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A New Improved Energy-Momentum Tensor*

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We show that the matrix elements of the conventional symmetric energy-momentum tensor are cut-off dependent in renormalized perturbation theory for most renormalizable field theories. However, we argue that, for any renormalizable field theory, it is possible to construct a new energy-momentum tensor, such that the new tensor defines the same four-momentum and Lorentz generators as the conventional tensor, and, further, has finite matrix elements in every order of renormalized perturbation theory. ("Finite" means independent of the cut-off in the limit of large cut-off.) We explicitly construct this tensor in the most general case.

The new tensor is an improvement over the old for another reason: the currents associated with scale transformations and conformal transformations have very simple expressions in terms of the new tensor, rather than the very complicated ones they have in terms of the old.

We also show how to alter general relativity in such a way that the new tensor becomes the source of the gravitational field, and demonstrate that the new gravitation theory obtained in this way meets all the experimental tests that have been applied to general relativity.

1. INTRODUCTION

In any local field theory, the energy-momentum tensor is an important object; knowledge of its matrix elements is needed to describe scattering in a weak external

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gravitational field. Therefore, it is desirable for the energy-momentum tensor to be renormalizable; that is to say, for its matrix elements to be cut-off independent.[1] (Of course, the matrix elements at zero momentum transfer are finite, since these are simply connected to the total energy and momentum. Likewise, the first derivatives at zero are finite, since these are connected to the Lorentz generators. But we know nothing *a priori* about higher derivatives).

Our principal result is that for any renormalizable field theory, it is possible to find an energy-momentum tensor whose matrix elements are finite. However, this energy-momentum tensor is not always the conventional one. For example, for the simplest renormalizable quantum field theory, described by the Lagrangian[2]

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \tfrac{1}{2}\mu_0{}^2\varphi^2 - \lambda_0\varphi^4, \tag{1.1}$$

the conventional tensor,

$$T_{\mu\nu} = \partial_\mu\varphi\,\partial_\nu\varphi - g_{\mu\nu}\mathscr{L}, \tag{1.2}$$

does not have finite matrix elements even to lowest order in λ. However, the modified tensor,

$$\Theta_{\mu\nu} = T_{\mu\nu} - \tfrac{1}{6}(\partial_\mu\partial_\nu - g_{\mu\nu}\Box^2)\,\varphi^2 \tag{1.3}$$

has finite matrix elements to all orders in λ. Note that this tensor defines the same four-momentum and Lorentz generators as the conventional tensor.

In Section 2 we prove that the matrix elements of the tensor (1.3) are finite to all orders in λ. In Section 3 we extend this result to an arbitrary renormalizable interaction of massive scalar, spinor, and vector field (we exclude massless fields to avoid considerations of infrared divergences). We have written Section 2 as clearly as we can, giving the details of every step in the proof. Section 3 is much more telegraphic in style, with many details suppressed.

Our proof is not rigorous.[1a] The reason is this: in order to disentangle efficiently overlapping divergences, we use the renormalization procedure of Bogoliubov, Parasiuk, and Hepp (BPH) [1]. The validity of this procedure has been rigorously proved only for a special cut-off, one so strong that any polynomial interaction (not just a renormalizable one) is made finite. Unfortunately, our proof also uses certain consequences (Ward identities) of the conservation of $\Theta_{\mu\nu}$,

$$\partial^\mu\Theta_{\mu\nu} = 0. \tag{1.4}$$

[1] Throughout this paper, "finite" and "cut-off independent" mean "possessing a finite limit as the cut-off goes to infinity, in every order of renormalized perturbation theory."

[2] We use a metric such that the signature of the metric tensor is $(+---)$.

[1a] *Note added in proof:* Indeed, our arguments have recently been criticized by K. Symanzik (private communications). In Appendix 4, we discuss the alterations needed in the definition of the energy-momentum tensor if these criticisms are valid.

This equation is not valid in the presence of the BPH cut-off. If there were a proof of the BPH theorem for the conventional Feynman cut-off,[3] our proof would be rigorous. This is because the Feynman cut-off can be derived from a Lagrangian, with the aid of regulator fields [2]. For this reason, the arguments given in Section 2 should properly be considered as merely heuristic, since the regulator fields are not explicitly taken into account. The correct arguments are given in Section 3, which treats theories involving an arbitrary number of fields, some of which may be regulator fields. The Lagrangian form enables us to write an energy-momentum tensor that is conserved, even in the presence of the cut-off. This tensor is not strictly given by (1.2), but contains additional terms involving the regulator fields. In Section 4 we check, by an explicit calculation, that the matrix elements of this tensor go to a finite limit as the regulator masses go to infinity, to lowest order in λ.

