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A Vierbein Formalism of Radiation Damping

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The equations of motion of a charged particle moving in a general Riemannian space are derived. A vierbein treatment is adopted in contrast to the tensorial procedure of DeWitt and Brehme [see Ann. Phys. (N. Y.) 9, 220 (1960)] with a subsequent simplification of computations referring to the world tube. The resulting equations of motion differ from those of DeWitt and Brehme by the inclusion of terms involving the Ricci tensor. This discrepancy appears to be due to an error on the part of the above authors, which is discussed in the text.

INTRODUCTION

Before embarking on any theory of the electron it is necessary to state to which model the analysis applies. In the work that follows we will be considering a point model for the electron in which we have the field equations holding all the way up to the electron's center, which would then appear as a point of singularity.

In this simple model the difficulty now arises that if we accept Maxwell's theory, the field in the immediate neighborhood of the electron has an infinite mass. A possible line of attack to overcome this difficulty is to modify Maxwell's theory so as to make the energy of the field around the singularity finite, however this leads to great complexity.

Proceeding from the opposite point of view, Dirac (*1*) retained Maxwell's theory to describe the field right up to the point of singularity, and then surmounted the difficulties associated with an infinite energy by a process of direct omission or subtraction of unwanted terms. Thus he obtained a theory that could be used to calculate all the results obtainable from experiment, rather than a model for the electron.

Hadamard (*2*) pointed out the fact that a plane or spherical sharp pulse of light, when propagating in a curved four-dimensional hyperbolic Riemannian manifold, does not, in general, remain a sharp pulse, but gradually develops a "tail." This effect originated from the fact that the vector and scalar wave equations had solutions not only on the null cone but also inside the timelike portions of the null cone.

In the covariant generalisation of Dirac's work, De Witt and Brehme (*3*) utilized this result of Hadamard's to show that in a general Riemannian manifold electrogravitic bremsstrahlung occurred in addition to the usual radiation-damping effects. This interaction having its origin in the failure of Huygens' principle, was then found to modify the ponderomotive equations to the extent of an integral over the whole past history of the particle. They interpreted the "tail" as a sort of scatter process, with the "bumps" in space–time playing the role of scatterers that allow the radiation field originating in the particle, which normally "outruns" the particle, to act directly back on the particle.

Although gravitational fields cannot be distinguished from inertial fields by any experiment conducted on a purely local basis, they can be distinguished over an extended region by experiments which measure field gradients, i.e., which measure the second derivative of the potential. In general relativity the potential is the space-time metric, and the second derivatives of the metric can be expressed uniquely by the components of the Riemann tensor, which describes the "true" gravitational field. It would therefore not be surprising if, when radiation reaction is included, we find the Riemann tensor entering explicitly into the dynamical equations of a charged particle moving in a gravitational field. The surprising result obtained by De Witt and Brehme was that the Riemann tensor made no such appearance.

This paper gives a vierbein formalism of radiation damping that possesses distinct advantages when compared with the tensorial treatment. With the aid of the propagated vierbein, a natural coordinate system is developed and all relevant tensorial equations are resolved into equations invariant with respect to coordinate transformations. This procedure tends to simplify the working especially in the case of computations referring to the world tube, and enables one to draw a striking analogy with the corresponding treatment in flat space–time. As we shall see later, the results obtained here differ from those of De Witt and Brehme in that the Riemann tensor makes an explicit appearance in the equations of motion which, as stated before, was a physically expected result. (This difference appears to be due to a computational error on their part and will be discussed later in section V.)

Section I is essentially a resumé of De Witt and Brehme's investigation using vierbein techniques. A quasi-Cartesian coordinate system with varying base point is developed, and these coordinates, rather than the derivative of the characteristic function, are used as the springboard for the covariant expansion techniques. In Section II we refer the vector wave equation to the components of the vierbein, and the resulting wave equation is solved in terms of invariant ("bein" index) Green's functions whose properties are then derived. The Lorentz invariant form of the equations of classical electrodynamics can be obtained in two ways. We can construct the Lagrangian for a structureless point particle in terms of invariant field quantities and apply the principle of stationary action for independent variations in the dynamical variables $x^{(\alpha)}$ and $A^{(\alpha)}$ (position and vector potential) referred to the vierbein, see Sciama (*4*). However in Section III we give a direct conversion of the well-known tensorial equations in order to preserve a direct comparison with the work of De Witt and Brehme. In Section IV we find that our choice of coordinate system enables us to give a very simple, essentially flat space, formalism for the construction of the world tube. Finally, in Section V, we compute the energy–momentum balance of the particle by integrating the stress tensor over the surface of the world tube. The equations of motion, including radiation damping, then follow after a classical mass renormalization.

I. CONSTRUCTION OF THE CHARACTERISTIC FUNCTION AND THE PROPAGATED VIERBEIN

Let $P(x)$ and $P'(x')$ be two points of space–time joined by a geodesic Γ. If, as we shall assume, Γ is unique for the points P and P', then the Characteristic function, denoted by $\sigma(x, x')$, is a function of these two points and defined by the equations

$$\tfrac{1}{2}g^{\mu\nu}\sigma_{\cdot\mu}\sigma_{\cdot\nu} = \sigma, \qquad \tfrac{1}{2}g^{\mu'\nu'}\sigma_{\cdot\mu'}\sigma_{\cdot\nu'} = \sigma, \tag{1.1}$$

$$\lim_{x'\to x} \sigma = [\sigma] = 0, \tag{1.2}$$

where $g^{\mu\nu}$ is the contravariant metric tensor at P, dots denote covariant differentiation, and dashes refer to quantities evaluated at P'. Hereafter the bracket notation of (1.2) will be used to denote the coincidence limit of the function contained [c.f. Synge (*5*)].

The coincidence limits of the covariant derivatives of the Characteristic function are obtained by repeatedly differentiating (1.1) and using (1.2), we find

$$[\sigma_{\cdot\mu}] = 0, \tag{1.3}$$

$$[\sigma_{\cdot\mu\nu}] = g_{\mu\nu}\,, \tag{1.4}$$

$$[\sigma_{\cdot\mu\nu\kappa}] = 0, \tag{1.5}$$

$$[\sigma_{\cdot\mu\nu\kappa\sigma}] = \tfrac{1}{3}(R_{\mu\kappa\nu\sigma} + R_{\mu\sigma\nu\kappa}), \tag{1.6}$$

our convention for the Riemann tensor being

$$R^{\mu}{}_{\nu\kappa\sigma} = \left\{ {\mu \atop \kappa\nu} \right\}_{,\sigma} - \left\{ {\mu \atop \nu\sigma} \right\}_{,\kappa} + \left\{ {\rho \atop \nu\kappa} \right\}\left\{ {\mu \atop \rho\sigma} \right\} - \left\{ {\rho \atop \nu\sigma} \right\}\left\{ {\mu \atop \rho\kappa} \right\}.$$

Using these results and covariant expansion techniques we can form the expansions

$$\sigma_{\cdot\mu\nu} = g_{\mu\nu} + \tfrac{1}{3}R_{\mu\ \nu}^{\ \kappa\ \rho}\sigma_{\cdot\kappa}\sigma_{\cdot\rho} + O(s^3), \tag{1.7}$$

$$\sigma_{\cdot\mu\nu\kappa} = \tfrac{1}{3}(R_{\mu\kappa\nu}{}^{\rho} + R_{\mu\ \nu\kappa}^{\ \rho})\,\sigma_{\cdot\rho} + O(s^2), \tag{1.8}$$

$$\sigma_{\cdot\mu\nu\kappa\rho} = \tfrac{1}{3}(R_{\mu\kappa\nu\rho} + R_{\mu\rho\nu\kappa}) + O(s), \tag{1.9}$$

where "S" is the measure of the geodesic PP'.

Consider now the timelike world line of the particle on which the proper time τ' acts as parameter. At a point τ_0 construct an orthonormal vierbein whose timelike component is the four-velocity, $\dot{z}_\mu(\tau_0)$, of the particle. Under Fermi–Walker transport, this component remains tangential to the curve [see Synge (5).] The combination of this with the orthogonality relation produces the result that our construction also provides us with an orthonormal triad orthogonal to the particle world line. If $\mu_\nu^{(\alpha)}$ are the vierbein components, the law of Fermi–Walker Transport along the particle world line can be written

$$D\mu_{(i)}^{\kappa} = \mu_{(i)\nu}\ddot{z}^{\nu}\dot{z}^{\kappa}, \qquad i = 1, 2, 3, \tag{1.10}$$

where $D = \delta/\delta\tau'$ is the absolute derivative. The boundary condition at τ_0 is

$$\lim_{\tau'\to\tau_0} \mu_\nu^{(\alpha)}(z(\tau')) = \mu_\nu^{(\alpha)}(z(\tau_0)). \tag{1.11}$$

At a given point, $z_\mu(\tau')$, on the world line, construct a geodesic to the point "x", (Fig. 1). Introduce on this geodesic an orthonormal vierbein, $\lambda_\nu^{(\alpha)}$, which is propagated under Fermi–Walker transport along the geodesic according to the laws

$$g^{\nu\sigma}\lambda_\mu^{(\alpha)}(s)_{\cdot\nu}\sigma_{\cdot\sigma} = 0, \tag{1.12}$$

$$\lim_{s\to\tau'} \lambda_\nu^{(\alpha)}(s) = \mu_\nu^{(\alpha)}(\tau'), \tag{1.13}$$

where "s" is the measure of the geodesic and "σ" is its Characteristic function. We have now induced an orthonormal vierbein at the field point "x" in terms of a given vierbein, $\mu_\nu^{(\alpha)}(z(\tau_0))$, and Fermi–Walker transport, first along the world line then along the connecting geodesic. We raise and lower the covariant components of the vierbein with use of the metric tensor in the following manner

$$\begin{aligned} \lambda_{(\alpha)\mu} &= g_{\mu\nu}\lambda_{(\alpha)}^{\nu}, \qquad & \mu_{(\alpha)\mu} &= g_{\mu\nu}\mu_{(\alpha)}^{\nu}, \\ \lambda_{(\alpha)}^{\mu} &= g^{\mu\nu}\lambda_{(\alpha)\nu}, \qquad & \mu_{(\alpha)}^{\mu} &= g^{\mu\nu}\mu_{(\alpha)} \end{aligned} \tag{1.14}$$

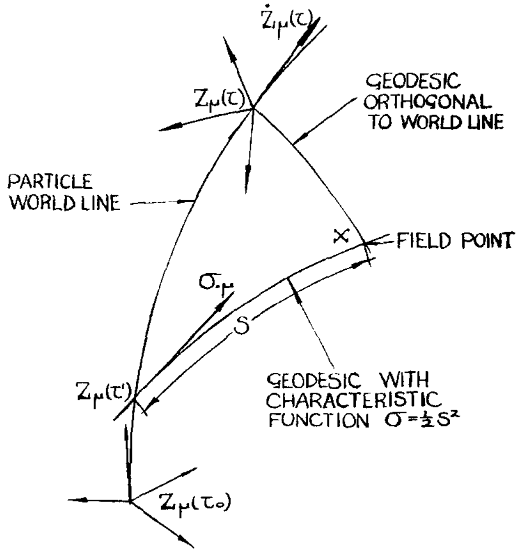

FIG. 1. Construction of the field point.

