in \mathfrak{D}. \tag{2.20}$$

Thus the NC vector fields in $\mathfrak{D}$ generates an infinite tower of the so-called polyvector fields [3]. Note that the leading term gives rise to the ordinary vector fields that will be identified with a frame basis associated to the tangent bundle $T\mathcal{M}$ of an emergent manifold $\mathcal{M}$. Since the leading term in (2.20) already generates the gravitational fields of spin 2, the higher-order terms correspond to higher-spin fields with spin ≥ 3.

Since we have started with a large N matrix model, it is natural to expect that the IKKT-type matrix model (2.1) is dual to a higher-dimensional gravity or string theory according to the large N duality or gauge/gravity duality [22]. The emergent gravity is realized via the gauge-gravity duality following the $d = 0$ case of the flowchart in Fig. 1 [3]:

$$\mathcal{A}_N \quad \Longrightarrow \quad \mathcal{A}_\theta \quad \Longrightarrow \quad \mathfrak{D}. \tag{2.21}$$

The gauge theory side of the duality is described by the set of large N matrices that consists of an associative, but NC, algebra $\mathcal{A}_N$. By choosing a proper vacuum such as Eq. (2.4), a matrix in $\mathcal{A}_N$ is regarded as a linear representation of an operator acting on a separable Hilbert space $\mathcal{H}$. That is, the

[2] In Part II, we will use the Greek letters to denote local indices of NC coordinates unlike the Part I indicating commutative ones as in Eq. (1.6).

matrix algebra $\mathcal{A}_N$ is realized as a representation of an operator algebra $\mathcal{A}_\theta$ on the Hilbert space $\mathcal{H}$, i.e., $\mathcal{A}_N \cong \mathrm{End}(\mathcal{H})$. Consequently the algebra $\mathcal{A}_N$ is isomorphically mapped to the NC $\star$-algebra $\mathcal{A}_\theta$, as Eq. (2.14) has clearly illustrated. The gravity side of the duality is defined by associating the derivation $\mathfrak{D}$ of the algebra $\mathcal{A}_\theta$ with a quantized frame bundle $\widehat{\mathfrak{X}}(\mathcal{M})$ of an emergent spacetime manifold $\mathcal{M}$. The noncommutativity of an underlying algebra is thus crucial to realize the emergent gravity. As we discussed in footnote 1, this is the reason why we need the Moyal-Heisenberg vacuum (2.5) instead of the conventional Coulomb branch vacuum [1]. If we choose the conventional vacuum, we will fail to realize the isomorphism between $\mathcal{A}_N$ and $\mathcal{A}_\theta$. After all, in order to describe a quantum geometry mathematically, we need to find a right NC algebra.[3]

It is important to perceive that the realization of emergent geometry through the duality chain in Eq. (2.21) is intrinsically local. Therefore it is necessary to consider patching or gluing together the local constructions to form a set of global quantities. For this purpose, the concept of sheaf may be essential because it makes it possible to reconstruct global data starting from open sets of locally defined data [23]. Let us explain this feature briefly since its extensive exposition was already given in Ref. [3]. Its characteristic feature becomes transparent when the commutative limit, i.e. $|\theta| \to 0$, is taken into account. In this limit, the NC $\star$-algebra $\mathcal{A}_\theta$ reduces to a Poisson algebra $\mathfrak{P}^{(i)} = (C^\infty(U_i), \{-,-\}_\theta)$ defined on a local patch $U_i \subset M$ in an open covering $M = \bigcup_{i \in I} U_i$. The Poisson algebra $\mathfrak{P}^{(i)}$ arises as follows. Let $L \to M$ be a line bundle over M whose connection is denoted by $\mathcal{A}$. We assume that the curvature $\mathcal{F}$ of the line bundle L is a *nondegenerate*, closed two-form. Therefore we identify the curvature two-form $\mathcal{F} = d\mathcal{A}$ with a symplectic structure of M. On an open neighborhood $U_i \subset M$, it is possible to represent $\mathcal{F}^{(i)} = B + F^{(i)}$ where $F^{(i)} = dA^{(i)}$ and B is the constant symplectic two-form already introduced in Eq. (2.5). Consider a chart $(U_i, \phi_{(i)})$ where $\phi_{(i)} \in \mathrm{Diff}(U_i)$ is a local trivialization of the line bundle L over the open subset U_i obeying $\phi^*_{(i)}(\mathcal{F}^{(i)}) = B$. Such a local chart always exists owing to the Darboux theorem or the Moser lemma in symplectic geometry [24] and the local coordinate chart obeying $\phi^*_{(i)}(\mathcal{F}^{(i)}) = B$ is called Darboux coordinates. Thus the line bundle $L \to M$ corresponds to a dynamical symplectic manifold $(M, \mathcal{F})$ where $\mathcal{F} = B + dA$. The dynamical system is locally described by the Poisson algebra $\mathfrak{P}^{(i)} = (C^\infty(U_i), \{-,-\}_\theta)$ in which the vector space $C^\infty(U_i)$ is formed by the set of Darboux transformations $\phi_{(i)} \in \mathrm{Diff}(U_i)$ equipped with the Poisson bracket defined by the Poisson bivector $\theta = B^{-1} \in \Gamma(\Lambda^2 TM)$.

Consider a collection of local charts to make an atlas $\{(U_i, \phi_{(i)})\}$ on $M = \bigcup_{i \in I} U_i$ and complete the atlas by gluing these charts on their overlap. To be precise, suppose that $(U_i, \phi_{(i)})$ and $(U_j, \phi_{(j)})$ are two coordinate charts and $F^{(i)} = dA^{(i)}$ and $F^{(j)} = dA^{(j)}$ are local curvature two-forms on U_i

[3]The explicit realization of the duality chain (2.21) depends on the data of the matrix algebra $\mathcal{A}_N$. In particular, the vacuum of the algebra $\mathcal{A}_N$ depends on the rank N and the number of linearly independent matrices. Given the data of $\mathcal{A}_N$, the vacuum will be specified by choosing a most primitive one so that more general solutions are generated by deforming the primitive vacuum as we already implemented in Eq. (2.6). For instance, for our particular choice given by $N \to \infty$ and *even* number of matrices, the Moyal-Heisenberg algebra (2.5) is the most primitive vacuum for quantum gravity. This statement may be regarded as a quantum version of the Darboux theorem in symplectic geometry.

and U_j, respectively. We choose the coordinate maps $\phi_{(i)} \in \text{Diff}(U_i)$ and $\phi_{(j)} \in \text{Diff}(U_j)$ such that $\phi^*_{(i)}(B + F^{(i)}) = B$ and $\phi^*_{(j)}(B + F^{(j)}) = B$. On an intersection $U_i \cap U_j$, the local data $(A^{(i)}, \phi_{(i)})$ and $(A^{(j)}, \phi_{(j)})$ on Darboux charts $(U_i, \phi_{(i)})$ and $(U_j, \phi_{(j)})$, respectively, are patched or glued together by [25]

$$A^{(j)} = A^{(i)} + d\lambda^{(ji)}, \tag{2.22}$$

$$\phi_{(ji)} = \phi_{(j)} \circ \phi^{-1}_{(i)}, \tag{2.23}$$

where $\phi_{(ji)} \in \text{Diff}(U_i \cap U_j)$ is a symplectomorphism on $U_i \cap U_j$ generated by a Himiltonian vector field $X_{\lambda^{(ji)}}$ satisfying $\iota(X_{\lambda^{(ji)}})B + d\lambda^{(ji)} = 0$. We sometimes denote the interior product ι_X by $\iota(X)$ for a notational convenience. Similarly, we can glue the local Poisson algebras $\mathfrak{P}^{(i)}$ to form a globally defined Poisson algebra $\mathfrak{P} = \bigcup_{i \in I} \mathfrak{P}^{(i)}$. The global vector fields $V_a = V^\mu_a(y)\frac{\partial}{\partial y^\mu} \in \Gamma(T\mathcal{M}),\ a = 1, \cdots, 2n$, in Eq. (2.20) can be obtained by applying a similar globalization to the derivation $\mathfrak{D}$, which form a linearly independent basis of the tangent bundle $T\mathcal{M}$ of a $2n$-dimensional emergent manifold $\mathcal{M}$. As a consequence, the set of global vector fields $\mathfrak{X}(\mathcal{M}) = \{V_a | a = 1, \cdots, 2n\}$ results from the globally defined Poisson algebra $\mathfrak{P}$ [3].

The vector fields $V_a \in \mathfrak{X}(\mathcal{M})$ are related to an orthonormal frame, the so-called vielbeins $E_a \in \Gamma(T\mathcal{M})$, in general relativity by the relation

$$V_a = \lambda E_a, \qquad a = 1, \cdots, 2n. \tag{2.24}$$

The conformal factor $\lambda \in C^\infty(\mathcal{M})$ is determined by imposing the condition that the vector fields V_a preserve a volume form

$$\nu = \lambda^2 v^1 \wedge \cdots \wedge v^{2n}, \tag{2.25}$$

where $v^a = v^a_\mu(y)dy^\mu \in \Gamma(T^*\mathcal{M})$ are coframes dual to V_a, i.e., $\langle v^a, V_b \rangle = \delta^a_b$. This means that the vector fields V_a obey the conditions

$$\mathcal{L}_{V_a}\nu = \big(\nabla \cdot V_a + (2 - 2n)V_a \ln \lambda\big)\nu = 0, \qquad \forall a = 1, \cdots, 2n, \tag{2.26}$$

where $\mathcal{L}_X = \iota_X d + d\iota_X$ is the Lie derivative with respect to a vector field X. Note that a symplectic manifold always admits such volume-preserving vector fields. (See Appendix B in [3].) Together with the volume-preserving condition (2.26), the relation (2.24) completely determines a $2n$-dimensional Riemannian manifold $\mathcal{M}$ whose metric is given by [3, 4, 5]

$$\begin{aligned} ds^2 &= \mathcal{G}_{\mu\nu}(x)dx^\mu \otimes dx^\nu = e^a \otimes e^a \\ &= \lambda^2 v^a \otimes v^a = \lambda^2 v^a_\mu(y) v^a_\nu(y) dy^\mu \otimes dy^\nu, \end{aligned} \tag{2.27}$$

where $e^a = e^a_\mu(x)dx^\mu = \lambda v^a \in \Gamma(T^*\mathcal{M})$ are orthonormal one-forms on $\mathcal{M}$. After all, the $2n$-dimensional Riemannian manifold $\mathcal{M}$ is emergent from the commutative limit of polyvector fields $\widehat{V}_a = V_a + \mathcal{O}(\theta^2) \in \mathfrak{D}$ derived from NC $U(1)$ gauge fields.

So far we have discussed the emergence of spaces only. However, the theory of relativity dictates that space and time must be coalesced into the form of Minkowski spacetime in a locally inertial frame. Hence, if general relativity is realized from a NC $\star$-algebra $\mathcal{A}_\theta$, it is necessary to put space and time on an equal footing in the NC $\star$-algebra $\mathcal{A}_\theta$. If space is emergent, so should time. Thus, an important problem is how to realize the emergence of "time." Quantum mechanics offers us a valuable lesson that the definition of (particle) time is strictly connected with the problem of dynamics. In quantum mechanics, the time evolution of a dynamical system is defined as an inner automorphism of NC algebra $\mathcal{A}_\hbar$ generated by the NC phase space

$$[x^i, x^j] = 0, \qquad [x^i, p_j] = i\hbar\delta^i_j, \qquad i, j = 1, \cdots, n. \tag{2.28}$$

The time evolution for an observable $f \in \mathcal{A}_\hbar$ is simply an inner derivation of $\mathcal{A}_\hbar$ given by

$$\frac{df}{dt} = \frac{i}{\hbar}[H, f]. \tag{2.29}$$

A remarkable picture, as observed by Feynman [26], Souriau, and Sternberg [27], is that the physical forces such as the electromagnetic, weak and strong forces, can be realized as the deformations of an underlying vacuum algebra such as Eq. (2.28). For example, the most general deformation of the Heisenberg algebra (2.28) within the *associative* algebra $\mathcal{A}_\hbar$ is given by

$$x^i \to x^i, \qquad p_i \to p_i + A_i(x,t), \qquad H \to H + A_0(x,t), \tag{2.30}$$

where $(A_0, A_i)(x,t)$ must be electromagnetic gauge fields. Then the time evolution of a particle system under a time-dependent external force is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{i}{\hbar}[H, f]. \tag{2.31}$$

Note that the construction of the NC algebra $\mathcal{A}_N$ or $\mathcal{A}_\theta$ bears a close parallel to quantum mechanics. The former is based on the NC space (1.2) while the latter is based on the NC phase space (2.28). The NC $U(1)$ gauge fields in Eq. (2.6) act as deformations of the vacuum algebra (2.5) in the matrix algebra $\mathcal{A}_N$, similarly to Eq. (2.30) in the quantum algebra $\mathcal{A}_\hbar$. Therefore we can apply the same philosophy to the NC algebra $\mathcal{A}_N$ or $\mathcal{A}_\theta$ to define a dynamical system based on the Moyal-Heisenberg algebra (2.5). In other words, we can consider a one-parameter family of deformations of zero-dimensional matrices which is parameterized by the coordinate t. Then the one-parameter family of deformations characterized by (2.6) and (2.7) can be regarded as the time evolution of a dynamical system. For this purpose, we extend the NC algebra $\mathcal{A}_\theta$ to $\mathcal{A}^1_\theta \equiv \mathcal{A}_\theta\big(C^\infty(\mathbb{R})\big) = C^\infty(\mathbb{R}) \otimes \mathcal{A}_\theta$ whose generic element takes the form

$$\widehat{f}(t,y) \in \mathcal{A}^1_\theta. \tag{2.32}$$

The matrix representation (2.14) is then replaced by

$$\widehat{f}(t,y) = \sum_{n,m=1}^{\infty} |n\rangle\langle n|\widehat{f}(t,y)|m\rangle\langle m| = \sum_{n,m=1}^{\infty} f_{nm}(t)|n\rangle\langle m| \tag{2.33}$$

where $f_{nm}(t) := [f(t)]_{nm}$ are elements of a matrix $f(t)$ in $\mathcal{A}_N^1 \equiv \mathcal{A}_N\big(C^\infty(\mathbb{R})\big) = C^\infty(\mathbb{R}) \otimes \mathcal{A}_N$ as a representation of the observable (2.32) on the Hilbert space (2.12). As the Heisenberg equation (2.31) in quantum mechanics suggests, the evolution equation for an observable $\widehat{f}(t,y) \in \mathcal{A}_\theta^1$ in the Heisenberg picture is defined by

$$\frac{d\widehat{f}(t,y)}{dt} = \frac{\partial \widehat{f}(t,y)}{\partial t} - i[\widehat{A}_0(t,y), \widehat{f}(t,y)]_\star \equiv \widehat{D}_0 \widehat{f}(t,y) \tag{2.34}$$

where we denoted the local Hamiltonian density by

$$\widehat{H}(t,y) \equiv -\widehat{A}_0(t,y) \in \mathcal{A}_\theta^1. \tag{2.35}$$

The definition (2.34) is intended for the following reason. Note that

$$-i[\phi_a, \widehat{f}(t)] = \partial_a \widehat{f}(t,y) - i[\widehat{A}_a(t,y), \widehat{f}(t,y)]_\star \equiv \widehat{D}_a \widehat{f}(t,y), \tag{2.36}$$

where the representation (2.33) has been employed. Then one can see that the inner automorphism $\mathrm{Inn}(\mathcal{A}_\theta)$ of $\mathcal{A}_\theta$ can be lifted to the automorphism of $\mathcal{A}_\theta^1$ given by

$$\widehat{A}_0(t,y) \to \widehat{U}(t,y) \star \frac{\partial \widehat{U}^{-1}(t,y)}{\partial t} + \widehat{U}(t,y) \star \widehat{A}_0(t,y) \star \widehat{U}^{-1}(t,y), \tag{2.37}$$

$$\widehat{A}_a(t,y) \to \widehat{U}(t,y) \star \frac{\partial \widehat{U}^{-1}(t,y)}{\partial y^a} + \widehat{U}(t,y) \star \widehat{A}_a(t,y) \star \widehat{U}^{-1}(t,y), \tag{2.38}$$

where $\widehat{U}(t,y) = e_\star^{i\widehat{\lambda}(t,y)}$ with $\widehat{\lambda}(t,y) \in \mathcal{A}_\theta^1$. It is obvious that the above automorphism is nothing but the gauge transformation for NC $U(1)$ gauge fields in $(2n+1)$-dimensions [28].

