$$= -\frac{1}{2}\int_{\substack{\text{all}\\ \text{space}}} \mathbf{P}\cdot\mathbf{E}_0\, dv$$

(since $\mathbf{P}=0$ outside v)

$$= \frac{1}{2}\int_{\substack{\text{all}\\ \text{space}}} \mathbf{P}\cdot\nabla V_f\, dv$$

(E_0 due to free charges only. $\mathbf{E}_0 = -\nabla V_f$)

$$= \frac{1}{2}\int_{\substack{\text{all}\\ \text{space}}} (\nabla\cdot V_f\mathbf{P})dv - \frac{1}{2}\int_{\substack{\text{all}\\ \text{space}}} (\nabla\cdot\mathbf{P})V_f\, dv$$

$$= \frac{1}{2}\int_{\substack{\text{surface}\\ \text{at infinity}}} (V_f\mathbf{P})\cdot n\, da + \frac{1}{2}\int_{\substack{\text{all}\\ \text{space}}} \rho_b V_f\, dv$$

(because $-\nabla\cdot\mathbf{P} = \rho_{\text{bound}} = \rho_b$)

$$= 0 + \frac{1}{2}\int_v \rho_b V_f\, dv$$

(since ρ_b and $\mathbf{P}$ are zero outside V)

$$= \frac{1}{2}\int_v \rho_b V_f\, dv = \frac{1}{2}\int_v \rho_f V_b\, dv$$

(since it is immaterial whether the free charges are considered as field generating and bound charges as field sensing or vice versa).

[a]Author for correspondence.

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Retardation and relativity: The case of a moving line charge

Oleg D. Jefimenko

Physics Department, West Virginia University, Morgantown, West Virginia 26506

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The electric and magnetic fields of a line charge of finite length uniformly moving along its axis are derived by using retarded field integrals and also by using transformation equations of the special relativity theory. Both derivations yield the same field equations, although the derivation methods are drastically different. In particular, whereas transformation equations generally associated with Lorentz length contraction are crucial for the relativistic derivation, Lorentz contraction is not used in the retarded field derivation. On the other hand, although the retarded field derivation is based on the idea that retardation in the propagation of electromagnetic effects is a fundamental electromagnetic phenomenon, the relativistic derivation does not take retardation into account. An examination of the two solutions indicates that the field equations obtained classically and relativistically are identical because retardation is implicit in relativistic transformations. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

It is well known that the electric and magnetic fields of an electric point charge moving with constant velocity in a vacuum can be derived by using classical electrodynamics[1–4] as well as by using the special relativity theory.[5–7] However, if one compares classical and relativistic derivations, one discovers a surprising discrepancy between them. First, whereas the most crucial physical effect upon which classical derivations are based is the retardation in the propagation of electric and magnetic fields (or potentials), relativistic derivations make no reference to retardation. Second, whereas an indispensable element in relativistic derivations are transformation equations generally associated with Lorentz length contraction, classical derivations do not take Lorentz contraction into account. Yet, at least for a point charge, both classical and relativistic derivations yield the same field equations.

There can be no doubt that retardation is a real and a most significant phenomenon. How, then, can it be ignored in relativistic derivations? And why can Lorentz length contraction be ignored in classical calculations involving linear dimensions of moving charges without making the results of the calculations incorrect?

Quite clearly, the answer to both questions should be obtainable from a sufficiently extensive analysis of the relativistic and classical methods of determining electric and magnetic fields of moving charge distributions. The following considerations can be used as a foundation for such an analysis.

It is well known that although in his famous 1905 article[8] Einstein based his special relativity theory on the principle of the independence of the speed of light on the motion of the emitting body (his "Second Postulate"), the theory can be based on Maxwell's electromagnetic equations.[9] On the other hand, Maxwell's equations (for fields in a vacuum) can be derived from the retarded integrals expressing electric and magnetic fields of time-variable charge and current distributions[10–12]

$$\mathbf{E}=\frac{1}{4\pi\epsilon_0}\int\left\{\frac{[\rho]}{r^2}+\frac{1}{rc}\frac{\partial[\rho]}{\partial t}\right\}\mathbf{r}_u\,dv' - \frac{1}{4\pi\epsilon_0 c^2}\int\frac{1}{r}\left[\frac{\partial \mathbf{J}}{\partial t}\right]dv' \tag{1}$$

and

$$\mathbf{H}=\frac{1}{4\pi}\int\left\{\frac{[\mathbf{J}]}{r^2}+\frac{1}{rc}\frac{\partial[\mathbf{J}]}{\partial t}\right\}\times\mathbf{r}_u\,dv', \tag{2}$$

