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Quantum Fields on Manifolds: PCT and Gravitationally Induced Thermal States

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We formulate an axiomatic scheme, designed to provide a framework for a general, rigorous theory of relativistic quantum fields on a class of manifolds, that includes Kruskal's extension of Schwarzschild space-time, as well as Minkowski space-time. The scheme is an adaptation of Wightman's to this class of manifolds. We infer from it that, given an arbitrary field (in general, interacting) on a manifold X, the restriction of the field to a certain open submanifold $X^{(+)}$, whose boundaries are event horizons, satisfies the Kubo–Martin–Schwinger (KMS) thermal equilibrium conditions. This amounts to a rigorous, model-independent proof of a generalised Hawking–Unruh effect. Further, in cases where the field enjoys a certain PCT symmetry, the conjugation governing the KMS condition is just the PCT operator. The key to these results is an analogue, that we prove, of the Bisognano–Wichmann theorem [*J. Math. Phys.* **17** (1976), Theorem 1]. We also construct an alternative scheme by replacing a regularity condition at an event horizon by the assumption that the field in $X^{(+)}$ is in a ground, rather than a thermal, state. We show that, in this case, the observables in $X^{(+)}$ are uncorrelated to those in its causal complement, $X^{(-)}$, and thus that the event horizons act as physical barriers. Finally, we argue that the choice between the two schemes must be dictated by the prevailing conditions governing the state of the field.

1. INTRODUCTION

The formulation of quantum theory in an algebraic form, that is applicable to systems with finite or infinite numbers of degrees of freedom, has provided the framework for rigorous versions of both field theory and statistical mechanics (cf. Refs. [2–7]). Moreover, a mathematical connection between quantum field theory and classical statistical mechanics has been established, whereby the problem of constructing quantum fields in Minkowski space is essentially reduced to a statistical mechanical one [8, 9].

A quite different interconnection between quantum theory, statistical mechanics and general relativity has been suggested on the basis of Hawking's idea that the gravitational field due to a Black Hole can thermalise ambient quantum fields [10–14]. This idea, though not yet rigorously established,[1] is particularly interesting

[1] The principal limitations in rigour and generality of Refs. [10–14], for example, are that (a) they are based on formulations of quantum fields, with infinite numbers of degrees of freedom, within the Fock–

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from the statistical mechanical standpoint, since it implies that a quantum field may be thermalised by secular forces, e.g., certain gravitational ones, whereas the traditional view is that thermalisation arises essentially from the action of stochastic forces.

The object of the present article is to construct a framework for a rigorous general theory of quantum fields on a class of manifolds that includes Kruskal's extension of Schwarzschild space-time, as well as Minkowski space-time: a preliminary sketch of this framework was given in Ref. [15]. We formulate it as a system of axioms, that are essentially adaptations of those proposed by Wightman [5, 6] for fields in flat space-time, and are thus designed to represent the minimal conditions for the existence of relativistic quantum fields on the manifolds concerned, together with certain regularity assumptions. From these axioms, we infer that, given an arbitrary, in general interacting, quantum field on a manifold X of the specified class, the restriction of the field to a certain open submanifold, $X^{(+)}$, whose boundaries are event horizons, satisfies the Kubo–Martin–Schwinger (KMS) thermal equilibrium conditions [16–18] (formally, $\langle A(\tau)B\rangle = \langle BA(\tau + i\hbar\beta)\rangle$, where A, B are arbitrary observables, and $A(\tau)$ is the time-translate of A). This amounts to a rigorous, model-independent generalisation of the Hawking–Unruh thermalisation effect [10, 11]. Furthermore, we show that, if the field on X enjoys a global PCT symmetry, then the PCT operator is simply the conjugation[2] which, in a standard formulation [18, 19], governs the KMS conditions in $X^{(+)}$. The key to these results is an analogue, that we prove, of the Bisognano–Wichmann Theorem [1b, Theorem 1]. Clearly, the results signify that the KMS conditions play a central role in determining the global properties of the field on X. In addition to the scheme just discussed, we also formulate an alternative one, in which a certain regularity condition at an event horizon is replaced by the assumption that the field in $X^{(+)}$ is in a ground state. This latter scheme is shown to imply that the observables in $X^{(+)}$ are uncorrelated to those in the causally complementary region, $X^{(-)}$, which signifies that the event horizons act, in a certain sense, as physical barriers. We argue that a choice between the two schemes must depend on the prevailing circumstances governing the state of the field.

We emphasise here that all our results are obtained on an axiomatic, rather than a constructive, basis. What we prove is that these results are consequences of axioms that represent general demands of quantum theory and relativity, as applied to fields on a specified class of manifolds. These axioms should therefore be applicable to quantum fields on a Black Hole background, for example. On the other hand, we do not address ourselves in this article to the problem of constructing field theoretic models that satisfy the axioms.

Let us now discuss both the geometry and the field quantisation in more specific

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Hilbert representation, that is generally applicable only to finite systems (cf. [3, Chap. 1]); and (b) they are restricted to free massless fields [10–13] and perturbative treatments [14], of interacting ones [14], even though the validity of such treatments is known to be extremely dubious [5, p. 166].

[2] A conjugation is an antilinear transformation, J, of a Hilbert space $\mathscr{H}$, such that $J^2 = I$ and $(Jf, Jg) \equiv (g, f)$.

terms. The space-time manifolds we consider are of the form $X = \mathbf{R}^2 \times Y$ (pointwise $x = (t, w; y)$), with metric given by the formula

$$ds^2 = A(t^2 - w^2, y)(dt^2 - dw^2) - B(t^2 - w^2, y)\, d\sigma^2(y), \tag{1.1}$$

where A, B are positive-valued, smooth functions on $R \times Y$ and $d\sigma^2(y)$ is a positive metric on Y. We define $X^{(\pm)}$ to be the open submanifolds on X given by $w > |t|$ and $w < -|t|$, respectively. Thus $X^{(\pm)}$ are isometric with $\mathbf{R}_\pm \times \mathbf{R} \times Y$ (pointwise $(\xi, \tau; y)$), with

$$w = \xi \cosh \tau, \qquad t = \xi \sinh \tau, \tag{1.2}$$

and

$$ds^2 = A(-\xi^2, y)(\xi^2\, d\tau^2 - dy^2) - B(-\xi^2, y)\, d\sigma^2(y). \tag{1.3}$$

One sees readily that our formulation of X, $X^{(\pm)}$ covers the case where $X^{(\pm)}$ are the exterior and interior Schwarzschild manifolds and X is their Kruskal extension (cf. [20, 21]) as well as the case where X is Minkowski space and $X^{(\pm)}$ are the Rindler wedges.

It follows from (1.1) that t is a time-coordinate for X, though the time-translations, $t \to t + \text{constant}$, are not, in general, isometries of the manifold. However, it follows from (1.1) that these translations do become isometries[3] when restricted to the surfaces E, E', given by $w \pm t = 0$, respectively, i.e., $\partial/\partial t$ is a Killing vector on E, E'; and further, these surfaces correspond to the past and future event horizons, respectively, for $X^{(+)}$ (cf. [20, 21]). From (1.3), we see that τ is a time coordinate for $X^{(\pm)}$, and that time-translations, $\tau \to \tau + \text{constant}$, are isometries of these submanifolds. Indeed, by Eqs. (1.1) and (1.2), they are the restrictions to $X^{(\pm)}$ of the isometries of X given by the generalised Lorentz transformations, $t \to t \cosh b + w \sinh b$, $w \to w \cosh b + t \sinh b$.

Our principal axioms for a real scalar field (for example) in X are that the field is a Hermitian operator-valued distribution φ in a Hilbert space $\mathscr{H}$, possessing a unit vector Ψ such that (a) $\mathscr{H}$ is generated by the application to Ψ of the polynomials in φ, smeared out against a suitable class of test-functions, (b) $\varphi(x)$ and $\varphi(x')$ intercommute if x and x' have spacelike separation, and (c) there is a strongly continuous unitary representation U, in $\mathscr{H}$, of a Lie group of isometries of X, such that

$$U(g)\Psi = \Psi \qquad \text{and} \qquad U(g)\,\varphi(x)\,U(g^{-1}) = \varphi(gx). \tag{1.4}$$

These specifications of the field have their analogues in the Wightman scheme for fields in Minkowski space. However, there is an important difference between the cases where G includes time-translations, as when G is the proper Poincaré group and X is Minkowski space, and those when it is not. For, in the former cases, one may

[3] Thus, putting $u = w - t$ and $v = w + t$, the surface E corresponds to $v = 0$ and *on this surface* the transformations $u \to u + \text{constant}$, or equivalently, $t \to t + \text{constant}$ are isometries.

define the Hamiltonian H as $-i\hbar \times$ the generator of the one-parameter subgroup of $U(G)$ that corresponds to time-translations; and then one may assume the *spectrum condition* that H is positive, i.e., that Ψ is a ground state. This condition plays a crucial role in the theory of fields in Minkowski space, since it permits certain analytic continuations of the Wightman distributions $(\Psi, \varphi(x_1) \cdots \varphi(x_k)\Psi)$ and thereby leads to powerful general results, e.g., the PCT and Spin-Statistics Theorems [5, 6]. On the other hand, in the general case where time-translations are not isometries of X, one cannot similarly define a Hamiltonian and therefore one lacks a spectrum condition that implies an analytic continuation[4] of the distributions $(\Psi, \varphi(x_1) \cdots \varphi(x_k)\Psi)$. However, the situation is greatly improved by the fact, noted above, that the restriction of time-translations to the event horizon E are isometries. This fact presents us with the opportunity of introducing an axiom to the effect that φ induces a field φ_E on E, for which time-translations are unitarily represented in an appropriate subspace of $\mathscr{H}$ and the associated Hamiltonian is positive (spectrum condition!). In this way, we obtain a field theory on E that is essentially analogous to the Wightman theory. The introduction of certain dynamical postulates then enables us to extend the field theory from E to $X^{(\pm)}$ and to X.