Our leitmotif is that a consistent theory of quantum gravity should be background-independent, so that it should not presuppose any spacetime background on which fundamental processes develop. Hence the background-independent theory must provide a mechanism of spacetime generation such that every spacetime structure including the flat spacetime arises as a solution of the theory itself. The most natural candidate for such a background-independent theory is a zero-dimensional matrix model such as Eq. (2.1) because it is not necessary to assume the prior existence of spacetime to define the theory. Hence a background spacetime also arises as a vacuum solution of an underlying theory. We emphasize again that the NC nature of the vacuum solution, e.g. Eq. (2.5), is essential to realize the large N duality via the duality chain (2.21). A profound feature is that the background-independent theory is intrinsically dynamical because the space of all possible solutions is extremely large, typically infinite-dimensional and generic deformations of a primitive vacuum such as Eq. (2.5) will span a large subspace, at least, in the Morita equivalent class of NC algebras [3]. We argued that the dynamics under the Moyal-Heisenberg vacuum (2.4) is described by the NC algebra $\mathcal{A}_N^1 = \mathcal{A}_N\big(C^\infty(\mathbb{R})\big) = C^\infty(\mathbb{R}) \otimes \mathcal{A}_N$. One may regard $\mathcal{A}_N^1$ as a one-parameter family of deformations of the algebra $\mathcal{A}_N$. In this case we can generalize the duality chain (2.21) to realize the "time-dependent" gauge/gravity duality as follows:

$$\mathcal{A}_N^1 \quad \Longrightarrow \quad \mathcal{A}_\theta^1 \quad \Longrightarrow \quad \mathfrak{D}^1. \tag{2.39}$$

It is well-known [29] that in the case of $\mathcal{A}^1_N$ or $\mathcal{A}^1_\theta$, the module of its derivations can be written as a direct sum of the submodules of horizontal and inner derivations:

$$\mathfrak{D}^1 = \mathrm{Hor}(\mathcal{A}^1_N) \oplus \mathfrak{D}(\mathcal{A}^1_N) \cong \mathrm{Hor}(\mathcal{A}^1_\theta) \oplus \mathfrak{D}(\mathcal{A}^1_\theta) \tag{2.40}$$

where horizontal derivation is a lifting of smooth vector fields on $\mathbb{R}$ onto $\mathcal{A}^1_N$ or $\mathcal{A}^1_\theta$ and is locally generated by a vector field

$$g(t,y)\frac{\partial}{\partial t} \in \mathrm{Hor}(\mathcal{A}^1_\theta). \tag{2.41}$$

The inner derivation $\mathfrak{D}(\mathcal{A}^1_\theta)$ is defined by lifting the NC vector fields in Eq. (2.19) onto $\mathcal{A}^1_\theta$ and generated by

$$\{\widehat{V}_a(t) \equiv \mathrm{ad}_{\widehat{\phi}_a} \in \mathfrak{D}(\mathcal{A}^1_\theta) | \widehat{\phi}_a(t,y) = p_a + \widehat{A}_a(t,y) \in \mathcal{A}^1_\theta, \ a = 1, \cdots, 2n\} \tag{2.42}$$

and

$$\Big\{\widehat{V}_0(t) - \frac{\partial}{\partial t} \equiv \mathrm{ad}_{\widehat{A}_0} \in \mathfrak{D}(\mathcal{A}^1_\theta) | \widehat{A}_0(t,y) \in \mathcal{A}^1_\theta\Big\}. \tag{2.43}$$

It might be remarked that the definition of the time-like vector field $\widehat{V}_0(t)$ is motivated by the quantum Hamilton's equation (2.34), i.e.,

$$\widehat{V}_0(t) := \frac{d}{dt}. \tag{2.44}$$

Consequently, the module of the derivations of the NC algebra $\mathcal{A}^1_\theta$ is given by

$$\mathfrak{D}^1 = \Big\{\widehat{V}_A(t) = \big(\widehat{V}_0, \widehat{V}_a\big)(t) | \widehat{V}_0(t) = \frac{\partial}{\partial t} + \mathrm{ad}_{\widehat{A}_0}, \ \widehat{V}_a(t) = \mathrm{ad}_{\widehat{\phi}_a}, \ A = 0, 1, \cdots, 2n\Big\}. \tag{2.45}$$

In the commutative limit, $|\theta| \to 0$, the time-dependent polyvector fields $\widehat{V}_A(t)$ in $\mathfrak{D}^1$ will take the following form

$$\widehat{V}_0(t) = \frac{\partial}{\partial t} + A^\mu_0(t,y)\frac{\partial}{\partial y^\mu} + \sum_{p=2}^\infty A^{\mu_1\cdots\mu_p}_0(t,y)\frac{\partial}{\partial y^{\mu_1}}\cdots\frac{\partial}{\partial y^{\mu_p}}, \tag{2.46}$$

$$\widehat{V}_a(t) = V^\mu_a(t,y)\frac{\partial}{\partial y^\mu} + \sum_{p=2}^\infty V^{\mu_1\cdots\mu_p}_a(t,y)\frac{\partial}{\partial y^{\mu_1}}\cdots\frac{\partial}{\partial y^{\mu_p}}. \tag{2.47}$$

Let us truncate the above polyvector fields to ordinary vector fields given by

$$\mathfrak{X}(\mathcal{M}) = \Big\{V_A = V^M_A(t,y)\frac{\partial}{\partial X^M} | A, M = 0, 1, \cdots, 2n\Big\} \tag{2.48}$$

where $V^0_A = \delta^0_A$ and $X^M = (t, y^\mu)$ are local coordinates on an emergent *Lorentzian* manifold $\mathcal{M}$ of $(2n+1)$-dimensions. The orthonormal vielbeins on $T\mathcal{M}$ are then obtained by the prescription [1]

$$(V_0, V_a) = (E_0, \lambda E_a) \in \Gamma(T\mathcal{M}) \tag{2.49}$$

or on $T^*\mathcal{M}$

$$(e^0, e^a) = (v^0, \lambda v^a) \in \Gamma(T^*\mathcal{M}). \tag{2.50}$$

The conformal factor $\lambda \in C^\infty(\mathcal{M})$ is similarly determined by the volume-preserving condition

$$\mathcal{L}_{V_A}\nu_t = \big(\nabla \cdot V_A + (2-2n)V_A \ln \lambda\big)\nu_t = 0, \qquad \forall A = 0, 1, \cdots, 2n. \tag{2.51}$$

The above condition explicitly reads as

$$\frac{\partial \rho}{\partial t} + \partial_\mu(\rho A_0^\mu) = 0 \quad \& \quad \partial_\mu(\rho V_a^\mu) = 0, \tag{2.52}$$

where $\rho = \lambda^2 \mathrm{det} v_\mu^a$ and

$$\nu_t \equiv dt \wedge \nu = \lambda^2 dt \wedge v^1 \wedge \cdots \wedge v^{2n} \tag{2.53}$$

is a $(2n+1)$-dimensional volume form on $\mathcal{M}$. If the structure equation of vector fields $V_A \in \Gamma(T\mathcal{M})$ is defined by

$$[V_A, V_B] = -g_{AB}{}^C V_C, \tag{2.54}$$

the volume-preserving condition (2.51) can equivalently be written as [5]

$$g_{BA}{}^B = V_A \ln \lambda^2. \tag{2.55}$$

In the end, the Lorentzian metric on a $(2n+1)$-dimensional spacetime manifold $\mathcal{M}$ is given by [3, 4, 5]

$$\begin{aligned} ds^2 &= \mathcal{G}_{MN}(X) dX^M \otimes dX^N = \eta_{AB} e^A \otimes e^B \\ &= -v^0 \otimes v^0 + \lambda^2 v^a \otimes v^a = -dt^2 + \lambda^2 v_\mu^a v_\nu^a (dy^\mu - \mathbf{A}^\mu)(dy^\nu - \mathbf{A}^\nu) \end{aligned} \tag{2.56}$$

where $\mathbf{A}^\mu := A_0^\mu(t, y) dt$.

It should be noted that the time evolution (2.44) for a general time-dependent system is not completely generated by an inner automorphism since $\mathrm{Hor}(\mathcal{A}_\theta^1)$ is not an inner but outer derivation. This happens since the time variable t is a bach. Thus one may extend the phase space by introducing a conjugate variable H of t so that the extended phase space becomes a symplectic manifold. Then it is well-known [24] that the time evolution of a time-dependent system can be defined by the inner automorphism of the extended phase space whose extended Poisson bivector is given by

$$\vartheta = \theta + \frac{\partial}{\partial t} \bigwedge \frac{\partial}{\partial H} \tag{2.57}$$

where

$$\theta = \frac{1}{2}\theta^{\mu\nu} \frac{\partial}{\partial y^\mu} \bigwedge \frac{\partial}{\partial y^\nu} \tag{2.58}$$

is the original Poisson bivector related to the NC space (1.2). As a result, one can see [5] that the temporal vector field (2.44) is realized as a generalized Hamiltonian vector field defined by

$$V_0 = \mathcal{X}_H = -\vartheta(dH) = \frac{\partial}{\partial t} + X_H \tag{2.59}$$

where $X_H = \theta(dA_0)$ is the original Hamiltonian vector field which is a classical part of the inner derivation $\mathrm{ad}_{\widehat{A}_0} = X_H + \mathcal{O}(\theta^2) \in \mathfrak{D}(\mathcal{A}^1_\theta)$. But we have to pay the price for the extension of phase space. In the extended phase space, the time t is now promoted to a dynamical variable whereas it was simply an affine parameter describing a Hamiltonian flow in the old phase space. Then the extended Poisson structure (2.57) raises a serious issue whether the time variable for a general time-dependent system might also be quantized; in other words, time also becomes an operator obeying the commutation relation $[t, H] = -i$. We want to be modest not to address this issue since it is a challenging open problem even in quantum mechanics.

We figure out the time issue in a less ambitious way. Suppose that $(M, B \equiv \theta^{-1})$ is the original symplectic manifold responsible for the emergence of spaces. Now we consider a contact manifold $(\mathbb{R} \times M, \widetilde{B})$ where $\widetilde{B} = \pi_2^* B$ is defined by the projection $\pi_2 : \mathbb{R} \times M \to M, \ \pi_2(t, x) = x$ [24]. We define the concept of (space)time in emergent gravity through the contact manifold $(\mathbb{R} \times M, \widetilde{B})$ in the sense that the derivations in Eq. (2.45) can be obtained by quantizing the contact manifold $(\mathbb{R} \times M, \widetilde{B})$. Indeed it is shown in Appendix A that the time-like vector field V_0 in Eq. (2.59) arises as a Hamiltonian vector field of a cosymplectic manifold whose particular class is a contact manifold. Note that the emergent geometry described by the metric (2.56) respects the (local) Lorentz symmetry. If one looks at the metric (2.56), one can see that the Lorentzian manifold $\mathcal{M}$ becomes the Minkowski spacetime on a local Darboux chart in which all fluctuations die out, i.e., $v^a_\mu \to \delta^a_\mu, \ \mathbf{A}^\mu \to 0$, so $\lambda \to 1$. We have to emphasize [1] that the vacuum algebra responsible for the emergence of the Minkowski spacetime is the Moyal-Heisenberg algebra (2.5). Many surprising results will immediately come out from this dynamical origin of the flat spacetime [4, 5, 30], which is absent in general relativity.

We close this section by observing that the quantized version of the contact manifold $(\mathbb{R} \times M, \widetilde{B})$ is described by the MQM whose action is given by

$$S = \frac{1}{g^2_{YM}} \int dt \mathrm{Tr} \left(\frac{1}{2} (D_0 \phi_a)^2 + \frac{1}{4} [\phi_a, \phi_b]^2 \right), \tag{2.60}$$

where $D_0 \phi_a = \frac{\partial \phi_a}{\partial t} - i[A_0, \phi_a]$. The equations of motion for the matrix action (2.60) are given by

$$D_0^2 \phi_a + [\phi_b, [\phi_a, \phi_b]] = 0, \tag{2.61}$$

which must be supplemented with the Gauss constraint

$$[\phi_a, D_0 \phi_a] = 0. \tag{2.62}$$

As we discussed before, we interpret the matrix model (2.60) as a Hamiltonian system of the IKKT matrix model whose action is given by Eq. (2.1). Note that the original BFSS matrix model [31] contains 9 adjoint scalar fields while the action (2.60) has even number of adjoint scalar fields. For the former case, on the one hand, we have no idea how to realize the adjoint scalar fields as a matrix representation of NC $U(1)$ gauge fields on a Hilbert space like as (2.33). Even it may be nontrivial to construct the Hilbert space because the M-theory is involved with a 3-form instead of symplectic 2-form. For the latter case, on the other hand, the previous Moyal-Heisenberg vacuum (2.4) is naturally extended to the vacuum configuration of $\mathcal{A}^1_N$ given by

$$\langle \phi_a \rangle_{\rm vac} = p_a, \qquad \langle \widehat{A}_0 \rangle_{\rm vac} = \mathcal{E}, \tag{2.63}$$

where the vacuum moduli $p_a \in \mathcal{A}^1_N$ satisfy the commutation relation (2.5) and $\mathcal{E}$ is a constant proportional to the identity matrix. We consider all possible deformations of the vacuum (2.63) and parameterize them as

$$\widehat{\phi}_A(t, y) = p_A + \widehat{A}_A(t, y) \in \mathcal{A}^1_\theta, \tag{2.64}$$

where the isomorphism (2.33) between $\mathcal{A}^1_N$ and $\mathcal{A}^1_\theta$ was used. Note that

$$[\widehat{\phi}_A, \widehat{\phi}_B]_\star = -i\big(B_{AB} - \widehat{F}_{AB}\big), \tag{2.65}$$

where

$$\widehat{F}_{AB} = \partial_A \widehat{A}_B - \partial_B \widehat{A}_A - i[\widehat{A}_A, \widehat{A}_B]_\star \in \mathcal{A}^1_\theta \tag{2.66}$$

and

$$B_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab} \end{pmatrix}.$$

Plugging the fluctuations (2.64) into the action (2.60) leads to a $(2n+1)$-dimensional NC $U(1)$ gauge theory with the action

$$\begin{aligned} S &= -\frac{1}{g_{YM}^2} \int dt \mathrm{Tr} \left(\frac{1}{2} (D_0 \phi_a)^2 - \frac{1}{4} [\phi_a, \phi_b]^2 \right) \\ &= -\frac{1}{4G_{YM}^2} \int d^{2n+1} y \big(\widehat{F}_{AB} - B_{AB} \big)^2, \end{aligned} \tag{2.67}$$

where $G_{YM}^2 = (2\pi)^n |\mathrm{Pf}\theta| g_{YM}^2$ is the $(2n+1)$-dimensional gauge coupling constant. By applying the duality chain (2.39) to time-dependent matrices in $\mathcal{A}^1_N$, it is straightforward to derive the module $\mathfrak{D}^1$ in Eq. (2.45) from the large N matrices or NC $U(1)$ gauge fields in the action (2.67). A Lorentzian spacetime described by the metric (2.56) corresponds to a classical geometry derived from the NC module $\mathfrak{D}^1$ [3].

