THE MOTION OF CHARGED PARTICLES IN KALUZA–KLEIN SPACE–TIME

J. GEGENBERG
Department of Mathematics, York University, Downsview, Ontario, Canada

and

G. KUNSTATTER
Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7

Received 23 July 1984

The equations of motion for charged particles are derived from the geodesic hypothesis in the five-dimensional Kaluza–Klein theory. It is shown that even within this purely classical framework the theory does not describe low mass charged particles, and that in the background of a Kaluza–Klein monopole, the long range scalar field has striking observable consequences for electron motion, even at very large distances.

Kaluza–Klein theory [1,2] (for a good recent review see ref. [3]) provides a geometrical framework for the unification of gravity with the other fundamental interactions. The basis for this is that the gauge fields A_μ can be viewed as the local expression for the connection one-form of a fibre bundle E on spacetime M_4, with fibre invariant under the action of the gauge group, G. In the Kaluza–Klein approach, the dynamics of the gravitational and gauge fields on M_4 emerge from the vacuum Einstein equations, $\hat{R}_{AB} = 0$, for the pseudo-riemannian metric $\hat{g}_{AB}$ on the (4 + N)-dimensional manifold, E. The unobservability (at least at the present time) of the extra dimensions is accounted for by postulating that the fibres of E are compact manifolds with mean curvature of the order of the Planck length ($\sim 10^{-33}$ cm). It is well known, however, that serious problems occur when one tries to incorporate low mass, charged matter fields within the Kaluza–Klein framework [4]. If one assumes that such matter fields couple minimally to the higher-dimensional metric (i.e. obey a higher-dimensional Klein–Gordon equation), then the fields acquire a huge mass (of the order of the Planck mass $\sim 10^{-5}$ g) after dimensional reduction. Despite much work [5], this problem has not as yet been satisfactorily solved.

The purpose of this letter is two-fold. First, we shall show that the "large mass problem" emerges in a purely classical context if one correctly analyzes the consequences of assuming that charged test particles traverse geodesics of the five-dimensional Kaluza–Klein metric. Our analysis clarifies the origin of this problem and shows it to be essentially kinematical: the rest frame (in five dimensions) of a particle with charge e is in relative motion to that of an electrically neutral observer, and the "Lorentz transformation" from the former frame to the latter transforms the invariant mass $\hat{m}$ of the test particle to an effective mass $(\hat{m}^2 + e^2/16\pi G)^{1/2}$. The large size of the effective mass is a direct consequence of the magnitude of the physically observed $e^2/16\pi G \sim (10^{-6}\ \mathrm{g})^2$. Note that this discussion does not require fixing the internal radius to be small, although this related result follows when one tries to explain charge quantization in terms of five-dimensional quantum mechanics [3] or field theory.

Secondly, we shall examine in some detail the motion of charged particles in the background of the recently discovered Kaluza–Klein monopole [6–8]. We will demonstrate that the long-range scalar field causes the effective mass of an electron to vary out to distances of the order of 10^3 km. Moreover, at even greater distances from the monopole, the scalar force dominates all other interactions by many orders of magnitude. Although it is well known [7] that this scalar field might cause problems if it does not acquire a mass due to quantum corrections, the huge ratio of the scalar effects to the other forces has not been previously pointed out.

In the following, we will consider only a five-dimensional Kaluza–Klein theory and defer the generalization to higher dimensions to subsequent work. Hence we start with a pseudo-riemannian manifold $(\hat{M}, \hat{g})$ where $\hat{M}$ is a smooth five-dimensional manifold and $\hat{g} = \hat{g}_{AB} dx^A \otimes dx^B$ is a metric with signature (−++++). The indices $A, B, ... \in \{0, 1, 2, 3, 5\}$, while Greek indices $\mu, \nu, ... \in \{0, 1, 2, 3\}$. Tensors on $\hat{M}$ will always be hatted. We will set $c = 1$, but preserve all other constants explicitly since they play a crucial role in the final results.