The conditions for orthonomality can be written as

$$\mu^{\nu}_{(\alpha)}\mu_{\nu(\beta)} = \lambda^{\mu}_{(\alpha)}\lambda_{\mu(\beta)} = \eta_{(\alpha\beta)}\,, \tag{1.15}$$

where

$$\eta_{(\alpha\beta)} = \eta^{(\alpha\beta)} = \mathrm{diag}(1, 1, 1, -1), \tag{1.16}$$

a diagonal invariant matrix with the elements indicated; it satisfies the relation

$$\eta^{(\alpha\gamma)}\,\eta_{(\beta\gamma)} = \delta_{\beta}{}^{\alpha}, \tag{1.17}$$

the right-hand side of (1.17) being the Kronecker delta.

The "bein" indices on the vectors have no tensorial meaning but nevertheless we shall raise and lower the indices by means of the η-matrix. Thus we define

$$\lambda^{(\alpha)\mu} = \eta^{(\alpha\beta)}\lambda^{\mu}_{(\beta)}\,, \qquad \lambda^{(\alpha)}_{\mu} = \eta^{(\alpha\beta)}\lambda_{(\beta)\mu}\,, \tag{1.18}$$

and deduce by (1.17)

$$\lambda^{\mu}_{(\alpha)} = \eta_{(\alpha\beta)}\lambda^{(\beta)\mu}, \qquad \lambda_{(\alpha)\mu} = \eta_{(\alpha\beta)}\lambda^{(\beta)}_{\mu}. \tag{1.19}$$

Now (1.15) may be written more neatly as

$$\lambda^{\mu}_{(\alpha)}\lambda^{(\beta)}_{\mu} = \delta_{\alpha}{}^{\beta}, \tag{1.20}$$

and it is an algebraic consequence of this that

$$\lambda^{\mu}_{(\alpha)}\lambda^{(\alpha)}_{\nu} = \delta_{\nu}{}^{\mu}. \tag{1.21}$$

We next define the invariant "coordinates" of the point "x" with respect to the vierbein at $z_{\mu}(\tau')$, by the relation

$$x^{(\alpha)} = -\sigma._{\mu}\lambda^{(\alpha)\mu}, \qquad (\alpha = 0, 1, 2, 3). \tag{1.22}$$

These quasi-Cartesian coordinates become Fermi and Optical coordinates when the geodesic is orthogonal to the particle world line and null, respectively.

Using (1.1), (1.21), and (1.22) we see that our coordinates satisfy the relation

$$x^{(\alpha)}x_{(\alpha)} = \sigma._{\mu}\lambda^{(\alpha)\mu}\sigma._{\nu}\lambda^{\nu}_{(\alpha)} = g^{\mu\nu}\sigma._{\mu}\sigma._{\nu} = 2\sigma. \tag{1.23}$$

Corresponding to (1.3)–(1.6), we can obtain knowledge of the vierbein propagation along the geodesic by differentiating (1.12); we find

$$\lambda^{(\alpha)}{}_{\mu.\nu\rho}\sigma.^{\nu} + \lambda^{(\alpha)}{}_{\mu.\nu}\sigma.^{\nu}{}_{\rho} = 0, \tag{1.24}$$

$$\lambda^{(\alpha)}{}_{\mu.\nu\rho\eta}\sigma.^{\nu} + \lambda^{(\alpha)}{}_{\mu.\nu\rho}\sigma.^{\nu}{}_{\eta} + \lambda^{(\alpha)}{}_{\mu.\nu\eta}\sigma.^{\nu}{}_{\rho} + \lambda^{(\alpha)}{}_{\mu.\nu}\sigma.^{\nu}{}_{\rho\eta} = 0, \tag{1.25}$$

from which it follows that

$$[\lambda^{(\alpha)}{}_{\mu.\nu}] = 0, \tag{1.26}$$

$$[\lambda^{(\alpha)}{}_{\mu.\nu\rho}] = \tfrac{1}{2}R_{\nu\rho\mu}{}^{\eta}\lambda_{\eta}{}^{(\alpha)}, \tag{1.27}$$

after use has been made of the identities satisfied by the Riemann tensor. The particle traces out a world line in space–time given by a set of functions $z_{\mu}(\tau')$. Dots over the z's will be used to denote absolute covariant differentiation with respect to the parameter τ'. Thus

$$\dot{z}^{\mu} = dz^{\mu}/d\tau', \tag{1.28}$$

$$\ddot{z}^{\mu} = d\dot{z}^{\mu}/d\tau' + \left\{ {\mu \atop \rho\sigma} \right\} \dot{z}^{\rho}\dot{z}^{\sigma}, \tag{1.29}$$

$$\dddot{z}^{\mu} = d\ddot{z}^{\mu}/d\tau' + \left\{ {\mu \atop \rho\sigma} \right\} \ddot{z}^{\rho}\dot{z}^{\sigma}, \text{ etc.} \tag{1.30}$$

Use of absolute covariant derivatives enables us to write the τ'-derivatives of any scalar function ϕ of the z's in the covariant forms

$$\dot{\phi} = \phi_{\cdot\mu}\dot{z}^{\mu}, \tag{1.31}$$

$$\ddot{\phi} = \phi_{\cdot\mu\nu}\dot{z}^{\mu}\dot{z}^{\nu} + \phi_{\cdot\mu}\ddot{z}^{\mu}, \tag{1.32}$$

$$\dddot{\phi} = \phi_{\cdot\mu\nu\sigma}\dot{z}^{\mu}\dot{z}^{\nu}\dot{z}^{\sigma} + 3\phi_{\cdot\mu\nu}\ddot{z}^{\mu}\dot{z}^{\nu} + \phi_{\cdot\mu}\dddot{z}^{\mu}, \tag{1.33}$$

$$\begin{aligned}\ddddot{\phi} &= \phi_{\cdot\mu\nu\sigma\rho}\dot{z}^{\mu}\dot{z}^{\nu}\dot{z}^{\sigma}\dot{z}^{\rho} + 5\phi_{\cdot\mu\nu\sigma}\ddot{z}^{\mu}\dot{z}^{\nu}\dot{z}^{\sigma} \\ &\quad + \phi_{\cdot\mu\nu\sigma}\dot{z}^{\mu}\dot{z}^{\nu}\ddot{z}^{\sigma} + 4\phi_{\cdot\mu\nu}\dddot{z}^{\mu}\dot{z}^{\nu} \\ &\quad + 3\phi_{\cdot\mu\nu}\ddot{z}^{\mu}\ddot{z}^{\nu} + \phi_{\cdot\mu}\ddddot{z}^{\mu}, \text{ etc.}\end{aligned} \tag{1.34}$$

The quantities (1.28), (1.29), and (1.30) will be referred to the vierbein components with the notation

$$\dot{z}^{(\alpha)} = \dot{z}^{\mu}\lambda_{\mu}^{(\alpha)}, \tag{1.35}$$

$$\ddot{z}^{(\alpha)} = \ddot{z}^{\mu}\lambda_{\mu}^{(\alpha)}, \tag{1.36}$$

$$\dddot{z}^{(\alpha)} = \dddot{z}^{\mu}\lambda_{\mu}^{(\alpha)}, \text{ etc.} \tag{1.37}$$

Using the expansions (1.7)–(1.9), (1.26), and (1.27), and Eqs. (1.31)–(1.34), and repeatedly differentiating (1.22), we can obtain expansions for the absolute τ' derivatives of the invariant "coordinates" of the point "x" as follows

$$Dx^{(\alpha)} = -\dot{z}^{(\alpha)} - \tfrac{1}{6}R_{\mu}{}^{\nu\rho\eta}\sigma_{\cdot\nu}\sigma_{\cdot\rho}\dot{z}^{\mu}\lambda_{\eta}^{(\alpha)} + O(s^3), \tag{1.38}$$

$$D^2x^{(\alpha)} = -\ddot{z}^{(\alpha)} + \tfrac{2}{3}R_{\mu\nu\rho}{}^{\eta}\sigma_{\cdot\eta}\dot{z}^{\nu}\dot{z}^{\rho}\lambda^{\mu(\alpha)} + O(s^2), \tag{1.39}$$

$$D^3x^{(\alpha)} = -\dddot{z}^{(\alpha)} + O(s). \tag{1.40}$$

Similarly from (1.37), using (1.26) and (1.27), we find

$$D\dot{z}^{(\alpha)} = \ddot{z}^{(\alpha)} + \tfrac{1}{2}R_{\nu}{}^{\rho}{}_{\mu}{}^{\kappa}\lambda_{\kappa}^{(\alpha)}\dot{z}^{\mu}\dot{z}^{\nu}\sigma_{\cdot\rho} + O(s^2), \tag{1.41}$$

$$D^2\dot{z}^{(\alpha)} = \dddot{z}^{(\alpha)} + O(s). \tag{1.42}$$

Of course, since τ' measures proper time along the particle world line, the following relations hold:

$$g_{\mu\nu}\dot{z}^{\mu}\dot{z}^{\nu} = -1; \tag{1.43}$$

$$g_{\mu\nu}\ddot{z}^{\mu}\dot{z}^{\nu} = 0; \tag{1.44}$$

$$g_{\mu\nu}\dot{z}^{\mu}\dddot{z}^{\nu} = -g_{\mu\nu}\ddot{z}^{\mu}\ddot{z}^{\nu} = -\ddot{z}^{2}. \tag{1.45}$$

A bi-scalar of fundamental importance in the theory of geodesics is given by

$$\begin{aligned} \Delta &= - \mid -\sigma._{\mu'\beta} \mid . \mid \lambda^{\mu'(\alpha)}(z)\, \lambda^{\gamma}_{(\alpha)}(x) \mid \\ &= -\bar{g}^{-1} \mid -\sigma._{\mu'\beta} \mid, \end{aligned} \tag{1.46}$$

where

$$\bar{g}(x, z) = g^{1/2}(x)\, g^{1/2}(z) = \bar{g}(z, x), \tag{1.47}$$

is a bi-scalar density of weight 1 at both x and z. The dashes in (1.46) refer to evaluation at the field point x. An important qualitative law concerning the bi-scalar Δ can be obtained by repeatedly differentiating (1.1), first with respect to the field point and then with respect to the particle point, thus

$$\sigma._{\mu'\alpha} = g^{\sigma'\rho'}\sigma._{\sigma'\alpha}\sigma._{\rho'\mu'} + g^{\sigma'\rho'}\sigma._{\sigma'}\sigma._{\rho'\mu'\alpha} . \tag{1.48}$$

If $\sigma.^{-1\mu'\alpha}$ is the inverse of $\sigma._{\mu'\alpha}$ satisfying

$$\sigma.^{-1\mu'\alpha}\sigma._{\nu'\alpha} = \delta^{\mu'}_{\nu'} , \qquad \sigma.^{-1\mu'\beta}\sigma._{\mu'\alpha} = \delta_{\alpha}^{\ \beta}, \tag{1.49}$$

then (1.48) can be written in the form

$$4 = \mid -\sigma._{\nu'\beta} \mid \{\mid -\sigma._{\mu'\alpha} \mid \sigma.^{\rho'}\}._{\rho'} . \tag{1.50}$$

Also, noting

$$\bar{g}._{\nu'} = \bar{g}_{,\nu'} - \left\{ {\mu' \atop \mu'\nu'} \right\} \bar{g} = 0, \tag{1.51}$$

we see from (1.46) and (1.50) that the bi-scalar Δ satisfies the equation

$$\Delta^{-1}(\Delta\sigma.^{\mu'})._{\mu'} = 4. \tag{1.52}$$

This equation determines the amount by which Δ decreases or increases along each geodesic according as the rate of divergence of neighboring geodesics, measured by $\sigma.^{\mu'}{}_{\mu'}$, is greater or less than 4, the rate in flat space-time.