3 Cosmic inflation as a time-dependent Hamiltonian system

In Part I [1], we observed that a NC spacetime is caused by the Planck energy condensate responsible for the generation of spacetime and the Planck energy condensate into vacuum must be a dynamical process. The cosmic inflation corresponds to the dynamical mechanism for the instantaneous condensation of vacuum energy to enormously spread out spacetime. Hence the cosmic inflation as a dynamical system is typically a time-dependent solution and must be described by a non-Hamiltonian dynamics. Now we will illuminate how the cosmic inflation can be described by the conformal Hamiltonian dynamics [15, 16] which appears in, for example, simple mechanical systems with friction. In Appendix A we briefly review generalized symplectic manifolds that correspond to a natural phase space describing the conformal Hamiltonian dynamics.

Let us consider the simplest case, namely when the symplectic manifold is $\mathbb{R}^{2n}$ with coordinates (q^i, p_i) and $\omega = dq^i \wedge dp_i = da$ where $a = \frac{1}{2}(q^i dp_i - p_i dq^i)$. The symplectic manifold $(\mathbb{R}^{2n}, \omega)$ corresponds to a local system of a locally conformal symplectic (LCS) manifold as reviewed in Appendix A. A conformal vector field X is defined by

$$\iota_X \omega = \kappa a + dH, \tag{3.1}$$

where $H : \mathbb{R}^{2n} \to \mathbb{R}$ is the Hamiltonian and κ is a nonzero constant. Note that Eq. (3.1) implies

$$\mathcal{L}_X \omega = \kappa \omega. \tag{3.2}$$

Therefore the vector field X is a Lie algebra generator of conformal infinitesimal transformations defined by Eq. (A.29). It is easy to solve Eq. (3.1) for the vector field X and the result is given by

$$X = \frac{\kappa}{2}\Big(q^i \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial p_i} \Big) + X_H, \tag{3.3}$$

where X_H is a usual Hamiltonian vector field obeying $\iota_{X_H}\omega = dH$. Thus the Hamilton's equations are given by

$$\frac{dq^i}{dt} = X(q^i) = \frac{\kappa}{2} q^i + \frac{\partial H}{\partial p_i}, \tag{3.4}$$

$$\frac{dp_i}{dt} = X(p_i) = \frac{\kappa}{2} p_i - \frac{\partial H}{\partial q^i}. \tag{3.5}$$

The equations of motion for the Hamiltonian $H = \frac{1}{2}p_i^2 + U(q)$ are reduced to the differential equations

$$\ddot{q}^i - \kappa \dot{q}^i + \frac{\partial V}{\partial q^i} = 0, \tag{3.6}$$

where $V(q) = U(q) + \frac{\kappa^2}{8} q_i^2$. To be specific, the integral curves for $U(q) = \frac{1}{2}\omega^2 q_i^2$ are given by[4]

$$q^i(t) = e^{\frac{\kappa}{2}t} q^i(\kappa = 0; t), \qquad p_i(t) = e^{\frac{\kappa}{2}t} p_i(\kappa = 0; t), \tag{3.7}$$

where $q^i(\kappa = 0; t) = A^i \sin(\omega t + \theta)$ and $p_i(\kappa = 0; t) = B_i \cos(\omega t + \theta)$ describe the usual harmonic oscillator with a closed orbit when $\kappa = 0$. Therefore we see that the flow generated by a conformal vector field has the property

$$\phi^* \omega = e^{\kappa t} \omega, \tag{3.8}$$

[4]Note that $a = b + d\lambda$ where $b = -p_i dq^i$ and $\lambda = \frac{1}{2} q^i p_i$. Thus one can also define the conformal vector field X by $\iota_X \omega = \kappa b + dH'$ where $H' = H + \kappa\lambda$. In this case $X = \kappa p_i \frac{\partial}{\partial p_i} + X_{H'}$ and the equations of motion are given by $\frac{dq^i}{dt} = \frac{\partial H'}{\partial p_i}$ and $\frac{dp_i}{dt} = \kappa p_i - \frac{\partial H'}{\partial q^i}$. For $H' = \frac{1}{2}(p_i^2 + \omega^2 q_i^2)$, the general solution is given by $q^i(t) = A^i e^{\frac{\kappa}{2}t} \sin\Big(\sqrt{\omega^2 - \frac{\kappa^2}{4}}\, t + \theta\Big)$. However the vector field defined by Eq. (3.3) is more convenient for our case.

which may be directly obtained by integrating Eq. (3.2).[5] This means that the volume of phase space exponentially expands (contracts) if $\kappa > 0$ ($\kappa < 0$).

The mathematical parallelism between quantum mechanics and NC spacetime suggests how to formulate the cosmic inflation as a dynamical system. First note that the NC space (1.2) in commutative limit becomes a phase space with the symplectic form

$$B = \frac{1}{2} B_{\mu\nu} dy^\mu \wedge dy^\nu. \tag{3.9}$$

The dynamics of Hamiltonian systems is characterized by the invariance of phase space volume under time evolution and the conservation of phase space volume for divergenceless Hamiltonian flows is known as the Liouville theorem [24]. However, the cosmic inflation means that the volume of spacetime phase space has to exponentially expand as we have seen from the above mechanical analogue. Hence the cosmic inflation as a dynamical system has to be regarded as a non-Hamiltonian system and a generalized Liouville theorem is necessary to describe the exponential expansion of spacetime. We have already explained above how such a non-Hamiltonian dynamics can be formulated in terms of a *conformal* Hamiltonian dynamics characterized by the (local) flow obeying Eq. (3.2). See Appendix A for a mathematical exposition of general time-dependent nonconservative dynamical systems.[6]

Let us apply the conformal Hamiltonian dynamics to the cosmic inflation. Recall that we have considered an atlas $\{(U_i, \phi_{(i)})\}$ on $M = \bigcup_{i \in I} U_i$ as a collection of local Darboux charts and complete it by gluing these local charts on their overlap. On each local chart, we have a local symplectic structure $\Omega_i = \frac{1}{2} B_{\mu\nu} dy^\mu_{(i)} \wedge dy^\nu_{(i)}$ where $\{y^\mu_{(i)}\}$ are Darboux coordinates on a local patch $U_i \subset M$. As was explained in Refs. [36, 37] and reproduced in Appendix A, the phase space coordinates $\{y^\mu_{(i)}\}_{U_i}$ of a conformal Hamiltonian system undergo a nontrivial time evolution even in a local Darboux frame. For example, look at the equations of motion (3.4) and (3.5) to recognize such a nontrivial time evolution even when $H = 0$. The dynamics in this case consists of the orbits of a conformal vector field X obeying the condition (A.29). The result is essentially the same as the previous mechanical system with negative-friction. To be specific, write $\Omega_i = da_{(i)}$ on a local patch $U_i \subset M$ where $a_{(i)} = -\frac{1}{2} p^{(i)}_\mu dy^\mu_{(i)}$ with $p^{(i)}_\mu = B_{\mu\nu} y^\nu_{(i)}$ and consider a conformal vector field X defined by

$$\iota_X \Omega_i = \kappa a_{(i)} + dH_i, \tag{3.10}$$

where $H_i : U_i \to \mathbb{R}$ is a local Hamiltonian and κ is a positive constant. Using the fact that $d\Omega_i = 0$, it is easy to derive the condition (A.29) from Eq. (3.10), i.e.,

$$\mathcal{L}_X \Omega_i = \kappa \Omega_i. \tag{3.11}$$

[5]The proof goes as follows. Let ϕ_t denote the flow of X. By the Lie derivative theorem [24], we have $\frac{d}{dt}(\phi_t^* \omega) = \phi_t^* \mathcal{L}_X \omega = \kappa \phi_t^* \omega$, which has the unique solution (3.8).

[6]We want to remark that such systems ubiquitously arise in, e.g., dynamical systems with friction and nonequilibrium statistical mechanics. Recently the statistical mechanics of non-Hamiltonian systems has been formulated using a generalized Liouville measure to study the simulation of molecular dynamics. See, for example, [32, 33, 34, 35]. We think that their formulation may be useful to understand the evolution of our early universe, especially, regarding to the issue of the cosmic Landau damping discussed in the last section.

The vector field X obeying Eq. (3.10) is given by

$$X = \frac{\kappa}{2} y^\mu_{(i)} \frac{\partial}{\partial y^\mu_{(i)}} + X_{H_i}, \tag{3.12}$$

where X_{H_i} is the ordinary Hamiltonian vector field satisfying $\iota(X_{H_i})\Omega_i = dH_i$. The conformal vector field $Z_{(i)} \equiv \frac{1}{2} y^\mu_{(i)} \frac{\partial}{\partial y^\mu_{(i)}}$ in Eq. (3.12) is known as the Liouville vector field [15, 16] and is generated by the open Wilson line (1.8) [1]. We will set $H_i = 0$ for simplicity. The time evolution of local Darboux coordinates is then determined by the equations

$$\frac{dy^\mu_{(i)}}{dt} = X(y^\mu_{(i)}) = \frac{\kappa}{2} y^\mu_{(i)}. \tag{3.13}$$

The solution is given by

$$y^\mu_{(i)}(t) = e^{\frac{\kappa}{2}t} y^\mu_{(i)}(0). \tag{3.14}$$

We may glue the local solutions (3.14) to have a global form

$$p_a(t) = B_{ab} y^b(t) = e^{\frac{\kappa}{2}t} p_a. \tag{3.15}$$

Then the time-dependent canonical one-form is given by

$$a(t) = -\frac{1}{2} p_a(t) dy^a(t) = -\frac{1}{2} e^{\kappa t} p_a dy^a \tag{3.16}$$

and thus

$$\Omega(t) = da(t) = e^{\kappa t} B. \tag{3.17}$$

The exterior derivative above acts only on $\mathbb{R}^{2n}$. One can show using the proof in footnote 5 that the result (3.17) is the integral form of Eq. (3.11). More generally, the result (3.17) is a particular case of the general Moser flow ϕ_t generated by a time-dependent vector field X_t for an LCS manifold which is given by [38]

$$\phi_t^* \Omega_t = \exp\Big(\int_0^t \phi_s^* \big(b_s(X_s) \big) ds \Big) \cdot \Omega, \tag{3.18}$$

where the one-form b is the Lee form of Ω [39]. The above result (3.17) is simply obtained from Eq. (3.18) when $b(X)$ is a constant κ.

We have motivated the cosmic inflation with the idea that the vacuum configuration (2.63) is a final state accumulating the vacuum energy [1]. Therefore, the cosmic inflation corresponds to a dynamical system describing the transition from the initial state referring to "absolutely nothing" to the final state. For this purpose, let us consider a symplectic manifold $\big(M, \Omega(t)\big)$ whose symplectic two-form is given by Eq. (3.17). It was shown in [1] that this symplectic manifold arises from a time-dependent vacuum solution given by

$$\langle \phi_a(t) \rangle_{\rm vac} = p_a(t) = e^{\frac{\kappa}{2}t} p_a, \qquad \langle \widehat{A}_0(t,y) \rangle_{\rm vac} = \widehat{a}_0(t,y). \tag{3.19}$$

Recall that the temporal gauge field in Eq. (3.19) is given by the non-local Hamiltonian (1.7). As was shown in Eq. (1.8), it is necessary to turn on a non-local Hamiltonian to satisfy the equations of motion (2.61) as well as the Gauss constraint (2.62) and it leads to the conformal vector field (3.12). However we will set $\widehat{a}_0(t,y) = 0$ to highlight the conformal Hamiltonian dynamics of cosmic inflation and compare its difference with the case $\widehat{a}_0(t,y) \neq 0$ later. Since the vacuum (3.19) is in highly non-equilibrium, it is expected that it will eventually evolve to the final state (2.63) through interactions with an environment (e.g., ubiquitous fluctuations) as we have learned from hydrodynamics and thermodynamics in non-equilibrium. The decay of exponentially growing modes via interactions with the environment is known as the reheating process in physical cosmology. However we do not know the precise mechanism for the reheating. We will speculate in Sec. 5 a plausible picture for the reheating mechanism. It turns out [1] that κ is identified with the inflationary Hubble constant H and the inflationary energy scale is given by

$$H = (n-1)\kappa \gtrsim 10^{11} \sim 10^{14} \text{ GeV}. \tag{3.20}$$

Let us first determine the vacuum geometry emergent from the vacuum configuration (3.19). In this case it is not necessary to glue Darboux charts because we have not introduced local fluctuations yet, so the Darboux coordinates in (3.19) are globally defined. Note that

$$\langle [\phi_a(t), \phi_b(t)] \rangle_{\text{vac}} = -ie^{\kappa t} B_{ab} = -i\Omega_{ab}(t), \tag{3.21}$$

and so we regard $\Omega(t) = \frac{1}{2}\Omega_{ab}(t) dy^a \wedge dy^b$ as the symplectic structure of the inflating vacuum (3.19). According to the definition (A.11), we get (omitting the symbol indicating the vacuum for a notational simplicity)

$$V_a(t) = \theta(t)\big(dp_a(t)\big) = e^{\frac{\kappa}{2}t} V_a(0) \tag{3.22}$$

where $V_a(0) = \delta^\mu_a \frac{\partial}{\partial y^\mu}$. Similarly,

$$V_0(t) = \frac{\partial}{\partial t} \tag{3.23}$$

since we set $\widehat{A}_0(t,y) = 0$. Thus the dual one-forms are given by

$$v^0(t) = dt, \qquad v^a(t) = e^{-\frac{\kappa}{2}t} v^a(0) \tag{3.24}$$

where $v^a(0) = \delta^a_\mu dy^\mu$. It is easy to calculate the Lie algebra defined by Eq. (2.54) for the time-dependent vector fields $V_A(t)$ where

$$g_{AB}{}^C = \begin{cases} g_{0a}{}^b = -g_{a0}{}^b = \frac{\kappa}{2}\delta^b_a, & a, b = 1, \cdots, 2n; \\ 0, & \text{otherwise.} \end{cases} \tag{3.25}$$

Thus $\lambda^2 = e^{n\kappa t}$ according to Eq. (2.55). Note that, if we include the temporal gauge field in Eq. (3.19), the conformal factor is enhanced to $\lambda^2 = e^{2n\kappa t}$ [1]. The invariant volume form of the vacuum manifold is then given by

$$\nu_t = \lambda^2 dt \wedge v^1(t) \wedge \cdots \wedge v^{2n}(t) = dt \wedge dy^1 \wedge \cdots \wedge dy^{2n}. \tag{3.26}$$

After applying the above results to the metric (2.56), we see that the vacuum configuration (3.19) determines the spacetime geometry with the metric

$$ds^2 = -dt^2 + e^{Ht} d\mathbf{y} \cdot d\mathbf{y}. \tag{3.27}$$

This is the de Sitter space in flat coordinates which covers half of the de Sitter manifold. Definitely the inflation metric (3.27) describes a homogeneous and isotropic Universe known as the Friedmann-Robertson-Walker metric in physical cosmology. By comparing this result with Eq. (1.9), we see that the temporal gauge field (1.8) enhances the inflation by the factor two, i.e. $H \to 2H$.