We make the standard assumption that $\hat{M}$ possesses a Killing vector $\hat{K} = \partial/\partial x^5$, and decompose the metric as follows

$$\hat{g}_{AB} dx^A dx^B = g_{\mu\nu} dx^\mu dx^\nu + \lambda^2 (A_\mu dx^\mu + dx^5)^2 , \quad (1)$$

where $\lambda^2 = \hat{g}_{55} = |\hat{K}|^2$ and $A_\mu = \hat{g}_{\mu 5}/\lambda^2$. Note that x^5 is taken to be a dimensionless angular variable of period 2π.

With the above ansatz, the five-dimensional gravitational action, $\hat{I}$, decomposes as follows

$$\hat{I} = \frac{1}{16\pi\hat{G}} \int d^5x \sqrt{-\hat{g}}\, \hat{R}$$

$$= \frac{2\pi}{16\pi\hat{G}} \int \lambda \sqrt{-g}\, d^4x (\hat{R} - \tfrac{1}{4}\lambda^2 F_{\mu\nu} F^{\mu\nu}) , \quad (2)$$

where $\hat{G}$ is the five-dimensional gravitational constant, $F_{\mu\nu} \equiv A_{\mu,\nu} - A_{\nu,\mu}$ and we have dropped a surface term. We now identify the physical constants and fields by invoking the following correspondence principle: when $\lambda(x) = \lambda_\infty$ = const. we must recover the Einstein–Maxwell theory, with action

$$I = \frac{1}{16\pi G} \int \sqrt{-g}\, d^4x [R - (16\pi G/4) \bar{F}_{\mu\nu} \bar{F}^{\mu\nu}] , \quad (3)$$

where $\bar{F}_{\mu\nu}$ is the physical electromagnetic field strength and G is the four-dimensional gravitation constant. This yields the standard associations

$$2\pi\lambda_\infty / 16\pi\hat{G} = 1/16\pi G \quad (4a)$$

and

$$\bar{F}_{\mu\nu} = (\lambda_\infty / \sqrt{16\pi G}) F_{\mu\nu} . \quad (4b)$$

Note that the above correspondence principle does *not* fix λ_∞, which can only be determined when charged matter fields are coupled to the gravitational action in eq. (2) [3,7].

For simplicity, we make the standard assumption [3,7] that test particles in the Kaluza–Klein theory follow five-dimensional geodesics. The relevant five-dimensional geodesic equation is

$$\hat{m} \left(\frac{d^2 x^A}{d\hat{s}^2} + \hat{\Gamma}^A_{BC} \frac{dx^B}{d\hat{s}} \frac{dx^C}{d\hat{s}} \right) = 0 , \quad (5)$$

where $\hat{m}$ is the invariant mass of the test particle, $\hat{\Gamma}^A_{BC}$ is the Christoffel symbol constructed from $\hat{g}_{AB}$, and $d\hat{s}$ is the five-dimensional element of arc length defined by

$$-\epsilon d\hat{s}^2 = \hat{g}_{AB} dx^A dx^B . \quad (6)$$

In the above $\epsilon = 1$ for timelike geodesics, 0 for null geodesics and −1 for spacelike geodesics. After dimensional reduction, we find with the help of eq. (1) that eq. (5) splits as follows:

$$\hat{m}(\lambda^2 A_\mu dx^\mu / d\hat{s} + \lambda^2 dx^5 / d\hat{s}) = \text{const} \equiv \hat{p} , \quad (7)$$

$$\frac{d^2 x^\mu}{d\hat{s}^2} + \Gamma^\mu_{\rho\nu} \frac{dx^\rho}{d\hat{s}} \frac{dx^\nu}{d\hat{s}}$$

$$= \frac{\hat{p}}{\hat{m}} \frac{\sqrt{16\pi G}}{\lambda_\infty} \bar{F}^\mu{}_\nu \frac{dx^\nu}{d\hat{s}} + \frac{\hat{p}^2}{\hat{m}^2} g^{\mu\nu} \frac{\lambda_{,\nu}}{\lambda^3} , \quad (8)$$