Finally, using (1.7), (1.26), and (1.27), we see from (1.46) that Δ has the expansion

$$\Delta = 1 - \tfrac{1}{6}R^{\mu'\nu'}\sigma._{\mu'}\sigma._{\nu'} + O(s^3), \tag{1.53}$$

our convention for the Ricci tensor being

$$R_{\mu\nu} = g^{\sigma\kappa}R_{\mu\sigma\nu\kappa} , \qquad R = g^{\mu\nu}R_{\mu\nu} . \tag{1.54}$$

II. SOLUTIONS OF THE INVARIANT WAVE EQUATION

In this section we look for solutions of the vector wave equation referred to the components of the propagated vierbein and given by

$$\lambda^{\mu}_{(\alpha)}\{A_{\mu.\nu\sigma}g^{\nu\sigma} + R_{\mu}{}^{\nu}A_{\nu}\} = 0. \tag{2.1}$$

To do this we first introduce the invariant components of A_μ evaluated on the vierbein and given by

$$A_{(\alpha)} = A_{\mu}\lambda^{\mu}_{(\alpha)}, \tag{2.2}$$

with the dual relations

$$A_{\mu} = A_{(\alpha)}\lambda^{(\alpha)}_{\mu}. \tag{2.3}$$

From (2.3) we have

$$g^{\nu\sigma}A_{\mu.\nu\sigma} = A_{(\alpha),(\beta\gamma)}\lambda^{(\gamma)\nu}\lambda^{(\beta)}_{\nu}\lambda^{(\alpha)}_{\mu} + A_{(\alpha)}\lambda^{(\alpha)}{}_{\mu.\nu}{}^{\nu}$$
$$+ A_{(\alpha),(\beta)}\{2\lambda^{(\beta)\nu}\lambda^{(\alpha)}{}_{\mu.\nu} + \lambda^{(\beta)}{}_{\nu.}{}^{\nu}\lambda^{(\alpha)}_{\mu}\}. \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.1) we can write the invariant wave equation in the alternative form

$$\eta^{(\beta\gamma)}A_{(\delta),(\beta\gamma)} + A_{(\alpha),(\beta)}\{2\lambda^{(\beta)\nu}\lambda^{\mu}_{(\delta)}\lambda^{(\alpha)}{}_{\mu.\nu} + \delta_{\delta}{}^{\alpha}\lambda^{(\beta)}{}_{\nu.}{}^{\nu}\}$$
$$+ A_{(\alpha)}\{\lambda^{\mu}_{(\delta)}\lambda^{(\alpha)}{}_{\mu.\nu}{}^{\nu} + R_{\mu}{}^{\nu}\lambda^{\mu}_{(\delta)}\lambda^{(\alpha)}_{\nu}\} = 0. \tag{2.5}$$

Following Hadamard (3) we look for an elementary solution of the form

$$G^{(1)}_{(\delta\epsilon)} = \frac{1}{(2\pi)^2}\left\{\frac{u_{(\delta\epsilon)}}{\sigma} + v_{(\delta\epsilon)}\log|\sigma| + w_{(\delta\epsilon)}\right\}, \tag{2.6}$$

and using (1.23) we compute

$$\eta^{(\beta\gamma)}G^{(1)}_{(\delta\epsilon),(\beta\gamma)} = \frac{1}{(2\pi)^2}\{-\sigma^{-2}(2\eta^{(\beta\gamma)}u_{(\delta\epsilon),(\beta)}\sigma_{,(\gamma)})$$
$$+ \sigma^{-1}(\eta^{(\beta\gamma)}u_{(\delta\epsilon),(\beta\gamma)} + 2v_{(\delta\epsilon)} + 2\eta^{(\beta\gamma)}v_{(\delta\epsilon),(\beta)}\sigma_{,(\gamma)})$$
$$+ \eta^{(\beta\gamma)}v_{(\delta\epsilon),(\beta\gamma)}\log|\sigma| + \eta^{(\beta\gamma)}w_{(\delta\epsilon),(\beta\gamma)}\}. \tag{2.7}$$

Also taking note of (1.12) we find

$$G^{(1)}_{(\alpha\epsilon),(\beta)}\{2\lambda^{(\beta)\nu}\lambda^{\mu}_{(\delta)}\lambda^{(\alpha)}{}_{\mu.\nu} + \delta_\delta{}^\alpha\lambda^{(\beta)}{}_{\nu.}{}^{\nu}\}$$

$$= \frac{1}{(2\pi)^2}\{-\sigma^{-2}u_{(\delta\epsilon)}\lambda^{(\beta)}{}_{\nu.}{}^{\nu}\sigma,_{(\beta)} + \sigma^{-1}v_{(\delta\epsilon)}\lambda^{(\beta)}{}_{\nu.}{}^{\nu}\sigma,_{(\beta)}$$
$$+ \sigma^{-1}(2u_{(\alpha\epsilon),(\beta)}\lambda^{(\beta)\nu}\lambda^{\mu}_{(\delta)}\lambda^{(\alpha)}{}_{\mu.\nu} + u_{(\delta\epsilon),(\beta)}\lambda^{(\beta)}{}_{\nu.}{}^{\nu})$$
$$+ v_{(\alpha\epsilon),(\beta)}(2\lambda^{(\beta)\nu}\lambda^{(\alpha)}{}_{\mu.\nu}\lambda^{\mu}_{(\delta)} + \delta_\delta{}^\alpha\lambda^{(\beta)}{}_{\nu.}{}^{\nu})\log|\sigma|$$
$$+ w_{(\alpha\epsilon),(\beta)}(2\lambda^{(\beta)\nu}\lambda^{(\alpha)}{}_{\mu.\nu}\lambda^{\mu}_{(\delta)} + \delta_\delta{}^\alpha\lambda^{(\beta)}{}_{\nu.}{}^{\nu})\}. \tag{2.8}$$

If we write the wave equation (2.5) in the form

$$H(A_{(\delta)}) = 0, \tag{2.9}$$

then combining (2.6)–(2.8) we see that the vector wave equation (2.5) can be written as

$$\frac{1}{(2\pi)^2}\{-\sigma^{-2}(2\eta^{(\beta\gamma)}u_{(\delta\epsilon),(\beta)}\sigma,_{(\gamma)} + u_{(\delta\epsilon)}\lambda^{(\beta)}{}_{\nu.}{}^{\nu}\sigma,_{(\beta)})$$
$$+ \sigma^{-1}(H(u_{(\delta\epsilon)}) + 2v_{(\delta\epsilon)} + 2\eta^{(\beta\gamma)}v_{(\delta\epsilon),(\beta)}\sigma,_{(\gamma)} + v_{(\delta\epsilon)}\lambda^{(\beta)}{}_{\nu.}{}^{\nu}\sigma,_{(\beta)})$$
$$+ H(v_{(\delta\epsilon)})\log|\sigma| + H(w_{(\delta\epsilon)})\} = 0. \tag{2.10}$$

In order for this expression to vanish the coefficient of the logarithmic factor must vanish everywhere, and the coefficients of the singular factors σ^{-2} and σ^{-1} must vanish at least on the light cone. This is achieved most simply by taking the coefficient of σ^{-2} to vanish everywhere so that we have

$$2\eta^{(\beta\gamma)}u_{(\delta\epsilon),(\beta)}\sigma,_{(\gamma)} + u_{(\delta\epsilon)}\lambda^{(\beta)}{}_{\nu.}{}^{\nu}\sigma,_{(\beta)} = 0, \tag{2.11}$$

$$H(v_{(\delta\epsilon)}) = 0, \tag{2.12}$$

$$H(u_{(\delta\epsilon)}) + 2v_{(\delta\epsilon)} + 2\eta^{(\beta\gamma)}v_{(\delta\epsilon),(\gamma)}\sigma,_{(\beta)} + v_{(\delta\epsilon)}\lambda^{(\beta)}{}_{\nu.}{}^{\nu}\sigma,_{(\beta)} + \sigma H(w_{(\delta\epsilon)}) = 0. \tag{2.13}$$

To find the complete solution to (2.11) we require some knowledge of the divergence of the vierbein. We find this in the following manner:

$$\lambda^{(\beta)}{}_{\nu.}{}^{\nu}\sigma,_{(\beta)} = (\lambda^{(\beta)}_{\nu}\sigma,_{(\beta)}) - \eta^{(\beta\gamma)}\sigma,_{(\beta\gamma)}$$
$$= \sigma.{}^{\nu}_{\nu} - 4, \tag{2.14}$$

from (1.23).

Comparision of this equation with (1.52), gives the identity

$$\lambda^{(\beta)}{}_{\nu.}{}^{\nu}\sigma,_{(\beta)} \equiv -\Delta^{-1}\Delta._{\nu}\sigma.^{\nu} \equiv -\Delta^{-1}\Delta,_{(\beta)}\sigma,^{(\beta)}. \tag{2.15}$$

This is a most interesting equation; it shows that we can use the propagated vierbein, instead of the bi-scalar Δ, to determine the rate of divergence of neighbouring geodesics. A result of this form was to be expected possibly, since each geodesic is determined uniquely by the vierbein and an associated direction, the equation (2.15) is notably of the second order in the geodesic distance. The appropriate normalisation condition for the invariant $u_{(\delta\epsilon)}$ is

$$[u_{(\delta\epsilon)}] = \eta_{(\delta\epsilon)} . \tag{2.16}$$

Equations (2.11) and (2.16) serve to define $u_{(\delta\epsilon)}$ uniquely. Noting the relation (2.15) we find

$$u_{(\delta\epsilon)} = \Delta^{1/2}\eta_{(\delta\epsilon)} . \tag{2.17}$$

The solutions, for $v_{(\delta\epsilon)}$ and $w_{(\delta\epsilon)}$, of Eqs. (2.12) and (2.13) are most easily obtained by expanding the functions in a power series

$$v_{(\delta\epsilon)} = \sum_{n=0}^{\infty} v_{n(\delta\epsilon)}\sigma^n, \qquad w_{(\delta\epsilon)} = \sum_{n=0}^{\infty} w_{n(\delta\epsilon)}\sigma^n, \tag{2.18}$$

and obtaining the recurrence formulas for the coefficients. However, it will suffice for our purposes to note that the limiting condition on $v_{(\delta\epsilon)}$, as obtained from (2.11) and (2.17) with

$$\eta^{(\beta\gamma)}u_{(\delta\epsilon),(\beta\gamma)} = -\tfrac{1}{6}R\eta_{(\delta\epsilon)} + O(s), \tag{2.19}$$

from (1.53) and (2.18), can be written, with use of (1.2), (1.3), (1.26), and (1.27), as

$$v_{(\delta\epsilon)} = -\tfrac{1}{2}\{R_\mu{}^\nu\lambda^\mu_{(\delta)}\lambda_{(\epsilon)\nu} - \tfrac{1}{6}R\eta_{(\delta\epsilon)}\} + O(s). \tag{2.20}$$