or from

$$\mathbf{E}=-\frac{1}{4\pi\epsilon_0}\int\frac{\left[\nabla'\rho+\frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t}\right]}{r}dv' \tag{3}$$

and

$$\mathbf{H}=\frac{1}{4\pi}\int\frac{[\nabla'\times\mathbf{J}]}{r}dv'. \tag{4}$$

In these equations, the square brackets are the retardation symbol indicating that the quantities between the brackets are to be evaluated for the retarded time $t'=t-r/c$, where t is the time for which $\mathbf{E}$ and $\mathbf{H}$ are evaluated, ρ is the electric charge density, $\mathbf{J}$ is the current density, r is the distance between the field point x, y, z (point for which $\mathbf{E}$ and $\mathbf{H}$ are evaluated) and the source point x', y', z' (volume element dv'), $\mathbf{r}_u$ is a unit vector directed from the source point to the field point, and c is the velocity of light.

It is clear therefore that the basic electromagnetic relations of the special relativity theory should be compatible with Eqs. (1)–(4).[13] However, as was first recognized by Liénard,[2] and as has been recently demonstrated directly,[3,14] the actual computation of the electric and magnetic fields of moving charges from retarded integrals requires the use of "retarded" length of the charge under consideration. The retarded length, also known as the "effective" length, is different from the actual length of the charge because the field signals received at a point of observation at a particular "present time" t are sent out from the trailing end and from the leading end of the moving charge (distances r_1 and r_2 from the point of observation) at different retarded times $t_1'=t-r_1/c$ and $t_2'=t-r_2/c$. As a result, the field of the charge appears to originate not from the actual volume occupied by the charge, but from its effective volume. But how is the effective volume of the moving charge taken into account in relativistic calculations?[15] The fact that relativistic and classical expressions for the fields of a uniformly moving point charge are identical (at least in their form and symbols) appears to indicate that the effective volume of the charge is implicit in the special relativity theory. On the other hand, the actual volume of the charge distribution constituting the "point" charge does not enter into relativistic calculations (although it is used in classical calculations). Therefore, a point charge may not be suitable for a definitive comparison of the relativistic and classical solutions. A more appropriate object for such a comparison is a charge of finite dimensions.

In accordance with the above considerations, the compatibility of classical and relativistic computations of the electric and magnetic fields of moving charge distributions is investigated in this paper by using a line charge of finite length uniformly moving along its axis.

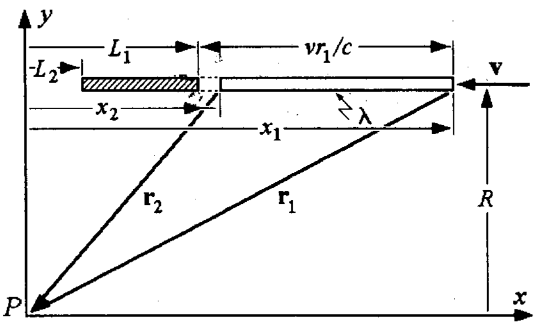

Fig. 1. A line charge of linear density λ is moving with constant velocity **v**. The retarded positions of the trailing and leading ends of the charge are x_1 and x_2, respectively. The present positions of the two ends are L_1 and L_2, respectively. The distance between the trajectory of the charge and the x axis is R. The point of observation P is at the origin. The retarded, or effective, length of the charge is longer than its true length.

II. DIRECT CLASSICAL SOLUTION

Consider a line charge of finite length, cross-sectional area S, uniform charge density ρ, and linear charge density $\lambda=\rho S$ moving with constant velocity $\mathbf{v}$ parallel to the x axis of a rectangular system of coordinates in the negative direction of the axis at a distance R above the axis (Fig. 1). Let the point of observation P be at the origin. What is the electric field at P at the time t when the leading end of the charge is at a distance L_2 from the y axis, and the trailing end is at a distance L_1 from the y axis?

To find the electric field of the moving charge by using Eq. (1) or Eq. (3), we need to know the retarded position of the charge corresponding to the time for which the field is computed.