where $\hat{p} = \hat{m} \hat{g}_{5A} dx^A / d\hat{s}$ is the momentum of the test particle in the fifth direction, $\Gamma^\mu_{\nu\rho}$ is the Christoffel symbol associated with $g_{\mu\nu}$ and $\bar{F}_{\mu\nu}$ is defined in eq. (4b). It is important to note that the geodesic postulate uniquely fixes the coupling of the test particle to the scalar field. One of the main successes of the original Kaluza–Klein theory is the fact that eq. (8) correctly reproduces the Lorentz force when $\lambda = \lambda_\infty$ provided that $\hat{p}$ is associated with the charge of the test particle as follows [2]:

$$\hat{p} = q\lambda_\infty / \sqrt{16\pi G} . \quad (9)$$

At this stage q and λ_∞ are still arbitrary. As Klein [2] originally pointed out, one can fix λ_∞ by requiring that charge quantization $q = ne$ follow from angular momentum quantization $\langle\hat{p}\rangle = n\hbar$ in the fifth dimension. This yields

$$\lambda_\infty = \hbar\sqrt{16\pi G}/e \sim 10^{-32}\ \text{cm}\,. \qquad (10)$$

We wish to emphasize that this is an additional assumption which is quite compelling but nonetheless unnecessary for the phenomenological description of classical charged particle motion.

In eq. (8) $d\hat{s}$ is the five-dimensional line element which is not in general the same as the four-dimensional line element defined by [‡1]

$$ds^2 = -g_{\mu\nu}dx^\mu dx^\nu\,. \qquad (11)$$

Moreover, a neutral clock, with $\hat{p} = 0$, necessarily measures ds and not $d\hat{s}$, so that as long as physical measurements are performed with neutral clocks and rods, s must be used to parametrize the motion of test particles [‡2]. From eqs. (1) and (7) we find that [‡3]

$$-\epsilon d\hat{s}^2 = -ds^2 + (\hat{p}^2/\lambda^2\hat{m}^2)d\hat{s}^2\,, \qquad (12)$$

so that eq. (8) transforms to the following:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\rho\nu}\frac{dx^\rho}{ds}\frac{dx^\nu}{ds} = \frac{q}{m_{\text{eff}}}\bar{F}^\mu{}_\nu\frac{dx^\nu}{ds} + \frac{q^2}{m^2_{\text{eff}}}\frac{\lambda^2_\infty}{16\pi G}\frac{\lambda_{,\rho}}{\lambda^3}\left(g^{\mu\rho} + \frac{dx^\mu}{ds}\frac{dx^\rho}{ds}\right). \qquad (13)$$

In eq. (13) we have defined

$$m_{\text{eff}} \equiv \hat{m}\,ds/d\hat{s} = [\epsilon\hat{m}^2 + (q^2/16\pi G)\lambda^2_\infty/\lambda^2]^{1/2}. \qquad (14)$$

Thus, the net effect of the change of parameters is to replace the invariant mass, $\hat{m}$, of the test particle by an effective mass: m_{eff}. This effective mass arises due to the "Lorentz transformation" relating the rest frame of a neutral observer to the rest frame of the charged particle. These two frames are in relative motion because charge is associated with momentum in the fifth dimension. Note that m_{eff} is not constant unless $\lambda = \lambda_\infty$ = const. This will play an important role in the later discussion, but for now let us assume that $m_{\text{eff}} \sim \text{const} = m_{\text{eff}}(\lambda_\infty)$. We now identify $m_{\text{eff}}(\lambda_\infty)$ with the observed mass, m_q, of the charged particle and set $q = e$, so that

$$m_q = (\epsilon\hat{m}^2 + e^2/16\pi G)^{1/2}\,. \qquad (15)$$

Since $e^2/16\pi G = \sqrt{\alpha}M_{\text{pl}} \approx 10^{-6}$ g, it is impossible to describe a test particle with charge e and effective mass less than the Planck mass, unless the test particle follows a *space-like* geodesic in five dimensions ($\epsilon = -1$). This is precisely the "large mass problem" of Kaluza–Klein field theory, although in a somewhat different and perhaps more intuitive form. Let us again stress that this result does not require fixing λ_∞.