The vector fields $V_A(t)$ form a solvable Lie algebra and the de Sitter space is its Lie group. The Lie algebra for Eq. (3.25) has the generators $V_0 = -\frac{\kappa}{2} L_{0(2n+1)},\ V_a = \frac{1}{2}(L_{0a} + L_{a(2n+1)})$, which is indeed a subalgebra of the de Sitter algebra where L_{AB} are the Lie algebra generators of $SO(2n+1,1)$ Lorentz symmetry. In this point of view, energy and momentum do not commute unlike in the Minkowski spacetime and are no longer conserved, as translations are no more a symmetry of the space.[7] Instead, energy generates scale transformations in momentum. This is the reason why the isometry of the de Sitter space is enhanced to $SO(2n+1,1)$ which combines $SO(2n,1)$ Lorentz transformations and translations together [40]. In the limit $\kappa \to 0$, we recover the Minkowski spacetime.

Important remarks are in order. First we see that the cosmic inflation is a typical example of an LCS manifold. The LCS manifold has a disparate property compared to symplectic manifolds. First of all, it is allowed a nontrivial conformal vector field defined by Eq. (3.11) even when an underlying Hamiltonian function identically vanishes. The so-called Liouville vector field $Z \equiv \frac{1}{2} y^\mu \frac{\partial}{\partial y^\mu}$ is still nontrivial [15] and it generates the exponential expansion of spacetime described by the metric (3.27).[8] If the one-form a in Eq. (3.10) is proportional to the Lee form b, X is called a Hamiltonian vector field of an LCS manifold. See the definition (A.10). Even in this case, the Hamiltonian vector field shows a peculiar property different from the symplectic case: If b is not exact, $X_H = 0$ only if $H = 0$. Therefore we see that the vector fields of an LCS manifold is in stark contrast to those of a symplectic manifold, in which $X_H = 0$ implies $H =$ constant only and, due to this property, the constant vacuum energy does not couple to gravity as was shown in Part I. Remarkably, if the cosmic inflation is described by an LCS (or more generally LCC) manifold, the vacuum energy rightly couples to gravity during the inflation. This is a desirable property since the cosmic inflation is triggered by the condensate of vacuum energy. Physically the reason is obvious since every quantity during the inflation is time-dependent due to the existence of the nontrivial Liouville vector field.

[7] One important consequence is that the energy will not be positive. Polyakov has suggested [41] that this makes de Sitter space unstable with respect to decay by creation of particle-antiparticle pairs.

[8] It would be worthwhile remarking that it is not possible to realize the Liouville vector field in terms of a local Hamiltonian function. Probably this situation becomes more transparent by the mechanical analogue described by Eq. (3.6). Thus the inflation is a dynamical system without any Hamiltonian. It may explain why even string theory faces many difficulties to realize the cosmic inflation. However we show in Appendix B that this situation can be cured by introducing a time-dependent Hamiltonian.

It may be instructive to understand the above situation more closely in comparison with the equilibrium case described by the metric (2.56). First note that the invariant volume form (2.53) can be written as

$$\nu_t = \lambda^{2-2n}\nu_g, \tag{3.28}$$

where $\nu_g = e^0 \wedge \cdots \wedge e^{2n} = \sqrt{-\mathcal{G}} d^{2n+1}x$ is the volume form of the metric. Therefore, the vector fields V_A do not necessarily preserve the Riemannian volume form ν_g although they preserve the volume form ν_t. However, since $\lambda^2 \to 1$ at spatial infinity according to Eq. (2.55), $\nu_t|_\infty = \nu_g|_\infty$ for the asymptotic volume forms denoted by $\nu_t|_\infty$ and $\nu_g|_\infty$. In other words, the flow generated by V_A leads to only local changes of the spacetime volume while it preserves the volume element at asymptotic regions. On the contrary, the conformal vector field changes the spacetime volume everywhere. Accordingly it definitely gives rise to the exponential expansion of the spacetime volume. After all, we see that a natural phase space for the cosmic inflation has to contain an LCS manifold replacing a standard symplectic manifold. Including time, it becomes an LCC manifold [37]. Our result shows that the matrix model (2.60) contains the LCC manifold as a solution.

As was summarized in Eq. (1.12), a general Lorentzian metric describing $(2n+1)$-dimensional inflating spacetime can be obtained by considering arbitrary fluctuations around the inflationary background (1.7). The fluctuations are given by Eq. (1.10) and form a time-dependent NC algebra ${}^t\mathcal{A}^1_\theta$. Let us denote the corresponding time-dependent matrix algebra by ${}^t\mathcal{A}^1_N$ which consists of a time-dependent solution of the action (2.60). Then the general Lorentzian metric describing a $(2n+1)$-dimensional inflationary universe is constructed by using the following duality chain [1]:

$${}^t\mathcal{A}^1_N \quad \Longrightarrow \quad {}^t\mathcal{A}^1_\theta \quad \Longrightarrow \quad {}^t\mathfrak{D}^1. \tag{3.29}$$

The module ${}^t\mathfrak{D}^1$ of derivations of the NC algebra ${}^t\mathcal{A}^1_\theta$ is given by Eq. (1.11). In the classical limit of the module, we get a general inflationary universe described by the metric (1.12). The chain of maps in (3.29) shows how to realize the large N duality in Fig. 1 and achieve the background-independent description of an inflationary universe. A remarkable picture is that the cosmic inflation arises as a time-dependent solution of MQM and describes the dynamical process of Planck energy condensate in vacuum without introducing any inflaton field as well as an *ad hoc* inflation potential [1]. In conclusion, the emergent spacetime is a completely new paradigm that enables the background-independent description of an inflationary universe [42].

4 NC spacetime as a second-quantized string

We know that quantum mechanics is the more fundamental description of nature than classical physics. The microscopic world is already quantum. Nevertheless, the quantization is necessary to find a quantum theoretical description of nature since we have understood our world starting with the classical description which we understand better. After quantization, the quantum theory is described by a fundamental NC algebra such as Eq. (2.28). A striking feature of the NC algebra $\mathcal{A}_\hbar$ is that every point in $\mathbb{R}^n$ is unitarily equivalent because translations in $\mathbb{R}^n$ are generated by an inner automorphism of $\mathcal{A}_\hbar$, i.e., $f(x+a) = U(a)f(x)U(a)^\dagger$ where $f(x) \in \mathcal{A}_\hbar$ and $U(a) = e^{ip_i a^i/\hbar} \in \mathrm{Inn}(\mathcal{A}_\hbar)$. Therefore, through the quantization, the concept of (phase) space is doomed. Instead the (phase) space is replaced by the algebra $\mathcal{A}_\hbar$ and its Hilbert space representation and dynamical variables become operators acting on the Hilbert space. Only in the classical limit, a phase space with the symplectic structure $\omega = dx^i \wedge dp_i$ is emergent from the quantum algebra $\mathcal{A}_\hbar$ such as (2.28).

Recall that the mathematical structure of NC spacetime is basically the same as the NC phase space in quantum mechanics [11]. Therefore essential features in quantum mechanics must be applied to the NC spacetime too. In particular, NC algebras $\mathcal{A}_\theta$ such as the NC space (1.2) also play a fundamental role and every points in the NC space are indistinguishable, i.e., unitarily equivalent because any two points are connected by an inner automorphism of $\mathcal{A}_\theta$. In other words, there is no space(time) for the same reason as quantum mechanics and a classical spacetime must be derived from the NC algebra $\mathcal{A}_\theta$. After all, an important lesson is that NC spacetime necessarily implies emergent spacetime.

Although spacetime at a microscopic scale, e.g. the Planck scale L_P, is intrinsically NC, we understand the NC spacetime through the quantization of a symplectic (or more generally Poisson) manifold. Let (M, B) be a symplectic manifold. On the one hand, the basic concept in symplectic geometry is an area defined by the symplectic two-form B that is a nondegenerate, closed two-form. On the other hand, the basic concept in Riemannian geometry determined by a pair (M, g) is a distance defined by the metric tensor g that is a nondegenerate, symmetric bilinear form. One may identify this distance with a geodesic worldline of a "particle" moving in M. Geodesic curves in M give us all information of Riemannian geometry (M, g). On the contrary, the area in symplectic geometry (M, B) may be regarded as a minimal worldsheet swept by a "string" moving in M. In this picture, the wiggly string, so a fluctuating worlsheet, corresponds to a deformation of symplectic structure in M. This picture becomes more transparent by the so-called pseudoholomorphic or J-holomorphic curve introduced by Gromov [43].

Let (M, J) be an almost complex manifold and (Σ, j) be a Riemann surface. By the compatibility of J to B, we have the relation $g(X, Y) = B(X, JY)$ for any vector fields $X, Y \in \mathfrak{X}(M)$. Let us also fix a Hermitian metric h of (Σ, j). A smooth map $f : \Sigma \to M$ is called pseudoholomorphic [17] if the differential $df : T\Sigma \to TM$ is a complex linear map with respect to j and J:

$$df \circ j = J \circ df. \tag{4.1}$$

This condition corresponds to the commutativity of the following diagram

$$\begin{tikzcd} T\Sigma \arrow[r, "j"] \arrow[d, "df"'] & T\Sigma \arrow[d, "df"] \\ TM \arrow[r, "J"'] & TM \end{tikzcd}$$

Since $J^{-1} = -J$, it is also equivalent to $\overline{\partial}_J f = 0$ where $\overline{\partial}_J f := \frac{1}{2}(df + J \circ df \circ j)$. For example, suppose that the Riemann surface is (Σ, i) where i is the standard complex structure. We can work in a chart $u_\epsilon : U_\alpha \to \mathbb{C}$ with local coordinate $z = \tau + i\sigma$ where $U_\epsilon \subset \Sigma$ is an open neighborhood. Define $f_\epsilon = f \circ u_\epsilon^{-1}$. In this case, we have

$$\overline{\partial}_J f = \frac{1}{2}\left[\Big(\frac{\partial f_\epsilon}{\partial \tau} + J(f_\epsilon)\frac{\partial f_\epsilon}{\partial \sigma}\Big)d\tau + \Big(\frac{\partial f_\epsilon}{\partial \sigma} - J(f_\epsilon)\frac{\partial f_\epsilon}{\partial \tau}\Big)d\sigma\right]. \tag{4.2}$$

Thus we see that $\overline{\partial}_J f = 0$ if

$$\frac{\partial f_\epsilon}{\partial \tau} + J(f_\epsilon)\frac{\partial f_\epsilon}{\partial \sigma} = 0. \tag{4.3}$$

Since J is B-compatible, every smooth map $f : \Sigma \to M$ satisfies [44, 45]

$$\frac{1}{2}\int_\Sigma ||df||_g^2 \, d\mathrm{vol}_\Sigma = \int_\Sigma ||\overline{\partial}_J f||_g^2 \, d\mathrm{vol}_\Sigma + \int_\Sigma f^* B, \tag{4.4}$$

where the norms are taken with respect to the metric g and $d\mathrm{vol}_\Sigma$ is a volume form on Σ. In terms of local coordinates, (σ^1, σ^2) on Σ and $f(\sigma) = (x^1, \cdots, x^{2n})$ on M,

$$||df||_g^2 = g_{\mu\nu}\big(f(\sigma)\big)\frac{\partial x^\mu}{\partial \sigma^a}\frac{\partial x^\nu}{\partial \sigma^b}h^{ab}(\sigma) \tag{4.5}$$

and $d\mathrm{vol}_\Sigma = \sqrt{h} d^2\sigma$. Therefore, the left-hand side of Eq. (4.4) is nothing but the Polyakov action in string theory. For a pseudoholomorphic curve $f : \Sigma \to M$ that obeys $\overline{\partial}_J f = 0$, we thus have the identity

$$S_P(f) \equiv \frac{1}{2}\int_\Sigma ||df||_g^2 \, d\mathrm{vol}_\Sigma = \int_\Sigma f^* B. \tag{4.6}$$

This means that any pseudoholomorphic curves minimize the "harmonic energy" $S_P(f)$ in a fixed homology class and so are harmonic maps. In other words, their symplectic area coincides with the surface area. Therefore, any pseudoholomorphic curve is a solution of the worldsheet Polyakov action $S_P(f)$. For instance, if $M = \mathbb{C}^n$ with complex coordinates $\phi^i = x^{2i-1} + \sqrt{-1}x^{2i}$ $(i = 1, \ldots, n)$ and $f_\epsilon(z, \bar{z}) \equiv \phi^i(z, \bar{z})$, Eq. (4.3) becomes

$$\frac{1}{2}\Big(\frac{\partial}{\partial \tau} + \sqrt{-1}\frac{\partial}{\partial \sigma}\Big)\phi^i(z, \bar{z}) = \partial_{\bar{z}}\phi^i(z, \bar{z}) = 0. \tag{4.7}$$

In this case, pseudoholomorphic curves coincide with holomorphic curves. Moreover such curves are harmonic and minimal surfaces.[9]

The pseudoholomorphic curve also provides us a useful tool to understand the emergent gravity picture. To demonstrate this aspect, let us include a boundary interaction in the sigma model (4.4) such that the open string action is given by

$$S_A(f) \equiv \frac{1}{2}\int_\Sigma ||df||_g^2 \, d\mathrm{vol}_\Sigma + \int_{\partial\Sigma} f^* A, \tag{4.8}$$

[9]In the topological A-model that is concerned with pseudoholomorphic maps from Σ to $M = T^*N$, there is a vanishing theorem [46] stating that $\int_\Sigma f^* B = 0$. In particular, the mappings from $\partial\Sigma$ to N are necessarily constant.

where the one-form A is the connection of a line bundle $L \to M$. Using the Stokes' theorem, the second term can be written as

$$\int_{\partial\Sigma} f^* A = \int_\Sigma f^* dA. \tag{4.9}$$

After combining the identities (4.4) and (4.9) together, we write the action

$$S_A(f) = \int_\Sigma ||\overline{\partial}_J f||_g^2 \, d\mathrm{vol}_\Sigma + \int_\Sigma f^* \mathcal{F}, \tag{4.10}$$

where $\mathcal{F} = B + F$ and $F = dA$. If one recalls the derivation of Eq. (4.4), one may immediately realize that the action $S_A(f)$ can equivalently be written as the form of the Polyakov action

$$S_P(\psi) \equiv \frac{1}{2} \int_\Sigma ||d\psi||_{\mathcal{G}}^2 \, d\mathrm{vol}_\Sigma, \tag{4.11}$$

where the differential $d\psi$ for a smooth map $\psi : \Sigma \to M$ has the norm taken with respect to some metric $\mathcal{G}$. For this purpose, let us assume that the almost complex structure J is also compatible with the deformed symplectic structure $\mathcal{F}$, i.e.,

$$\mathcal{G}(X, Y) = \mathcal{F}(X, JY), \qquad \forall X, Y \in \mathfrak{X}(M) \tag{4.12}$$

is a Riemannian metric on M. An explicit representation of the Polyakov action (4.11) can be made by introducing local coordinates $\psi(\sigma) = (X^1, \cdots, X^{2n})$ on an open set $U_i \subset M$ so that

$$||d\psi||_{\mathcal{G}}^2 = \mathcal{G}_{\mu\nu}\big(\psi(\sigma)\big) \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} h^{ab}(\sigma). \tag{4.13}$$