In the remainder of the paper we discuss some properties of the new energy momentum tensor (1.3). In Section 5 we show that this tensor is a particularly natural one to use if one is interested in the scale transformation properties of fields, and explain how this is connected with the criterion of renormalizability. We also make some remarks on the representation of generators of conformal transformations; these are irrelevant to the main body of our work, but flow naturally from our formalism, and may be of interest in another context.[4] In Section 6 we discuss some consequences of assuming the new tensor is in fact the source of gravitation, and show how to alter the gravitational Lagrangian so this is indeed the case.

Two appendices contain fuller discussions of points treated scantily in Section 4. A third appendix is a brief discussion of the equal-time commutation relations of the components of the new energy-momentum tensor.

2. A Single Scalar Field with Quartic Self-Interaction

1. *The Trace of* $\Theta_{\mu\nu}$

In the theory described by the Lagrangian

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \tfrac{1}{2}\mu_0{}^2\varphi^2 - \lambda_0\varphi^4, \tag{2.1}$$

we define

$$\Theta_{\mu\nu} = \partial_\mu\varphi\,\partial_\nu\varphi - g_{\mu\nu}\mathscr{L} - \tfrac{1}{6}(\partial_\mu\partial_\nu - g_{\mu\nu}\square^2)\,\varphi^2. \tag{2.2}$$

[3] Actually, the cut-off we use in Section 4, when we do explicit calculations, is a slight modification of the Feynman cut-off; the regularized propagator falls off as k^{-6}, rather than k^{-4}.

[4] The possibility that scale and conformal transformations are approximate symmetries of the S matrix is a recurrent theme in high-energy physics. Some modern treatments of this idea are: G. Mack and A. Salam, Ann. Phys. **53**, 174 (1969); D. J. Gross and J. Wess, Cern Preprint Th 1076 (1969); and L. S. Brown and M. Gell-Mann (in preparation). We would like to thank Dr. Gross for several enlightening discussions of conformal invariance.

The equation of motion

$$\Box^2\varphi + \mu_0{}^2\varphi + 4\lambda_0\varphi^3 = 0, \tag{2.3}$$

implies that

$$\Theta_\mu{}^\mu = \mu_0{}^2\varphi^2. \tag{2.4}$$

Notice that this object is much "softer" than the trace of the conventional energy-momentum tensor; it involves fewer fields and fewer derivatives. This "softness" will be important to our proof. We remark that Eq. (2.4) remains true if we redefine both the right and the left hand sides by subtracting their vacuum expectation values. (This corresponds to neglecting all disconnected Feynman diagrams for which one component is a vacuum-expectation-value diagram.) We shall always make such a redefinition in everything that follows; however, to avoid complicating our notation, we shall not indicate it explicitly.

2. *Ward Identities and Trace Identities*

We define the position and momentum-space Green's functions by

$$G^{(n)}(x_1 \cdots x_n) = \langle 0 \mid T\varphi'(x_1) \cdots \varphi'(x_n) \mid 0\rangle, \tag{2.5a}$$

and

$$\begin{aligned} &G^{(n)}(p_1 \cdots p_n)(2\pi)^4\, \delta^4(p_1 + \cdots + p_n) \\ &\quad = \int d^4x_1 \cdots d^4x_n\, e^{ip_1\cdot x_1} \cdots e^{ip_n\cdot x_n} G^{(n)}(x_1 \cdots x_n), \end{aligned} \tag{2.5b}$$

where the prime denotes the renormalized fields. To study the matrix elements of the energy-momentum tensor we add to the Lagrangian (2.1) an additional term:

$$\mathscr{L} \to \mathscr{L} + \Theta_{\mu\nu}\mathscr{J}^{\mu\nu} + \mu_0{}^2\varphi^2\mathscr{J}, \tag{2.6}$$

where $\mathscr{J}_{\mu\nu}$ and $\mathscr{J}$ are arbitrary c-number functions of space and time. We define

$$i\Gamma_{\mu\nu}^{(n)}(x; x_1 \cdots x_n) = \frac{\delta}{\delta\mathscr{J}^{\mu\nu}(x)}\, G^{(n)}(x_1 \cdots x_n) \mid_{\mathscr{J}^{\mu\nu}=0}, \tag{2.7a}$$

and

$$i\Gamma^{(n)}(x; x_1 \cdots x_n) = \frac{\delta}{\delta\mathscr{J}(x)}\, G^{(n)}(x_1 \cdots x_n) \mid_{\mathscr{J}=0}. \tag{2.7b}$$