The Kaluza–Klein monopole is a static solution of the five-dimensional Einstein equations $\hat{R}_{AB} = 0$. For our purposes, it is most convenient to use coordinates $(t, r, \theta, \varphi, x^5)$ in which the metric $\hat{g}_{AB}$ is given by [‡4]

$$\hat{g}_{AB}dx^A dx^B = -dt^2 + (1 + M/\rho)^2dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + [4Mr/(\rho + M)]^2(\tfrac{1}{2}\cos\theta\, d\varphi + dx^5)^2\,, \qquad (16)$$

where $\rho^2 = r^2 + M^2$ with M a constant. We can easily read off the four-dimensional space–time metric $g_{\mu\nu}$, the electromagnetic field strength tensor $\bar{F}_{\mu\nu}$, and the scalar field λ from the above five-metric:

$$g_{\mu\nu} = \text{diag}(-1, (1 + M/\rho)^2, r^2, r^2, r^2\sin^2\theta)\,, \qquad (17)$$

$$\bar{F}_3{}^2 = -\sin^2\theta\,\bar{F}_2{}^3 = g\sin\theta/r^2\,,$$

$$\bar{F}_\mu{}^\nu = 0\,, \quad \text{otherwise}\,, \qquad (18)$$

$$\lambda = 4Mr/(\rho + M)\,, \qquad (19)$$

where the magnetic charge $g = \frac{1}{2}\lambda_\infty/\sqrt{16\pi G}$, and from eq. (19) we see that $\lambda_\infty = 4M$. The parameter M, which determines the "mass" of the monopole is therefore fixed by λ_∞. Note that the standard choice for λ_∞

‡1 In the following we consider only geodesics which are time-like when projected to four dimensions.

‡2 We would like to thank K. Kuchar for clarifying this point. See also ref. [9].

‡3 Alternatively, let $dx^A/d\hat{s}$ be the tangent vector field to the geodesics of $\hat{g}_{AB}$, such that $\hat{\epsilon}\hat{g}_{AB}(dx^A/d\hat{s})(dx^B/d\hat{s}) = 1$, where $\hat{\epsilon} \in \{-1, 0, 1\}$ is the indicator of $dx^A/d\hat{s}$. Then if we write $dx^A/d\hat{s} = (f dx^\mu/ds, dx^5/d\hat{s})$ and insist that $g_{\mu\nu}(dx^\mu/ds)(dx^\nu/ds) = -1$, it follows that f is (up to a sign) uniquely determined to be $(\hat{p}/\lambda^2 - \hat{\epsilon})^{1/2}$, which is equivalent to eq. (12) above.

‡4 In our notation, x^5 replaces $\psi/2$ in ref. [6].

given in eq. (10) above guarantees that the Dirac charge quantization condition, $eg = n\hbar/2$, is satisfied with $n = 1$.

We now wish to examine the geodesic motion of an electron in the background of a Kaluza–Klein monopole. To accomplish this we must choose $\epsilon = -1$ in eq. (15) and "fine tune" $\hat{m}$ so that $m_q = m_e = 0.5$ MeV. Although this procedure poses grave conceptual difficulties, it is nonetheless worthwhile to examine the resulting phenomenology. When eqs. (17)–(19) are substituted into eq. (13) the resulting equations of motions are:

$$m_{\mathrm{eff}} \mathrm{d}t/\mathrm{d}s = \text{const} , \tag{20}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} - \frac{Mr}{\rho^2(M+\rho)}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 - \frac{r\rho}{M+\rho}\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2\right] = \frac{q^2}{m_{\mathrm{eff}}^2}\frac{M}{16\pi G}\frac{\rho}{r^3}\left[1 + \left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 \frac{(\rho+M)^2}{\rho^2}\right] , \tag{21}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 = -\frac{q}{m_{\mathrm{eff}}}\frac{g\sin\theta}{r^2}\frac{\mathrm{d}\varphi}{\mathrm{d}s} + \frac{q^2}{m_{\mathrm{eff}}^2}\frac{M}{16\pi G}\frac{(\rho+M)^2}{\rho r^3}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} , \tag{22}$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\varphi}{\mathrm{d}s} - \frac{\cos\theta}{\sin\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{q}{m_{\mathrm{eff}}}\frac{q}{r^2\sin^2\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s} + \frac{q^2}{m_{\mathrm{eff}}^2}\frac{M}{16\pi G}\frac{(\rho+M)^2}{\rho r^3}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\varphi}{\mathrm{d}s} . \tag{23}$$