The material of this article will be organised as follows. Section 2 is devoted to a brief summary of the algebraic formulation of quantum statistical mechanics, in a form that is directly applicable to our later requirements. This includes a specification of the KMS conditions in terms of a conjugation operator, as well as a discussion of their physical significance.

In Section 3, we shall formulate Wightman fields in flat space-time from the standpoint of uniformly accelerated observers. In view of Einstein's Principle of Equivalence, this is tantamount to formulating quantum fields subjected to gravitational forces, that correspond to uniform accelerations. We show that these forces induce an Unruh thermalisation effect. This result can be understood from the following simple argument. The Bisognano–Wichmann (BW) theorem [1b, Theorem 1] for an arbitrary Wightman field in Minkowski space, X, tells us that the restriction of the vacuum state Ψ to the Rindler wedge, $X^{(+)}$, satisfies the KMS condition, at a specified temperature, w.r.t. the Lorentz boosts $t \to t \cosh b + w \sinh b$, $w \to w \cosh b + t \sinh b$. Therefore, as restrictions of these boosts to $X^{(+)}$ correspond to time-translations, $\tau \to \tau + \text{constant}$, for uniformly accelerated observers, it follows easily that the BW theorem implies the Unruh effect. Furthermore, one sees from the explicit form of that theorem that the conjugation involved in the KMS condition, for the state relative to the accelerated observer, is just the PCT operator, corrected by a spatial rotation through π about the axis $0w$.

In Section 4, we shall specify our axioms for the field on X, for the general case when time-translations are not necessarily isometries of the manifold. As explained above, these axioms lead to a field φ_E, induced by φ on E, that carries the essential properties of a Wightman field, including a spectrum condition.

[4] In our previous note on the subject, we made an *ad hoc* assumption, no longer needed in our present treatment, of an analytic continuation of these distributions [15, condition (C.2)].

In Section 5, we prove analogues of the Reeh–Schlieder, PCT and BW theorems for φ_E by means of adaptations of the methods used to prove those original theorems for Wightman fields.

In Section 6, we derive the Unruh effect for the field in $X^{(+)}$, as a consequence of the results of the previous Section, supplemented by *either* an assumption concerning the stability of the state in $X^{(+)}$ *or* a dynamical one, signifying essentially that the field in $X^{(+)}$ is determined by that on E, i.e., that E behaves as a characteristic[5] surface for the dynamics in $X^{(+)}$. We then show that, if φ enjoys PCT symmetry w.r.t. the space-time inversion $(t, w; y) \to (-t, -w; y)$, and if E corresponds to a characteristic surface (in the sense indicated above) for the global dynamics of the field, then the PCT conjugation is just the one governing the KMS condition for the Hawking–Unruh effect in $X^{(+)}$.

In Section 7, we consider an alternative quantisation scheme, based on the replacement of axiom (A.5) of Section 4, which essentially postulates the regularity of the field at the event horizon E, by the hypothesis that the restriction of Ψ to $X^{(+)}$ is a ground state w.r.t. the Hamiltonian governing time-translations $\tau \to \tau +$ constant. We show that, in this case, the observables in the regions $X^{(+)}$ and $X^{(-)}$ are mutually uncorrelated. In the case of a Schwarzschild Black Hole, this would mean that the surface of the Schwarzschild sphere was not only an event horizon but also a physical barrier separating its interior and exterior regions.

In Section 8, we summarise our results and also argue that the choice between the quantisation scheme of Section 7 and that of Section 4 must generally be based on considerations of the circumstances governing the state of the system.

Throughout this article, we shall use the standard symbol $\mathbf{R}$, $\mathbf{R}_\pm$ and $\mathbf{C}$ to denote the real line, the positive and negative reals, and the complex numbers, respectively. The symbols $\mathscr{D}$ and $\mathscr{S}$ will denote the usual spaces of L. Schwartz, and $C_0(\mathbf{R})$ will denote the set of complex-valued, continuous functions on $\mathbf{R}$ that vanish outside some bounded region. We shall use the symbol $:=$ to signify equality by definition.

2. Quantum Statistical Preliminaries

We shall devote this Section to a summary of standard definitions and results, that serve to generalise quantum statistical mechanics to systems with possibly infinite numbers of degrees of freedom. For general expositions of the subject, see Refs. [3, 4], and also [23–25] for treatments of bounded observables.

Thus, we take the observables of a quantum system to be the self-adjoint elements of a *-algebra $\mathscr{A}$ of operators (i.e., an algebra closed w.r.t. the adjoint mapping $A \to A^*$) in a Hilbert space $\mathscr{H}$, with common dense domain $\mathscr{K}$ that is stable under $\mathscr{A}$. It should be noted that, in the case of an infinite system, this representation of the observables is generally inequivalent to the standard one of Fock [3, 4]. The states of

[5] We use the term "characteristic" rather than "Cauchy" here because E is a null surface (cf. Ref. [22, Theorem 3.2.1]).

the system are assumed to correspond to the density matrices in $\mathscr{H}$, the vector states being the particular ones for which the associated density matrices are one-dimensional projectors. The dynamics of the system is taken to be given by a strongly continuous unitary representation V of the additive reals, $\mathbf{R}$, in $\mathscr{H}$, such that $\mathscr{K}$ is stable under $V(\mathbf{R})$ and $\mathscr{A}$ is closed w.r.t. the transformations $A \to V(t)AV(-t)$. The Hamiltonian of the system is then $\hbar K$, where iK is the generator of $V(\mathbf{R})$.

Let Ψ be a unit vector in $\mathscr{H}$ that is invariant under $V(\mathbf{R})$ and cyclic w.r.t. $\mathscr{A}$, i.e. $\mathscr{A}\Psi$ is dense in $\mathscr{H}$. The invariance of Ψ under $V(R)$ implies that $K\Psi = 0$. Ψ is termed a *ground state*, or *zero temperature state*, if K is positive: it is termed a *thermal state*, for (non-zero) temperature $T = (k\beta)^{-1}$, if it satisfies the KMS condition, which may be expressed in the following form [18, 19, 26]:

$$(\exp(-\tfrac{1}{2}\beta\hbar K)A\Psi, \exp(-\tfrac{1}{2}\beta\hbar K)B\Psi) = (B^*\Psi, A^*\Psi), \qquad \forall A, B \in \mathscr{A}. \tag{2.1}$$

Equivalently, this condition may be written as

$$J\exp(-\tfrac{1}{2}\beta\hbar K)A\Psi = A^*\Psi, \qquad \forall A \in \mathscr{A}, \tag{2.2}$$

where J is a (unique) conjugation of $\mathscr{H}$, i.e., an antilinear transformation of $\mathscr{H}$ such that $J^2 = 1$ and $(Jf, Jg) \equiv (g, f)$. We note here that, once it is given that Ψ is a thermal state, Eq. (2.2) determines J and β uniquely [19], and furthermore a ground state can never satisfy that formula.

The ground and thermal states comprise the *equilibrium states*. We remark here that the characterisation of equilibrium according to the prescriptions given here is founded on a variety of stability arguments that serve to justify the original hypotheses of Gibbs [27, 28]. We also point out that even thermal states, which carry entropy, do indeed correspond to vectors in appropriate representation spaces [3, 4].

3. Fields in Minkowski Space

In this section, we take X to be Minkowski space (pointwise $x = (x^{(0)} = ct, x^{(1)}, x^{(2)}, x^{(3)})$) and G to be the proper Poincaré group of transformations of X. In particular, we define $T(\mathbf{R})$ and $L(\mathbf{R})$ to be the one-parameter subgroups of G, corresponding to time-translations and Lorentz boosts, respectively, according to the formulae

$$T(t)(x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}) = (x^{(0)} + ct, x^{(1)}, x^{(2)}, x^{(3)}) \tag{3.1}$$

and

$$L(\tau)(x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}) = (x^{(0)} \cosh\tau + x^{(1)} \sinh\tau, x^{(1)} \cosh\tau + x^{(0)} \sinh\tau, x^{(2)}, x^{(3)}). \tag{3.2}$$

We define $X^{(\pm)}$ to be the open submanifolds of X given by $x^{(1)} > |x^{(0)}|$ and $x^{(1)} < -|x^{(0)}|$, respectively. Thus $X^{(\pm)}$ are both stable under $L(\mathbf{R})$.

We formulate quantum fields in X according to Wightman's scheme [5, 6]. For simplicity, we confine ourselves here to real scalar fields, though an analogous treatment of other fields would lead to the same results (cf. Comment (4) at the end of this section). Thus, we represent a field in X by a quadruple $(\mathscr{H}, \varphi, \Psi, U)$, where $\mathscr{H}$ is a Hilbert space, φ a linear mapping[6] of the Schwartz space $\mathscr{S}(X)$ into the operators in $\mathscr{H}$, Ψ a unit vector in $\mathscr{H}$ that is cyclic w.r.t. the algebra $\mathscr{A}$ of polynomials in $\{\varphi(f) \mid f \in \mathscr{S}(X)\}$, and U a strongly continuous representation of G in $\mathscr{H}$, such that the following axioms[7] are fulfilled.

(W.1) $\varphi(f_1) \cdots \varphi(f_k)\Psi$ is strongly continuous w.r.t. each of the elements $f_1, ..., f_k$ of $\mathscr{S}(X)$.

(W.2) $\varphi(f)^* = \varphi(\bar{f})$ on $\mathscr{A}\Psi$ (Hermiticity!).

(W.3) $U(g)\Psi = \Psi$ and $U(g)\varphi(f)U(g^{-1}) = \varphi(f_g)$, with $f_g := f(g^{-1}x)$, $\forall f \in \mathscr{S}(X)$, $g \in G$ (*covariance*!).

(W.4) $\varphi(f_1)$ and $\varphi(f_2)$ commute on $\mathscr{A}\Psi$ if the supports of f_1, f_2 have space-like separation (locality!).

(W.5) Ψ is a ground state w.r.t. the time-translational group $\hat{T}(\mathbf{R}) := U(T(\mathbf{R}))$; i.e., the infinitessimal generator of $\hat{T}(\mathbf{R})$ is $i \times a$ positive operator (spectrum condition!).