One can then apply the same derivation of Eq. (4.4) to the action (4.11) to derive the identity

$$\frac{1}{2} \int_\Sigma ||d\psi||_{\mathcal{G}}^2 \, d\mathrm{vol}_\Sigma = \int_\Sigma ||\overline{\partial}_J \psi||_{\mathcal{G}}^2 \, d\mathrm{vol}_\Sigma + \int_\Sigma \psi^* \mathcal{F}. \tag{4.14}$$

For pseudoholomorphic curves $\psi : \Sigma \to M$ satisfying $\overline{\partial}_J \psi = 0$, we finally get the result

$$S_P(\psi) = \frac{1}{2} \int_\Sigma ||d\psi||_{\mathcal{G}}^2 \, d\mathrm{vol}_\Sigma = \int_\Sigma \psi^* \mathcal{F}. \tag{4.15}$$

The above argument reveals a nice picture that dynamical $U(1)$ gauge fields in a line bundle L over M deform an underlying symplectic structure (M, B) and this deformation is transformed into the dynamics of gravity [3]. This is a reincarnation of the duality chain in Fig. 1 indicating the gauge-gravity duality. As we observed before, the symplectic geometry is probed by strings while the Riemannian geometry is probed by particles. We note that the NC space (1.2) defines only a minimal area whereas the concept of point is doomed as if $\hbar$ in quantum mechanics introduces a minimal area in the NC phase space (2.28). The minimal area (surface) in the NC space behaves like the smallest unit of spacetime blob and acts as a basic building block of string theory. The concept of pseudo-holomorphic or J-holomorphic curves in symplectic geometry plays a role of such minimal surfaces.

It is known [17] that there is a nonlinear Fredholm theory which describes the deformations of a given pseudoholomorphic curve $f : \Sigma \to (M, J)$ and the deformations are parameterized by a finite-dimensional moduli space. (This moduli space may be enriched by considering pseudoholomorphic curves in an LCS manifold.) When a symplectic manifold is probed with a string or pseudoholomorphic curve, the notion of a wiggly string in this probe picture corresponds to the deformation of a symplectic structure. Hence the emergence of gravity from symplectic geometry or more precisely NC $U(1)$ gauge fields may not be surprising because we know from string theory that a Riemannian geometry (or general relativity) is emergent from the wiggly string.

We can think of the integral $A(f) = \int_\Sigma f^* B$ in two ways if f is a pseudoholomorphic curve. On the one hand, the pointwise compatibility between the structures (B, J) means that $A(f)$ is essentially the area of the image of f, measured in the Riemannian metric g. On the other hand, the condition that B is closed means that $A(f)$ is a topological (homotopy) invariant of the map f since it depends only on the evaluation of a closed 2-form B on the 2-chain defined by $f(\Sigma)$. Hence we can use the curves in two main ways [17]. The first way is as geometrical probes to explore a symplectic manifold, as we advocated above. The second way is as the source of numerical invariants known as the Gromov-Witten invariants. Using the pseudoholomorphic curves, Gromov proved a surprising non-squeezing theorem [43, 44, 45] stating that a ball $B_{2n}(r)$ of radius r in a symplectic vector space $\mathbb{R}^{2n}$ with the standard symplectic form B cannot be mapped by a symplectomorphism into any cylinder $B_2(R) \times \mathbb{R}^{2n-2}$ of radius R if $R < r$. It is possible to replace $\mathbb{R}^{2n-2}$ by a $(2n-2)$-dimensional compact symplectic manifold V with $\pi_2(V) = 0$.

Now we will discuss how a NC space provides us an important clue for a background-independent formulation of string theory. The NC spacetime is defined by the quantization of a symplectic manifold (M, B). One may try to lift the notion of the pseudoholomorphic curve to a quantized symplectic manifold, namely, a NC space such as Eq. (1.2). The quantization of a symplectic manifold leads to a radical change of classical concepts such as spaces and observables. The classical space is replaced by a Hilbert space and dynamical observables become operators acting on the Hilbert space. Then the NC spacetime provides a more elegant framework for the background-independent formulation of quantum gravity in terms of matrix models, which is still elusive in string theory. We explained how the dynamical Lorentzian spacetime (2.56) emerges from a classical solution of the matrix model (2.60). Remarkably, the cosmic inflation described by the metric (1.9) also arises as a vacuum solution of the time-dependent matrix model.

In order to grasp how a pseudoholomorphic curve looks like in NC spacetime, let us consider the simplest case in Eq. (4.7). After quantization, the coordinates of $\mathbb{C}^n$ denoted by $\phi^i(z, \bar{z})$ become operators in a NC $\star$-algebra $\mathcal{A}^2_\theta \equiv \mathcal{A}_\theta\big(C^\infty(\mathbb{R}^2)\big) = C^\infty(\mathbb{R}^2) \otimes \mathcal{A}_\theta$, i.e., $\phi^i(z, \bar{z}) \to \widehat{\phi}^i(z, \bar{z}) \in \mathcal{A}^2_\theta$. The worldsheet $\mathbb{R}^2$ may be replaced by $\mathbb{T}^2$ or $\mathbb{R} \times \mathbb{S}^1$. Let us clarify the notation $\mathcal{A}^2_\theta$ after the Wick rotation of the worldsheet coordinate $\tau = it$, so $\mathbb{R}^2 \to \mathbb{R}^{1,1}$. Consider a generic element in the NC $\star$-algebra $\mathcal{A}^2_\theta$ given by

$$\widehat{f}(t, \sigma, y) \in \mathcal{A}^2_\theta. \tag{4.16}$$

The matrix representation (2.33) is now generalized to

$$\widehat{f}(t,\sigma,y) = \sum_{n,m=1}^{\infty} |n\rangle\langle n|\widehat{f}(t,\sigma,y)|m\rangle\langle m| = \sum_{n,m=1}^{\infty} f_{nm}(t,\sigma)|n\rangle\langle m| \tag{4.17}$$

where the coefficients $f_{nm}(t,\sigma) := [f(t,\sigma)]_{nm}$ are elements of a matrix $f(t,\sigma)$ in $\mathcal{A}_N^2 \equiv \mathcal{A}_N\big(C^\infty(\mathbb{R}^{1,1})\big) = C^\infty(\mathbb{R}^{1,1}) \otimes \mathcal{A}_N$ as a representation of the observable (4.16) on the Hilbert space (2.12). Then we have an obvious generalization of the duality chain (2.39) as follows:

$$\mathcal{A}_N^2 \quad \Longrightarrow \quad \mathcal{A}_\theta^2 \quad \Longrightarrow \quad \mathfrak{D}^2. \tag{4.18}$$

The module of derivations is similarly a direct sum of the submodules of horizontal and inner derivations [29]:

$$\mathfrak{D}^2 = \mathrm{Hor}(\mathcal{A}_N^2) \oplus \mathfrak{D}(\mathcal{A}_N^2) \cong \mathrm{Hor}(\mathcal{A}_\theta^2) \oplus \mathfrak{D}(\mathcal{A}_\theta^2), \tag{4.19}$$

where horizontal derivations are locally generated by a vector field

$$k(t,\sigma,y)\frac{\partial}{\partial t} + l(t,\sigma,y)\frac{\partial}{\partial \sigma} \in \mathrm{Hor}(\mathcal{A}_\theta^2). \tag{4.20}$$

It can be shown [3, 5] that the matrix model for the duality chain (4.18) is given by

$$S = -\frac{1}{g_s^2}\int d^2\sigma \mathrm{Tr}\left(\frac{1}{4}F_{\alpha\beta}^2 + \frac{1}{2}(D_\alpha\phi_a)^2 - \frac{1}{4}[\phi_a,\phi_b]^2\right), \tag{4.21}$$

where $a = 2, \cdots, 2n+1$ and $\sigma^\alpha = (t,\sigma)$, $\alpha = 0,1$ and $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta]$. The $n = 4$ case is known as the matrix string theory that is supposed to describe a nonperturbative type IIA string theory in light-cone gauge [19]. The matrix string theory can be obtained from the BFSS matrix model via compactification on a circle [22]. To achieve this goal, the BFSS matrix model has to have 9 adjoint scalar fields, $\phi_a(t)$ $(a = 1, \cdots, 9)$, unlike the action (2.60) with even number of adjoint scalar fields. The reason why we consider only even number of adjoint scalar fields is to realize the equivalence (2.67). In this case, the action (2.60) can be understood as a Hilbert space representation of certain NC gauge theory under a symplectic vacuum such as (2.5) with rank$(B) = 2n$. However we do not know a corresponding NC gauge theory whose Hilbert space representation precisely reproduces the BFSS matrix model. We will further comment on this issue later. Fortunately the matrix string theory (4.21) has 8 adjoint scalar fields for $n = 4$. Thus it is possible to realize it as the Hilbert space representation of $(9+1)$-dimensional NC $U(1)$ gauge theory with rank$(B) = 8$ [3, 5]. Therefore it will be interesting to understand how to derive the matrix string theory (4.21) from the MQM (2.60) as if the latter has been derived from a contact structure of the zero-dimensional matrix model (2.1).

The basic idea is similar to the previous scheme to construct the one-dimensional matrix model (2.60) through the contact structure of zero-dimensional matrices. A difference is that we start with the one-dimensional matrix model (2.60) and introduce an additional contact structure along a spatial direction whose coordinate is called σ in our case. Ultimately, the matrix string theory (4.21) can be realized as the quantization of a regular 2-contact manifold. See Ref. [36] for a general k-contact manifold. First let us consider the projection $\pi_2 : \mathbb{R}^{1,1} \times M \to M,\ \pi_2(\sigma^\alpha, x) = x$ where M is a symplectic manifold with the symplectic form B.[10] The regular 2-contact $(2n+2)$-dimensional manifold is defined by a quartet $(\mathbb{R}^{1,1} \times M, \widetilde{B}, \eta^\alpha),\ \alpha = 0, 1$, where $\widetilde{B} = \pi_2^* B$, such that

$$\eta^0 \wedge \eta^1 \wedge B^n \neq 0 \tag{4.22}$$

everywhere and $d\eta^\alpha = \gamma^\alpha B$ with constants γ^α and $dB = 0$. Moreover there are uniquely defined two Reeb vectors R_α $(\alpha = 0, 1)$ satisfying

$$\iota_{R_\alpha} \eta^\beta = \delta_\alpha^\beta, \qquad \iota_{R_\alpha} B = 0, \qquad \alpha, \beta = 0, 1. \tag{4.23}$$

The above relations imply

$$\mathcal{L}_{R_\alpha} \eta^\beta = 0, \qquad \mathcal{L}_{R_\alpha} B = 0, \qquad [R_0, R_1] = 0. \tag{4.24}$$

For example, the contact forms for the matrix string theory (4.21) are given by

$$\eta^0 = dt - \frac{1}{2} p_a dy^a, \qquad \eta^1 = d\sigma - \frac{1}{2} p_a dy^a, \tag{4.25}$$

which determines the corresponding Reeb vectors

$$R_0 = \frac{\partial}{\partial t}, \qquad R_1 = \frac{\partial}{\partial \sigma}. \tag{4.26}$$

These Reeb vectors span the space of horizontal derivations in Eq. (4.20).

Since there are two independent contact structures, each contact structure generates its own Hamiltonian vector field defined by (A.42). For the contact structures in Eq. (4.25), they are given by

$$V_\alpha = \frac{\partial}{\partial \sigma^\alpha} + A_\alpha^\mu(t, \sigma, y) \frac{\partial}{\partial y^\mu}. \tag{4.27}$$

The quantization of the 2-contact manifold $(\mathbb{R}^{1,1} \times M, \widetilde{B}, \eta^\alpha)$ is simple because it is performed using the Darboux coordinates (σ^α, y^a). It is basically defined by the quantization of the symplectic manifold (M, B) in which σ^α are regarded as classical variables like the time coordinate in the algebra $\mathcal{A}_\theta^1$. After quantization, a generic element of the NC $\star$-algebra $\mathcal{A}_\theta^2$ takes the form (4.16). Then the module $\mathfrak{D}^2$ in Eq. (4.19) is generated by

$$\mathfrak{D}^2 = \Big\{ \widehat{V}_A(t, \sigma) = \big(\widehat{V}_\alpha, \widehat{V}_a\big)(t, \sigma) | \widehat{V}_\alpha(t, \sigma) = \frac{\partial}{\partial \sigma^\alpha} + \mathrm{ad}_{\widehat{A}_\alpha},\ \widehat{V}_a(t, \sigma) = \mathrm{ad}_{\widehat{\phi}_a} \Big\}, \tag{4.28}$$

[10] It is possible to replace $\mathbb{R}^{1,1} \times M$ by a general $(2n+2)$-dimensional manifold N as far as there is a well-defined two-dimensional foliation $\mathcal{V}$ such that the corresponding space of leaves $N/\mathcal{V} = M$ is a Hausdorff differentiable manifold [36]. See (A.24) for a relevant discussion. We will keep the maximal simplicity for a plain argument.

where $A = 0, 1, \cdots, 2n+1$ and the adjoint operations are inner derivations of $\mathcal{A}^2_\theta$. Finally the corresponding Lorentzian metric dual to the matrix string theory (4.21) is given by [3, 5]

$$ds^2 = \lambda^2 \eta_{AB} v^A \otimes v^B = \lambda^2 \big(\eta_{\alpha\beta} d\sigma^\alpha d\sigma^\beta + v^a_\mu v^a_\nu (dy^\mu - \mathbf{A}^\mu)(dy^\nu - \mathbf{A}^\nu)\big), \tag{4.29}$$

where $\mathbf{A}^\mu := A^\mu_\alpha(t,\sigma,y) d\sigma^\alpha$ and $\lambda^2 = \nu_{(t,\sigma)}(V_0, V_1, \cdots, V_{2n+1})$ is determined by the volume preserving condition, $\mathcal{L}_{V_A}\nu_{(t,\sigma)} = 0$, with respect to a given volume form

$$\nu_{(t,\sigma)} = dt \wedge d\sigma \wedge \nu = \lambda^2 dt \wedge d\sigma \wedge v^1 \wedge \cdots \wedge v^{2n}. \tag{4.30}$$

Let us come back to our previous question about the generalization of pseudoholomorphic curves to a quantized spacetime. In order to address this issue, let us consider the Wick rotation $t = -i\tau$ again to return to the Euclidean space. If the quantum version of pseudoholomorphic curves exists, Eq. (4.3) suggests that it will also obey the first-order partial differential equations. It is well-known [47] that the matrix action (4.21) admits such a first-order system. For simplicity, assume that adjoint scalar fields mostly vanish except $(\phi_2, \phi_3) \neq 0$. It is convenient to use the complex variables

$$\phi = \frac{1}{2}(\phi_2 - i\phi_3), \qquad \phi^\dagger = \frac{1}{2}(\phi_2 + i\phi_3). \tag{4.31}$$

It is not difficult to show that the Euclidean action with $\phi_a = 0$ for $a = 4, \cdots, 9$ can be written as the Bogomol'nyi-type, i.e.,

$$\begin{aligned} S &= \frac{1}{g_s^2}\int d^2\sigma \mathrm{Tr}\left(\frac{1}{4}F^2_{\alpha\beta} + \frac{1}{2}(D_\alpha\phi_a)^2 - \frac{1}{4}[\phi_a, \phi_b]^2\right) \\ &= \frac{2}{g_s^2}\int d^2\sigma \mathrm{Tr}\left(\big(iF_{z\bar{z}} - [\phi, \phi^\dagger]\big)^2 + |D_{\bar{z}}\phi|^2 - i\partial_\alpha\big(\varepsilon^{\alpha\beta}\phi^\dagger D_\beta\phi\big)\right). \end{aligned} \tag{4.32}$$

Since the last term is a topological number, the minimum of the action is achieved in the configurations obeying

$$F_{z\bar{z}} + i[\phi, \phi^\dagger] = 0, \qquad D_{\bar{z}}\phi = 0. \tag{4.33}$$

Note that the above equations recover Eq. (4.7) in a very commutative limit where $[\phi^\dagger, \phi] = 0$. Therefore it is reasonable to identify Eq. (4.33) with the quantum version of pseudoholomorphic curves.