The Fourier transforms of these objects, $\Gamma_{\mu\nu}(k; p_1 \cdots p_n)$ and $\Gamma(k; p_1 \cdots p_n)$, are defined as in Eq. (2.5b).

Ordinary perturbation theory gives all these functions as sums of Feynman diagrams; Nambu [3] has shown that such sums can be written as T^* products; in our case

$$\Gamma^{(n)}_{\mu\nu} = \langle 0 \mid T^* \Theta_{\mu\nu}(x)\, \varphi'(x_1) \cdots \varphi'(x_n) \mid 0 \rangle \tag{2.8a}$$

and

$$\Gamma^{(n)} = \langle 0 \mid T^* \mu_0{}^2 \varphi^2(x)\, \varphi'(x_1) \cdots \varphi'(x_n) \mid 0 \rangle. \tag{2.8b}$$

The T^* product is the same as the T product for a set of undifferentiated fields; for an expression involving derivatives, however, the T^* product is defined to be the T product with the derivatives placed *outside* the time-ordering symbol. Thus, to give a definite example,

$$T^*(\partial_\mu \varphi(x)\, \partial_\nu \varphi(y)) = \partial_\mu{}^x \partial_\nu{}^y T(\varphi(x)\, \varphi(y)). \tag{2.9}$$

We are now in a position to derive Ward identities—expressions for $k_\mu \Gamma^{\mu\nu}$. These are obtained in the standard way, by calculating the divergence of (2.8a), using the canonical commutators,

$$[\partial_0 \varphi(x, t), \varphi(y, t)] = -i\delta^3(\mathbf{x} - \mathbf{y}), \tag{2.10}$$

and taking the Fourier transform. Note that the extra term in Eq. (2.2) does not contribute to these identities, because of the definition of the T^* product. The result is

$$k^\mu \Gamma^{(n)}_{\mu\nu}(k; p_1 \cdots p_n) = -i \sum_P (p_1 + k)_\nu G^{(n)}(p_1 + k, p_2 \cdots p_n) \tag{2.11}$$

where the sum is on cyclic permutations of the indices 1 to n.

A useful corollary of the Ward identities is obtained by differentiating with respect to k and setting all momenta equal to zero:

$$\Gamma^{(n)}_{\mu\nu}(0; 0 \cdots 0) = -i(n-1)\, g^{\mu\nu} G^{(n)}(0 \cdots 0). \tag{2.12}$$

Equation (2.4) also places restrictions of the Γ's, which we call "trace identities". Since it is necessary to bring the derivatives inside the time-ordering symbol before one can use the equations of motion, these identities also have commutator terms. Their form is

$$g^{\mu\nu} \Gamma^{(n)}_{\mu\nu}(k; p_1 \cdots p_n) = \Gamma^{(n)}(k; p_1 \cdots p_n) - i \sum_P G^{(n)}(k + p_1, p_2 \cdots p_n). \tag{2.13}$$

3. *BPH Renormalization*

The main result of Bogoliubov, Parasiuk, and Hepp [1] can be stated in the following way: Suppose one begins with a Lagrangian that is a polynomial in fields and their derivatives, and calculates Green's functions in the usual Feynman perturbation theory. Eventually, at some stage in the expansion, one will encounter divergent (cut-off dependent) one-particle-irreducible[5] diagrams, of superficial degree of divergence d.[6] At this stage, one adds extra terms to the Lagrangian, the so-called counter-terms. These terms will involve as many fields as the diagram has external lines, and up to d derivatives. (For example, in a theory with quadratically divergent meson self-energy diagrams, the extra terms will be of the form φ^2 and $\partial_\mu \varphi \, \partial^\mu \varphi$.)[7] The coefficients of these terms are chosen to cancel the first d terms in the Taylor expansion of the divergent diagrams about the point zero (the point at which all external momenta vanish). When this is done, the resultant perturbation expansion has a finite limit in every order. This completes our statement of the BPH theorem.