The following classic theorems are consequences of axioms (W.1–W.5).

PCT Theorem [5, 6]. *There is a unique conjugation J_0 of $\mathscr{H}$ such that*

$$J_0\varphi(f_1) \cdots \varphi(f_k)\Psi = \varphi(f_k^\dagger) \cdots \varphi(f_1^\dagger)\Psi,$$
$$\forall f_1, ..., f_k \in \mathscr{S}(X),$$

where

$$f^\dagger(x) := \bar{f}(-x).$$

Reeh–Schlieder Theorem [5, 6, 29]. *Let Λ be a arbitrary open subset of X, and let $\mathscr{A}(\Lambda)$ be the subalgebra of $\mathscr{A}$ given by the polynomials in $\{\varphi(f) \mid f \in \mathscr{S}(X);$ $\operatorname{supp} f \subset \Lambda\}$. Then $\mathscr{A}(\Lambda)\Psi$ is dense in $\mathscr{H}$.*

Bisognano–Wichmann Theorem [1a, b Theorem 1]. *Let iK be the infinitessimal generator of $\hat{L}(\mathbf{R}) := U(L(\mathbf{R}))$. Then*

$$J_0\hat{\rho}e^{-\pi K}A\Psi = A^*\Psi, \qquad \forall A \in \mathscr{A}(X^{(+)}),$$

[6] Here $\varphi(f)$ corresponds to what may heuristically be regarded as a pointwise-defined field, $\varphi(x)$, smoothed out against f, i.e., $\varphi(f) = \int dx\, \varphi(x) f(x)$.

[7] We omit the axioms of clustering and of uniqueness of the vacuum, since they will not be needed here.

where J_0 is the PCT *conjugation, $\hat{\rho} = U(\rho)$ and ρ is the partial inversion $x^{(2)} \to -x^{(2)}$, $x^{(3)} \to -x^{(3)}$.*

We now seek to characterise the state Ψ from the standpoint of a uniformly accelerated observer, O_{acc}. For this purpose, we represent $X^{(+)}$ in Rindler coordinates $(\xi, \tau, x^{(2)}, x^{(3)})$, where ξ, τ run through $\mathbf{R}_+$ and $\mathbf{R}$, respectively, and

$$x^{(0)} = \xi \sinh \tau; \qquad x^{(1)} = \xi \cosh \tau. \tag{3.3}$$

The Minkowski metric for $X^{(+)}$, when expressed in these coordinates, takes the form

$$ds^2 = \xi^2 d\tau^2 - d\xi^2 - (dx^{(2)})^2 - (dx^{(3)})^2. \tag{3.4}$$

Thus, τ is a temporal coordinate. Moreover, the curves on which ξ, $x^{(2)}$, $x^{(3)}$ are constant correspond to trajectories of uniform acceleration α along $Ox^{(1)}$ [20, Chap. 6], with

$$\alpha = c^2/\xi. \tag{3.5}$$

By (3.4) and (3.5), the proper time τ_α on such a trajectory is given by

$$\tau_\alpha = \xi\tau/c = c\tau/\alpha. \tag{3.6}$$

The asymptotes to the trajectory lie in the boundaries, $x^{(1)} \pm x^{(0)} = 0$, of $X^{(+)}$, which are the past and future event horizons, respectively, for an observer O_{acc} that accelerates uniformly in the $Ox^{(1)}$ direction.

Let $L^{(+)}(\mathbf{R})$ be the one-parameter group of isometries of $X^{(+)}$ corresponding to time-translations for O_{acc}, i.e.,

$$L^{(+)}(\tau)(\xi, \tau', x^{(2)}, x^{(3)}) = (\xi, \tau' + \tau, x^{(2)}, x^{(3)}). \tag{3.7}$$

Thus, by (3.2), (3.3) and (3.7), $L^{(+)}(\tau)$ is the restriction of $L(\tau)$ to $X^{(+)}$. On combining this observation with the Reeh–Schlieder and BW theorems, together with the definition of thermal states in Section 2, we immediately obtain the following result.

PROPOSITION 1. *The restriction of the state Ψ to $\mathscr{A}(X^{(+)})$ is a thermal one satisfying the following* KMS *condition w.r.t. the time-translation group $\hat{L}^{(+)}(\mathbf{R})$ $(:=U(L(\mathbf{R}))$ for O_{acc}.*

$$J \exp(-\pi K^{(+)}) A \Psi = A^* \Psi, \qquad \forall A \in \mathscr{A}(X^{(+)}), \tag{3.8}$$

where $iK^{(+)}$ is the generator of $\hat{L}^{(+)}(\mathbf{R})$,

$$J = J_0 \hat{\rho},$$

J_0 is the PCT *operator, and $\hat{\rho}$ the unitary representative of the partial inversion $x^{(2)} \to -x^{(2)}$, $x^{(3)} \to -x^{(3)}$.*

Comments. (1) This Proposition signifies that Ψ, which is a ground state for an inertial observer (axiom (W.5)), corresponds to a thermal one for O_{acc}. Further, on comparing Eqs. (2.2) and (3.8), one sees that the temperature of the state is $\hbar/2\pi k$. This is not the observed temperature, however, since it is based on the time τ, rather than the proper time τ_α. In fact, the temperature observed by O_{acc} will take the value T_α, corresponding to a rescaling of time from τ to τ_α, i.e., by (3.6)

$$T_\alpha = \hbar\alpha/2\pi kc. \tag{3.9}$$

This is a rigorous generalisation of Unruh's result [11] for fields in flat space-time.

(2) In view of Einstein's Principle of Equivalence, one can interpret this result as signifying that a gravitational field, corresponding to a uniform acceleration α in flat space-time, thermalises the field to a temperature $\hbar\alpha/2\pi kc$.

(3) The conjugation J, arising in the KMS condition for the field observed by O_{acc}, is simply the PCT operator J_0, corrected by the partial inversion $\hat{\rho}$. Thus J is the PCT conjugation associated with the restricted inversion $x^{(0)} \to -x^{(0)}$, $x^{(1)} \to -x^{(1)}$.

(4) The results discussed here stem directly from the PCT, RS and BW theorems and therefore may be generalised, like those theorems (cf. [5, 6, 1b]) to arbitrary quantum fields in Minkowski space.

4. Fields in Curved Space-Time

Let X ($=\mathbf{R}^2 \times Y$) be a manifold which, together with its submanifolds $X^{(\pm)}$ and E, conforms to the specifications of Section 1 (cf. Eqs. (1.1)–(1.3)). We define $L(\mathbf{R})$, $L^{(\pm)}(\mathbf{R})$ to be the one-parameter groups of isometries of X, $X^{(\pm)}$, corresponding to Lorentz-like boosts and Rindler-like time translations, respectively, according to the formula

$$L(\tau)(t, w; y) = (t \cosh \tau + w \sinh \tau, w \cosh \tau + t \sinh \tau; y) \tag{4.1}$$

and

$$L^{(\pm)}(\tau)(\xi, \tau'; y) = (\xi, \tau + \tau'; y). \tag{4.2}$$

Thus, by (1.2), (4.1) and (4.2), $L^{(\pm)}(\tau)$ is the restriction of $L(\tau)$ to $X^{(\pm)}$. We note again that, in view of (1.1), the time-translations, $t \to t +$ constant, are not generally isometries of X.

Since the event horizon E is the submanifold of X on which $t + w = 0$, it follows that $E = \mathbf{R} \times Y$ and that the points of E are given by the coordinates (t, y). By (1.1) and (4.1), E is stable under $L(\mathbf{R})$, and the restriction $L_E(\tau)$, of $L(\tau)$ to E, is given by the formula

$$L_E(\tau)(t, y) = (te^{-\tau}, y). \tag{4.3}$$

Since, by (1.1), $\partial/\partial t$ is a Killing vector on E, this manifold is also equipped with a one-parameter group of isometries, $T_E(\mathbf{R})$, coresponding to time-translations:

$$T_E(t)(t', y) = (t + t', y). \tag{4.4}$$

Let $E^{(\pm)}$ be the subsets of E in which $t \lessgtr 0$, i.e., the regions of E lying on the boundaries of $X^{(\pm)}$, respectively. Then it follows from (4.3) and (4.4) that $E^{(\pm)}$ are both stable under $L_E(\mathbf{R})$, but not under $T_E(\mathbf{R})$.

We formulate a quantum field on X by a natural adaptation of the Wightman scheme. For simplicity, we again confine the analysis to a real scalar field, but in fact our main results may be generalised to arbitrary fields in X (cf. remarks at the end of Section 8). Thus, we take a quantum field on X to correspond to a quintuple $(\mathscr{F}, \mathscr{H}, \varphi, \Psi, \hat{L})$, where $\mathscr{F}$ is the space $\mathscr{S}(\mathbf{R}^2) \otimes \mathscr{D}(Y)$ of test-functions on X $(=\mathbf{R}^2 \times Y)$, where $\mathscr{S}$ and $\mathscr{D}$ are the Schwartz spaces; $\mathscr{H}$ is a Hilbert space; φ is a linear mapping of $\mathscr{F}$ into the operators in $\mathscr{H}$; Ψ is a unit vector in $\mathscr{H}$ which is cyclic w.r.t. the algebra $\mathscr{A}$ of polynomials in $\{\varphi(F) \mid F \in \mathscr{F}\}$; and $\hat{L}(\mathbf{R})$ is a strongly continuous unitary representation of $\mathbf{R}$ in $\mathscr{H}$, such that the following axioms are satisfied.