Mathematically Eq. (4.33) is equivalent to the Hitchin equations describing a Higgs bundle [48]. A Higgs bundle is a system composed of a connection A on a principal G-bundle or simply a vector bundle E over a Riemann surface Σ and a holomorphic endomorphism ϕ of E satisfying Eq. (4.33). The Hitchin equations describe four-dimensional Yang-Mills instantons on $\Sigma \times \mathbb{R}^2$ which are invariant with respect to the translation group $\mathbb{R}^2$. (This $\mathbb{R}^2$ is transverse to the Riemann surface, so independent of the worldsheet $\mathbb{R}^2$.) Using the translation invariance, the Yang-Mills instantons can be dimensionally reduced to the Riemann surface Σ in which Yang-Mills gauge fields along the isometry directions become an adjoint Higgs field ϕ. In our case the gauge group G is $U(N)$. In particular, we are interested in the large N limit, i.e., $N \to \infty$. In this limit, the action (4.32) can be mapped to four-dimensional NC $U(1)$ gauge theory under the Coulomb branch vacuum $\langle \phi_a \rangle_{\rm vac} = p_a, \ a = 2, 3$ obeying the commutation relation $[p_2, p_3] = -iB_{23}$. Then the Hitchin equations (4.33) precisely become the self-duality equation for NC $U(1)$ instantons on $\mathbb{R}^2(\text{or } \Sigma) \times \mathbb{R}^2_\theta$ [49, 50]. The corresponding gravitational metric for the case $n = 1$ was already identified in Eq. (4.29) with the analytic continuation $t = -i\tau$. It was shown in [51, 52, 53] that the solution of the Hitchin equations (4.33) is dual to four-dimensional gravitational instantons which are hyper-Kähler manifolds. In particular, the real heaven is governed by the $su(\infty)$ Toda equation and the self-duality equation for the real heaven exactly reduces to the commutative limit of the Hitchin equations (4.33). See eq. (4.31) in Ref. [51]. Thus the Hitchin system with the gauge group $G = U(N \to \infty)$ may be closely related to the Toda field theory. Indeed this interesting connection was already analyzed in [54]. In sum, Hitchin equations, NC $U(1)$ instantons, gravitational instantons and pseudoholomorphic curves may be only the tip of the iceberg in the matrix string theory (4.21) that have barely shown themselves.

Let us conclude this section by drawing an invaluable insight. So far we have understood NC spacetimes too easily. However the NC spacetime is much more radical and mysterious than we thought. It is fair to say that we have not yet fully understood the mathematically precise sense in which spacetime should be NC. Indeed we have observed at the outset of this section that NC spacetime necessarily implies emergent spacetime if spacetime at microscopic scales should be viewed as NC. This means that classical spacetime is somehow a derived concept.[11] Since we form our picture of the world by recognizing the NC spacetime as a small deformation of classical symplectic or Poisson manifolds, we need an efficient tool to explore the symplectic geometry. The most natural object to probe symplectic manifolds is a pseudoholomorphic curve which is a stringy generalization of a geodesic worldline in Riemannian geometry [17]. Recall that the pseudoholomorphic curve is basically a minimal surface or a string worldsheet embedded into spacetime. However, to make sense of the emergent spacetime proposal, we need a mathematically precise framework for describing strings in a background-independent way. If it is so, the background-independent theory does not have to assume from the outset that strings are vibrating in a preexisting spacetime. In this section we have aimed at clarifying how the pseudoholomorphic curves can be lifted to a NC spacetime by the matrix string theory. The matrix string theory naturally extends the first-quantized string theory so that it also describes the perturbative interactions of splitting and joining of strings, producing surfaces with nontrivial topology [19]. That is, the matrix string theory is a second-quantized theory in which spacetime emerges from the collective behavior of matrix strings. Thus we argue that the NC spacetime can be viewed as a second-quantized string for the background-independent formulation of quantum gravity, which is still elusive in ordinary string theory.

[11] This prospect has been recently advocated by Moore in (especially, Sec. 9) "Physical mathematics and the future" (available at http://www.physics.rutgers.edu/∼gmoore/). See also Segal in "Space and spaces" (available at http://www.lms.ac.uk/sites/lms.ac.uk/files/files/About_Us/AGM_talk.pdf) and [55].

5 Discussion

We want to emphasize again that NC spacetime necessarily implies emergent spacetime if spacetime at microscopic scales should be viewed as NC. The NC spacetime is much more radical and mysterious than we previously thought. (See Sec. 1 in Ref. [3] for the discussion of this aspect.) In order to understand NC spacetime correctly, we need to deactivate the thought patterns that we have installed in our brains and taken for granted for so many years. As we argued in Part I, the background-independent formulation of quantum gravity requires the concept of emergent spacetime that may open a new perspective to resolve the notorious problems in theoretical physics such as the cosmological constant problem, hierarchy problem, dark energy, dark matter, and cosmic inflation. In particular, the emergent spacetime picture admits a background-independent formulation of inflationary cosmology so that the inflation simply arises as a time-dependent solution of a large N matrix model without introducing any inflaton field as well as an *ad hoc* inflation potential. Therefore it brings about radical changes of physics, especially, regarding to physical cosmology.

In Part II, we have explored the mathematical foundation for the large N duality in Fig. 1 in order to elucidate how the large N duality can be applied to physical cosmology. The most remarkable aspect of the background-independent formulation for inflationary cosmology is that the cosmic inflation is described by large N matrices only without introducing any inflaton field and an *ad hoc* inflation potential. Thus an urgent question is how to make a successful exit from inflation with no help of the inflaton field.

We certainly live in the universe where the inflationary epoch had lasted only for a very tiny period in very early times although it is currently in an accelerating phase driven by the dark energy. Therefore there should be some relaxation mechanism for the (first-order) phase transition from the inflating universe to a radiation-dominated universe. We showed that the former is described by the metric (1.12) whereas the latter is described by (2.56) and both arise as solutions of the background-independent matrix model (2.60). In scalar field inflation scenarios, the relaxation mechanism is known as the reheating in which the scalar field switches from being overdamped to being underdamped and begins to oscillate at the bottom of the potential to transfer its energy to a radiation dominated plasma at a temperature sufficient to allow standard nucleosythesis [56]. For this purpose, it is necessary to introduce a very *ad hoc* potential for the inflaton. In our case, however, we have introduced neither an inflaton field nor an inflation potential. Nevertheless, the inflation was possible since an LCS manifold admits a rich variety of vector fields, in particular, the Liouville vector field which generates the inflation

We do not know the precise mechanism for the graceful exit. Thereby we will briefly speculate a plausible scenario only. Let us start with a naive observation. The Lorentzian metric (1.12) describes general scalar-tensor perturbations on the inflating spacetime. Since the fluctuations have been superposed on the inflating background, we suspect that there may be some nonlinear damping mechanism through the interactions between the background and the density fluctuations. To be precise, there may be a cosmic analogue of the Landau damping in plasma physics originally applied to longitudinal oscillations of an electron plasma. The Landau damping in a plasma occurs due to the energy exchange between an electromagnetic wave and particles in the plasma with velocity approximately equal to phase velocity of the wave and leads to exponentially decaying collective oscillations.[12] The Landau damping may be intuitively understood by considering how a surfer gains energy from the sea wave. If the suffer is slightly slower than the wave mode, the mode loses energy to the suffer. For the wave to be damped, the wave velocity and the surfer velocity must be similar and then the surfer is trapped by the wave. A similar situation may happen in the inflating spacetime (1.12). Local fluctuations (suffers) on the inflating spacetime (the wave mode) are given by Eq. (1.10). Note that these local fluctuations carry an additional localized energy and this local energy will cause a slight delay of the drift of local lumps compared to the inflating background. Moreover these drift delays will occur everywhere since (quantum) fluctuations are everywhere. Then this is precisely the condition for the Landau damping to occur. If this is true, the inflating mode will transfer its kinetic energy to ubiquitous local fluctuations, ending the inflation through an exponential damping and entering to a radiation dominated era via the reheating at a sufficiently high temperature for the standard Big Bang.

The above speculation may be too good to be true. However, it may not be so absurd, considering the fact that the cosmic inflation is described by a conformal Hamiltonian system [15, 16] which often appears in dynamical systems with friction and the transition of such dynamical systems in nonequilibrium into equilibrium is induced by interactions with environment. For the cosmic inflation, ubiquitous fluctuations over the inflating spacetime will play a role of the environment. Furthermore it seems to be a reasonable clue since the underlying theory for emergent gravity is the Maxwell's electromagnetism on NC spacetime and the Landau damping can be realized even at a nonlinear level [57]. Therefore it will be important to verify whether the innocent idea can work or not. Probably the cosmic Landau damping may be closely related to the instability of de Sitter space suggested by Polyakov [41].

Our real world, $\mathbb{R}^{1,3} \cong \mathbb{R} \times \mathbb{R}^3$, is mystic as ever because the spatial 3-manifold $\mathbb{R}^3$ does not belong to the family of (almost) symplectic manifolds. We thus finally want to list possible stairways to our real world - the four-dimensional Lorentzian spacetime $\mathcal{M}$:

A. Analytic continuation or Wick rotation from $\mathbb{R}^4$.

B. Kaluza-Klein compactification $\mathcal{M} \times \mathbb{S}^1$.

C. Constact manifold $(\mathbb{R}^3, \eta)$.

D. Nambu structure $(\mathbb{R}^3, C)$.

[12] There is a nice exposition on the Landau damping by Werner Herr, "Physics of Landau Damping: An introduction (to a mysterious topic)," available at https://indico.cern.ch/event/216963/contribution/41/material/slides/0.pdf. Recently the Landau damping has been mathematically established even at the non-linear level [57].

Here η is a contact form on $\mathbb{R}^3$ and $C = \frac{1}{3!} C_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda$ is a nondegenerate, closed three-form on $\mathbb{R}^3$. In the case (A), the Lorentzian metric is obtained from Eq. (2.27) with $n = 2$ by the Wick rotation $y^4 = iy^0$. We used this boring method to evaluate the dark energy in Ref. [4]. It is also straightforward to compactify the $(4+1)$-dimensional Lorentzian metric (2.56) onto $\mathbb{S}^1$ to get the result (B). Since the time is also defined as a contact structure, the case (C) has two contact structures as the matrix string theory discussed in Sec. 4. It may be interesting to briefly explore some clue for the cosmic inflation in the context (C). Let $N = \mathbb{R} \times \mathbb{R}^3$ and $t \in \mathbb{R}$ be the time coordinate and $f_t = f(t)$ be a positive monotonic function. Define a time-dependent closed two-form on N by

$$B_t = d\lambda_t = f_t (dT \wedge \eta + d\eta) \tag{5.1}$$

where $\lambda_t = f_t \eta$ and $T = \ln f_t$. Since $B_t^2 = e^{2T} dT \wedge \eta \wedge d\eta$ is nowhere vanishing, B_t is a symplectic structure on N. Consider a time-dependent Hamiltonian $H : N \to \mathbb{R}$ such that $dH = -e^T dT$ and denote the Hamiltonian vector field of H by X_H. Let R be the Reeb vector field associated with the contact form η. Then it is easy to show that

$$\iota_R B_t = dH, \tag{5.2}$$

that is, $R = X_H$. A very interesting property is that

$$Z = \frac{\partial}{\partial T} \tag{5.3}$$

is the Liouville vector field of the symplectic form B_t, i.e., $\mathcal{L}_Z B_t = B_t$ or $\iota_Z B_t = \lambda_t$. This condition can be written as $\mathcal{L}_Z \lambda_t = \lambda_t$. One can regard the Liouville vector field Z as the Reeb vector field associated with the contact form dT. Since $\iota_Z (B_t^2) = e^{2T} \eta \wedge d\eta$, the one-form λ_t gives rise to a contact form on every three-dimensional submanifold $M \subset N$ transverse to Z. Thus we expect that the conformal vector field Z will generate an inflationary metric given by

$$ds^2 = -dT^2 + e^{2T} d\mathbf{x} \cdot d\mathbf{x}. \tag{5.4}$$

It will be interesting to have a microscopic derivation of the above inflation metric from the matrix string theory (4.21). The approach in [58] may be useful for this case. Since we have no idea how to formulate emergent gravity based on the Nambu structure (D), the last case would remain to be our dream. It may be of M-theory origin because it is involved with the 3-form C instead of symplectic 2-form B.

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A Locally conformal cosymplectic manifolds

In this Appendix we briefly review the mathematical foundation of locally conformal cosymplectic (LCC) manifolds. It was shown in [37] that an LCC manifold can be seen as a generalized phase space of time-dependent Hamiltonian system. Thus we argue that the LCC manifold is also a natural phase space describing the cosmic inflation of our universe as a direct application of the results in Refs. [36, 37] to emergent gravity.

First let us consider locally conformal symplectic (LCS) manifolds. An LCS manifold is a triple (M, Ω, b) where b is a closed one-form and Ω is a nondegenerate (but not closed) two-form satisfying

$$d\Omega - b \wedge \Omega = 0. \tag{A.1}$$

The dimension of M will be assumed to be at least 4 and the one-form b is called the Lee form [39]. If the Lee form b is exact, the manifold is globally conformal symplectic (GCS). A symplectic manifold corresponds to the case with $b = 0$. Locally by choosing $b = d\lambda^{(\alpha)}$ for a local function $\lambda^{(\alpha)} : U_\alpha \to \mathbb{R}$ on an open neighborhood U_α, Eq. (A.1) is equivalent to $d(e^{-\lambda^{(\alpha)}}\Omega) = 0$, so the local geometry of LCS manifolds is exactly the same as that of symplectic manifolds. Thus an LCS form on a manifold M is a non-degenerate two-form Ω that is locally conformal to a symplectic form. In other words, on an LCS manifold (M, Ω, b), there exists an open covering $\{U_\alpha\}$ of M and a smooth positive function f_α on each U_α such that $f_\alpha \Omega|_{U_\alpha}$ is symplectic on U_α. Two LCS forms Ω and Ω' are said to be (conformally) equivalent if there exists some positive function f such that $\Omega' = f\Omega$, where the Lee form of Ω' is just $b' = b + d\ln f$. An interesting example [59] is provided by the Hopf manifolds that are diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{2n-1}$ and have a locally conformal Kähler metric while they admit no Kähler metric.