For an ordinary renormalizable theory, the counter-terms required are of the same form as the terms in the original Lagrangian. The total Lagrangian (the original Lagrangian plus the BPH counter-terms) is therefore of the same form as the original Lagrangian. If we rescale the fields so that the kinetic term is of standard form, we can identify the coefficients of the other terms as bare masses, coupling constants, etc. Thus, in this case, the content of the BPH theorem is the comforting statement that the Green's functions for appropriately rescaled fields can be made cut-off independent, provided the bare masses and coupling constants are chosen in an appropriate cut-off-dependent way. The coefficients of the corresponding terms in the original Lagrangian are much like the physical masses and coupling constants as usually defined, except that they are the values of one-particle-irreducible Green's functions at the point zero, rather than at some astutely chosen mass-shell point.

For our purposes, it will be useful to define the concept of "a set of operators closed under renormalization". In any renormalizable field theory, let $A_1 \cdots A_n$ be a set of monomials in the fundamental fields and their derivatives. Let us change the original Lagrangian of the theory by adding to it an additional term:

$$\mathscr{L} \to \mathscr{L} + \sum_a A_a \mathscr{J}_a , \tag{2.14}$$

where the $\mathscr{J}$'s are arbitrary functions of space and time. Let us define $\Gamma_a^{(n)}$ as the Fourier transform of the variational derivative of the n-point Green's function in the fundamental fields with respect to $\mathscr{J}_a$, evaluated at $\mathscr{J}_a$ equals zero. ($\Gamma_a^{(n)}$ will

[5] A Feynman diagram is said to be one-particle irreducible if it is connected and if it remains connected if any one internal line is cut.

[6] The superficial degree of divergence of a diagram is given by $d = 2I - 4V + P + 4$, where I is the number of internal lines, V the number of vertices, and P the number of momenta appearing in the numerator of the Feynman integral, either from propagators or from derivative interactions.

[7] The BPH theorem would require $\varphi \Box^2 \varphi$ in addition, but this is equivalent, upon integration, to $\partial_\mu \varphi \partial^\mu \varphi$.

be a function of particle labels, spin indices, and $n + 1$ external momenta, of which n are independent.) The BPH theorem tells us that the Γ's can be made finite if we add appropriate counter-terms to the Lagrangian. If these counterterms are also in the set $A_1 \cdots A_n$, we say the set is closed under renormalization.

It follows from the BPH theorem that, given a set of operators closed under renormalization, we can find a set of cut-off-independent functions $R_a^{(n)}$, such that

$$\Gamma_a^{(n)} = \sum_b c_{ab} R_b^{(n)}, \tag{2.15}$$

where the c's are constant, possibly cut-off-dependent, coefficients. Further, the c's are completely determined if we know the terms of order d or less in the Taylor expansions about the point zero of these Γ's to which there correspond diagrams of superficial degree of divergence d. In particular, we can choose the R's such that for each R_a, only one of these Taylor coefficients is nonzero.

4. *The Finiteness of the Trace*

Simple power-counting shows that the operator φ^2 is, all by itself, closed under renormalization. This can also be seen by remembering that, in the $\lambda\varphi^4$ the only superficially divergent diagrams are those with two external lines, which are quadratically divergent, and those with four external lines, which are logarithmically divergent. Adding a coupling with $\mathscr{J}$, as in Eq. (2.6), to any internal line introduces an extra propagator, and reduces the degree of divergence by two.

Thus, Eq. (2.15) degenerates to

$$\Gamma^{(n)} = cR^{(n)}. \tag{2.16}$$

We normalize such that $R^{(2)}$ is one at the point zero. Thus

$$\Gamma^{(2)}(0; 0, 0) = c. \tag{2.17}$$

However, it follows from Eqs. (2.12) and (2.13) that

$$\Gamma^{(2)}(0; 0, 0) = -2iG^{(2)}(0, 0). \tag{2.18}$$

Since G is finite, c is finite, and all $\Gamma^{(n)}$ are finite. (Notice that the "softness" of the trace was crucial here. If we had used the trace of the conventional energy-momentum tensor, we would have had to add many terms to form a set of operators closed under renormalization. We then would have had too few Ward identities to determine all the unknown coefficients.)