(A.1) $\varphi(f_1 \otimes g_1) \cdots \varphi(f_k \otimes g_k)\Psi$ is strongly continuous w.r.t. $f_1,\ldots, f_k \in \mathscr{S}(\mathbf{R}^2)$ and $g_1,\ldots, g_k \in \mathscr{D}(Y)$; and hence, by the nuclearity of the Schwartz spaces, there is a unique sequence of continuous linear functionals $\{W^{(k)} \mid k \in N\}$ on $\{\mathscr{S}(\mathbf{R}^{2k}) \otimes \mathscr{D}(Y^k)\}$, define by the formula

$$\begin{aligned} &W^{(k)}(f_1 \otimes f_2 \cdots \otimes f_k; g_1 \otimes g_2 \cdots \otimes g_k) \\ &\quad = (\Psi, \varphi(f_1 \otimes g_1) \cdots \varphi(f_k \otimes g_k)\Psi)\ \forall f_1,\ldots, f_k \in \mathscr{S}(\mathbf{R}^2),\ g_1,\ldots, g_k \in \mathscr{D}(Y). \end{aligned} \tag{4.5}$$

(A.2) $\varphi(F)^* = \varphi(\bar{F})$ on $\mathscr{A}\Psi$ (Hermiticity!).

(A.3) $\hat{L}(\tau)\Psi = \Psi$ and $\hat{L}(\tau)\varphi(F)\hat{L}(-\tau) = \varphi(F_\tau)$, (4.6)

with

$$F_\tau(x) := F(L(-\tau)x)(\text{covariance w.r.t. } L(R)). \tag{4.7}$$

(A.4) $\varphi(F_1)$ and $\varphi(F_2)$ intercommute on $\mathscr{A}\Psi$ if the supports of F_1, F_2 have space-like separation (locality!).

These axioms have already been proposed, and discussed in some detail for linear fields on globally hyperbolic manifolds, by Isham [31].

In general, we cannot supplement (A.1)–(A.4) by a spectrum condition analogous to (W.5): for, unless the time-translations, $t \to t +$ constant, are isometries of X, there is no possibility of even defining a unitary representation of a time-translation group and thereby obtaining a Hamiltonian. In order to compensate for this deficiency, we introduce the following axiom, designed to yield a field on the submanifold E, which does have a time-translational isometry group, $T_E(R)$.

(A.5) Let $\{h_{1,n}\},\ldots, \{h_{k,n}\}$ be k arbitrary sequences of positive $\mathscr{D}(\mathbf{R})$-class functions such that $\int dt h_{j,n}(t) = 1$ and $\mathrm{supp}\, h_{j,n} \to \{0\}$ as $n \to \infty$; and for $F_j \in \mathscr{F}_E := \mathscr{S}(\mathbf{R}) \otimes \mathscr{D}(Y)$, let $\tilde{F}_{j,n}$ be the element of $\mathscr{F}$ defined by

$$\tilde{F}_{j,n}(t, w; y) = F_i^{(1)}(\tfrac{1}{2}(t - w); y)\, h_{j,n}(t + w), \tag{4.8}$$

where

$$F_j^{(1)}(t, y) = \frac{\partial}{\partial t} F_j(t; y). \tag{4.9}$$

Then $\varphi(\tilde{F}_{1,n}) \cdots \varphi(\tilde{F}_{k,n}) \Psi$ converges strongly, as $n \to \infty$, to a multilinear vector function $\Phi_E(F_1, \ldots, F_k)$, such that

(a) $\Phi_E(e_1 \otimes g_1, \ldots, e_k \otimes g_k)$ is strongly continuous w.r.t. $e_1, \ldots, e_k \in \mathscr{S}(\mathbf{R})$ and $g_1, \ldots, g_k \in \mathscr{D}(Y)$; and

(b) there exist $F_1, \ldots, F_k \in \mathscr{F}_E$ such that $\Phi_E(F_1, \ldots, F_j, F_{j+1,\tau}, \ldots, F_{k,\tau})$ is not constant w.r.t. τ where

$$F_{l,\tau}(x_E) := F_l(L_E(-\tau) x_E), \qquad \forall F \in \mathscr{F}_E,\ x_E \in E. \tag{4.10}$$

Comments. (1) This axiom essentially defines a sense in which φ induces a field φ_E on E, in accordance with the formal prescription $\varphi_E(t, y) = -(\partial/\partial t)\, \varphi(t, -t; y)$ (cf. Proposition 2, below). Here the derivative w.r.t. t stems from the presence of $F_j^{(1)}$ in (4.8). $F_j^{(1)}$, rather than F_j, was used in this latter equation so as render it invariant under the isometries $x \to L(\tau)x$, $x_E \to L_E(\tau)x_E$ of X and E, respectively (cf. proof of (E.3) in Proposition 2, below).

(2) In view of the difficulty of constructing a non-trivial field in a curved space-time, for which (A.5) could be checked, it is perhaps worth mentioning that this assumption can be confirmed by direct calculation for a free massive field in two-dimensional Minkowski space.

PROPOSITION 2. *Assuming* (A.2–A.5), *there is a field on E, specified by the quintuple* $(\mathscr{F}_E, \mathscr{H}_E, \varphi_E, \Psi, \hat{L}_E)$, *where* $\mathscr{F}_E = \mathscr{S}(\mathbf{R}) \otimes \mathscr{D}(Y)$, $\mathscr{H}_E(\ni \Psi)$ *is a subspace of* $\mathscr{H}$, φ_E *is a linear mapping of* $\mathscr{F}_E$ *into the operators in* $\mathscr{H}_E$, Ψ *is cyclic w.r.t. the algebra* $\mathscr{A}_E$ *of polynomials in* $\{\varphi_E(F) \mid F \in \mathscr{F}_E\}$, *and* $\hat{L}_E(\mathbf{R})$ *is a strongly continuous, non-trivial unitary representation of* $\mathbf{R}$ *in* $\mathscr{H}_E$, *such that*

$$\Phi_E(F_1, \ldots, F_k) = \varphi_E(F_1) \cdots \varphi_E(F_k) \Psi, \qquad \forall F_1, \ldots, F_k \in \mathscr{F}_E, \tag{4.11}$$

and the following conditions are fulfilled.

(E.1) $\varphi_E(e_1 \otimes g_1) \cdots \varphi_E(e_k \otimes g_k) \Psi$ *is strongly continuous w.r.t.* $e_1, \ldots, e_k \in \mathscr{S}(\mathbf{R})$ *and* $g_1, \ldots, g_k \in \mathscr{D}(Y)$.

(E.2) $\varphi_E(F)^* = \varphi_E(\bar{F})$ *on* $\mathscr{A}_E \Psi$.

(E.3) $\hat{L}(\tau)$ *is the restriction to* $\mathscr{H}_E$ *of* $\hat{L}(\tau)$, *and further*

$$\hat{L}_E(\tau)\, \varphi_E(F)\, \hat{L}_E(-\tau) = \varphi_E(F_\tau), \qquad \forall F \in \mathscr{F}_E. \tag{4.12}$$

(E.4) *$\varphi_E(F_1)$ and $\varphi_E(F_2)$ intercommute on $\mathscr{A}_E \Psi$ if the supports of F_1, F_2 have space-like separation.*

(E.4)′ *$\varphi_E(e_1 \otimes g_1)$ and $\varphi_E(e_2 \otimes g_2)$ intercommute on $\mathscr{A}_E \Psi$ if the supports of e_1 and e_2 are disjoint (further locality condition!).*

Before proving this Proposition, we supplement (E.1)–(E.4)′ by the following two assumptions, that are made feasible by the fact that $\partial/\partial t$ is a Killing vector on E and thus $T_E(R)$ is an isometry group for that manifold.

(E.5) There is a strongly continuous unitary representation, $\hat{T}_E$, of $\mathbf{R}$ in $\mathscr{H}_E$ such that

$$\hat{T}_E(t)\,\Psi = \Psi \qquad \text{and} \qquad \hat{T}_E(t)\,\varphi_E(F)\,\hat{T}_E(-t) = \varphi_E(F^{(t)}) \tag{4.13}$$

with

$$F^{(t)}(x_E) := F(T_E(-t)x_E) \qquad \text{(covariance w.r.t. } T_E(\mathbf{R}))!). \tag{4.14}$$

(E.6) The self-adjoint operator K_E, in $\mathscr{H}_E$, given by $(-i) \times$ generator of $\hat{T}_E(\mathbf{R})$, is positive (spectrum condition!).

Note. In the special case where X is Minkowski space, (E.5) and (E.6) follow from (A.5) and the Wightman axioms, together with a mild continuity assumption. In this case, K_E is the restriction to $\mathscr{H}_E$ of $\hbar^{-1}(H + cP^{(1)})$, where H is the Hamiltonian and $P^{(1)}$ the momentum along $Ox^{(1)}$.

Proof of Proposition 2. Let $\mathscr{K}_E$ be the linear span of the vectors $\Phi_E(F_1,..., F_k)$, and let $\mathscr{H}_E$ be the closure of $\mathscr{K}_E$. Since Φ_E is linear in each of its arguments, we may define a linear mapping φ_E of $\mathscr{F}_E$ into the operators in $\mathscr{H}_E$, with common domain $\mathscr{K}_E$, according to the formula

$$\varphi_E(F_1)\,\Phi_E(F_2,..., F_k) = \Phi_E(F_1,..., F_k); \qquad \varphi_E(F_1)\,\Psi = \Phi_E(F_1). \tag{4.15}$$

Hence (4.10) is satisfied, $\mathscr{A}_E \Psi (\equiv \mathscr{K}_E)$ is dense in $\mathscr{H}_E$, and (E.1) follows immediately from (A.5) and (4.10).

By (A.5) and (4.10),

$$\begin{aligned}(\varphi_E(F_1) \cdots \varphi_E(F_k)\,\Psi, \varphi_E(F)\,\varphi_E(F_{k+1}) \cdots \varphi_E(F_{k+l})\,\Psi)\\ = \lim_{n\to\infty} (\varphi(\tilde{F}_{1,n}) \cdots \varphi(\tilde{F}_{k,n})\,\Psi, \varphi(\tilde{F}_n)\,\varphi(\tilde{F}_{k+1,n}) \cdots \varphi(\tilde{F}_{k+l,n})\,\Psi)\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}[\varphi_E(F_1), \varphi_E(F_2)]\,\varphi_E(F_3) \cdots \varphi_E(F_k)\,\Psi\\ = s\text{-}\lim_{n\to\infty} [\varphi(\tilde{F}_{1,n}), \varphi(\tilde{F}_{2,n})]\,\varphi(\tilde{F}_{3,n}) \cdots \varphi(\tilde{F}_{k,n})\,\Psi.\end{aligned} \tag{4.17}$$

Condition (E.2) follows easily from (A.2) and (4.16). (E.4), on the other hand, follows from (A.4) and (4.17), together with the fact that, since $\operatorname{supp} h_{j,n} \to \{0\}$ as $n \to \infty$, Eq. (4.8) signifies that the space-like separation of the supports of F_1 and F_2 ensures the same for those of $F_{1,n}$ and $F_{2,n}$ for large enough n.