First, let us determine the retarded position x_2 of the leading end of the charge corresponding to the time t, that is, the position from which the leading end sends out its field signal which arrives at P at the time t. If the retarded distance between P and the leading end is r_2, then the time it takes for the signal to travel from the leading end to P is r_2/c. During this time the charge travels a distance $v(r_2/c)$. Therefore, at the moment when the leading end sends out its field signal, the position of the leading end is

$$x_2=L_2+r_2v/c. \tag{5}$$

Next, let us find the retarded position x_1 of the trailing end of the charge corresponding to the time t. If the retarded distance between P and the trailing end is r_1, then the time it takes for the signal to travel from the trailing end to P is r_1/c. During this time the charge travels a distance $v(r_1/c)$. Hence, at the moment when the trailing end sends out its signal, the position of the trailing end is

$$x_1=L_1+r_1v/c. \tag{6}$$

A. The x component of the electric field

We are now ready to find the electric field of the charge by using Eq. (1) or Eq. (3). The easiest way to find the x component of the electric field of the charge under consideration from retarded integrals is to use Eq. (3). According to this equation, the x component of the field is due to the x components of $[\nabla' \rho]$ and $[\partial \mathbf{J}/\partial t]$ of the moving charge. For the line charge under consideration, these components exist only at the leading and trailing ends of the charge and are the same as for the moving charged prism discussed in two recent publications:[3,14] $[\nabla' \rho]_x = \rho/w$ for the leading end, and $[\nabla' \rho]_x = -\rho/w$ for the trailing end, $[\partial \mathbf{J}/\partial t]_x = -v^2\rho/w$ for the leading end, and $[\partial \mathbf{J}/\partial t]_x = v^2 \rho/w$ for the trailing end, where w is the thickness of the surface layer of the charge (this is the actual thickness, not the retarded one; for an explanation of the difference between the two thicknesses see Ref. 3). Since the surface layer of the charge may be assumed as thin as one wishes, the retarded volume integral in Eq. (3), as far as the x component of the field is concerned, reduces to the product of the integrand and the volume of the surface layers of the leading and trailing ends of the charge at their retarded positions. For the leading end, this volume is, using the asterisk to indicate values evaluated at retarded positions,[3,14]

$$w_2^* S = \frac{wS}{1-(\mathbf{r}_2 \cdot \mathbf{v})/r_2 c}, \tag{7}$$

and for the trailing end it is

$$w_1^* S = \frac{wS}{1-(\mathbf{r}_1 \cdot \mathbf{v})/r_1 c}. \tag{8}$$

The x component of the electric field is therefore

$$E_x = -\frac{\rho S(1-v^2/c^2)}{4\pi\epsilon_0}\left(\frac{1}{r_2[1-(\mathbf{r}_2\cdot\mathbf{v})/r_2 c]} - \frac{1}{r_1[1-(\mathbf{r}_1\cdot\mathbf{v})/r_1 c]}\right), \tag{9}$$

or

$$E_x = -\frac{\lambda(1-v^2/c^2)}{4\pi\epsilon_0}\left(\frac{1}{r_2 - x_2 v/c} - \frac{1}{r_1 - x_1 v/c}\right). \tag{10}$$

Equation (10) gives the electric field in terms of the retarded position of the charge. We shall now convert it to the present position of the charge (that is, the actual position of the charge at the time t). First, we note that, by Eq. (5),

$$L_2^2 = x_2^2 - 2x_2 r_2 v/c + r_2^2 v^2/c^2. \tag{11}$$

Next, we write the denominator of the first fraction inside the parentheses of Eq. (10) as

$$r_2 - x_2 v/c = [(r_2 - x_2 v/c)^2]^{1/2} = (r_2^2 - 2r_2 x_2 v/c + x_2^2 v^2/c^2)^{1/2}. \tag{12}$$

Adding and subtracting x^2 and $r_2^2 v^2/c^2$ to the right side of Eq. (12), we then have

$$r_2 - x_2 v/c = (r_2^2 - 2r_2 x_2 v/c + x_2^2 v^2/c^2 + x_2^2 - x_2^2 + r_2^2 v^2/c^2 - r_2^2 v^2/c^2)^{1/2}. \tag{13}$$

Let us now collect the terms on the right side of Eq. (13) into three groups

$$x_2^2 - 2r_2 x_2 v/c + r_2^2 v^2/c^2, \tag{14}$$

$$r_2^2 - x_2^2, \tag{15}$$

and

$$x_2^2 v^2/c^2 - r_2^2 v^2/c^2. \tag{16}$$

By Eq. (11), the first group represents L_2^2; the second group is simply R^2 (see Fig. 1); and the third group is $-R^2 v^2/c^2$.