An LCS manifold can be seen as a generalized phase space of Hamiltonian dynamical systems since the form of the Hamilton's equations is preserved by homothetic canonical transformations. Let us recapitulate how the LCS manifolds naturally arise from the Hamiltonian dynamics of particles. Consider a dynamical system with n degrees of freedom so that its phase space is a $2n$-dimensional differentiable manifold M endowed with an open covering of coordinate neighborhoods $\{U_\alpha\}_{\alpha \in I}$ with local coordinates $\big(q^i_{(\alpha)}, p_i^{(\alpha)}\big)$, $i = 1, \cdots, n$. Then we know that the dynamics consists of the orbits of a Hamiltonian vector field X_H. Every point of M has an open neighborhood U_α with the local Darboux coordinates $\big(q^i_{(\alpha)}, p_i^{(\alpha)}\big)$. One can restrict the Hamiltonian H and a nondegenerate two-form ω to each U_α to have a local Hamiltonian $H_\alpha = H_\alpha\big(q^i_{(\alpha)}, p_i^{(\alpha)}\big)$ and a symplectic structure $\omega_\alpha = dq^i_{(\alpha)} \wedge dp_i^{(\alpha)}$. Similarly the globally defined Hamiltonian vector field X_H is restricted to U_α which is precisely given by X_{H_α}. Then the orbits are defined by the Hamilton's equations

$$\frac{dq^i_{(\alpha)}}{dt} = \frac{\partial H_\alpha}{\partial p_i^{(\alpha)}}, \qquad \frac{dp_i^{(\alpha)}}{dt} = -\frac{\partial H_\alpha}{\partial q^i_{(\alpha)}}. \tag{A.2}$$

When one takes the coordinate chart definition of symplectic manifolds, there is no compulsory reason why one should require the two-form ω to be closed. Indeed, the Hamiltonian formulation of particle dynamics consists in asking the local forms ω_α and local functions H_α to glue up to a global symplectic form ω and a global Hamiltonian H. However, since the dynamical information is given by a global vector field, it is more natural to only require that the transition functions

$$q^i_{(\beta)} = q^i_{(\beta)}\big(q^i_{(\alpha)}, p^{(\alpha)}_i\big), \qquad p^{(\beta)}_i = p^{(\beta)}_i\big(q^i_{(\alpha)}, p^{(\alpha)}_i\big) \tag{A.3}$$

on an overlap $U_\alpha \cap U_\beta \neq \emptyset$ preserve the form of the Hamilton's equations (A.2). This happens not only if Eq. (A.3) implies

$$\omega_\beta = dq^i_{(\beta)} \wedge dp^{(\beta)}_i = dq^i_{(\alpha)} \wedge dp^{(\alpha)}_i = \omega_\alpha, \qquad H_\beta = H_\alpha, \tag{A.4}$$

where $H_\alpha : U_\alpha \to \mathbb{R},\ \alpha \in I$, but also if it implies

$$\omega_\beta = \lambda_{\beta\alpha}\omega_\alpha, \qquad H_\beta = \lambda_{\beta\alpha}H_\alpha, \tag{A.5}$$

where $\lambda_{\beta\alpha} = \text{constant} \neq 0$. Since $\iota(X_{H_\alpha})\omega_\alpha = dH_\alpha$, from Eq. (A.5) we obtain

$$X_{H_\alpha} = X_{H_\beta}, \tag{A.6}$$

so the integral curves of X_{H_α} and X_{H_β} are the same. Furthermore, Eq. (A.5) implies the cocycle condition

$$\lambda_{\gamma\beta}\lambda_{\beta\alpha} = \lambda_{\gamma\alpha} \tag{A.7}$$

as the gluing condition. We know that the cocycle condition (A.7) implies the existence of the local functions $\sigma_\alpha : U_\alpha \to \mathbb{R}$ satisfying

$$\lambda_{\beta\alpha} = \frac{e^{\sigma_\alpha}}{e^{\sigma_\beta}}. \tag{A.8}$$

Thus Eq. (A.5) shows that

$$\omega = e^{\sigma_\alpha}\omega_\alpha, \qquad H = e^{\sigma_\alpha}H_\alpha \tag{A.9}$$

are globally defined on M. Moreover a Hamiltonian vector field is globally defined, i.e. $X_H = X_{H_\alpha}$, as was indicated in Eq. (A.6). Hence we have a basic line bundle L over M and a Hamiltonian H as a cross-section of L (a "twisted Hamiltonian") instead of a simple function. Therefore (M, ω) is an LCS manifold that can be considered as a natural phase space of Hamiltonian dynamical systems, more general than the symplectic manifolds.

As we explained in Sec. 2, the realization of emergent geometry is intrinsically local too. The emergent geometry is constructed by gluing local Darboux charts and their local Poisson algebras. Therefore the construction of an LCS manifold as a generalized phase space for particle dynamics should also be applied to the emergent geometry. Therefore it is helpful to briefly review infinitesimal automorphisms of an LCS manifold (M, Ω, b). The infinitesimal automorphism (IA) will be denoted by $\mathfrak{A}_\Omega$. Let $C^\infty(M)$ denote the associative algebra of smooth functions on M and $f : M \to \mathbb{R}$ be such a globally defined function. The Hamiltonian vector field X_f of $f \in C^\infty(M)$ with respect to the LCS form Ω is defined by

$$\iota(X_f)\Omega = df - fb. \tag{A.10}$$

As we observed above, there is a well-defined line bundle L over M in which local functions $f_\alpha \equiv e^{-\sigma_\alpha} f$ on a patch $U_\alpha \subset M$ correspond to sections of $L \to U_\alpha$. If we take the Lee form on U_α as $b|_{U_\alpha} = d\sigma_\alpha$, Eq. (A.10) refers to the usual (local) Hamiltonian vector field $X_{f_\alpha} = X_f$ defined by

$$\iota(X_{f_\alpha})\Omega_\alpha = df_\alpha \tag{A.11}$$

where $\Omega_\alpha = e^{-\sigma_\alpha}\Omega$. Using the Cartan formula for the Lie derivative

$$\mathcal{L}_X = d\iota_X + \iota_X d, \tag{A.12}$$

one can immediately deduce from Eqs. (A.1) and (A.10) that

$$\mathcal{L}_{X_f}\Omega = b(X_f)\Omega, \tag{A.13}$$

$$\mathcal{L}_{X_f} b = db(X_f). \tag{A.14}$$

Therefore, unlike the symplectic case, the Hamiltonian vector field X_f is in general not an IA of LCS manifolds.

Using the Hamiltonian vector fields defined by Eq. (A.10), we define the Poisson bracket

$$\{f, g\}_\Omega = \iota(X_f)\iota(X_g)\Omega = -\Omega(X_f, X_g) = e^{\sigma_\alpha}\iota(X_{f_\alpha})\iota(X_{g_\alpha})\Omega_\alpha = e^{\sigma_\alpha}\{f_\alpha, g_\alpha\}_{\Omega_\alpha}. \tag{A.15}$$

Then we can calculate the double Poisson bracket

$$\{\{f, g\}_\Omega, h\}_\Omega = X_h\big(\Omega(X_f, X_g)\big) - b(X_h)\Omega(X_f, X_g). \tag{A.16}$$

Using this result, it is easy to check the Jacobi identity of the Poisson bracket:

$$\{\{f, g\}_\Omega, h\}_\Omega + \{\{g, h\}_\Omega, f\}_\Omega + \{\{h, f\}_\Omega, g\}_\Omega = \big(d\Omega - b \wedge \Omega\big)(X_f, X_g, X_h) = 0. \tag{A.17}$$

Let $\mathfrak{P} = (C^\infty(M), \{-,-\}_\Omega)$ be the Poisson-Lie algebra of (M, Ω) and $\mathfrak{X}(M)$ the Lie algebra of vector fields of M. The result (A.15) shows that the mapping $\mathfrak{H} : \mathfrak{P} \to \mathfrak{X}(M)$ given by $f \mapsto X_f$ is a Lie algebra homomorphism because one can derive the relation

$$X_{\{f,g\}_\Omega} = [X_f, X_g] \tag{A.18}$$

from the Jacobi identity (A.17). However, if (M, Ω) is a (connected) LCS manifold that is not GCS, then $\mathfrak{H}$ must be a monomorphism, i.e., an injective homomorphism. See the Proposition 2.1 in [36] for the proof. This means that $X_f = 0$ implies $f = 0$. This is in stark contrast to symplectic manifolds, in which $X_f = 0$ just implies $f = \mathrm{constant}$. Since we argue that the phase space for cosmic inflation is a locally conformal (co)symplectic manifold, this implies a desirable property that vacuum energy couples to gravity and triggers cosmic inflation. However, it does not mean that the cosmological constant problem threatens the emergent gravity because physical quantities during inflation are not constant but time-dependent as we noted before.

Let us denote the IA of (M, Ω) by $\mathfrak{X}_\Omega(M)$ whose elements obey $\mathcal{L}_X \Omega = 0$. Then we have $\mathcal{L}_X b = 0$ by Eq. (A.1) which implies the condition $b(X) = \mathrm{constant}$. In particular, if $X, Y \in \mathfrak{X}_\Omega(M)$, then $b(X) = \mathrm{constant},\ b(Y) = \mathrm{constant}$ and $db(X, Y) = 0$ yields $b([X, Y]) = 0$ using the formula

$$db(X, Y) = X\big(b(Y)\big) - Y\big(b(X)\big) - b([X, Y]). \tag{A.19}$$

Hence, the application $l : \mathfrak{X}_\Omega(M) \to \mathbb{R}$ defined by $l(X) = b(X)$ is a Lie algebra homomorphism, called the Lee homomorphism of $\mathfrak{X}_\Omega(M)$. The kernel $\ker(l)$ is the Lie algebra of the horizontal elements of $\mathfrak{X}_\Omega(M)$, denoted by $\mathfrak{X}^{\mathrm{hor}}_\Omega(M)$. The IA $X \in \mathfrak{X}_\Omega(M)$ with $l(X) \neq 0$ is called transversal IA and an LCS manifold M is called the first kind if it has a transversal IA. Otherwise, M is of the second kind and the Lee homomorphism is trivial. Note that, if (M, Ω) is of the first kind and $f : M \to \mathbb{R}$ is a function such that $df|_{x_0} = b(x_0)$, then $(M, e^{-f}\Omega)$ has the Lee form $b - df$ with a vanishing point, so it becomes an LCS manifold of the second kind.

There is a special vector field A defined by $\iota_A \Omega = b$. Then it is easy to see

$$\iota_A b = 0, \qquad \mathcal{L}_A b = 0, \qquad \mathcal{L}_A \Omega = 0. \tag{A.20}$$

We do have $X_f \in \mathfrak{X}_\Omega(M)$ if and only if $b(X_f) = 0$ according to Eq. (A.13) or equivalently $b(X_f) = \iota_{X_f} \iota_A \Omega = -\iota_A(df - fb) = -A(f) = 0$. Let us fix an element $B \in l^{-1}(1) \subset \mathfrak{X}_\Omega(M)$. Then every element Y in $\mathfrak{X}_\Omega(M)$ has a unique decomposition

$$Y = X + l(Y)B, \qquad X \in \mathfrak{X}^{\mathrm{hor}}_\Omega(M). \tag{A.21}$$

Now, put $a \equiv -\iota_B \Omega$, so $a(B) = 0$ and $a(A) = \iota_B \iota_A \Omega = b(B) = 1$. Since $\mathcal{L}_B \Omega = (\iota_B d + d\iota_B)\Omega = 0$, this yields a particular expression for Ω given by

$$\Omega = da - b \wedge a = d_b a, \tag{A.22}$$

where d_b is the Lichnerowicz differential defined by $d_b \beta = d\beta - b \wedge \beta$ for any k-form β and satisfies $d_b^2 = 0$. Furthermore, using the formula $[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$ for vector fields X and Y, we have $\mathcal{L}_B a = 0$, hence $\iota_B da = 0$ that means rank $da < 2n$. Since $\Omega^n \neq 0$, one can deduce from Eq. (A.22) the condition

$$b \wedge a \wedge (da)^{n-1} \neq 0 \tag{A.23}$$

everywhere. This yields the Proposition 2.2 in Ref. [36] that a manifold M of dimension $2n$ admits an LCS structure of the first kind if and only if it admits two one-forms a, b such that $db = 0$, rank $da < 2n$ and Eq. (A.23) holds at every point of M. Note also that $\iota_A da = \iota_A(\Omega + b \wedge a) = b - a(A)b = 0$. This means that $[A, B] = 0$ because $\iota_A da = \mathcal{L}_A a = -\mathcal{L}_A \iota_B \Omega = -\iota_{[A,B]}\Omega = 0$. In sum, there exist particular vector fields A and B in $\mathfrak{X}_\Omega(M)$ that obey

$$[A, B] = 0, \qquad a(A) = b(B) = 1, \qquad a(B) = b(A) = 0. \tag{A.24}$$

Thus one can obtain on M the vertical foliation $\mathcal{V} = \mathrm{span}\{A, B\}$, whose leaves are the orbits of a natural action of $\mathbb{R}^2$.

Suppose that (M,Ω) is an LCS manifold of the first kind and B is a basic transversal IA. Let $\mathfrak{X}^{\rm hor}_\Omega(M,B)$ be the Lie subalgebra of $\mathfrak{X}^{\rm hor}_\Omega(M)$ whose automorphisms also preserve B. It turns out that $X\in\mathfrak{X}^{\rm hor}_\Omega(M,B)$ if and only if $\mathcal{L}_X\Omega=0,\ b(X)=0$ and $[X,B]=0$. Similarly consider the subset of $C^\infty(M)$ that consists of functions satisfying $A(f)=B(f)=0$ and is denoted by $C^\infty_{\mathcal{V}}(M)$. Then one can show that $\mathfrak{P}_{\mathcal{V}}=\big(C^\infty_{\mathcal{V}}(M),\{-,-\}_\Omega\big)$ is a Poisson-Lie subalgebra of $\mathfrak{P}$ and $\mathfrak{H}:\mathfrak{P}_{\mathcal{V}}\to\mathfrak{X}^{\rm hor}_\Omega(M,B)$ is an isomorphism. A striking fact is that a semi-simple Lie group G cannot act transitively on a nonsymplectic LCS manifold.