The elementary Green's function, given by (2.6), therefore takes the form

$$G^{(1)}_{(\delta\epsilon)} = \frac{1}{(2\pi)^2}\left\{\frac{\Delta^{1/2}\eta_{(\delta\epsilon)}}{\sigma} + v_{(\delta\epsilon)}\log|\sigma| + w_{(\delta\epsilon)}\right\}, \tag{2.21}$$

from which the symmetric Green's function, which is a solution of the inhomogeneous wave equation, can be obtained by moving into the complex plane. We introduce the Feynman propagator

$$G^F_{(\delta\epsilon)} = \frac{1}{(2\pi)^2}\left\{\frac{\Delta^{1/2}\eta_{(\delta\epsilon)}}{\sigma + i0} + v_{(\delta\epsilon)}\log(\sigma + i0) + w_{(\delta\epsilon)}\right\}, \tag{2.22}$$

and then separate it into real and imaginary parts

$$G^F_{(\delta\epsilon)} = G^{(1)}_{(\delta\epsilon)} - 2i\bar{G}_{(\delta\epsilon)} . \tag{2.23}$$

Using the formal identities

$$(\sigma + i0)^{-1} = \mathscr{P}\sigma^{-1} - \pi i\delta(\sigma), \tag{2.24}$$

$$\log(\sigma + i0) = \log|\sigma| + \pi i\theta(-\sigma), \tag{2.25}$$

where

$$\theta(\sigma) = \begin{Bmatrix} 0, & \sigma < 0 \\ 1, & \sigma > 0 \end{Bmatrix}, \tag{2.26}$$

we find for the "symmetric" Green's function, $\bar{G}_{(\delta\epsilon)}$,

$$\bar{G}_{(\delta\epsilon)} = (8\pi)^{-1}\{\Delta^{1/2}\eta_{(\delta\epsilon)}\delta(\sigma) - v_{(\delta\epsilon)}\theta(-\sigma)\}. \tag{2.27}$$

We now define the various Green's functions

$$G^{\text{ret}}_{(\delta\epsilon)}(x, z) = 2\theta[\Sigma(x), z]\,\bar{G}_{(\delta\epsilon)}(x, z), \tag{2.28}$$

$$G^{\text{adv}}_{(\delta\epsilon)}(x, z) = 2\theta[z, \Sigma(x)]\,\bar{G}_{(\delta\epsilon)}(x, z), \tag{2.29}$$

$$G_{(\delta\epsilon)} = G^{\text{adv}}_{(\delta\epsilon)} - G^{\text{ret}}_{(\delta\epsilon)}, \tag{2.30}$$

where $\Sigma(x)$ is an arbitrary space-like hypersurface containing x, and $\theta[\Sigma(x), z] = 1 - \theta[z, \Sigma(x)]$ is equal to 1 when z lies to the past of $\Sigma(x)$ and vanishes when z lies in the future. These various Green's functions satisfy the equations

$$\bar{G}_{(\delta\epsilon)} = \tfrac{1}{2}(G^{\text{ret}}_{(\delta\epsilon)} + G^{\text{adv}}_{(\delta\epsilon)}), \tag{2.31}$$

$$H(\bar{G}_{(\delta\epsilon)}) = H(G^{\text{ret}}_{(\delta\epsilon)}) = H(G^{\text{adv}}_{(\delta\epsilon)}) = -\eta_{(\delta\epsilon)}\bar{g}^{-1/2}\delta^{(4)}, \tag{2.32}$$

$$H(G_{(\delta\epsilon)}) = 0, \tag{2.33}$$

and the symmetry properties

$$G_{(\delta\epsilon)}(x, z) = -G_{(\epsilon\delta)}(z, x), \tag{2.34}$$

$$G^{\text{ret}}_{(\delta\epsilon)}(x, z) = G^{\text{adv}}_{(\epsilon\delta)}(z, x), \tag{2.35}$$

$$\bar{G}_{(\delta\epsilon)}(x, z) = \bar{G}_{(\epsilon\delta)}(z, x), \tag{2.36}$$

$$v_{(\delta\epsilon)}(x, z) = v_{(\epsilon\delta)}(z, x). \tag{2.37}$$

Finally it should be pointed out that (2.13) and (2.18) leave $w_{0(\delta\epsilon)}$ arbitrary in the solution for $w_{(\delta\epsilon)}$ which corresponds to adding to $G^{(1)}$ any singularity free solution of the wave equation. However this arbitrariness disappears in the solution for the symmetric Green's function $\bar{G}_{(\delta\epsilon)}$.

III. THE VECTOR POTENTIAL

The dynamical equations for a structureless point particle of charge "e" and "bare" mass m_0, interacting with an electromagnetic field $F_{\mu\nu}$ in a space–time of arbitrary fixed metric, are given in nonrationalized relativistic units by

$$m_0 \ddot{z}^\mu = e F^\mu{}_\nu \dot{z}^\nu, \tag{3.1}$$

$$g^{1/2} F^{\mu\nu}{}_{\cdot\nu} = 4\pi j^\mu, \tag{3.2}$$

and the corresponding law for the stress density of the system

$$T^{\mu\nu}{}_{\cdot\nu} = 0, \tag{3.3}$$

where j^μ is the current density and

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}\,. \tag{3.4}$$

In the Lorentz gauge,

$$g^{\mu\nu} A_{\mu.\nu} = 0, \tag{3.5}$$

substitution of (3.4) into (3.2) yields the equation

$$-4\pi j^\mu = g^{1/2}(g^{\nu\sigma} A^\mu{}_{\cdot\nu\sigma} + R_\nu{}^\mu A^\nu), \tag{3.6}$$

or, upon introduction of the vierbein propagated between z and x,

$$-4\pi j^{(\alpha)} = g^{1/2} H(A^{(\alpha)}), \qquad j^{(\alpha)} = j^\mu \lambda_\mu^{(\alpha)}, \tag{3.7}$$

of which particular solutions are given by

$$A^{\text{ret}}_{(\alpha)}(x) = 4\pi \int G^{\text{ret}}_{(\alpha\beta)}(x, x')\, j^{(\beta)}(x')\, d^4x', \tag{3.8}$$

$$A^{\text{adv}}_{(\alpha)}(x) = 4\pi \int G^{\text{adv}}_{(\alpha\beta)}(x, x')\, j^{(\beta)}(x')\, d^4x', \tag{3.9}$$

yielding the retarded and advanced proper fields of the particle

$$\begin{aligned} F^{\text{ret}}_{(\alpha\beta)} &= (A^{\text{ret}}_{\nu,\mu} - A^{\text{ret}}_{\mu,\nu})\, \lambda^\mu_{(\alpha)} \lambda^\nu_{(\beta)} \\ &= A^{\text{ret}}_{(\beta).(\alpha)} - A^{\text{ret}}_{(\alpha).(\beta)} + A^{\text{ret}}_{(\gamma)} \lambda^\mu_{(\alpha)} \lambda^\nu_{(\beta)} \{\lambda^{(\gamma)}{}_{\nu.\mu} - \lambda^{(\gamma)}{}_{\mu.\nu}\}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} F^{\text{adv}}_{(\alpha\beta)} &= (A^{\text{adv}}_{\nu.\mu} - A^{\text{adv}}_{\mu.\nu})\, \lambda^\mu_{(\alpha)} \lambda^\nu_{(\beta)} \\ &= A^{\text{adv}}_{(\beta).(\alpha)} - A^{\text{adv}}_{(\alpha).(\beta)} + A^{\text{adv}}_{(\gamma)} \lambda^\mu_{(\alpha)} \lambda^\nu_{(\beta)} \{\lambda^{(\gamma)}{}_{\nu.\mu} - \lambda^{(\gamma)}{}_{\mu.\nu}\}. \end{aligned} \tag{3.11}$$

It can be easily shown that the solutions (3.8) and (3.9) satisfy the Lorentz condition corresponding to (3.5).

The total field may be expressed in the alternative forms

$$F_{(\alpha\beta)} = F^{\text{in}}_{(\alpha\beta)} + F^{\text{ret}}_{(\alpha\beta)} = F^{\text{out}}_{(\alpha\beta)} + F^{\text{adv}}_{(\alpha\beta)}, \tag{3.12}$$

which serve as definitions for the fields $F^{\text{in}}_{(\alpha\beta)}$ and $F^{\text{out}}_{(\alpha\beta)}$. Another useful form is

$$F_{(\alpha\beta)} = \bar{F}^{\text{free}}_{(\alpha\beta)} + \bar{F}_{(\alpha\beta)}, \tag{3.13}$$

where

$$\bar{F}_{(\alpha\beta)} = \tfrac{1}{2}(F^{\text{ret}}_{(\alpha\beta)} + F^{\text{adv}}_{(\alpha\beta)}), \tag{3.14}$$

$$\bar{F}^{\text{free}}_{(\alpha\beta)} = \tfrac{1}{2}(F^{\text{in}}_{(\alpha\beta)} + F^{\text{out}}_{(\alpha\beta)}) = F^{\text{in}}_{(\alpha\beta)} + \tfrac{1}{2}F^{\text{rad}}_{(\alpha\beta)} = F^{\text{out}}_{(\alpha\beta)} - \tfrac{1}{2}F^{\text{rad}}_{(\alpha\beta)}, \tag{3.15}$$

and

$$F^{\text{rad}}_{(\alpha\beta)} = F^{\text{ret}}_{(\alpha\beta)} - F^{\text{adv}}_{(\alpha\beta)}. \tag{3.16}$$

The fields $\bar{F}_{(\alpha\beta)}$ and $F^{\text{rad}}_{(\alpha\beta)}$ may be expressed in terms of potentials $\bar{A}_{(\alpha)}$ and $A^{\text{rad}}_{(\alpha)}$ which are defined by integral expressions of the form (3.8) and (3.9), involving the functions $\bar{G}_{(\alpha\beta)}$ and $G_{(\alpha\beta)}$, respectively. The various fields thus defined satisfy the equations

$$\begin{aligned}
&\bar{F}^{(\alpha\beta)}{}_{,(\beta)} + \bar{F}^{(\gamma\beta)}\lambda_{(\gamma)}{}^{\mu}{}_{.\nu}\lambda^{\nu}_{(\beta)}\lambda^{(\alpha)}_{\mu} + \bar{F}^{(\alpha\beta)}\lambda_{(\beta)}{}^{\nu}{}_{.\nu} \\
&= F^{\text{ret}(\alpha\beta)}{}_{,(\beta)} + F^{\text{ret}(\gamma\beta)}\lambda_{(\gamma)}{}^{\mu}{}_{.\nu}\lambda^{\nu}_{(\beta)}\lambda^{(\alpha)}_{\mu} + F^{\text{ret}(\alpha\beta)}\lambda_{(\beta)}{}^{\nu}{}_{.\nu} \\
&= F^{\text{adv}(\alpha\beta)}{}_{,(\beta)} + F^{\text{adv}(\gamma\beta)}\lambda_{(\gamma)}{}^{\mu}{}_{.\nu}\lambda^{\nu}_{(\beta)}\lambda^{(\alpha)}_{\mu} + F^{\text{adv}(\alpha\beta)}\lambda_{(\beta)}{}^{\nu}{}_{.\nu} \\
&= 4\pi g^{-1/2} j^{(\alpha)},
\end{aligned} \tag{3.17}$$

$$F^{\begin{bmatrix}\text{in}\\ \text{out}\\ \text{free}\\ \text{rad}\end{bmatrix}(\alpha\beta)}{}_{,(\beta)} + F^{\begin{bmatrix}\text{in}\\ \text{out}\\ \text{free}\\ \text{rad}\end{bmatrix}(\gamma\beta)}\lambda_{(\gamma)}{}^{\mu}{}_{.\nu}\lambda^{\nu}_{(\beta)}\lambda^{(\alpha)}_{\mu} + F^{\begin{bmatrix}\text{in}\\ \text{out}\\ \text{free}\\ \text{rad}\end{bmatrix}(\alpha\beta)}\lambda_{(\beta)}{}^{\nu}{}_{.\nu} = 0. \tag{3.18}$$