Similar relations hold for the denominator of the second fraction inside the parentheses of Eq. (10). Therefore, Eq. (10) transforms to:

$$E_x = \frac{\lambda(1-v^2/c^2)}{4\pi\epsilon_0 R}\left[\frac{1}{(L_1^2/R^2 + 1 - v^2/c^2)^{1/2}} - \frac{1}{(L_2^2/R^2 + 1 - v^2/c^2)^{1/2}}\right], \tag{17}$$

where only the present time quantities appear.

B. The y component of the electric field

The easiest way to find the y component of the electric field of the charge under consideration by means of retarded field integrals is to use Eq. (1). Only the first integral of Eq. (1) makes a contribution to the y component of the field, because $\partial \mathbf{J}/\partial t$ has no y component.[3,14] Separating this integral into two integrals, we then have

$$E_y = -\frac{1}{4\pi\epsilon_0}\int \frac{[\rho]}{r^3} R\, dv' - \frac{1}{4\pi\epsilon_0}\int \frac{1}{r^2 c}\frac{\partial[\rho]}{\partial t} R\, dv'. \tag{18}$$

The first integral in Eq. (18) is the same as for a stationary charge, except that the integration must be extended over the retarded (effective) length of the charge. Designating the contribution of the first integral as E_{1y} and noting that $r = (x'^2 + R^2)^{1/2}$, we obtain

$$E_{1y} = -\frac{1}{4\pi\epsilon_0}\int \frac{\rho}{r^3} R\, dv' = -\frac{\rho S}{4\pi\epsilon_0}\int_{x_2}^{x_1} \frac{R}{(x'^2 + R^2)^{3/2}}\, dx' \tag{19}$$

or

$$E_{1y} = -\frac{\lambda}{4\pi\epsilon_0 R}\left[\frac{x_1}{(x_1^2 + R^2)^{1/2}} - \frac{x_2}{(x_2^2 + R^2)^{1/2}}\right] = -\frac{\lambda}{4\pi\epsilon_0 R}\left(\frac{x_1}{r_1} - \frac{x_2}{r_2}\right). \tag{20}$$

In order to evaluate the second integral of Eq. (18), we must determine the value of the derivative $\partial[\rho]/\partial t$.

Let us suppose that some stationary charge distribution ρ is given as a function of position coordinates x', y', and z'. If this charge distribution moves with velocity $\mathbf{v}$, it becomes the same function of $x' - v_x t$, $y' - v_y t$, and $z' - v_z t$. If the charge distribution moves and is expressed in terms of the retarded time $t' = t - r/c$, it becomes the same function of $x' - v_x(t - r/c)$, $y' - v_y(t - r/c)$, and $z' - v_z(t - r/c)$, or

$$[\rho] = \rho[x' - v_x(t - r/c), y' - v_y(t - r/c), z' - v_z(t - r/c)], \tag{21}$$

where v_x, v_y, and v_z are now the components of the velocity of the charge at the retarded time (and therefore at the retarded position) of the charge. It is this latter function whose derivative appears in the second integral of Eq. (18).

In order to evaluate this derivative, we must take into account that the distance r changes as the charge moves. The rate of change of this distance at the retarded time is

$$\frac{\partial r}{\partial t'}=\frac{\partial(\mathbf{r}\cdot\mathbf{r})^{1/2}}{\partial t'}=\frac{2\mathbf{r}\cdot(\partial\mathbf{r}/\partial t')}{2(\mathbf{r}\cdot\mathbf{r})^{1/2}}=-\frac{\mathbf{r}\cdot\mathbf{v}}{r}. \tag{22}$$

(Note, with increasing t', r increases, so that for the charge under consideration $\partial\mathbf{r}$ is opposite to $\mathbf{v}$). As we shall presently see, to find the derivative appearing in Eq. (18), we also need the derivative $\partial t'/\partial t$ expressed in terms of $\mathbf{r}$ and $\mathbf{v}$. This derivative can be found as follows. Differentiating $t'=t-r/c$, we have

$$\frac{\partial t'}{\partial t}=\frac{\partial(t-r/c)}{\partial t}=1-\frac{1}{c}\frac{\partial r}{\partial t'}\frac{\partial t'}{\partial t}, \tag{23}$$

or

$$\frac{\partial t'}{\partial t}=\frac{1}{1+(\partial r/\partial t')/c}, \tag{24}$$

which, with Eq. (22) becomes

$$\frac{\partial t'}{\partial t}=\frac{1}{1-(\mathbf{r}\cdot\mathbf{v})/rc}. \tag{25}$$