By (1.1), the points $(t_1, w_1; y_1)$ and $(t_2, w_2; y_2)$ have space-like separation if $|w_1 - w_2| > |t_1 - t_2|$. Hence, as $w = -t$ on E, one may easily infer from (4.8) that, if the supports of e_1 and e_2 are disjoint, then $h_{1,n}$ and $h_{2,n}$ may be chosen so that, for $F_1 = e_1 \otimes g_1$ and $F_2 = e_2 \otimes g_2$, the supports of $\tilde{F}_{1,n}$ and $\tilde{F}_{2,n}$ have space-like separation for every finite n. Hence, (A.4) and (4.17) imply (E.4)'.

In order to establish (E.3), we note that, by (1.2), (4.7), (4.9) and (4.10),

$$\tilde{F}_{j,n,\tau}(t, w; y) = (F_{j,\tau})^{(1)}(\tfrac{1}{2}(t - w); y)\, \tilde{h}_{n,\tau}(t + w), \tag{4.18}$$

where

$$\tilde{h}_{n,\tau}(t) := e^{\tau} h_n(te^{\tau}). \tag{4.19}$$

Therefore, for each $\tau \in \mathbf{R}$, the sequence $\{\tilde{h}_{n,\tau}\}$ has all the properties required of h_n in (A.5), and thus (4.16) implies that we may replace F_j, $\tilde{F}_{j,n}$ by $F_{j,\tau}$, $\tilde{F}_{j,n,\tau}$, respectively, in (A.5) and (4.10). Hence, using (A.3),

$$\begin{aligned} \varphi_E(F_{1,\tau}) \cdots \varphi_E(F_{k,\tau})\,\Psi &= \operatorname*{s\text{-}lim}_{n \to \infty} \hat{L}(\tau)\, \varphi(\tilde{F}_{1,n}) \cdots \varphi(\tilde{F}_{k,n})\,\Psi \\ &= \hat{L}(\tau)\, \varphi_E(F_1) \cdots \varphi_E(F_k)\,\Psi. \end{aligned}$$

Consequently, as $\mathscr{A}_E \Psi$ is dense in $\mathscr{H}_E$, this latter subspace is stable under $\hat{L}(R)$, and the restriction $\hat{L}_E(\mathbf{R})$ of $\hat{L}(\mathbf{R})$ to $\mathscr{H}_E$ satisfies (E.3). Finally, it follows from (4.11) and condition (b) of (A.5) that $\hat{L}_E(\mathbf{R})$ does not reduce to the identity, i.e. this group is non-trivial. ∎

5. Fields on E

In this section, we shall assume the properties (E.1–E.6) for φ_E and thence deduce the following three theorems, which are analogues of the Reeh–Schlieder, PCT and Bisognano–Wichmann theorems, in that order.

THEOREM 3. *Let I be an open interval in* $\mathbf{R}$, *and let $\mathscr{A}_E(I)$ be the algebra of polynomials in $\{\varphi_E(e \otimes g) \mid e \in \mathscr{S}(\mathbf{R}),\ \operatorname{supp} e \subset I;\ g \in \mathscr{D}(Y)\}$. Then $\mathscr{A}_E(I)\,\Psi$ is dense in $\mathscr{H}_E$.*

THEOREM 4. *There is a (unique) conjugation J_E of $\mathscr{H}_E$ such that*

$$J_E \varphi_E(F_1) \cdots \varphi_E(F_k)\,\Psi = \varphi_E(F_k^{\dagger}) \cdots \varphi_E(F_1^{\dagger})\,\Psi, \quad \forall F_1, \dots, F_k \in \mathscr{F}_E, \tag{5.1}$$

where

$$F^{\dagger}(t, y) := \bar{F}(-t, y). \tag{5.2}$$

THEOREM 5. *Let* $\mathscr{A}_E^{(\pm)}$ *be the algebra of polynomials in* $\{\varphi_E(F) F \in \mathscr{F}_E;$ $\operatorname{supp} F \subset E^{(\pm)}\}$. *Then* Ψ *is cyclic w.r.t.* $\mathscr{A}_E^{(\pm)}$ *in* $\mathscr{H}_E$, *and*

$$J_E \exp(\mp \pi K_E) A_E \Psi = A_E^* \Psi, \qquad \forall A_E \in \mathscr{A}_E^{(\pm)}. \tag{5.3}$$

Comments. (1) Since $E^{(+)}$ is stable under $L_E(R)$ (by (4.3)), it follows from Theorem 5 that Ψ satisfies the KMS condition w.r.t. $\hat{L}_E(R)$ for the restriction of φ_E to $E^{(+)}$, at temperature $\hbar/2\pi k$.

(2) The conjugation J_E appearing in this condition is just the PCT operator of Theorem 4, without any correction corresponding to a spatial inversion for Y, since this does not arise here.

The proofs of these theorems follow similar lines to the corresponding ones for fields in Minkowski space. They are based on a treatment of the continuous Wightman-like functionals $W_E^{(k)}$, on $\mathscr{S}(\mathbf{R}^k) \otimes \mathscr{D}(Y^k)$, which may be defined, in view of (E.1) and the nuclearity of the Schwartz spaces, by the formula

$$\begin{aligned} &W_E^{(k)}(e_1 \otimes e_2 \cdots \otimes e_k; g_1 \otimes g_2 \cdots \otimes g_k) \\ &\quad = (\Psi, \varphi_E(e_1 \otimes g_1) \cdots \varphi_E(e_k \otimes g_k) \Psi), \qquad \forall e_1, \dots, e_k \in \mathscr{S}(\mathbf{R}); g_1, \dots, g_k \in \mathscr{D}(Y). \end{aligned} \tag{5.4}$$

For fixed $g \in \mathscr{D}(Y^k)$, we define $W_{E,g}^{(k)}$ to be the tempered distribution given by

$$W_{E,g}^{(k)}(e) = W_E^{(k)}(g; e), \qquad \forall e \in \mathscr{S}(\mathbf{R}^k). \tag{5.5}$$

It follows by standard arguments [5, 6] from the $\hat{T}_E(R)$ invariance of Ψ that $W_{E,g}^{(k)}$ corresponds to a tempered distribution $\mathscr{W}_{E,g}^{(k-1)}$ $(\in \mathscr{S}'(\mathbf{R}^{k-1}))$ according to the formula

$$\mathscr{W}_{E,g}^{(k-1)}(t_1, \dots, t_{k-1}) = W_{E,g}^{(k)}(t_1 + t_2 \cdots + t_k, t_2 + \cdots + t_k, \dots, t_k). \tag{5.6}$$

Furthermore, the spectrum condition (E.6) implies that $\mathscr{W}_{E,g}^{(k-1)}$ is the boundary value of a function analytic in the tube $T_{k-1} := \{(z_1, \dots, z_{k-1}) \in \mathbf{C}^{k-1} \mid \operatorname{Im} z_1, \operatorname{Im} z_2, \dots, \operatorname{Im} z_{k-1} < 0\}$, i.e.,

$$\mathscr{W}_{E,g}^{(k-1)}(e) = \lim_{\lambda_1, \dots, \lambda_{k-1} \downarrow 0} \int dt_1 \cdots dt_{k-1} \mathscr{W}_{E,g}^{(k-1)}(t_1 + i\lambda_1, \dots, t_{k-1} + i\lambda_{k-1})\, e(t_1, \dots, t_{k-1}),$$

$$\forall e \in \mathscr{S}(\mathbf{R}^{k-1}). \tag{5.7}$$

Likewise, for $\tilde{\Psi} \in \mathscr{H}_E$ and $g_1, \dots, g_k \in \mathscr{D}(Y)$, we define the tempered distribution $\tilde{W}_{E,g}^{(k)} \in \mathscr{S}'(\mathbf{R}^k)$ by the formula

$$\tilde{W}_{E,g}^{(k)}(e_1 \otimes e_2 \cdots \otimes e_k) = (\tilde{\Psi}, \varphi_E(e_1 \otimes g_1) \cdots \varphi_E(e_k \otimes g_k) \Psi), \tag{5.8}$$

and note that $\tilde{W}^{(k)}_{E,g}$ is the boundary value of a function that is analytic in $\tilde{T}_k := \{(z_1,..., z_k) \in \mathbf{C}^k \mid (-z_1, z_1 - z_2,..., z_{k-1} - z_k) \in T_k\}$.