The formula (A.13) proves that a Hamiltonian vector field is a conformal infinitesimal transformation (CIT) of (M,Ω). In general, a vector field X is a CIT if

$$\mathcal{L}_X\Omega=\alpha_X\Omega \tag{A.25}$$

where α_X is a function on M. The CIT forms a Lie algebra denoted by $\mathfrak{X}^c_\Omega(M)$. By differentiating Eq. (A.25), one can derive that $\mathcal{L}_Xb=d\alpha_X$, which implies

$$\alpha_X=b(X)+\kappa,\qquad \kappa=\text{constant}. \tag{A.26}$$

One can rewrite Eq. (A.25) as

$$\kappa\Omega=d_b(\iota_X\Omega). \tag{A.27}$$

Thus an LCS form Ω is d_b-exact if there is a CIT X. Or it can be written in terms of a local symplectic form $\Omega_\alpha=e^{-\sigma_\alpha}\Omega$ as

$$\mathcal{L}_X\Omega_\alpha=\big(\alpha_X-b(X)\big)\Omega_\alpha. \tag{A.28}$$

That is, the local form of the CIT is given by

$$\mathcal{L}_X\Omega_\alpha=\kappa\Omega_\alpha. \tag{A.29}$$

If we write $\Omega_\alpha=dA_{(\alpha)}$ on an open neighborhood U_α according to the Poincaré lemma, Eq. (A.29) can be written as the form [16]

$$\iota_X\Omega_\alpha=\kappa A_{(\alpha)}+df_\alpha, \tag{A.30}$$

where $f_\alpha:U_\alpha\to\mathbb{R}$ is a smooth function on U_α. If the conditions (A.29) and (A.30) hold either locally or globally, we will call X a conformal vector field which plays an important role in our discussion. If $H^1(M)=0$, the conformal vector field X has a unique decomposition given by

$$X=\kappa Z+X_f, \tag{A.31}$$

where $\iota_Z\Omega=A$ and $\iota_{X_f}\Omega=df$. The vector field Z is called the Liouville vector field [15]. Note that, even though $f=0$ identically, the conformal vector field $X=\kappa Z$ is nontrivial and it is generated by the open Wilson line (1.8) in our case [1]. We observed in Sec. 3 that this remarkable property leads to a desirable consequence for the cosmic inflation.

We can extend the Lee homomorphism to $l:\mathfrak{X}^c_\Omega(M)\to\mathbb{R}$ by defining $l(X)=b(X)-\alpha_X=-\kappa$. If $X,Y\in\mathfrak{X}^c_\Omega(M)$, we get $\alpha_{[X,Y]}=X\big(b(Y)\big)-Y\big(b(X)\big)$ from $\mathcal{L}_{[X,Y]}\Omega=\alpha_{[X,Y]}\Omega$ and so

$l([X,Y]) = b([X,Y]) - \alpha_{[X,Y]} = -db(X,Y) = 0$ using the formula (A.19). Hence the extended l is also a Lie algebra homomorphism. Its kernel is denoted by $\ker l = \mathfrak{X}^l_{\rm Ham}(M)$ and consists of vector fields X to obey $\mathcal{L}_X \Omega_\alpha = 0$, i.e., of locally Hamiltonian vector fields. Note that $\widetilde{l}(X)$ for $\widetilde{\Omega} = e^\varphi \Omega$ is equal to $l(X)$ for Ω. Thus the Lee homomorphism l is conformally invariant. If we fix an element $C \in l^{-1}(1)$, we can get for every $Y \in \mathfrak{X}^c_\Omega(M)$ the unique decomposition

$$Y = X + l(Y)C, \qquad X \in \mathfrak{X}^l_{\rm Ham}(M). \tag{A.32}$$

Then, if $c = -\iota_C \Omega$, we can solve $\mathcal{L}_C \Omega = (\iota_C d + d\iota_C)\Omega = \alpha_C \Omega$ to get a particular expression for Ω given by

$$\Omega = dc - b \wedge c = d_b c. \tag{A.33}$$

In a conservative dynamical system described by a Hamiltonian vector field, time coordinate t is not a phase space coordinate but an affine parameter on particle trajectories. But, for a general time-dependent system, it is necessary to include the time coordinate as an extra phase space coordinate. The corresponding $(2n+1)$-dimensional manifold is known as an almost cosymplectic manifold which is a triple (M, Ω, η) where Ω and η are a two-form and a one-form on M such that $\eta \wedge \Omega^n \neq 0$. If Ω and η are closed, i.e., $d\Omega = d\eta = 0$, then M is said to be a cosymplectic manifold. Thus an odd-dimensional counterpart of a symplectic manifold is given by a cosymplectic manifold, which is locally a product of a symplectic manifold with a circle or a line. A contact manifold constitutes a subclass of cosymplectic manifolds with $\Omega = d\eta$. Then the one-form η is called a contact structure or a contact one-form. Given a contact one-form η, there is a unique vector field R such that $\iota_R \eta = 1$ and $\iota_R \Omega = 0$. This vector field R is known as the Reeb vector field of the contact form η. Two contact forms η and η' on M are equivalent if there is a smooth positive function ρ on M such that $\eta' = \rho\eta$, since $\eta' \wedge (d\eta')^n = \rho^{n+1} \eta \wedge (d\eta)^n \neq 0$. The contact structure $C(\eta)$ determined by η is the equivalence class of η.

The Darboux theorem for a contact manifold (M, η) states that, in an open neighborhood of each point of M, it is always possible to find a set of local (Darboux) coordinates $(x^1, \cdots, x^n, y_1, \cdots, y_n, z)$ such that the one-form η can be written as

$$\eta = dz - \sum_{i=1}^n y_i dx^i \tag{A.34}$$

and the Reeb vector field is given by

$$R = \frac{\partial}{\partial z}. \tag{A.35}$$

To understand the contact one-form η more closely, first let us denote by $\mathcal{D}$ the contact distribution or subbundle defined by the kernel of η. If X, Y are (local) vector fields in $\mathcal{D}$, we have

$$d\eta(X,Y) = X\big(\eta(Y)\big) - Y\big(\eta(X)\big) - \eta([X,Y]) = -\eta([X,Y]). \tag{A.36}$$

This says that the distribution is integrable if and only if $d\eta$ is zero on $\mathcal{D}$. However the condition $\eta \wedge (d\eta)^n \neq 0$ means that the kernel of $d\eta$ is one-dimensional and everywhere transverse to $\mathcal{D}$.

Consequently, $d\eta$ is a linear symplectic form on $\mathcal{D}$ and the largest integral submanifolds of $\mathcal{D}$ are n-dimensional, so maximally non-integrable. In other words, a contact structure is nowhere integrable. In the above Darboux coordinate system, the contact subbundle $\mathcal{D}$ is spanned by

$$X_i = \frac{\partial}{\partial x^i} + y_i \frac{\partial}{\partial z}, \qquad Y^i = \frac{\partial}{\partial y_i}, \qquad i = 1, \cdots, n, \tag{A.37}$$

so they obey the bracket relations

$$[X_i, Y^j] = -\delta_i^j R, \qquad [X_i, R] = [Y^i, R] = 0. \tag{A.38}$$

Since $d\eta = \sum_{i=1}^n dx^i \wedge dy_i$ is a symplectic form with rank $2n$, the kernel of $d\eta$ is one-dimensional and generated by the Reeb vector R. Therefore every vector field X on M can be uniquely written as $X = fR + Y$ where $f \in C^\infty(M)$ and Y is a section of $\mathcal{D}$. A contact structure is regular if R is regular as a vector field, that is, every point of the manifold has a neighborhood such that any integral curve of the vector field passing through the neighborhood passes through only once.

Given a $(2n-1)$-dimensional contact manifold M with a contact form a, i.e. $a \wedge (da)^{n-1} \neq 0$, one can construct an LCS manifold by considering a principal bundle $p : V \to M$ with group $\mathbb{S}^1$ over M. Consider $V = \mathbb{S}^1 \times M$ endowed with the form $\Omega = da - b \wedge a = d_b a$, where b is the canonical one-form on $\mathbb{S}^1$. Clearly, Ω is nondegenerate and b is closed but not exact. And it obeys $d\Omega - b \wedge \Omega = d_b \Omega = d_b^2 a = 0$. Hence, (V, Ω) is an LCS manifold having b as its Lee form but it is not GCS. More generally, let $p : V \to M$ be an arbitrary principal bundle with group $\mathbb{S}^1$ over a $(2n-1)$-dimensional manifold M. And let a be the connection one-form on this principal bundle and $F = da$ be the corresponding curvature two-form. Then, if $b \wedge a \wedge F^{n-1} \neq 0$, the form $\Omega = F - b \wedge a$ defines an LCS structure on V which is not GCS.

Let $\mathfrak{X}(M)$ and $\Lambda^1(M)$ be the $C^\infty(M)$-modules of differentiable vector fields and one-forms on M, respectively. If (M, Ω, η) is a cosymplectic manifold, then there exists an isomorphism of $C^\infty(M)$-modules

$$\Upsilon : \mathfrak{X}(M) \to \Lambda^1(M) \tag{A.39}$$

defined by

$$\Upsilon(X) = \iota_X \Omega + \eta(X)\eta. \tag{A.40}$$

The Reeb vector field is given by $R = \Upsilon^{-1}(\eta)$. Let $f : M \to \mathbb{R}$ be a smooth function on M. The Hamiltonian vector field X_f is then defined by

$$\Upsilon(X_f) = df - R(f)\eta + \eta. \tag{A.41}$$

In other words, X_f is the vector field characterized by the identities

$$\iota(X_f)\Omega = df - R(f)\eta, \qquad \eta(X_f) = 1. \tag{A.42}$$

Then one can check that the time-like vector field V_0 in Eq. (2.59) is a Hamiltonian vector field for a cosymplectic manifold $(\mathbb{R} \times M, \pi_2^* B, dt)$ where $\pi_2 : \mathbb{R} \times M \to M$ and (M, B) is a symplectic manifold.

An almost cosymplectic manifold (M, Ω, η) is said to be LCC, if there exist an open covering $\{U_\alpha\}_{\alpha \in I}$ and local functions $\sigma_\alpha : U_\alpha \to \mathbb{R}$ such that

$$d(e^{-\sigma_\alpha}\Omega) = 0, \qquad d(e^{-\sigma_\alpha}\eta) = 0. \tag{A.43}$$

The local one-forms $d\sigma_\alpha$ glue up to a closed one-form b satisfying

$$d\Omega - b \wedge \Omega = d_b\Omega = 0, \qquad d\eta - b \wedge \eta = d_b\eta = 0. \tag{A.44}$$

Two LCC structures (Ω', η') and (Ω, η) are equivalent if $\Omega' = f\Omega$ and $\eta' = f\eta$ for a positive function f on M where the Lee form of Ω' is given by $b' = b + d\ln f$. An LCC manifold reduces to a cosympletic manifold if the Lee form b vanishes while it becomes an LCS manifold if $\eta = 0$ identically. The isomorphism (A.40) can be generalized to LCC manifolds and the corresponding Hamiltonian vector field is defined by

$$X_f = \Upsilon^{-1}\big(df - R(f)\eta + \eta\big) + fS \tag{A.45}$$

where S is called the canonical vector field defined by

$$\Upsilon(S) = b(R)\eta - b. \tag{A.46}$$

Therefore, X_f is characterized by the identities

$$\iota(X_f)\Omega = df - R(f)\eta + f\big(b(R)\eta - b\big), \qquad \eta(X_f) = 1. \tag{A.47}$$

It was shown in [37] that an LCC manifold can be seen as a generalized phase space of time-dependent Hamiltonian systems. Hence we argue that an LCC manifold also corresponds to a generalized phase space for an inflationary universe and its quantization realizes a background-independent formulation of the cosmic inflation, in particular, in the context of emergent spacetime.

B Harmonic oscillator with time-dependent mass

We observed that the NC spacetime $\mathbb{R}^{2n}_\theta$ in equilibrium is described by the Hilbert space of an n-dimensional harmonic oscillator while the inflating spacetime in nonequilibrium is described by the n-dimensional harmonic oscillator with a negative friction. The corresponding harmonic oscillator of constant frequency ω and friction coefficient α satisfies the equation

$$\ddot{q}^i + 2\alpha\dot{q}^i + \omega^2 q^i = 0, \qquad i = 1, \cdots, n. \tag{B.1}$$

The inflationary coordinates (3.14) correspond to the case $\alpha = -\frac{\kappa}{2} < 0$. It is known that the above second-order equation of motion cannot be directly derived from the Euler-Lagrange equation of any Lagrangian. However, there is an equivalent second-order equation

$$e^{2\alpha t}(\ddot{q}^i + 2\alpha\dot{q}^i + \omega^2 q^i) = 0, \tag{B.2}$$

for which a variational principle can be found [60]. Although Eq. (B.1) is traditionally considered to be non-Lagrangian, there exists an action principle for the equation of motion (B.2) in terms of the Lagrangian

$$L = \frac{1}{2} m(\dot{q}^2 - \omega^2 q^2) e^{2\alpha t}. \tag{B.3}$$

The corresponding Hamiltonian is given by

$$H = \frac{1}{2m}(e^{-2\alpha t} p^2 + e^{2\alpha t} m^2 \omega^2 q^2) \tag{B.4}$$

where $p_i = m\dot{q}^i e^{2\alpha t}$.

It is interesting to notice that the equation of motion (B.2) can be derived from an n-dimensional harmonic oscillator with a time-dependent mass $m(t)$ whose action is given by

$$S = \frac{1}{2} \int dt \big(m(t)\dot{q}^2 - k(t) q^2 \big) \tag{B.5}$$

where $k(t) = m(t)\omega^2$ with constant frequency ω. The variational principle, $\delta S = 0$, with respect to arbitrary variations δq^i leads to the equation of motion

$$m(t) \Big(\ddot{q}^i + \frac{\dot{m}(t)}{m(t)} \dot{q}^i + \omega^2 q^i \Big) = 0. \tag{B.6}$$

The second-order equation (B.2) corresponds to the case

$$\frac{\dot{m}(t)}{m(t)} = 2\alpha \quad \Rightarrow \quad m(t) = m_0 e^{2\alpha t}. \tag{B.7}$$

Recall that the equation of motion for the inflaton field corresponds to the case with the time-dependent mass $m(t) = m_0 e^{3Ht}$.

There is also the first-order formalism for the dynamical system (B.5). The action has the form

$$S = \frac{1}{2} \int dt \big(y\dot{x} - x\dot{y} - (y^2 + 2\alpha xy + \omega^2 x^2) \big) e^{2\alpha t}. \tag{B.8}$$

The equations of motion derived from the action (B.8) are given by

$$(\dot{y} + 2\alpha y + \omega^2 x) e^{2\alpha t} = 0, \qquad (\dot{x} - y) e^{2\alpha t} = 0. \tag{B.9}$$

The above action (B.8) describes a singular system with second-class constraints

$$\phi_x = p_x - \frac{1}{2} y e^{2\alpha t}, \qquad \phi_y = p_y + \frac{1}{2} x e^{2\alpha t} \tag{B.10}$$

with the Hamiltonian

$$H(x, y, t) = \frac{1}{2}(y^2 + 2\alpha xy + \omega^2 x^2) e^{2\alpha t}. \tag{B.11}$$

Even though the constraints are explicitly time-dependent, it is still possible to apply the Hamiltonian formalism with the help of Dirac brackets and perform the canonical quantization of the system. It was shown in [60] that the classical and quantum description of the harmonic oscillator described by the action (B.5) is equivalent to the first-order approach given in terms of the constraint system described by the action (B.8). Furthermore it can be proved that the dynamical system described by Eq. (B.2) is locally (i.e., $|t| < \infty$) equivalent to the system with the equation of motion (B.1).

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