We are now ready to find $\partial[\rho]/\partial t$. Differentiating Eq. (21) and taking into account that in the case under consideration the only component of the velocity is v_x, we have

$$\frac{\partial[\rho]}{\partial t}=\frac{\partial[\rho]}{\partial(t-r/c)}\frac{\partial(t-r/c)}{\partial t}$$
$$=\frac{\partial[\rho]}{\partial(t-r/c)}\frac{\partial t'}{\partial t}=-\frac{\partial[\rho]}{\partial x'}v_x\frac{\partial t'}{\partial t}, \tag{26}$$

or, replacing v_x by $-v$ and using Eq. (25),

$$\frac{\partial[\rho]}{\partial t}=\left[\frac{v}{1-(\mathbf{r}\cdot\mathbf{v})/rc}\right]\frac{\partial[\rho]}{\partial x'}. \tag{27}$$

We can now compute E_{2y}. Substituting Eq. (27) into Eq. (18), we have

$$E_{2y}=-\frac{1}{4\pi\epsilon_0 c}\int\frac{v}{r^2-r(\mathbf{r}\cdot\mathbf{v})/c}\frac{\partial[\rho]}{\partial x'}R\,dv', \tag{28}$$

where all the quantities refer to the retarded position of the charge. Since ρ is constant within the charge, the derivative $\partial[\rho]/\partial x'=0$ within the charge, so that the only contribution to $\partial[\rho]/\partial x'$ comes from the surface layer of the charge, where ρ changes from ρ (inside the charge) to 0 (outside the charge). Let the thickness of the surface layer as seen at the retarded position of the charge be w_1^* for the trailing end and w_2^* for the leading end.[3,14] We then have $\partial[\rho]/\partial x'=-\rho/w_1^*$ for the trailing end and $\partial[\rho]/\partial x'=\rho/w_2^*$ for the leading end of the moving line charge. The electric field E_{2y} is therefore

$$E_{2y}=-\frac{R}{4\pi\epsilon_0 c}\int\frac{v\rho/w_2^*}{r_2^2-r_2(\mathbf{r}_2\cdot\mathbf{v})/c}\,dv_2'$$
$$+\frac{R}{4\pi\epsilon_0 c}\int\frac{v\rho/w_1^*}{r_1^2-r_1(\mathbf{r}_1\cdot\mathbf{v})/c}\,dv_1', \tag{29}$$

where the integration is over the surface layers of the leading and trailing ends of the charge at the retarded positions of the charge. Since the thickness of the surface layers is much smaller than r_1 and r_2, we can replace the integrals, as before for E_x, by the products of the integrands and the volumes of integration (the volumes of the respective surface layers). Using Eqs. (7) and (8), we then have

$$E_{2y}=-\frac{R}{4\pi\epsilon_0 c}\left[\frac{v\rho/w_2^*}{r_2^2-r_2(\mathbf{r}_2\cdot\mathbf{v})/c}w_2^*S-\frac{v\rho/w_1^*}{r_1^2-r_1(\mathbf{r}_1\cdot\mathbf{v})/c}w_1^*S\right],$$
$$=-\frac{\lambda vR}{4\pi\epsilon_0 c}\left[\frac{1}{r_2(r_2-x_2v/c)}-\frac{1}{r_1(r_1-x_1v/c)}\right]. \tag{30}$$

Adding Eqs. (20) and (30), we obtain for the y component of the field

$$E_y=-\frac{\lambda}{4\pi\epsilon_0 R}\left[\frac{x_1}{r_1}-\frac{R^2v/c}{r_1(r_1-x_1v/c)}-\frac{x_2}{r_2}+\frac{R^2v/c}{r_2(r_2-x_2v/c)}\right],$$
$$=-\frac{\lambda}{4\pi\epsilon_0 R}\left[\frac{x_1(r_1-x_1v/c)-R^2v/c}{r_1(r_1-x_1v/c)}-\frac{x_2(r_2-x_2v/c)-R^2v/c}{r_2(r_2-x_2v/c)}\right], \tag{31}$$

or

$$E_y=-\frac{\lambda}{4\pi\epsilon_0 R}\left[\frac{x_1r_1-x_1^2v/c-R^2v/c}{r_1(r_1-x_1v/c)}-\frac{x_2r_2-x_2^2v/c-R^2v/c}{r_2(r_2-x_2v/c)}\right]. \tag{32}$$

But $x_1^2v/c+R^2v/c=r_1^2v/c$ and $x_2^2v/c+R^2v/c=r_2^2v/c$. Therefore,

$$E_y=-\frac{\lambda}{4\pi\epsilon_0 R}\left(\frac{x_1-r_1v/c}{r_1-x_1v/c}-\frac{x_2-r_2v/c}{r_2-x_2v/c}\right). \tag{33}$$