Proof of Theorem 3. In view of Eq. (5.8), it suffices to prove that, if the restriction of $\tilde{W}^{(k)}_{E,g}$ to $\mathscr{S}(I^k)$ is zero for all $g_1,..., g_k \in \mathscr{D}(Y)$, then $\tilde{\Psi} = 0$. In fact, this result may be inferred, using precisely the method of proof of the Reeh–Schlieder theorem [5, 6, 29], from the above-noted fact that $\tilde{W}^{(k)}_{E,g}$ is the boundary value of a function that is analytic in $\tilde{T}_k$. ∎

In preparation for the proofs of Theorems 4 and 5, we note that, by (4.3),

$$L_E(\tau)(t, y) = (\mathscr{L}(\tau)t, y), \tag{5.9}$$

where $\mathscr{L}(\mathbf{R})$ is the one-parameter group of transformations of $\mathbf{R}$ given by

$$\mathscr{L}(\tau)t = te^{-\tau}. \tag{5.10}$$

We define $\mathscr{L}(\mathbf{C})$ to be the group of transformations of $\mathbf{C}$, corresponding to the complexification of $\mathscr{L}(\mathbf{R})$, i.e.,

$$\mathscr{L}(z)z' = z'e^{-z}. \tag{5.11}$$

We then define the extended tube $T'_{k-1} := \{(Az_1,..., Az_{k-1}) \mid A \in \mathscr{L}(\mathbf{C}), (z_1,..., z_{k-1}) \in T_{k-1}\}$, and we define $\mathscr{T}_{k-1}$ to be the subset of T'_{k-1} consisting of elements $(r_1,..., r_{k-1})$ for which all the r_j's are real. Thus $\mathscr{T}_{k-1}$ corresponds to the set of Jost points of Wightman theory, and one can readily show that $\mathscr{T}_{k-1} = R^{k-1}_+ \cup R^{k-1}_-$. For $(t_1,..., t_k) \in \mathbf{R}^k$, with $(t_1 - t_2,..., t_{k-1} - t_k) \in \mathscr{T}_{k-1}$, we define

$$W^{(k)}_E(t_1,..., t_k; g_1,..., g_k) = \mathscr{W}^{(k-1)}_{E,g}(t_1 - t_2,..., t_{k-1} - t_k), \quad \text{with} \quad g = g_2 \otimes \cdots \otimes g_k. \tag{5.12}$$

Thus $W^{(k)}_E$ is the function obtained by unsmearing the distribution, denoted by the same symbol (cf. (5.4)) w.r.t. the test functions e. Hence it follows from (E.4)′ and the definition of $\mathscr{T}_{k-1}$ that $W^{(k)}_E$ is invariant under permutations $(t_i, g_i) \rightleftharpoons (t_j, g_j)$.

We shall require the following lemma, which corresponds to that of Ref. [5, p. 66].

LEMMA 6. *If* $(z_1,..., z_{k-1})$ *and* $Az_1,..., Az_{k-1})$ *both belong to* T_{k-1}, *with* $A \in \mathscr{L}(\mathbf{C})$, *then there exists a continuous mapping* Λ *from* $[0, 1]$ *into* $\mathscr{L}(\mathbf{C})$ *such that* $\Lambda(0) = I$, $\Lambda(1) = A$ *and* $(\Lambda(t)z_1,..., \Lambda(t)z_{k-1}) \in T_{k-1}\ \forall t \in [0, 1]$.

Proof. By (5.11), $\mathscr{L}(\mathbf{C})$ may be identified with the multiplicative group $\mathbf{C}\backslash\{0\}$. Hence it follows from the definition of T_{k-1} that, if $(z_1,..., z_{k-1})$ and $(Az_1,..., Az_{k-1})$ both belong to T_{k-1}, then $A(\in\mathbf{C}\backslash\{0\})$ does not lie in R_-. Hence, it follows that $\Lambda(t) := t + (1-t)A$ satisfies all the requirements of the lemma, since, on the one hand, the convexity of T_{k-1} ensures that, if $(z_1,..., z_{k-1})$ and $(Az_1,..., Az_{k-1})$ lie in T_{k-1}, then so does $(\Lambda(t)z_1,..., \Lambda(t)z_k)$ for all $t \in [0, 1]$; while, on the other hand,

$\Lambda(t) \in \mathbf{C}\backslash\{0\}$ $\forall t \in [0, 1]$, since the fact that $\Lambda \notin \mathbf{R}_-$ precludes the possibility that $\Lambda(t) = 0$ for some t in this interval. ∎

Proof of Theorem 4. This is analogous to the proof of the PCT theorem for fields in Minkowski space.

Thus we start by noting that, in view of Lemma 6, the following analogue of the Bargmann–Hall–Wightman theorem may be obtained, by means of the argument of Refs. 5, 6, 30]: *$\mathscr{W}_{E,g}^{(k-1)}$ has an analytic continuation to T'_{k-1}, and this is invariant under $(z_1,..., z_{k-1}) \rightarrow (\Lambda z_1,..., \Lambda z_{k-1})$.*

Hence, by (5.12), if $(t_1,..., t_k) \in \mathbf{R}^k$ and $(t_1 - t_2,..., t_{k-1} - t_k) \in \mathscr{T}_{k-1}$, then

$$W_E^{(k)}(t_1,..., t_k; g_1,..., g_k) = W_E^{(k)}(-t_1,..., -t_k; g_1,..., g_k).$$

Further, as noted after Eq. (5.12), the L.H.S. of this equation is invariant under permutations $(t_i, g_i) \rightleftharpoons (t_j, g_j)$ and is therefore also equal to $W_E^{(k)}(t_k,..., t_1; g_k,..., g_1)$, and thus

$$W_E^{(k)}(t_k,..., t_1 : g_k,..., g_1) = W_E^{(k)}(-t_1,..., -t_k; g_1,..., g_k), \tag{5.13}$$

which corresponds to the Weak Local Commutativity (WLC) condition for Wightman fields.

The required result now follows by direct analogy with the method used [5, 6] to infer PCT from WLC for fields in Minkowski space. ∎

The following lemma, which will be needed for the proof of Theorem 5, is analogous to Ref. [1a, Lemma 10].

LEMMA 7. *Let $\{I_n\}$ be the sequence of non-overlapping open intervals in $\mathbf{R}$ given by $I_n = (b + (n-1)a,\ c + (n-1)a)$, where a, b, c are chosen so that $0 < b < c < a$. Then*

$$e^{\pi K_E} c(K_E)\, \varphi_E(F_1) \cdots \varphi_E(F_k)\, \Psi = c(K_E)\, \varphi_E(F_1^{(-)}) \cdots \varphi_E(F_k^{(-)})\, \Psi,$$
$$\forall c \in C_0(\mathbf{R});\ \operatorname{supp} F_j \in I_j \times Y;\ j = 1,..., k, \tag{5.14}$$

where

$$F_j^{(-)}(t, y) := F_j(-t, y). \tag{5.15}$$

Proof. This is directly analogous to that of Ref. [1a, Lemma 10], with (E.4)′ playing the role of the Wightman axiom (W.4) and $I_j \times Y$ corresponding to the domain R_j of Ref. [1a]. ∎

Proof of Theorem 5. This is directly analogous to that of Ref. [1a, Theorem 1], with $I_j \times Y$ corresponding to the domain R_j of Ref. [1a] and Theorems 3 and Lemma 7 playing the roles of the Reeh–Schlieder theorem and Ref. [1a, Lemma 10], respectively. ∎

6. Fields on $X^{(\pm)}$, X

Let $\varphi^{(\pm)}$ be the restrictions of φ to $\mathscr{F}^{(\pm)} := \{f \in \mathscr{F} \mid \operatorname{supp} f \subset X^{(\pm)}\}$, where $X^{(\pm)}$ are the open submanifolds of X defined in Section 1. Recalling that $L^{(\pm)}(\mathbf{R})$ are time-translational isometry groups of $X^{(\pm)}$ (cf. Eq. (1.3), (4.2)), we shall show that the axioms (A), supplemented by certain further conditions, imply that the restriction of Ψ to $\varphi^{(+)}$ is a thermal state, whose temperature is determined by the gravitational field associated with the metric of $X^{(+)}$.

We take the fields on $X^{(\pm)}$ to be given by $(\mathscr{H}^{(\pm)}, \varphi^{(\pm)}, \Psi, \hat{L}^{(\pm)}(\mathbf{R}))$ where $\mathscr{H}^{(\pm)}$ is the closure of $\mathscr{A}^{(\pm)}\Psi$, $\mathscr{A}^{(\pm)}$ is the algebra of polynomials in $\{\varphi^{(\pm)}(f) \mid f \in \mathscr{F}^{(\pm)}\}$, and $\hat{L}^{(\pm)}$ is the unitary representation of $\mathbf{R}$ in $\mathscr{H}^{(\pm)}$ given by the formula

$$\hat{L}^{(\pm)}\Psi = \Psi; \quad \hat{L}^{(\pm)}(\tau)\,\varphi^{(\pm)}(f)\,\hat{L}^{(\pm)}(-\tau) = \varphi^{(\pm)}(f_\tau); \quad f_\tau(x) := f(L^{(\pm)}(-\tau)x). \tag{6.1}$$

Thus, by (4.6), (4.7) and (6.1), $\hat{L}^{(\pm)}(\tau)$ is the restriction of $\hat{L}(\tau)$ to $\mathscr{H}^{(\pm)}$. Hence, denoting the generators of $\hat{L}(\mathbf{R})$, $\hat{L}^{(\pm)}(\mathbf{R})$ by iK, $iK^{(\pm)}$, it follows from this observation and (E.3) that $K^{(\pm)}$, K_E are the restrictions of K to $\mathscr{H}^{(\pm)}$, $\mathscr{H}_E$, respectively.

Since, by Theorem 5, $\mathscr{A}_E^{(+)}\Psi$ is dense in $\mathscr{H}_E$ and further, by (A.5), the elements of $\mathscr{A}_E^{(+)}\Psi$ may be approximated arbitrarily closely by vectors in $\mathscr{A}^{(+)}\Psi$, it follows that $\mathscr{H}_E \subset \mathscr{H}^{(+)}$. We consider the possibility of assuming some of the following conditions, whose significance will be discussed presently.

(C.1) $\mathscr{H}_E = \mathscr{H}$.

(C.2) $\mathscr{H}_E = \mathscr{H}^{(+)}$.

(C.3) The restriction of Ψ to $\mathscr{A}^{(+)}$ corresponds to an equilibrium state of the field on $X^{(+)}$ at some unspecified temperature (possibly zero).

Comments. (1) In order to relate (C.1) to the dynamics of the system, we first note that E is a characteristic surface for classical relativistic wave equations on X, i.e., the solutions of such equations are functions of the values of the wave fields on E (cf. [22, Theorem 3.2.1]). We now point out that (C.1) would follow from the assumption of a corresponding property of E in relation to the dynamics of the quantum field φ, namely that this is determined by φ_E in such a way that $\mathscr{A}_E' \subset \mathscr{A}'$, where $\mathscr{A}_E', \mathscr{A}'$ are the weak commutants of $\mathscr{A}_E, \mathscr{A}$, respectively. For since $\mathscr{A}_E' \ni P_E$, the projection operator from $\mathscr{H}$ onto $\mathscr{H}_E$, this latter assumption implies that, for all $A \in \mathscr{A}$,

$$\begin{aligned}\|(I-P_E)A\Psi\|^2 &\equiv ((I-P_E)A\Psi, A\Psi) = (A^*A\Psi, (I-P_E)\Psi) \qquad (\text{as } P_E \in \mathscr{A}_E' \subset \mathscr{A}')\\ &= 0 \qquad (\text{as } P_E\Psi = \Psi),\end{aligned}$$

and therefore, in view of the density of $\mathscr{A}\Psi$ in $\mathscr{H}$, $P_E = I$, i.e., $\mathscr{H}_E = \mathscr{H}$.