Substituting the explicit forms (2.26)–(2.28) and for the current density

$$j^{(\alpha)} = e\int \delta^{1/2}\dot{z}^{(\alpha)}\,\delta^{(4)}\,d\tau', \qquad \delta = |\,\lambda^{(\alpha)}_{\mu}(x)\,\lambda^{\nu}_{(\alpha)}(z)|, \tag{3.19}$$

and designating retarded and advanced by "$-$" and "$+$" respectively, we get

$$A^{\pm}_{(\alpha)} = 4\pi e \int_{-\infty}^{+\infty} G^{\pm}_{(\alpha\beta)} \dot{z}^{(\beta)}\, d\tau'$$

$$= \pm e \int_{\tau_{\Sigma}}^{\pm\infty} \{\Delta^{1/2}\eta_{(\alpha\beta)}\, \delta(\sigma) - v_{(\alpha\beta)}\theta(-\sigma)\}\, \dot{z}^{(\beta)}\, d\tau', \tag{3.20}$$

where τ_{Σ} is the value of the proper time at the point of intersection of the world line of the particle with arbitrary spacelike hypersurface $\Sigma(X)$ containing x.

Changing the variable of integration from τ to σ, noting that

$$\sigma_{\Sigma} = \sigma(x, \tau_{\Sigma}) > 0, \tag{3.21}$$

$$\sigma(x, z(\pm\infty)) = -\infty \text{ for non-"runaway" trajectories,} \tag{3.22}$$

$$d\sigma = \sigma_{,(\alpha)}\dot{z}^{(\alpha)} d\tau' = \dot{\sigma} d\tau', \tag{3.23}$$

and defining the advanced and retarded proper time $\tau_{\pm}$, of the particle relative to the point x by

$$\sigma(x, z(\tau_{\pm})) = 0,$$

$$\tau_{+} > \tau_{\Sigma}, \qquad \tau_{-} < \tau_{\Sigma}, \tag{3.24}$$

we find

$$A^{\pm}_{(\alpha)} = \mp e \left| \frac{\Delta^{1/2}\eta_{(\alpha\beta)}\dot{z}^{(\beta)}}{\dot{\sigma}} \right|_{\tau'=\tau_{\pm}} \mp e \int_{\tau_{\pm}}^{\pm\infty} v_{(\alpha\beta)}\dot{z}^{(\beta)}\, d\tau'. \tag{3.25}$$

These are the invariant forms of the covariant Lienard-Wiechert potentials referred to the components of the propagated vierbein.

The corresponding field strengths can be obtained from (3.10) and (3.11), with use of (3.20) and the properties of the Dirac delta function, we find

$$F^{\pm}_{(\alpha\beta)} = \mp e \left| \frac{\Delta^{1/2}{}_{,(\beta)}\dot{z}_{(\alpha)} - \Delta^{1/2}{}_{,(\alpha)}\dot{z}_{(\beta)}}{D\sigma} - \frac{1}{D\sigma} D \left\{ \frac{\Delta^{1/2}(\dot{z}_{(\beta)}\sigma_{,(\alpha)} - \dot{z}_{(\alpha)}\sigma_{,(\beta)})}{D\sigma} \right\} \right.$$

$$\left. + \frac{v_{(\beta\gamma)}\dot{z}^{(\gamma)}\sigma_{,(\alpha)} - v_{(\alpha\gamma)}\dot{z}^{(\gamma)}\sigma_{,(\beta)}}{D\sigma} \right|_{\tau'=\tau_{\pm}} \mp e \int_{\tau_{\pm}}^{\pm\infty} (v_{(\beta\gamma).(\alpha)} - v_{(\alpha\gamma).(\beta)})\, \dot{z}^{(\gamma)}\, d\tau'$$

$$\mp e\lambda^{\mu}_{(\alpha)}\lambda^{\nu}_{(\beta)}(\lambda^{(\gamma)}{}_{\nu.\mu} - \lambda^{(\gamma)}{}_{\mu.\nu}) \left| \frac{\Delta^{1/2}\dot{z}_{(\gamma)}}{D\sigma} \right|_{\tau'=\tau_{\pm}}$$

$$\mp e\lambda_{(\alpha)}\lambda^{\nu}_{(\beta)}(\lambda^{(\gamma)}{}_{\nu.\mu} - \lambda^{(\gamma)}{}_{\mu.\nu}) \int_{\tau_{\pm}}^{\pm\infty} v_{(\gamma\delta)}\dot{z}^{(\delta)}\, d\tau'. \tag{3.26}$$

Finally, it should be made clear that (3.25) and (3.26), were derived under the assumption that σ is single-valued. Even though this may be true for the leading terms, which involve the behaviour of the particle only at the retarded and advanced proper times, it will not generally be true for the "tail" terms involving integrations over the whole past or future history of the particle. However, since the wave equation is linear, the superposition principle holds, and it is clear that the appropriate $v_{(\alpha\beta)}$ to use in the "tail" terms is the sum of the v's for all different geodesics between x and z, each term in the sum representing the contribution of an elementary wave.

IV. THE WORLD TUBE

In this section we construct a three-dimensional hypersurface about the world line of the particle, the world tube, generated by a small sphere surrounding the particle as time varies. In the vierbein notation of Section I, the generating sphere of radius ϵ, as time varies, produces a hypersphere defined by

$$2\sigma = \eta_{(\alpha\beta)} x^{(\alpha)} x^{(\beta)} = \epsilon^2, \tag{4.1}$$

$$\eta_{(\alpha\beta)} x^{(\alpha)} \dot{z}^{(\beta)} = 0, \tag{4.2}$$

the second equation arising from the orthogonality of the spatial components of

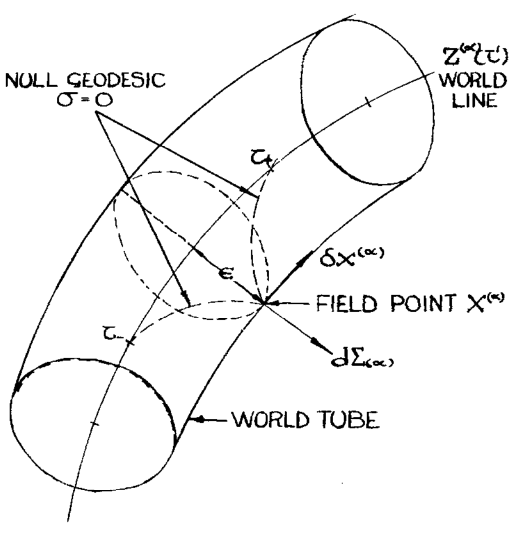

FIG. 2. The world tube.

the vierbein with respect to the four velocity of the particle at time τ. The $x^{(\alpha)}$ used here are the special coordinates defined in section 1 by the relation

$$x^{(\alpha)} = -\sigma_{.\mu}\lambda^{\mu(\alpha)}. \tag{4.3}$$

Let $\delta x^{(\alpha)}$ be an arbitrary displacement, of the point $x^{(\alpha)}$, within the hypersphere defined above; see Fig. 2. This variation produces a corresponding displacement in the proper time τ', which we denote by $\delta\tau'$. Writing $D = \delta/\delta\tau'$, the variation of Eqs. (4.1) and (4.2) are

$$\eta_{(\alpha\beta)}x^{(\alpha)}\delta x^{(\beta)} + \eta_{(\alpha\beta)}x^{(\alpha)}Dx^{(\beta)}\delta\tau' = 0, \tag{4.4}$$

$$\eta_{(\alpha\beta)}\delta x^{(\alpha)}\dot{z}^{(\beta)} + \eta_{(\alpha\beta)}\dot{z}^{(\beta)}Dx^{(\alpha)} + \eta_{(\alpha\beta)}x^{(\alpha)}D\dot{z}^{(\beta)}\delta\tau' = 0. \tag{4.5}$$

By (4.2) we also have

$$\eta_{(\alpha\beta)}x^{(\alpha)}Dx^{(\beta)} = \tfrac{1}{2}D(\eta_{(\alpha\beta)}x^{(\alpha)}x^{(\beta)}) = D\sigma = \sigma_{.\mu}\dot{z}^{\mu} = -x_{(\alpha)}\dot{z}^{(\alpha)} = 0, \tag{4.6}$$

and hence (4.4) and (4.5) may be written in the form

$$\eta_{(\alpha\beta)}x^{(\alpha)}\delta x^{(\beta)} = 0, \tag{4.7}$$

$$\eta_{(\alpha\beta)}\dot{z}^{(\alpha)}\delta x^{(\beta)} = -\{\eta_{(\alpha\beta)}\dot{z}^{(\beta)}Dx^{(\alpha)} + \eta_{(\alpha\beta)}x^{(\alpha)}D\dot{z}^{(\beta)}\}\, d\tau'. \tag{4.8}$$

Now (4.7) shows that the normal to the surface is in the "direction" of the invariant bein vector $x^{(\alpha)}$ and (4.8) shows that if we split up "δx" into a part $\delta_1 x$ orthogonal to $\dot{z}^{(\alpha)}$ and a part $\delta_2 x$ parallel to $\dot{z}^{(\alpha)}$ the magnitude of $\delta_2 x$ is

$$|\, \delta_2 x \,| = -\{\eta_{(\alpha\beta)}\dot{z}^{(\beta)}Dx^{(\alpha)} + \eta_{(\alpha\beta)}x^{(\alpha)}D\dot{z}^{(\beta)}\}\, d\tau'. \tag{4.9}$$

The three-dimensional "area" of an element of the surface is equal to the two-dimensional area, dS say, of an element of a section of the surface by a three dimensional plane orthogonal to $\dot{z}^{(\alpha)}$, multiplied by the element $|\, \delta_2 x \,|$ parallel to $\dot{z}^{(\alpha)}$. Thus the "directed" invariant element of the surface of the hypersphere can be written

$$d\Sigma^{(\alpha)} = |\, x^{(\alpha)} \,|^{-1}\, |\, \delta_2 x \,|\, x^{(\alpha)} dS. \tag{4.10}$$