Now, by Eq. (6), $x_1-r_1v/c=L_1$, and by Eq. (5), $x_2-r_2v/c=L_2$. Substituting L_1 and L_2 into Eq. (33) and transforming the denominators to the present position quantities by means of Eqs. (11)–(16), just as we did in Eq. (10), we finally obtain

$$E_y=\frac{\lambda}{4\pi\epsilon_0 R^2}\left[\frac{L_2}{(L_2^2/R^2+1-v^2/c^2)^{1/2}}-\frac{L_1}{(L_1^2/R^2+1-v^2/c^2)^{1/2}}\right]. \tag{34}$$

C. The magnetic field

Although we could find the magnetic field of the moving line charge from Eq. (2) or from Eq. (4), it is much simpler to find it from the electric field of the charge. As is shown in Ref. 3 on the basis of Eqs. (3) and (4), the magnetic flux density field $\mathbf{B}$ of any uniformly moving charge distribution is always

$$\mathbf{B}=(\mathbf{v}\times\mathbf{E})/c^2, \tag{35}$$

where $\mathbf{E}$ is the electric field of the moving charge distribution. Therefore, the magnetic field of our moving line charge can be easily found from Eqs. (17), (34), and (35). Since the computation is trivial, we shall omit it here.

III. DIRECT RELATIVISTIC SOLUTION

We shall now find the electric field of the moving line charge by using transformation equations of the special relativity theory. Let us suppose that the charge is at rest in a reference frame Σ' which is moving with velocity v relative to a reference frame Σ along their common x axis in the negative direction of the x axis. Let the origins Σ and Σ' coincide at $t=0$, and let the position of the charge in Σ' be the same as the "present position" of the charge shown in Fig. 1. Viewed from the Σ frame, the charge appears then to move with velocity v in the negative direction of the x axis. In the Σ' frame, where the charge is at rest, the x component of the electric field of the line charge is[16]

$$E'_x=\frac{\lambda'}{4\pi\epsilon_0 R'}\left[\frac{1}{(L_1'^2/R'^2+1)^{1/2}}-\frac{1}{(L_2'^2/R'^2+1)^{1/2}}\right], \tag{36}$$

where the primes indicate that all the values are those measured in Σ'.

To find the corresponding electric field in the Σ frame, we use the following transformation equations[17] for E_x, R, ρ, and L (observe that v is in the negative direction of the x axis):

$$E'_x=E_x, \tag{37}$$

$$\rho=\gamma(\rho'-vJ'_x/c^2), \tag{38}$$

$$R'=R, \tag{39}$$

$$L'_1=\gamma(L_1+vt), \tag{40}$$

and

$$L'_2=\gamma(L_2+vt), \tag{41}$$

where

$$\gamma=1(1-v^2/c^2)^{1/2}. \tag{42}$$

Choosing $t=0$ for the time of observation in Σ, and noting that $J'_x=0$ because the charge is stationary in Σ', we then promptly obtain for the x component of the electric field in Σ

$$E_x=\frac{\lambda(1-v^2/c^2)}{4\pi\epsilon_0 R}\left[\frac{1}{(L_1^2/R^2+1-v^2/c^2)^{1/2}}-\frac{1}{(L_2^2/R^2+1-v^2/c^2)^{1/2}}\right], \tag{43}$$

which is the same as Eq. (17) that we obtained by using classical retarded field calculations.

To find the y component of the field of the moving line charge, we first find the y component of the charge in the Σ' frame, where the charge is stationary. From Eq. (34) or by a direct calculation using electrostatic equations,[18] we have

$$E'_y=\frac{\lambda'}{4\pi\epsilon_0 R'^2}\left[\frac{L'_2}{(L_2'^2/R'^2+1)^{1/2}}-\frac{L'_1}{(L_1'^2/R'^2+1)^{1/2}}\right]. \tag{44}$$

Using again the transformation given by Eqs. (38)–(41) and also the transformation equation[17]

$$E_y=\gamma(E'_y-vB'_z), \tag{45}$$

and taking into account that the charge produces no magnetic field in Σ' (because the charge is at rest there) we immediately obtain from Eq. (44)

$$E_y=\frac{\lambda}{4\pi\epsilon_0 R^2}\left[\frac{L_2}{(L_2^2/R^2+1-v^2/c^2)^{1/2}}-\frac{L_1}{(L_1^2/R^2+1-v^2/c^2)^{1/2}}\right], \tag{46}$$

which also is the same as Eq. (34) obtained from classical calculations.