(2) Condition (C.2) is clearly a weaker condition than (C.1), and can similarly be related to an assumption that the field on $X^{(+)}$ is determined by that on E in such a way that $\mathscr{A}_E' \subset \mathscr{A}^{(+)\prime}$, the weak commutant of $\mathscr{A}^{(+)}$.

(3) Condition (C.3) may be regarded as a consequence of the assumption of appropriate stability properties on the state of the field on $X^{(+)}$ (cf. [27, 28] and the discussion at the end of Section 2).

THEOREM 8. *Under the assumption of axioms* (A.2–A.5), (E.5, E.6) *and either* (C.2) *or* (C.3), *the restriction of* Ψ *to the field on* $X^{(+)}$ *corresponds to a thermal state of temperature* $\hbar/2\pi k$, *w.r.t. the dynamical group* $\hat{L}^{(+)}(\mathbf{R})$; *i.e., there is a conjugation* J *of* $\mathscr{H}$ *such that*

$$J \exp(-\pi K^{(+)}) A\Psi = A^*\Psi, \qquad \forall A \in \mathscr{A}^{(+)}. \tag{6.2}$$

Comment. Since, by Eq. (1.3), the proper time for a local observer in $X^{(+)}$ is $c^{-1}(A(-\xi^2, y))^{1/2}\tau$, this theorem implies that the observed local temperature would be

$$T = \hbar c/2\pi k(A(-\xi^2, y))^{1/2}, \tag{6.3}$$

which is a generalised form of the Hawking temperature [10]. One may interpret this result as signifying that the gravitational field associated with the metric of the manifold $X^{(+)}$ thermalises the field $\varphi^{(+)}$ so that its local temperature is T. In fact, this result is a generalised Hawking–Unruh effect [10, 11].

PROPOSITION 9. *Assume* (A.2–A.5), (E.5, E.6), (C.1) *and the global PCT condition, that there is a conjugation* J' *of* $\mathscr{H}$ *such that*

$$J'\varphi(f_1) \cdots \varphi(f_k)\Psi = \varphi(f_k^\dagger) \cdots \varphi(f_1^\dagger)\Psi, \qquad \forall f_1,..., f_k \in \mathscr{F}, \tag{6.4}$$

where

$$f^\dagger(t, w; y) := \bar{f}(-t, -w; y). \tag{6.5}$$

Then J' *is equal to the conjugation* J *governing the KMS condition* (6.2) *for the state of the field on* $X^{(+)}$.

Comment. The PCT condition (6.4) appears to be feasible as the transformation $(t, w; y) \to (-t, -w; y)$ is an isometry of X. Further, in the special case where X is Minkowski space, the result given by Proposition 9 reduces to the relationship between the PCT and KMS conjugations, specified in Proposition 1 (cf. Comment (3), following the latter Proposition).

Proof of Theorem 8. (1) Assume (C2) and denote J_E by J. Then note that it follows from (A.5) that, for $B \in \mathscr{A}_E^{(-)}$, $\exists$ a sequence $\{B_n\} \in \mathscr{A}^{(-)}$ such that

$$s\text{-}\lim_{n\to\infty} B_n\Psi = B\Psi \qquad \text{and} \qquad s\text{-}\lim_{n\to\infty} B_n^*\Psi = B^*\Psi. \tag{6.6}$$

Further, as $X^{(+)}$ and $X^{(-)}$ have space-like separation from one another, if follows from (A.4) that, for $A \in \mathscr{A}^{(+)}$,

$$(B_n^*\Psi, A\Psi) = (A^*\Psi, B_n\Psi), \tag{6.7}$$

and hence, by (6.6),

$$(B^*\Psi, A\Psi) = (A^*\Psi, B\Psi), \qquad \forall A \in \mathscr{A}^{(+)}, \qquad B \in \mathscr{A}_E^{(-)}. \tag{6.8}$$

Thus, by Theorem 5 and the fact that K reduces to K_E on $\mathscr{H}_E$,

$$(A^*\Psi, B\Psi) = (JA\Psi, e^{\pi K}B\Psi), \qquad \forall B \in \mathscr{A}_E^{(-)}, \ A \in \mathscr{A}^{(+)}. \tag{6.9}$$

In view of the stability of $\mathscr{A}_E^{(-)}$ under $B \to \hat{L}(\tau) B \hat{L}(-\tau)$ and the invariance of Ψ under $\hat{L}(R)$, we may replace $B\Psi$ by $\hat{L}(\tau) B\Psi$ in (6.5), and thus obtain the equation

$$\begin{aligned}(A^*\Psi, \hat{L}(\tau) B\Psi) &= (JA\Psi, e^{\pi K}\hat{L}(\tau) B\Psi)\\ &= (JA\Psi, \hat{L}(\tau) e^{\pi K}B\Psi), \qquad \forall A \in \mathscr{A}^{(+)}, \ B \in \mathscr{A}_E^{(-)}, \ \tau \in \mathbf{R}.\end{aligned} \tag{6.10}$$

Let $\hat{c}$ be the Fourier transform of a C_0-class function c on $\mathbf{R}$. Then, on multiplying (6.10) by $\hat{c}(\tau)$ and integrating w.r.t. τ, we obtain the formula

$$(A^*\Psi, c(K) B\Psi) = (JA\Psi, c(K) e^{\pi K}B\Psi).$$

Hence, if $\Delta := \{c(K) B\Psi, \ B \in \mathscr{A}_E^{(-)}, \ c \in C_0(\mathbf{R})\}$, then

$$(A^*\Psi, \Psi') = (JA\Psi, e^{\pi K}\Psi'), \qquad \forall \Psi' \in \Delta. \tag{6.11}$$

Since, by Theorem 5 and (C.2), $\mathscr{A}_E^{(-)}\Psi$ is dense in $\mathscr{H}^{(+)}$, it follows from our definition of Δ that this is a dense domain of analytic vectors, and thus is a core, for $e^{\pi K}$ in $\mathscr{H}^{(+)}$. Therefore, by Eq. (6.11), $JA\Psi$ lies in the domain of $e^{\pi K}$ and

$$e^{\pi K}JA\Psi = A^*\Psi, \qquad \forall A \in \mathscr{A}^{(+)},$$

i.e.,

$$Je^{-\pi K}A^*\Psi = A\Psi, \qquad \forall A \in \mathscr{A}^{(+)},$$

and this is equivalent to the desired result in view of the fact that K reduces to $K^{(+)}$ on $\mathscr{H}^{(+)}$.

(2) Assume (C.3), i.e., that Ψ is a ground or thermal state for $\varphi^{(+)}$. We rule out the former possibility by the following reductio ad absurdum argument. If Ψ were a ground state for $\varphi^{(+)}$, then K would be positive in $\mathscr{H}^{(+)}$ and thus also in $\mathscr{H}_E$ ($\subset \mathscr{H}^{(+)}$) since, by Proposition 2, $\mathscr{H}_E$ is stable under $\hat{L}(\mathbf{R})$ and the restriction of K to that space is non-zero (i.e., $\hat{L}_E(\mathbf{R})$ is non-trivial). Thus, if Ψ were a ground state for $\varphi^{(+)}$, it would also be one for φ_E; but this is not possible, as Ψ is a thermal state for φ_E in $E^{(+)}$, by Theorem 5.

We are left, then, with the alternative that satisfies the KMS condition

$$Je^{-\alpha K^{(+)}}A\Psi \equiv Je^{-\alpha K}A\Psi = A^*\Psi, \qquad \forall A \in \mathscr{A}^{(+)}, \tag{6.12}$$

where α is some positive number and J is a conjugation of $\mathscr{H}^{(+)}$. We note now that, by (4.5), if $B \in \mathscr{A}_E^{(+)}$, then we may choose a sequence $\{B_n\}$ in $\mathscr{A}^{(+)}$ such that

$$s\text{-}\lim_{n \to \infty} B_n \Psi = B\Psi \qquad \text{and} \qquad s\text{-}\lim_{n \to \infty} B_n^* \Psi = B^* \Psi. \tag{6.13}$$

On substituting B_n for A in (6.12) and using (6.13), one infers from the fact that $e^{-\alpha K}$ is self-adjoint, and therefore closed, that $B\Psi$ lies in the domain of this operator and that

$$Je^{-\alpha K} B\Psi \equiv Je^{-\alpha K_E} B\Psi = B^*\Psi, \qquad \forall B \in \mathscr{A}_E^{(+)}. \tag{6.14}$$

This equation implies that $JB^*\Psi = e^{-\alpha K_E} B\Psi$, from which it follows, in view of the stability of $\mathscr{H}_E$ under $\hat{L}(\mathbf{R})$ and the density of $\mathscr{A}_E^{(+)}\Psi$ in $\mathscr{H}_E$ that this latter space is stable under J. Thus, Eq. (6.14) is a KMS conditions for Ψ, as a state of φ_E in $E^{(+)}$. On comparing this equation with (5.3), it follows from the uniqueness of the temperature and conjugation operator associated with a KMS state that $\alpha = \pi$ and J_E is the restriction of J to $\mathscr{H}_E$. ∎

Proof of Proposition 9. Since $\mathscr{H}_E = \mathscr{H}$ (by (C.1)), it suffices to prove that J' coincides with J_E on the dense subset $\mathscr{A}_E\Psi$ of $\mathscr{H}_E$.