We shall present a detailed analysis of these equations elsewhere but for the present we wish only to emphasize the more striking features which emerge. Firstly, eq. (20), which is the Kaluza–Klein analogue of particle energy conservation, reinforces the interpretation of m_{eff} as an effective mass. Moreover, using eq. (14) we can express m_{eff} to first order in (λ_∞/r) as

$$m_{\mathrm{eff}} = m_e[1 + 2\gamma^2(M/r^2)(\rho+M)]^{1/2}$$

$$\approx m_e\{1 + (\gamma^2\lambda_\infty/2r)[1 + O(\lambda_\infty/r)]\}^{1/2} , \tag{24}$$

where we have defined the dimensionless quantity

$$\gamma \equiv (e/m_e)/\sqrt{16\pi G} \sim 10^{20} . \tag{25}$$

We therefore find the remarkable result that m_{eff} is not well approximated by its constant asymptotic value, m_e, until

$$r \gg \gamma^2\lambda_\infty/2 \sim 10^3 \text{ km} . \tag{26}$$

A second interesting result can be obtained by examining the static acceleration due to the scalar force which appears on the right-hand side of eq. (21). We consider only the limit defined by eq. (26), so that this acceleration is approximately:

$$a_{\mathrm{s.c}} \sim \gamma^2 M/r^2 . \tag{27}$$

At a distance of 10^5 km from the monopole, this acceleration is still approximately 10^9 cm/s^2. Again we find that the large number $\gamma = (e/m_e)/\sqrt{16\pi G}$ causes the scalar field to have a huge effect far from the monopole, despite the fact that λ is very small.

To summarize, we have shown that the large mass problem in Kaluza–Klein theory is present even in the classical limit of test particle motion. It is a direct consequence of the local five-dimensional Lorentz structure of the theory and the empirical fact that $(q/m_q)/\sqrt{16\pi G}$ is a large number for elementary particles. We have also shown that this same large number leads to bizarre effects in the presence of a Kaluza–Klein monopole. Even at very large distances from the monopole, the effective mass of an electron is not constant, and the scalar force swamps all other forces by many orders of magnitude. This analysis emphasizes the need for caution when dealing with numerics in higher-dimensional field theories. Despite the small size of the internal dimensions, they can nonetheless have macroscopically observable consequences at the classical level.

We would like to thank K. Kuchar, J.W. Moffat, M. Perry and R. Sorkin for helpful discussions. One of us (G.K.) acknowledges the financial support of the Natural Sciences and Engineering Research Council of Canada.

References

[1] Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Math. Phys. (1921)966;
O. Klein, Z. Phys. 37 (1926) 37.
[2] D. Toms, in: An introduction to Kaluza–Klein theories, ed. H.C. Lee (World Scientific, Singapore, 1984).
[3] O. Klein, Nature 118 (1927) 516.
[4] E. Witten, Nucl. Phys. B186 (1981) 412.
[5] S. Weinberg, Phys. Lett. 138B (1984) 47.
[6] R.D. Sorkin, Phys. Rev. Lett. 51 (1983) 87.
[7] D.J. Gross and M.J. Perry, Nucl. Phys. B226 (1983) 29;
M.J. Perry, Phys. Lett. 137B (1984) 171.
[8] D. Pollard, J. Phys. A16 (1983) 565.
[9] M.A. Tonnelat, Les théories unitaires de l'électromagnétisme et de la gravitation (Gauthier-Villars, Paris, 1965) p. 179.