As in the case of classical derivations, there is no need to do a separate derivation of the magnetic field of the charge, because Eq. (35) is correct also relativistically.[19]

IV. SOLUTION IN TERMS OF POINT-CHARGE FIELD

In 1888, Heaviside,[1] by using his operational calculus, obtained the following equation for the electric field produced by a point charge q moving with constant velocity v:

$$\mathbf{E}=\frac{q}{4\pi\epsilon_0 r^3}\frac{(1-v^2/c^2)}{[1-(v^2/c^2)\sin^2\theta]^{3/2}}\mathbf{r}, \tag{47}$$

where r is the distance between the point of observation and the charge, c is the velocity of light, and θ is the angle between $\mathbf{r}$ and $\mathbf{v}$. The same equation results from the retarded potential integral,[2] from the retarded field integral, Eq. (3),[3] and from the relativistic transformation of the electric field of a stationary point charge.[5–7] We can use Eq. (47) for deriving the electric field of the moving line charge as a test of the solutions obtained in Secs. II and III.[20,21] Integrating the x component of Eq. (47) between L_2 and L_1, we obtain for E_x

$$\begin{aligned}E_x&=-\frac{\lambda}{4\pi\epsilon_0}\int_{L_2}^{L_1}\frac{(1-v^2/c^2)}{r^3[1-(v^2/c^2)\sin^2\theta]^{3/2}}x\,dx\\&=-\frac{\lambda(1-v^2/c^2)}{4\pi\epsilon_0}\int_{L_2}^{L_1}\frac{x\,dx}{[x^2+R^2(1-v^2/c^2)]^{3/2}}\\&=\frac{\lambda(1-v^2/c^2)}{4\pi\epsilon_0 R}\left[\frac{1}{(L_1^2/R^2+1-v^2/c^2)^{1/2}}-\frac{1}{(L_2^2/R^2+1-v^2/c^2)^{1/2}}\right].\end{aligned} \tag{48}$$

For E_y we similarly obtain

$$\begin{aligned}E_y&=-\frac{\lambda}{4\pi\epsilon_0}\int_{L_2}^{L_1}\frac{(1-v^2/c^2)}{r^3[1-(v^2/c^2)\sin^2\theta]^{3/2}}R\,dx\\&=-\frac{\lambda(1-v^2/c^2)R}{4\pi\epsilon_0}\int_{L_2}^{L_1}\frac{dx}{[x^2+R^2(1-v^2/c^2)]^{3/2}}\\&=\frac{\lambda}{4\pi\epsilon_0 R^2}\left[\frac{L_2}{(L_2^2/R^2+1-v^2/c^2)^{1/2}}-\frac{L_1}{(L_1^2/R^2+1-v^2/c^2)^{1/2}}\right].\end{aligned} \tag{49}$$

These are exactly the same equations that were obtained in Secs. II and III.

V. CONCLUSIONS

The fact that our classical and relativistic derivations of the electric and magnetic fields of the moving line charge yield the same field equations clearly shows that retardation in the propagation of electromagnetic effects is implicit in relativistic electrodynamics. However, the reason why Lorentz contraction can be ignored in the classical solution presented in Sec. II as well as in the solution presented in Sec. IV remains unclear.[22] A further analysis of the obtained equations is therefore needed.

[1]Oliver Heaviside, "The electromagnetic effects of a moving charge," The Electrician **22**, 147–148 (1888); Oliver Heaviside, "On the electromagnetic effects due to the motion of electricity through a dielectric," Philos. Mag. **27**, 324–339 (1889).

[2]A. Liénard, "Champ électrique et magnetique produit par une charge électrique concentré en un point et animé d'un mouvement quelconque," L'Éclariage élect. **16**, 5–14, 53–59, 106–112 (1898). See also E.Wiechert, "Elektrodynamische elementargesetze," Archives Néerlanaises, (2) **5**, 549–573 (1900) (Wiechert calculates potentials but not the fields).

[3]Oleg D. Jefimenko, "Direct calculation of the electric and magnetic fields of an electric point charge moving with constant velocity," Am. J. Phys. **62**, 79–85 (1994).

[4]There are many textbooks presenting classical derivations of the electric and magnetic fields of a moving point charge from retarded potentials. See, for example, David J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Engelwood Cliffs, 1989), 2nd ed., pp. 416–426.