Using (4.3) and noting that the determinant of the propagated vierbein is $g^{1/2}$, we have

$$|\, x^{(\alpha)} \,| = g^{1/2}\epsilon, \tag{4.11}$$

which when combined with (4.9) gives for the "directed" invariant surface element

$$d\Sigma^{(\alpha)} = -\epsilon^{-1}g^{-1/2}\,(\eta_{(\beta\gamma)}x^{(\beta)}D\dot{z}^{(\gamma)} + \eta_{(\beta\gamma)}\dot{z}^{(\beta)}Dx^{(\gamma)})\, x^{(\alpha)}dS\, d\tau'. \tag{4.12}$$

Since we shall be interested in the case when the tube radius ϵ becomes infinitesimally small it will suffice to use expansions in powers of ϵ in evaluating expressions (4.9) and (4.12). Writing

$$\kappa^2 = -\sigma_{.\mu\nu}\dot{z}^\mu\dot{z}^\nu - \sigma_{.\mu}\ddot{z}^\mu, \tag{4.13}$$

we have from (1.7)

$$\kappa^2 = 1 + x_{(\alpha)}\ddot{z}^{(\alpha)} - \tfrac{1}{3}R_{\mu\ \rho}^{\ \nu\ \kappa}\dot{z}^\mu\dot{z}^\rho\sigma_{.\nu}\sigma_{.\kappa} + O(\epsilon^3), \tag{4.14}$$

and using this with (1.38)–(1.41), and (1.43) we find that (4.12) has the expansion[1]

$$\begin{aligned} d\Sigma^{(\alpha)} &= -g^{-1/2}\epsilon^{-1}(1 + x_{(\alpha)}\ddot{z}^{(\alpha)} - \tfrac{1}{3}R_{\nu\ \mu}^{\ \rho\ \kappa}\dot{z}^\mu\dot{z}^\nu\sigma_{.\rho}\sigma_{.\kappa})\, x^{(\alpha)}\, dS\, d\tau' + O(\epsilon^5) \\ &= -g^{-1/2}\epsilon^{-1}\kappa^2 x^{(\alpha)}\, dS\, d\tau' + O(\epsilon^5). \end{aligned} \tag{4.15}$$

In the calculations of the next section we shall not actually integrate over the

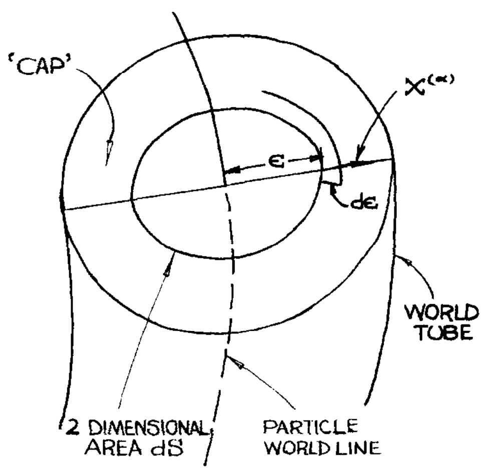

FIG. 3. The "cap" of the world tube.

[1] This is linked with the expression obtained by DeWitt and Brehme (*3*) through a length scale factor, which can be derived by defining the variation in the field point, from (4.3) by the relation

$$\delta x^{(\alpha)} = -\sigma,^{(\alpha)}{}_{(\beta)},\ \delta\bar{x}^{(\beta)},$$

the overbar relating to their coordinate system. Then, since we are dealing with arbitrary variations, the ratio of lengths in the different coordinate systems will be given by

$$\frac{|\delta x^{(\alpha)}|}{|\delta\bar{x}^{(\alpha)}|} = \Delta, \qquad \text{where} \qquad \Delta = |-\sigma,^{(\alpha)}{}_{(\beta)}|.$$

Thus (4.15) will contain the factor Δ^{-1}.

entire world tube. In fact we shall integrate only over an infinitesimal portion of it, plus the "caps" at the ends given by the geodetic cross sections $\tau' =$ constant. We shall therefore need an expression for the surface elements of the caps. This three-dimensional invariant bein-vector surface area can be broken down into the two-dimensional area dS, defined by the intersection of the hypersphere with the three-dimensional surface $\tau' =$ constant, multiplied by the increase, $d\epsilon$, in the radius of the hypersphere. The normal to this area, as given by (4.6), is $Dx^{(\alpha)}$. Thus for the "directed" area of the caps we have

$$d\Sigma^{(\alpha)} = \pm \mid Dx^{(\alpha)} \mid^{-1} Dx^{(\alpha)} dS\, d\epsilon. \tag{4.16}$$

The plus and minus sign is attached to the caps lying in the future or past of "τ" in order to preserve continuity of direction of the surface element relative to the interior of the tube.

We shall also need expansions for the retarded and advanced proper times at which the quantities appearing in (3.26) are to be evaluated. Introducing

$$\delta_\pm = \tau - \tau_\pm = O(\epsilon), \tag{4.17}$$

and recalling the defining equation (3.24), we have

$$0 = \sigma + \delta_\pm \dot{\sigma} + \tfrac{1}{2}\delta_\pm{}^2 \ddot{\sigma} + \tfrac{1}{6}\delta_\pm{}^3 \dddot{\sigma} + \tfrac{1}{24}\delta_{\pm,}\ddddot{\sigma} + O(\epsilon^5), \tag{4.18}$$

where σ and its derivatives are here to be evaluated at the points x and z. Making use of equations (1.31)–(1.34) and the expansions (1.7)–(1.9) and taking note of (1.44), (1.45), (4.1), (4.2), (4.13) and the symmetry properties of the Riemann tensor we find that Eq. (4.18) becomes

$$0 = \tfrac{1}{2}\epsilon^2 - \tfrac{1}{2}\delta_\pm{}^2\kappa^2 + \tfrac{1}{6}\delta_\pm{}^3 \sigma_{,(\alpha)} \dddot{z}^{(\alpha)} - \tfrac{1}{24}\delta_\pm{}^4 \ddot{z}^2 + O(\epsilon^5), \tag{4.19}$$

from which we obtain, on inverting the series,

$$\delta_\pm = \pm\epsilon\kappa^{-1}(1 \pm \tfrac{1}{6}\epsilon\sigma_{,(\alpha)}\ddot{z}^{(\alpha)} - \tfrac{1}{24}\epsilon^2\ddot{z}^2) + O(\epsilon^4). \tag{4.20}$$

Finally we record the expansions

$$\begin{aligned} \mid D\sigma \mid_{\tau'=\tau_\pm} &= \dot{\sigma} + \delta_\pm\ddot{\sigma} + \tfrac{1}{2}\delta_\pm{}^2\dddot{\sigma} + \tfrac{1}{6}\delta_\pm{}^3\ddddot{\sigma} + O(\epsilon^4) \\ &= -\delta_\pm\kappa^2(1 - \tfrac{1}{2}\delta_\pm\sigma_{,(\alpha)}\dddot{z}^{(\alpha)} + \tfrac{1}{6}\delta_\pm{}^2\ddot{z}^2) + O(\epsilon^4), \end{aligned} \tag{4.21}$$

$$\mid D\sigma \mid^{-1}_{\tau'=\tau_\pm} = -\delta_\pm^{-1}\kappa^{-2}(1 + \tfrac{1}{2}\delta_\pm\sigma_{,(\alpha)}\dddot{z}^{(\alpha)} - \tfrac{1}{6}\delta_\pm{}^2\ddot{z}^2) + O(\epsilon^2). \tag{4.22}$$

V. THE EQUATIONS OF MOTION

The conservation law (3.3) expresses the energy balance between the field and the particle. For practical application this differential characterization must be replaced by an integral one. The difficulty arising from the fact that $\int T^{\mu\nu} d^4x$ is not an invariant, nor even a vector, can be overcome in a natural way by replacing (3.3) by the invariant equation

$$\lambda_\mu^{(\alpha)} T^{\mu\nu}{}_{.\nu} = 0. \tag{5.1}$$

The integral representation, $\int \lambda^{(\alpha)} T^{\mu\nu}{}_{,\nu} d^4x$, arising from (5.1), is an invariant "bein" vector and consequently it is permissable to apply Gauss theorem and embark on a theory parallel to that of flat space. If we now let Σ denote the surface of the world tube between two proper times τ_1 and τ_2, and denote by Σ_1 and Σ_2 the corresponding "caps," and by V the interior of the tube enclosed by Σ, Σ_1 and Σ_2, we may write

$$\begin{aligned} 0 &= \int_V \lambda_\mu^{(\alpha)} T^{\mu\nu}{}_{.\nu}\, d^4x \\ &= \left(\int_\Sigma + \int_{\Sigma_1} + \int_{\Sigma_2}\right) \lambda_\mu^{(\alpha)} T^{\mu\nu}\, d\Sigma_\nu - \int_V \lambda^{(\alpha)}{}_{\mu.\nu} T^{\mu\nu}\, d^4x. \end{aligned} \tag{5.2}$$

Let us now take the limit $\epsilon \to 0$. The integrals over Σ_1, Σ_2 and V will then retain contributions only from the particle stress density; and if, furthermore, we take the fixed point z to lie on the particle world line at a proper time τ between τ_1 and τ_2, we then have, assuming $\tau_1 < \tau_2$ and making use of (4.16) and

$$\lambda_\mu^{(\alpha)} \lambda_\nu^{(\beta)} T_P^{\mu\nu} = m_0 \int \delta^{1/2} \dot{z}^{(\alpha)} \dot{z}^{(\beta)}\, \delta^{(4)}\, d\tau', \tag{5.3}$$

the expression for the particle stress density, from (5.2)

$$0 = \operatorname*{Lim}_{\epsilon\to 0} \left\{ \int_{\tau_1}^{\tau_2} \int_{4\pi} \lambda_\mu^{(\alpha)} T^{\mu\nu}\, d\Sigma_\nu \right\} + m_0 [\lambda_\mu^{(\alpha)} \dot{z}^\mu]_{\tau'=\tau_1}^{\tau'=\tau_2} - m_0 \int_{\tau_1}^{\tau_2} \lambda^{(\alpha)}{}_{\mu.\nu} \dot{z}^\mu \dot{z}^\nu\, d\tau'. \tag{5.4}$$

The next step is to let τ_1 and τ_2 approach τ. Denoting their infinitesimal separation by $d\tau$ we see that (5.4) becomes

$$0 = m_0 \ddot{z}^{(\alpha)}\, d\tau + \operatorname*{Lim}_{\epsilon\to 0} \int_{4\pi} \lambda_\mu^{(\alpha)} T^{\mu\nu}\, d\Sigma_\nu\,, \tag{5.5}$$

which can be written, using the notation

$$T^{(\alpha\beta)} = T^{\mu\nu} \lambda_\mu^{(\alpha)} \lambda_\nu^{(\beta)}, \tag{5.6}$$

as

$$0 = m_0 \ddot{z}^{(\alpha)}\, d\tau + \operatorname*{Lim}_{\epsilon\to 0} \int_{4\pi} T^{(\alpha\beta)}\, d\Sigma_{(\beta)} \,. \tag{5.7}$$