5. *The Last Step in the Proof*

Simple power-countings shows that the following set of symmetric tensors is closed under renormalization:

$$\{g_{\mu\nu}\varphi^2,\ g_{\mu\nu}\partial_\lambda\varphi\ \partial^\lambda\varphi,\ g_{\mu\nu}\varphi\Box^2\varphi,\ \partial_\mu\varphi\ \partial_\nu\varphi,\ \varphi\partial_\mu\partial_\nu\varphi,\ g_{\mu\nu}\varphi^4\}. \tag{2.19}$$

This can also be seen by observing that these operators involve the same fields, and the same number of derivatives, as the terms in the Lagrangian (2.1). Since only these properties, and not the scalar or tensor character of the operators, enter into power-counting, the same calculation that shows that the theory is renormalizable shows that this set is closed under renormalization.

$\Theta_{\mu\nu}$ is a linear combination of the operators in the set (2.19). Therefore, by Eq. (2.15), there exists six sets of finite functions, $R^{(n)}_{a\mu\nu}$, such that

$$\Gamma^{(n)}_{\mu\nu} = \sum_a c_a R^{(n)}_{a\mu\nu} . \tag{2.20}$$

Furthermore, we may choose the R's such that

$$\Gamma^{(2)}_{\mu\nu}(k; p_1, p_2) = g_{\mu\nu}(c_1 + c_2 q^2 + c_3 k^2) + c_4(k_\mu k_\nu - g_{\mu\nu} k^2) + c_5 q_\mu q_\nu + \cdots, \tag{2.21}$$

where the triple dots indicate cubic terms and higher, and

$$q = p_1 - p_2 . \tag{2.22}$$

Also,

$$\Gamma^{(4)}_{\mu\nu}(0; 0, 0, 0, 0) = c_6 g_{\mu\nu} . \tag{2.23}$$

We will now show that all the c's are finite, and complete our proof of the renormalizability of $\Theta_{\mu\nu}$. The Ward identity (2.11) tells us that $k^\mu \Gamma^{(n)}_{\mu\nu}$ is finite. Therefore,

$$(c_1 + c_2 q^2 + c_3 k^2)\, k_\mu + c_5 (q \cdot k)\, q_\mu \tag{2.24}$$

is finite, as is

$$c_6 k_\mu . \tag{2.25}$$

Extracting coefficients of independent vectors, we find that all the c's are finite, except perhaps for c_4.

However, the results of the preceding subsection, together with the trace identity (2.13), implies that $g^{\mu\nu}\Gamma^{(n)}_{\mu\nu}$ is finite. Therefore,

$$4(c_1 + c_2 q^2 + c_3 k^2) - 3c_4 k^2 + c_5 q^2 \tag{2.26}$$

is finite. This, together with the preceding result, enables us to deduce the finiteness of c_4.

3. A General Renormalizable Field Theory

1. *Lagrangians*

In this section, we shall extend the results of the preceding section to the most general renormalizable field theory. Our methods will be essentially the same as before: we construct an energy-momentum tensor with a "soft" trace, and use Ward and trace identities to show that it is renormalizable. The algebra will be much more complicated, though, because we have to keep track of a larger number of fields.

The most general renormalizable field theory involves a set of spinless fields, a set of Dirac bispinor fields, and a set of vector fields. We will denote these by φ^a, ψ^a, and $A_\mu{}^a$, respectively, where the index a runs through the appropriate range in each case. The Lagrangian is the sum of a free part and an interaction part,

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_I. \tag{3.1}$$

The free part is given by

$$\begin{aligned}\mathscr{L}_0 &= \tfrac{1}{2}\partial_\mu\varphi^a\,\partial^\mu\varphi^a - \tfrac{1}{2}(\mu_0{}^2)^a\,\varphi^a\varphi^a + \bar{\psi}^a(i\partial^\mu\gamma_\mu - m_0{}^a)\,\psi^a \\ &\quad - \tfrac{1}{4}(\partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a)^2 + \tfrac{1}{2}(M_0{}^2)^a\,A_\mu{}^a A^{\mu a},\end{aligned} \tag{3.2}$$