[5]See, for example, W. G. V. Rosser, *Classical Electromagnetism via Relativity* (Plenum, New York, 1968), pp. 29–42.

[6]For an alternative derivation see, for example, Edward M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1985), 2nd ed., pp. 181–184.

[7]An extensive discussion of relativistic transformations and their use in electrodynamics is given in (a) H. Arzelliès, *Électricité* (Gauther-Villars, Paris, 1963) and in (b) H. Arzelliès, *Relativistic Point Dynamics* (Pergamon, New York, 1972). The point charge fields are derived on pp. 190–191 in (a) and on pp. 157–161 in (b).

[8]A. Einstein, "Zur elektrodynamik bewegter Körper," Ann. Phys. **17**, 891–921 (1905). An English language translation can be found in *The Principle of Relativity* (Dover, New York, 1952) pp. 37–65. A more accurate translation is given in Arthur I. Miller, *Albert Einstein's Special Relativity Theory* (Addison-Wesley, Reading, 1981), pp. 392–415.

[9]See, for example, Ref. 5, pp. 4, 154 and references cited there.

[10]Oleg D. Jefimenko, *Electricity and Magnetism* (Electret Scientific, Star City, 1989), 2nd ed., pp. 514–516.

[11]Tran-Cong Ton, "On the time-dependent, generalized Coulomb and Biot–Savart laws," Am. J. Phys. **59**, 520–528 (1991).

[12]David J. Griffiths and Mark A. Heald, "Time-dependent generalization of the Biot–Savart and Coulomb laws," Am. J. Phys. **59**, 111–117 (1991).

[13]In fact, the Lorentz–Einstein transformations of relativistic electrodynamics can be derived from Eqs. (3) and (4). See Oleg D. Jefimenko, "Retardation and relativity: The derivation of Lorentz–Einstein transformations from retarded integrals for electric and magnetic fields," Am. J. Phys. **63**, 267–272 (1995).

[14]Oleg D. Jefimenko, "Direct derivation of the electric and magnetic fields of an accelerating point charge," Electret Scientific Co., Star City, WV, Preprint No. 94-214.

[15]It is important to emphasize that the retarded length is not at all the Lorentz-contracted length. Moreover, if the charge approaches the observer, its retarded length is *larger* than its actual length, whereas if the charge moves away from the observer its retarded length is *smaller* than its actual length.

[16]This field can be obtained from Eq. (17) by setting $v=0$. For a direct derivation using electrostatic equations see, for example. Ref. 10, p. 58.

[17]See, for example, Ref. 5, pp. 153, 157, and 168.

[18]See, for example, Ref. 10, p. 98.

[19]See, for example, Ref. 5, p. 39.

[20]Equation (47) has been used by various authors for obtaining the electric field of an *infinitely long* charge moving at constant velocity along its axis. See, for example, A. P. French, *Special Relativity* (Norton, New York, 1968), pp. 251–253 and Alan M. Portis, *Electromagnetic Fields, Sources, and Media* (Wiley, New York, 1978), pp. 199–200.

[21]In B. L. Blackford, "Electric field of a slowly moving rectangular current loop: a microscopic approach," Am. J. Phys. **62**(11), 1005–1008 (1994), Eq. (47) is used for obtaining the electric field of a moving current-carrying loop. The charge density, but not the loop itself, is assumed to be affected by Lorentz contraction.

[22]It is important to note that some authors consider that Lorentz contraction must be superimposed onto the retarded shape of the moving body. See, for example, Roy Weinstein, "Observation of length by a single observer," Am. J. Phys. **28**, 607–610 (1960) [observe that (a), (b), and (c) in Fig. 4 of that article should be read from top to bottom instead of from left to right].

MARCONI'S ACHIEVEMENT

Hats off to these inventors in bearskins and bark shoes. The man who first used deftly inserted rollers to move a stone whose weight seemed forever to defy the giant fists of his fellows surely experienced no less satisfaction than Marconi on perceiving the first airborne trans-oceanic telegraphic signal, provided of course that what the papers report about it is wholly true.

Ludwig Boltzmann, "On the Principles of Mechanics," 1902, reprinted in *Theoretical Physics and Philosophical Problems*, edited by Brian McGuiness (Reidel, Boston, 1974), pp. 147.

THE LIMITATIONS OF PHYSICS

What is surely impossible is that a theoretical physicist, given unlimited computing power, should deduce from the laws of physics that a certain complex structure is aware of its own existence.

Brian Pippard, quoted in Steven Weinberg, *Dreams of a Final Theory* (Pantheon Books, New York, 1992), p. 44.