The remainder of this section will be devoted to the computation of the second term in (5.7). We must first get the retarded and advanced proper fields (3.26) in the form of expansions. Noting

$$\sigma_{,\mu'} \lambda^{\mu'}_{(\alpha)}(x) = -\sigma_{,\mu} \lambda^{\mu}_{(\alpha)}(z) = x_{(\alpha)} \,, \tag{5.8}$$

we begin by computing

$$\begin{aligned}
| \sigma_{,\mu'} \lambda^{\mu'}_{(\alpha)}(x) |_{\tau'=\tau_\pm} &= x_{(\alpha)} + \delta_\pm D x_{(\alpha)} + \tfrac{1}{2}\delta_\pm{}^2 D^2 x_{(\alpha)} + \tfrac{1}{6}\delta_\pm{}^3 D^3 x_{(\alpha)} + O(\epsilon^4) \\
&= x_{(\alpha)} - \delta_\pm \dot{z}_{(\alpha)}\,,\, - \tfrac{1}{2}\delta_\pm{}^2 \ddot{z}_{(\alpha)}\,,\, - \tfrac{1}{6}\delta_\pm{}^3 \dddot{z}_{(\alpha)} \\
&\quad - \tfrac{1}{6}\delta_\pm R_\mu{}^{\nu\rho\kappa} \sigma_{,\nu} \sigma_{,\rho} \dot{z}^\mu \lambda_{(\alpha)\kappa} \\
&\quad + \tfrac{1}{3}\delta_\pm{}^2 R_{\mu\nu\rho}{}^{\kappa} \sigma_{,\kappa} \dot{z}^\nu \dot{z}^\rho \lambda^\mu_{(\alpha)}\,,\, + O(\epsilon^4).
\end{aligned} \tag{5.9}$$

Here we have used (1.38)–(1.40) with the understanding that all quantities refer to the world tube and comma's separate terms of different orders of magnitude. Also, from (1.41) and (1.42),

$$\begin{aligned}
| \dot{z}^{(\beta)} |_{\tau=\tau_\pm} &= \dot{z}^{(\beta)} + \delta_\pm D \dot{z}^{(\beta)} + \tfrac{1}{2}\delta_\pm{}^2 D^2 \dot{z}^{(\beta)} + O(\epsilon^3) \\
&= \dot{z}^{(\beta)}\,,\, + \delta_\pm \ddot{z}^{(\beta)}\,,\, + \tfrac{1}{2}\delta_\pm{}^2 \dddot{z}_{(\beta)} \\
&\quad + \tfrac{1}{2}\delta_\pm R_\nu{}^\rho{}_\mu{}^\kappa \dot{z}^\mu \dot{z}^\nu \sigma_{,\rho} \lambda^{(\beta)}_\kappa\,,\, + O(\epsilon^3).
\end{aligned} \tag{5.10}$$

Using (4.2) we have, from (1.53), (4.22), (5.9), and (5.10),

$$\begin{aligned}
&\left| \frac{1}{D\sigma} D \left\{ \frac{\Delta^{1/2}(\dot{z}_{(\beta)}\sigma_{,(\alpha)} - \dot{z}_{(\alpha)}\sigma_{,(\beta)})}{D\sigma} \right\} \right|_{\tau'=\tau_\pm} \\
&= (\eta_{(\alpha\kappa)}\eta_{(\beta\omega)} - \eta_{(\alpha\omega)}\eta_{(\beta\kappa)}) \times \{ -\delta_\pm^{-3} \kappa^{-4} x^{(\kappa)} \dot{z}^{(\omega)}\,, \\
&\quad + \tfrac{1}{2}\delta_\pm^{-1} \kappa^{-4} \ddot{z}^{(\kappa)} \dot{z}^{(\omega)}\,,\, - \tfrac{1}{2}\delta_\pm^{-2} \sigma_{,(\gamma)} \dddot{z}^{(\gamma)} x^{(\kappa)} \dot{z}^{(\omega)} + \tfrac{1}{2}\delta_\pm^{-1} x^{(\kappa)} \dddot{z}^{(\omega)} \\
&\quad + \tfrac{1}{12}\delta_\pm^{-3} R^{\mu\nu} \sigma_{,\mu} \sigma_{,\nu} x^{(\kappa)} \dot{z}^{(\omega)} - \tfrac{1}{12}\delta_\pm^{-1} R_{\mu\nu} \dot{z}^\mu \dot{z}^\nu x^{(\kappa)} \dot{z}^{(\omega)} \\
&\quad - \tfrac{1}{6}\delta_\pm^{-1} R_\nu{}^\rho{}_\mu{}^\kappa \lambda^{(\omega)}_\kappa \dot{z}^\mu \dot{z}^\nu \sigma_{,\rho} \dot{z}^{(\kappa)} - \tfrac{2}{3} \dot{z}^{(\kappa)} \dddot{z}^{(\omega)}\,, \} + O(\epsilon).
\end{aligned} \tag{5.11}$$

From (2.20) and (4.22) we have the expansion

$$\left| \frac{v_{(\beta\gamma)}\dot{z}^{(\gamma)}\sigma_{,(\alpha)} - v_{(\alpha\gamma)}\dot{z}^{(\gamma)}\sigma_{,(\beta)}}{D\sigma} \right|_{\tau'=\tau_\pm}$$

$$= (\eta_{(\alpha\kappa)}\eta_{(\beta\omega)} - \eta_{(\alpha\omega)}\eta_{(\beta\kappa)}) \times \{,, + \tfrac{1}{2}\delta_\pm^{-1} R_\mu{}^\nu \lambda^{\mu(\omega)} \dot{z}^\nu x^{(\kappa)}$$

$$- \tfrac{1}{12}\delta_\pm^{-1} R\dot{z}^{(\omega)} x^{(\kappa)} - \tfrac{1}{2} R_\mu{}^\nu \lambda^{\mu(\omega)} \dot{z}_\nu \dot{z}^{(\kappa)},\} + O(\epsilon), \tag{5.12}$$

and from (1.26), (1.27), (1.49), and (4.22) we have

$$\lambda^\mu_{(\alpha)}\lambda^\nu_{(\beta)}\{\lambda^{(\gamma)}{}_{\nu.\mu} - \lambda^{(\gamma)}{}_{\mu.\nu}\} \left| \frac{\Delta^{1/2}\dot{z}_{(\gamma)}}{D\sigma} \right|_{\tau'=\tau_\pm} = (\eta_{(\alpha\kappa)}\eta_{(\beta\omega)} - \eta_{(\alpha\omega)}\eta_{(\beta\kappa)})$$

$$\times \{,, - \tfrac{1}{2}\delta_\pm^{-1} R_\mu{}^\rho{}_\nu{}^\xi \sigma_{.\rho} \dot{z}_\xi \lambda^{\mu(\kappa)} \lambda^{\nu(\omega)},\} + O(\epsilon). \tag{5.13}$$

Finally, using (1.49) and (4.22), we have the expansion

$$\left| \frac{\Delta^{1/2}{}_{,(\beta)}\dot{z}_{(\alpha)} - \Delta^{1/2}{}_{,(\alpha)}\dot{z}_{(\beta)}}{D\sigma} \right|_{\tau'=\tau_\pm} = (\eta_{(\alpha\kappa)}\eta_{(\beta\omega)} - \eta_{(\alpha\omega)}\eta_{(\beta\kappa)})$$

$$\times \{,, - \tfrac{1}{6}\delta_\pm R_\nu{}^\mu \sigma_{.\mu} \lambda^{\nu(\kappa)} \dot{z}^{(\omega)} - \tfrac{1}{6} R_{\mu\nu} \lambda^{\mu(\kappa)} \dot{z}^\nu \dot{z}^{(\omega)},\} + O(\epsilon). \tag{5.14}$$

Combining (5.11)–(5.14) and using (4.20), we find for the retarded and advanced field strengths (3.26)

$$F^\pm_{(\alpha\beta)} = e(\eta_{(\alpha\kappa)}\eta_{(\beta\omega)} - \eta_{(\alpha\omega)}\eta_{(\beta\kappa)}) \times \Big\{ -\epsilon^{-3}\kappa^{-1} x^{(\kappa)} \dot{z}^{(\omega)},$$

$$+ \tfrac{1}{2}\epsilon^{-1}\kappa^{-3} \ddot{z}^{(\kappa)} \dot{z}^{(\omega)}, - \tfrac{1}{8}\epsilon^{-1} \ddot{z}^2 x^{(\kappa)} \dot{z}^{(\omega)} + \tfrac{1}{6}\epsilon^{-1} R_\nu{}^\mu \sigma_{.\mu} \lambda^{\nu(\kappa)} \dot{z}^{(\omega)}$$

$$+ \tfrac{1}{12}\epsilon^{-3} R^{\mu\nu} \sigma_{.\mu} \sigma_{.\nu} x^{(\kappa)} \dot{z}^{(\omega)} - \tfrac{1}{12}\epsilon^{-1} R_{\mu\nu} \dot{z}^\mu \dot{z}^\nu x^{(\kappa)} \dot{z}^{(\omega)}$$

$$- \tfrac{1}{6}\epsilon^{-1} R_\nu{}^\rho{}_\mu{}^\kappa \lambda^{(\omega)}_\kappa \dot{z}^\mu \dot{z}^\nu \sigma_{.\rho} \dot{z}^{(\kappa)} + \tfrac{1}{2}\epsilon^{-1} x^{(\kappa)} \dddot{z}^{(\omega)} + \tfrac{1}{12}\epsilon^{-1} R\dot{z}^{(\omega)} x^{(\kappa)}$$

$$- \tfrac{1}{2}\epsilon^{-1} R_\mu{}^\nu \lambda^{\mu(\omega)} \dot{z}_\nu x^{(\kappa)} + \tfrac{1}{2}\epsilon^{-1} R_\mu{}^\rho{}_\nu{}^\xi \sigma_{.\rho}\sigma_{.\xi} \lambda^{\mu(\kappa)} \lambda^{\nu(\omega)}$$

$$\mp \tfrac{1}{3} R_{\mu\nu} \lambda^{\mu(\kappa)} \dot{z}^\nu \dot{z}^{(\omega)} \mp \tfrac{2}{3} \dot{z}^{(\kappa)} \dddot{z}^{(\omega)} \pm \frac{1}{2} \int_\tau^{\pm\infty} f^{(\alpha\beta)}{}_{(\gamma)} \dot{z}^{(\gamma)}\, d\tau',\Big\} + O(\epsilon), \tag{5.15}$$

where

$$f_{(\alpha\beta\gamma)} = v_{(\alpha\gamma),(\beta)} - v_{(\beta\gamma),(\alpha)}\,. \tag{5.16}$$

From Eq. (5.15) it follows that the radiation field is everywhere finite. At the location of the particle itself we have, in fact

$$F^{\mathrm{rad}(\alpha\beta)} = F^{-(\alpha\beta)} - F^{+(\alpha\beta)}$$
$$= \tfrac{4}{3}e(\dot{z}^{(\alpha)}\dddot{z}^{(\beta)} - \dot{z}^{(\beta)}\dddot{z}^{(\alpha)}) + \tfrac{2}{3}eR_{\mu\nu}\dot{z}^{\nu}(\lambda^{\mu(\alpha)}\dot{z}^{(\beta)} - \lambda^{\mu(\beta)}\dot{z}^{(\alpha)})$$
$$+ e\int_{-\infty}^{+\infty} \epsilon(\tau - \tau')\, f^{(\alpha\beta)}{}_{(\gamma)}\, \dot{z}^{(\gamma)}(\tau')\, d\tau', \tag{5.17}$$

where

$$\epsilon(\tau) = \tau \mid \tau \mid^{-1} = \theta(\tau) - \theta(-\tau). \tag{5.18}$$

