12.3 Relativistic Electrodynamics

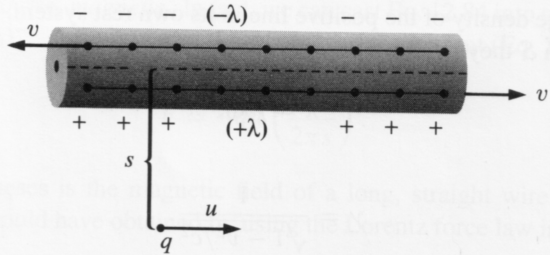
12.3.1 Magnetism as a Relativistic Phenomenon

Unlike Newtonian mechanics, classical electrodynamics is *already* consistent with special relativity. Maxwell's equations and the Lorentz force law can be applied legitimately in any inertial system. Of course, what one observer interprets as an electrical process another may regard as magnetic, but the actual particle motions they predict will be identical. To the extent that this did *not* work out for Lorentz and others, who studied the question in the late nineteenth century, the fault lay with the nonrelativistic mechanics they used, not with the electrodynamics. Having corrected Newtonian mechanics, we are now in a position to develop a complete and consistent formulation of relativistic electrodynamics. But I emphasize that we will not be changing the rules of electrodynamics in the slightest—rather, we will be *expressing* these rules in a notation that exposes and illuminates their relativistic character. As we go along, I shall pause now and then to rederive, using the Lorentz transformations, results obtained earlier by more laborious means. But the main purpose of this section is to provide you with a deeper understanding of the structure of electrodynamics—laws that had seemed arbitrary and unrelated before take on a kind of coherence and inevitability when approached from the point of view of relativity.

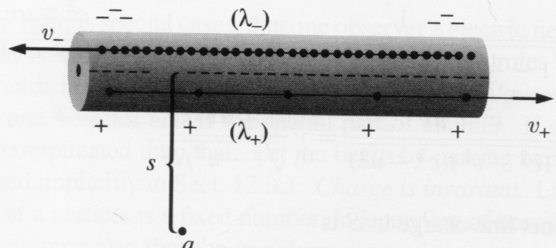
To begin with I'd like to show you why there *had* to be such a thing as magnetism, given electrostatics and relativity, and how, in particular, you can calculate the magnetic force between a current-carrying wire and a moving charge without ever invoking the laws of magnetism.[14] Suppose you had a string of positive charges moving along to the right at speed v. I'll assume the charges are close enough together so that we may regard them as a continuous line charge λ. Superimposed on this positive string is a negative one, $-\lambda$ proceeding to the left at the same speed v. We have, then, a net current to the right, of magnitude

$$I = 2\lambda v. \tag{12.75}$$

[14]This and several other arguments in this section are adapted from E. M. Purcell's *Electricity and Magnetism*, 2d ed. (New York: McGraw-Hill, 1985).



(a)



(b)

Figure 12.34

Meanwhile, a distance s away there is a point charge q traveling to the right at speed $u < v$ (Fig. 12.34a). Because the two line charges cancel, there is *no electrical force on* q in this system ($\mathcal{S}$).

However, let's examine the same situation from the point of view of system $\bar{\mathcal{S}}$, which moves to the right with speed u (Fig. 12.34b). In this reference frame q is at rest. By the Einstein velocity addition rule, the velocities of the positive and negative lines are now

$$v_{\pm} = \frac{v \mp u}{1 \mp vu/c^2}. \tag{12.76}$$

Because v_- is greater than v_+, the Lorentz contraction of the spacing between negative charges is more severe than that between positive charges; *in this frame*, therefore, *the wire carries a net negative charge!* In fact,

$$\lambda_{\pm} = \pm(\gamma_{\pm})\lambda_0, \tag{12.77}$$

where

$$\gamma_{\pm} = \frac{1}{\sqrt{1 - v_{\pm}^2/c^2}}. \tag{12.78}$$

and λ_0 is the charge density of the positive line in its own rest system. That's not the same as λ, of course—in $\mathcal{S}$ they're already moving at speed v, so

$$\lambda = \gamma\lambda_0, \tag{12.79}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{12.80}$$

It takes some algebra to put $\gamma_\pm$ into simple form:

$$\begin{aligned}\gamma_\pm &= \frac{1}{\sqrt{1 - \frac{1}{c^2}(v \mp u)^2(1 \mp vu/c^2)^{-2}}} = \frac{c^2 \mp uv}{\sqrt{(c^2 \mp uv)^2 - c^2(v \mp u)^2}} \\ &= \frac{c^2 \mp uv}{\sqrt{(c^2 - v^2)(c^2 - u^2)}} = \gamma\frac{1 \mp uv/c^2}{\sqrt{1 - u^2/c^2}}.\end{aligned} \tag{12.81}$$

Evidently, then, the net line charge in $\bar{\mathcal{S}}$ is

$$\lambda_{\text{tot}} = \lambda_+ + \lambda_- = \lambda_0(\gamma_+ - \gamma_-) = \frac{-2\lambda uv}{c^2\sqrt{1 - u^2/c^2}}. \tag{12.82}$$

Conclusion: As a result of unequal Lorentz contraction of the positive and negative lines, a current-carrying wire that is electrically neutral in one inertial system will be charged in another.

Now, a line charge λ_{tot} sets up an *electric* field

$$E = \frac{\lambda_{\text{tot}}}{2\pi\epsilon_0 s},$$

so *there is an electrical force on* q *in* $\bar{\mathcal{S}}$, to wit:

$$\bar{F} = qE = -\frac{\lambda v}{\pi\epsilon_0 c^2 s}\frac{qu}{\sqrt{1 - u^2/c^2}}. \tag{12.83}$$

But if there's a force on q in $\bar{\mathcal{S}}$, there must be one in $\mathcal{S}$; in fact, we can *calculate* it by using the transformation rules for forces. Since q is at rest $\bar{\mathcal{S}}$, and $\bar{F}$ is perpendicular to u, the force in $\mathcal{S}$ is given by Eq. 12.68:

$$F = \sqrt{1 - u^2/c^2}\,\bar{F} = -\frac{\lambda v}{\pi\epsilon_0 c^2}\frac{qu}{s}. \tag{12.84}$$

The charge is attracted toward the wire by a force that is purely electrical in $\bar{\mathcal{S}}$ (where the wire is charged, and q is at rest), but distinctly *non*electrical in $\mathcal{S}$ (where the wire is neutral). Taken together, then, electrostatics and relativity imply the existence of another force. This "other force" is, of course, *magnetic*. In fact, we can cast Eq. 12.84 into more familiar form by using $c^2 = (\epsilon_0\mu_0)^{-1}$ and expressing λv in terms of the current (Eq. 12.75):

$$F = -qu\left(\frac{\mu_0 I}{2\pi s}\right). \tag{12.85}$$

The term in parentheses is the magnetic field of a long, straight wire, and the force is precisely what we would have obtained by using the Lorentz force law in system $\mathcal{S}$.