where the sum over repeated indices is implied, and where we have assumed that the fields have been redefined by a linear transformation, if necessary, to put (3.2) into standard form. The interaction part is given by

$$\begin{aligned}\mathscr{L}_I &= \alpha^a\varphi^a + \beta^{abc}\varphi^a\varphi^b\varphi^c + \lambda^{abcd}\varphi^a\varphi^b\varphi^c\varphi^d \\ &\quad + g^{abc}\bar{\psi}^a\psi^b\psi^c + ih^{abc}\bar{\psi}^a\gamma_5\psi^b\varphi^c + e^{abc}\bar{\psi}^a\gamma^\mu\psi^b A_\mu{}^c \\ &\quad + 2f^{abc}(\partial^\mu\varphi^a)\,\varphi^b A_\mu{}^c + f^{abc}f^{ade}\varphi^b\varphi^d A_\mu{}^c A^{\mu e}.\end{aligned} \tag{3.3}$$

The numerical coefficients in this expression are the unrenormalized coupling constants. In general, the only restriction on these couplings is that the vector mesons must couple to commuting conserved currents; thus, the transformations

$$\begin{aligned}\delta^a\psi^b &= -ie^{bca}\psi^c, \\ \delta^a\varphi^b &= f^{bca}\varphi^c,\end{aligned} \tag{3.4}$$

must be a symmetry of the Lagrangian for every a, and two such transformations must commute. Of course, in special cases, symmetry principles, like isospin conservation or parity invariance, may place further restrictions on these couplings.

Some of these fields will be regulator fields. Depending on the regulator formalism one uses, these fields will either have imaginary coupling constants in their interactions or, alternatively, contribute to (3.2) with a sign opposite to that which we have displayed. Either of these devices serves to insure that the propagator for the regulator field has the desired negative sign. It is crucial that the regulator fields, and their contribution to the energy-momentum tensor, be included at every step. Otherwise, the Ward and trace identities, vital to our calculation, would not be true in the presence of the cut-off.

Although the Lagrangian (3.1) describes a renormalizable theory, the vector meson part is not in a good form for systematic renormalization theory; the meson propagator it implies,

$$D_{\mu\nu} = \frac{g_{\mu\nu} - \dfrac{k_\mu k_\nu}{M^2}}{k^2 - M^2}, \tag{3.5}$$

is badly behaved for high k. This difficulty can be eliminated by introducing auxiliary scalar fields. For simplicity, we will review this procedure for the case of a single vector meson interacting with a single Fermion; the generalization is trivial. We begin with an initial Lagrangian (in the sense of Section 2.3) of the form

$$\mathscr{L} = -\tfrac{1}{2}\partial_\mu A\,\partial^\mu A + \tfrac{1}{2}\mu^2 A_\mu A^\mu - \tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - m\bar{\psi}\psi + \bar{\psi}\gamma^\mu(i\partial_\mu + e\mathscr{A}_\mu)\,\psi, \tag{3.6}$$

where A is the auxiliary field, and

$$\mathscr{A}_\mu = A_\mu + \mu^{-1}\partial_\mu A. \tag{3.7}$$

It is possible to show, as a consequence of current conservation, that this Lagrangian defines the same on-the-mass-shell S matrix as does the Lagrangian without the auxilliary field A. The effective vector propagator, the propagator for $\mathscr{A}_\mu$, is

$$D_{\mu\nu} = \frac{g_{\mu\nu} - k_\mu k_\nu/k^2}{k^2 - \mu^2}. \tag{3.8}$$

This is well-behaved at large k.

Another consequence of current conservation will also be of use to us: the only counter-terms generated by the Lagrangian (3.6) are of the form of the last three terms—the first two terms in (3.6) receive no additions.