On the other hand, for the average of the retarded and advanced fields we have

$$\bar{F}_{(\alpha\beta)} = e(\eta_{(\alpha\kappa)}\eta_{(\beta\omega)} - \eta_{(\alpha\omega)}\eta_{(\beta\kappa)}) \times \Big\{-\epsilon^{-3}\kappa^{-1}x^{(\kappa)}\dot{z}^{(\omega)}\,,$$
$$+ \tfrac{1}{2}\kappa^{-3}e^{-1}\ddot{z}^{(\kappa)}\dot{z}^{(\omega)}\,, - \tfrac{1}{8}\epsilon^{-1}\ddot{z}^{2}x^{(\kappa)}\dot{z}^{(\omega)} + \tfrac{1}{2}\epsilon^{-1}x^{(\kappa)}\dddot{z}^{(\omega)}$$
$$+ \text{ terms linear and cubic in the } x\text{'s involving the Riemann tensor}$$
$$+ \frac{1}{4}\int_{-\infty}^{+\infty} f^{(\alpha\beta)}{}_{(\gamma)}\, \dot{z}^{(\gamma)}(\tau')\, d\tau'\,,\Big\} + O(\epsilon). \tag{5.19}$$

By breaking the total electromagnetic field up in the manner of equation (3.13) we may now use Eq. (5.19) to compute the stress density on the world tube. Noting that $\bar{F}^{\mathrm{free}}_{(\alpha\beta)}$ is singularity-free, or at any rate has no singularities arising from the particle itself, we may write

$$T^{(\alpha\beta)}\, d\Sigma_{(\beta)} = (4\pi)^{-1}\, g^{1/2}\{(\bar{F}^{(\alpha)}{}_{(\beta)}\bar{F}^{(\gamma\beta)} + \bar{F}^{\mathrm{free}(\alpha)}{}_{(\beta)}\bar{F}^{(\gamma\beta)} + \bar{F}^{(\alpha)}{}_{(\beta)}\bar{F}^{\mathrm{free}(\gamma\beta)})\, d\Sigma_{(\gamma)}$$
$$- (\tfrac{1}{4}\bar{F}_{(\beta\gamma)}\bar{F}^{(\beta\gamma)} + \tfrac{1}{2}\bar{F}^{\mathrm{free}}_{(\beta\gamma)}\bar{F}^{(\beta\gamma)})\, d\Sigma^{(\alpha)}\} + O(\epsilon). \tag{5.20}$$

Using (1.43)–(1.45), (4.14), and (4.15) we find, by straightforward computation,

$$g^{1/2}(\bar{F}^{(\alpha)}{}_{(\beta)}\bar{F}^{(\gamma\beta)}\, d\Sigma_{(\gamma)} - \tfrac{1}{4}\bar{F}_{(\beta\gamma)}\bar{F}^{(\beta\gamma)}\, d\Sigma^{(\alpha)})$$
$$= e^{2}\Big\{-\tfrac{1}{2}\epsilon^{-3}x^{(\alpha)}\,, + \tfrac{1}{2}\epsilon^{-1}\ddot{z}^{(\alpha)}\,, - \tfrac{3}{4}\epsilon^{-1}\ddot{z}^{(\alpha)}x_{(\beta)}\ddot{z}^{(\beta)} + \tfrac{1}{2}\epsilon^{-1}\ddot{z}^{2}x^{(\alpha)}$$
$$+ \text{ terms of odd degree in the } x\text{'s involving the Riemann tensor}$$
$$- \tfrac{1}{2}\dot{z}^{(\beta)}\int_{-\infty}^{+\infty} f^{(\alpha)}{}_{(\beta\gamma)}\dot{z}^{(\gamma)}(\tau')\, d\tau'\Big\}\, \epsilon^{-2}\, dS\, d\tau + O(\epsilon), \tag{5.21}$$

$$g^{1/2}\bar{F}^{\mathrm{free}(\alpha)}{}_{(\beta)}\bar{F}^{(\gamma\beta)}\, d\Sigma_{(\gamma)} = -e\bar{F}^{\mathrm{free}(\alpha)}{}_{(\beta)}\dot{z}^{(\beta)}\epsilon^{-2}\, dS\, d\tau + O(\epsilon), \tag{5.22}$$

$$g^{1/2}(\bar{F}^{(\alpha)}{}_{(\beta)}\bar{F}^{\mathrm{free}(\gamma\beta)}\, d\Sigma_{(\gamma)} - \tfrac{1}{2}\bar{F}^{\mathrm{free}}_{(\beta\gamma)}\bar{F}^{(\beta\gamma)}\, d\Sigma^{(\alpha)}) = O(\epsilon). \tag{5.23}$$

It is noteworthy that all terms of odd degree in the x's will be eliminated when integration over the surface S is performed. Carrying out the integration we get, in fact,

$$\int_{4\pi} T^{(\alpha\beta)}\, d\Sigma_{(\beta)} = \left\{ \frac{e^2}{2\epsilon} \ddot{z}^{(\alpha)} - \frac{1}{2} e^2 \dot{z}^{(\beta)} \int_{-\infty}^{+\infty} f^{(\alpha)}{}_{(\beta\gamma)} \dot{z}^{(\gamma)}(\tau')\, d\tau' \right.$$
$$\left. - e\bar{F}^{\text{free}(\alpha)}{}_{(\beta)} \dot{z}^{(\beta)} \right\} d\tau + O(\epsilon). \tag{5.24}$$

The divergent term in (5.24) has the same kinematical structure as the mass term in Eq. (5.7). It therefore has the effect of an unobservable mass renormalisation, and with the introduction of the "observed mass"

$$m = m_0 + \operatorname*{Lim}_{\epsilon\to 0}[\tfrac{1}{2}\epsilon^{-1}e^2], \tag{5.25}$$

Eq. (5.7) takes the form

$$m\ddot{z}^{(\alpha)} = e\bar{F}^{\text{free}(\alpha)}{}_{(\beta)}\dot{z}^{(\beta)} + \frac{1}{2} e^2 \dot{z}^{(\beta)} \int_{-\infty}^{+\infty} f^{(\alpha)}{}_{(\beta\gamma)} \dot{z}^{(\gamma)}(\tau')\, d\tau'. \tag{5.26}$$

For the purposes of application to physically set boundary conditions in the remote past it is more appropriate to work with the field $F^{\text{in}}_{(\alpha\beta)}$. Referring to Eqs. (3.15) and (5.17), we see that Eq. (5.26) becomes

$$m\ddot{z}^{(\alpha)} = eF^{\text{in}(\alpha)}{}_{(\beta)}\dot{z}^{(\beta)} + \frac{2e^2}{3}(\dddot{z}^{(\alpha)} - \dot{z}^{(\alpha)}\ddot{z}^2)$$
$$- \tfrac{1}{3}e^2 R_{\mu\nu}\dot{z}^{\nu}(\mu^{\mu(\alpha)} + \dot{z}^{(\alpha)}\dot{z}^{\mu})$$
$$+ e^2\dot{z}^{(\beta)} \int_{-\infty}^{\tau} f^{(\alpha)}{}_{(\beta\gamma)}\dot{z}^{(\gamma)}(\tau')\, d\tau'. \tag{5.27}$$

The vector wave equation corresponding to (5.27) can be obtained from the application of the vierbein transport law (1.10) and the definitions (1.35), (1.36) and (1.37). Writing "c" for the velocity of light and applying dimensional analysis to the resulting equation, we find (5.27) becomes

$$m\ddot{z}^{\alpha} = ec^{-1}F^{\text{in}\alpha}{}_{\beta}\dot{z}^{\beta} + \frac{2e^2}{3c^3}(\dddot{z}^{\alpha} - c^{-2}\dot{z}^{\alpha}\ddot{z}^2)$$
$$- \frac{e^2}{3c}(R_{\beta}{}^{\alpha}\dot{z}^{\beta} + c^{-2}\dot{z}^{\alpha}R_{\beta\gamma}\dot{z}^{\beta}\dot{z}^{\gamma})$$
$$+ e^2c^{-1}\dot{z}^{\beta}\int_{-\infty}^{\tau} f^{\alpha}{}_{\beta\gamma'}\dot{z}^{\gamma'}(\tau')\, d\tau', \tag{5.28}$$

where

$$f^{\alpha}{}_{\beta\gamma'} = f^{(\alpha)}{}_{(\beta\gamma)}\mu^{\alpha}_{(\alpha)}(z)\, \mu^{(\beta)}_{\beta}(z)\, \mu^{(\gamma)}_{\gamma'}(z'). \tag{5.29}$$

[The explicit appearance of the Ricci tensor in the radiation field gives rise to a different equation of motion to that obtained by DeWitt and Brehme (*1*). This is due to an incorrect computation on their part. See (*1*) page 225 equation (5.11). The left hand side of this equation should be evaluated at the retarded and advanced proper times. If we carry out this evaluation, using their notation, we have

$$\begin{aligned}
&[(u_{\mu\alpha.\nu} + v_{\mu\alpha}\sigma_{.\nu})\,\dot{z}^{\alpha}\,]_{\tau=\tau_{\pm}} \\
&\quad = \epsilon\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}\{-\tfrac{1}{2}\dot{z}^{\delta}R_{\delta}{}^{\alpha\beta\gamma}\Omega_{\gamma} - \tfrac{1}{6}\dot{z}^{\alpha}R^{\beta\gamma}\Omega_{\gamma} - \tfrac{1}{2}\Omega^{\beta}R_{\gamma}{}^{\alpha}\dot{z}^{\gamma} \\
&\qquad + \tfrac{1}{12}R\dot{z}^{\alpha}\Omega^{\beta} \pm \tfrac{1}{6}\kappa^{-1}\dot{z}^{\alpha}R_{\gamma}{}^{\beta}\dot{z}^{\gamma} \mp \kappa^{-1}\dot{z}^{\alpha}\dot{z}^{\beta} \\
&\qquad \pm \tfrac{1}{2}\kappa^{-1}\dot{z}^{\delta}R_{\delta}{}^{\alpha\beta}{}_{\gamma}\dot{z}^{\gamma} \pm \tfrac{1}{2}\kappa^{-1}\dot{z}^{\beta}R_{\gamma}{}^{\alpha}\dot{z}^{\gamma}\} + O(\epsilon^{2}).
\end{aligned}$$

The last four terms in this expression effect only the radiation field quantities, which then take the modified form given here in (5.17).]

The second term on the right-hand side of (5.28) is the familiar classical radiation damping term, while the third and fourth terms arise from the scattering effects of space–time curvature. When the incident field vanishes it is evident that radiation damping still occurs which prevents the physical solution of (5.28) being $\ddot{z}^{\alpha} = 0$, that is geodesic motion. It will be shown later that this is the case even in spaces conformal to flat space, with the notable exception of steady-state cosmology. Finally we note that the terms involving the Ricci tensor are multiplied by the first differentials of the $\dot{z}$'s only, and can thus be considered as a direct modification of the incident field, rather than an additional contribution to the radiation-damping effect.

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