Let $F_1, ..., F_k \in \mathscr{F}_E$ and let $\tilde{F}_{1,n}, ..., \tilde{F}_{k,n}$ be elements of $\mathscr{F}$ conforming to the specifications of (A.5). Then, on putting $f_j = \tilde{F}_{j,n}$ for $j = 1, ..., k$ in (6.4) and passing to the strong limit $n \to \infty$, one sees from (A.5) and (4.11) that

$$J'\varphi_E(F_1) \cdots \varphi_E(F_k)\Psi = \varphi_E(F_k^\dagger) \cdots \varphi_E(F_1^\dagger)\Psi$$

and therefore, by (5.1), J' coincides with J_E on $\mathscr{A}_E\Psi$, as required. ∎

7. An Alternative Field Quantisation

We now consider the quantisation of φ according to an alternative scheme, in which the regularity axiom (A.5) is dropped and, instead, the following two are assumed.

(B.1) Ψ is a ground state of the field $\varphi^{(+)}$ on $X^{(+)}$ w.r.t. the dynamical group $\hat{L}^{(+)}(\mathbf{R})$.

(B.2) Ψ is the only stationary state, w.r.t. $\hat{L}^{(+)}(\mathbf{R})$, of the field $\varphi^{(+)}$.

Note here that (B.1) meets the requirements of dynamical and thermodynamical stability for the field on $X^{(+)}$, while (B.2) is an ergodic hypothesis [3] and corresponds to the assumption of a unique 'vacuum state' for that field.

THEOREM 10. *Under the assumption of* (A.1–A.4) *and* (B), *the observables in* $X^{(+)}$ *and* $X^{(-)}$ *are uncorrelated in the state* Ψ, *i.e.*,

$$(\Psi, AB\Psi) = (\Psi, A\Psi)(\Psi, B\Psi), \qquad \forall A \in \mathscr{A}^{(+)},\ B \in \mathscr{A}^{(-)}.$$

Comment. Since, by (A.4), the algebras $\mathscr{A}^{(+)}$ and $\mathscr{A}^{(-)}$ intercommute, it follows that the field algebra $\tilde{\mathscr{A}}$ for $X^{(+)} \cup X^{(-)}$, consisting of polynomials in $\{\varphi(f) \mid f \in \mathscr{F};\ \operatorname{supp} f \subset X^{(+)} \cup X^{(-)}\}$, is isomorphic with $\mathscr{A}^{(+)} \otimes \mathscr{A}^{(-)}$. Consequently, Theorem 10 asserts that, under the specified conditions, the restriction of Ψ to $\tilde{\mathscr{A}}$ corresponds to a product state $\Psi^{(+)} \otimes \Psi^{(-)}$ on $\mathscr{A}^{(+)} \otimes \mathscr{A}^{(-)}$. This means that the event horizons bounding $X^{(+)}$ and $X^{(-)}$ now act as physical boundaries. Indeed, in the particular case where X is the Kruskal manifold and $X^{(\pm)}$ are the exterior and interior Schwarschild space-times, respectively, Theorem 10 signifies that the surface of the Schwarzschild sphere acts as a physical boundary that prevents correlations between the observables of the interior and those of the exterior of that sphere.

Proof of Theorem 10. Since $X^{(+)}$ and $X^{(-)}$ have spacelike separation from one another, it follows from (A.4) that

$$(A^*\Psi, B\Psi) = (B^*\Psi, A\Psi), \qquad \forall A \in \mathscr{A}^{(+)},\ B \in \mathscr{A}^{(-)}. \tag{7.1}$$

In view of the stability of $\mathscr{A}^{(+)}$ under transformations $A \to \hat{L}^{(+)}(\tau) A \hat{L}^{(+)}(-\tau)$ and the invariance of Ψ under $\hat{L}^{(+)}(\mathbf{R})$, we may replace $A\Psi$, $A^*\Psi$ by $\hat{L}^{(+)}(\tau) A\Psi$, $\hat{L}^{(+)}(\tau) A^*\Psi$, respectively, in (7.1). Thus

$$(\hat{L}^{(+)} A^*\Psi, B\Psi) = (B^*\Psi, \hat{L}^{(+)}(\tau) A\Psi), \qquad \forall A \in \mathscr{A}^{(+)}, B \in \mathscr{A}^{(-)}, \tau \in \mathbf{R}. \tag{7.2}$$

Since, by (B.1), Ψ is a ground state, w.r.t. $\hat{L}^{(+)}(\mathbf{R})$, of the field on $X^{(+)}$, the Fourier transforms of the left- and right-hand sides of (7.2), considered as tempered distributions, have supports in $\mathbf{R}_- \cup \{0\}$ and $\mathbf{R}_+ \cup \{0\}$, respectively. Hence, each side of (7.2) is a polynomial in τ, and this can only be constant since $|(\hat{L}^{(+)}(\tau) A^*\Psi, B\Psi)| \leqslant \|A^*\Psi\| \, \|B\Psi\|$ and is thus bounded. Therefore

$$(\hat{L}^{(+)}(\tau) A^*\Psi, B\Psi) = (A^*\Psi, B\Psi), \qquad \forall A \in \mathscr{A}^{(+)}, B \in \mathscr{A}^{(-)}, \tau \in \mathbf{R}. \tag{7.3}$$

Since $\mathscr{H}^{(+)}$ is stable under $\hat{L}^{(+)}(\mathbf{R})$, we may write the L.H.S. of (7.3) as the $\mathscr{H}^{(+)}$ inner product $(\hat{L}^{(+)}(\tau) A^*\Psi, P^{(+)}B\Psi) \equiv (A^*\Psi, \hat{L}^{(+)}(-\tau) P^{(+)} B\Psi)$, where $P^{(+)}$ is the projection operator from $\mathscr{H}$ onto $\mathscr{H}^{(+)}$. Consequently, as $\mathscr{A}^{(+)}\Psi$ is dense in $\mathscr{H}^{(+)}$, Eq. (7.3) signifies that $P^{(+)}B\Psi$ is invariant under $\hat{L}^{(+)}(\mathbf{R})$ and is therefore, by (B.2), a scalar multiple of Ψ, i.e.,

$$P^{(+)}B\Psi = \lambda\Psi, \quad \text{with} \quad \lambda = (\Psi, P^{(+)}B\Psi) \equiv (P^{(+)}\Psi, B\Psi) \equiv (\Psi, B\Psi).$$

Therefore, as the R.H.S. of (7.3) is equal to $(A^*\Psi, P^{(+)}B\Psi)$, it follows that

$$(A^*\Psi, B\Psi) = (A^*\Psi, \Psi)(\Psi, B\Psi),$$

i.e., $(\Psi, AB\Psi) = (\Psi, A\Psi)(\Psi, B\Psi)$, as required. ∎

8. Conclusion

Our principal results are the following.

(I) In the case of a Wightman field in Minkowski space, the state that corresponds to the vacuum for an inertial observer is seen by a uniformly accelerated one to be a thermal one at the Unruh temperature, for which the conjugation governing the associated KMS condition is simply the PCT operator, modified by a certain partial inversion (Proposition 1). In view of Einstein's Principle of Equivalence, this result may be interpreted as signifying that the gravitational field, associated with a uniform acceleration, serves to thermalise a quantum field.

(II) The formulation of a field on a curved space-time X, of the given class, according to a scheme analogous to that of Wightman (cf. axioms (A.1–A.4)) is limited by the fact that the non-stationarity of X w.r.t. time-translations, $t \to t +$ constant, precludes the assumption of a corresponding spectrum condition. However, the fact that $\partial/\partial t$ is a Killing vector on the event horizon, E, permits the introduction of further axioms, (A.5) and (E.5, E.6), which, together with (A.1–A.4), implies that the field on X induces one on E that satisfies the natural analogues of the Wightman assumptions, including the spectrum condition (Proposition 2). Thus, we are able to infer that this induced field satisfies analogues of the Reeh–Schlieder, PCT and Bisognano–Wichmann theorems (Theorems 3–5).

(III) From these results, we deduce that the field on the open submanifold $X^{(+)}$ of X is thermalised to the Hawking–Unruh temperature (Theorem 8), subject to the additional assumption that *either* the dynamics of the system determines this field in terms of that on E (cf. (C.2)) *or* the field on $X^{(+)}$ enjoys suitable stability properties (cf. (C.3)). Moreover, under the further assumptions that the field on X satisfies a global PCT condition and that the dynamics of the system determines this field in terms of its value on E (cf. (C.1)), the PCT conjugation is identical with that associated with the KMS condition for the thermal state in $X^{(+)}$ (Proposition 9). Thus, the quantisation of the field in $X^{(+)}$ and X are essentially governed by this KMS condition.

(IV) We provide an alternative scheme for field quantisation on the manifold by replacing (A.5) and subsequent assumptions by axioms to the effect that the field on $X^{(+)}$ is in a vacuum-type state (cf. (B.1, B.2)). According to this scheme, the observables in $X^{(+)}$ and $X^{(-)}$ are mutually uncorrelated (Theorem 10). In the case of a Schwarzschild Black Hole, this would mean that the surface of the Schwarzschild sphere behaved not only as an event horizon but also as a physical boundary separating the systems formed by the fields on its interior and exterior.

The two quantisation schemes constructed here are clearly based on different assumptions for the state Ψ, and thus the choice between them, in specific cases, must be dictated by the prevailing conditions governing the state of the field. In cases where the scheme described in (II) and (III) is employed, the construction of a field on X may be reduced to that of a field on $X^{(+)}$ (or E) in a thermal state at the

Hawking–Unruh temperature, together with a dynamical law governing a global specification of the field on X in terms of that on $X^{(+)}$ (or E). In the case where quantisation is based on the procedure described in (IV), global properties of the field depend on the states in $X^{(+)}$ and $X^{(-)}$ (vacuum on the former submanifold) and again a dynamical law governing the global specification of the field in terms of its values in $X^{(+)}$ and $X^{(-)}$.

Finally, it may be seen that all our results may be generalised, like the original Reeh–Schlieder, PCT and Bisognano–Wichmann theorems [5, 6, 1b], to arbitrary fields on the manifolds concerned.

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