2. *Energy-Momentum Tensors and Their Traces*

The conventional energy-momentum tensor is most simply expressed if we assemble all of the fields in the theory into a vector $\boldsymbol{\varphi}$. The momentum vector, $\boldsymbol{\pi}^\mu$ is defined by

$$\boldsymbol{\pi}^\mu = \partial\mathscr{L}/\partial(\partial_\mu\boldsymbol{\varphi}). \tag{3.9}$$

The spin matrix, $\boldsymbol{\Sigma}_{\mu\nu}$, is defined by the transformation properties of the fields under an infinitesimal Lorentz transformation,

$$\delta\boldsymbol{\varphi} = x_\mu\partial_\nu\boldsymbol{\varphi} - x_\nu\partial_\mu\boldsymbol{\varphi} + \boldsymbol{\Sigma}_{\mu\nu}\boldsymbol{\varphi}. \tag{3.10}$$

With these definitions, the conventional symmetric energy-momentum tensor [4] is given by

$$T^{\mu\nu} = \boldsymbol{\pi}^{\mu} \cdot \partial^{\nu}\boldsymbol{\varphi} - g^{\mu\nu}\mathscr{L} + \tfrac{1}{2}\partial_{\lambda}(\boldsymbol{\pi}^{\lambda} \cdot \boldsymbol{\Sigma}^{\mu\nu}\boldsymbol{\varphi} - \boldsymbol{\pi}^{\mu} \cdot \boldsymbol{\Sigma}^{\lambda\nu}\boldsymbol{\varphi} - \boldsymbol{\pi}^{\nu} \cdot \boldsymbol{\Sigma}^{\lambda\mu}\boldsymbol{\varphi}). \quad (3.11)$$

We are now in a position to state our main result: There exist constants ϵ^a such that

$$\Theta^{\mu\nu} = T^{\mu\nu} - \tfrac{1}{6}(\partial^{\mu}\partial^{\nu} - g^{\mu\nu}\Box^2)(\varphi^a + \epsilon^a)(\varphi^a + \epsilon^a) \quad (3.12)$$

has finite matrix elements in all orders of renormalized perturbation theory.[8] In general, the ϵ's are cutoff-dependent. Of course, in particular cases, symmetry principles may tell us that some of the ϵ's vanish. For example, for a parity-conserving Lagrangian, the ϵ's associated with pseudoscalar fields vanish. Likewise, for an isospin-conserving Lagrangian, those associated with fields on nonzero isospin vanish.

Straightforward (and very tedious) calculation shows that

$$\Theta_{\lambda}{}^{\lambda} = (\mu_0{}^2)^a\, \varphi^a\varphi^a + m_0{}^a\bar{\psi}^a\psi^a + (M_0{}^2)^a\, A_{\mu}{}^a A^{\mu a} - \beta^{abc}\varphi^a\varphi^b\varphi^c - 3\alpha^a\varphi^a + \epsilon^a\Box^2\varphi^a. \quad (3.13)$$

Note that we do *not* use the equations of motion to rewrite the last term.

Equation (3.13) is valid for the Lagrangian (3.1); it does not include the contributions from the scalar fields auxiliary to the vector meson fields. However, in the remainder of this section, we will ignore there terms, and, when counting divergences, treat $A_{\mu}{}^a$ as if it were $\mathscr{A}_{\mu}{}^a$. We show in Appendix I that we make no errors in doing this.

3. *Ward Identities, Trace Identities, and Supplementary Conditions*

Just as before, we add an additional term to the Lagrangian (3.1),

$$\mathscr{L} \to \mathscr{L} + \Theta_{\mu\nu}\mathscr{J}^{\mu\nu} + \Theta_{\lambda}{}^{\lambda}\mathscr{J}, \quad (3.14)$$

where $\Theta_{\lambda}{}^{\lambda}$ is given by Eq. (3.13). As in Section 2.2, we can define Green's functions, which we denote by $\Gamma_{\mu\nu}^{(r,s,t)}$ and $\Gamma^{(r,s,t)}$, where r, s, and t, respectively, denote the numbers of spinless, Dirac, and vector fields in the corresponding T^* product. These functions depend on $(r + s + t + 1)$ four-vectors, of which $(r + s + t)$ are independent, on $(r + s + t)$ field-labeling indices, on s spinor indices, and on t tensor indices. To avoid monstrous equations, we will, in the remainder of our discussion, only display explicitly the dependence on those variables which are

[8] This is not, of course, the unique solution to the problem of finding a finite energy-momentum tensor. For example, another solution is $P(\Box^2)\,\Theta_{\mu\nu}$, where P is any polynomial that has the value one at zero.

relevant to the argument. For these functions, we can derive Ward and trace identities. The explicit form of these equations is as complicated as it is unenlightening;[9] therefore, we will merely state here that they can be used to show that $k^\mu \Gamma_{\mu\nu}$ is finite, as is the difference between $g^{\mu\nu}\Gamma_{\mu\nu}$ and Γ. Also, by differentiating the Ward identities, we can show that all the $\Gamma_{\mu\nu}$'s are finite at the point zero.

None of this depends on the values we assign to the ϵ's. We now fix ϵ^a by demanding that

$$\Gamma^{(1,0,0)}(k;\, k) = \Gamma^{(1,0,0)}(0,\, 0) + 0(k^4). \tag{3.15}$$

We will refer to this set of equations as supplementary conditions.

4. *Finiteness of the Trace*

Straightforward power counting shows that the set of operators

$$\{\varphi^a\varphi^b\varphi^c,\ \varphi^a\varphi^b,\ \varphi^a,\ \bar\psi^a\psi^b,\ A_\mu{}^a A^{\mu b},\ \Box^2\varphi^a\} \tag{3.16}$$

is closed under renormalization. (The last term is necessary to close the set, for a cubic interaction can produce a quadratic divergence in $\Gamma^{(1,0,0)}$. This is the reason why we need, in general, the ϵ terms in the energy-momentum tensor.) Thus, just as in Section 2, we can establish that the matrix elements of $\Theta_\lambda{}^\lambda$ are finite if we can show that (1) $\Gamma^{(1,0,0)}$, $\Gamma^{(2,0,0)}$, $\Gamma^{(3,0,0)}$, $\Gamma^{(0,2,0)}$, and $\Gamma^{(0,0,2)}$ are finite at the point zero, and (2) the second-order terms in the Taylor expansion of $\Gamma^{(1,0,0)}$, about the point zero are finite. The first of these is a consequence of the Ward and trace identities; the second follows from the supplementary conditions.

5. *The Final Step in the Proof*

Let us consider the set of all tensor operators constructed from (1) four Boson[10] fields and no derivatives, (2) three Boson fields and up to one derivative, (3) two Boson fields and up to two derivatives, (4) one Boson field and up to two derivatives (5) two Fermion fields, one Boson field, and no derivatives, (6) two Fermion fields and up to one derivative, or (7) one vector field and three derivatives. Simple power-counting shows that this set of operators is closed under renormalization. The energy-momentum tensor is a linear combination of operators in this set. Thus, by arguments which should, by now, be familiar, we can prove the finiteness of the matrix elements of $\Theta_{\mu\nu}$ if we can show that certain coefficients in the Taylor expansions of certain $\Gamma_{\mu\nu}$'s about the point zero are finite. We can show this if we can express these coefficients in terms of $k^\mu\Gamma_{\mu\nu}^{(r,s,t)}$ and $g^{\mu\nu}\Gamma_{\mu\nu}^{(r,s,t)}$, which we already

[9] For the benefit of the curious, we give them in Appendix 2.

[10] As usual, "Boson" means either a spinless field or a vector field.

know to be finite. To show that such an expression exists, it suffices to show that the corresponding system of linear equations has a unique solution.

Thus the proof reduces to solving the following homogeneous problem: Given that

$$k^\mu \Gamma_{\mu\nu}^{(r,s,t)} = 0, \tag{3.17}$$

and

$$g^{\mu\nu} \Gamma_{\mu\nu}^{(r,s,t)} = 0, \tag{3.18}$$

prove that the necessary Taylor coefficients in the expansion of $\Gamma_{\mu\nu}^{(r,s,t)}$ about the point zero vanish.

The problem is readily solved for those terms which are of zero-th or first order in k. Let

$$\Gamma_{\mu\nu}^{(r,s,t)} = A_{\mu\nu} + A_{\mu\nu\lambda} k^\lambda + 0(k^2). \tag{3.19}$$

By Eq. (3.18)

$$k^\mu A_{\mu\nu} = 0. \tag{3.20}$$

Since k is arbitrary, this implies that

$$A_{\mu\nu} = 0. \tag{3.21}$$

Likewise,

$$A_{\mu\nu k} k^\nu k^\lambda = 0. \tag{3.22}$$

This implies that

$$A_{\mu\nu\lambda} = -A_{\mu\lambda\nu}\,. \tag{3.23}$$

However, we also know that

$$A_{\mu\nu\lambda} = A_{\nu\mu\lambda}\,. \tag{3.24}$$

These two equations imply that

$$\begin{aligned} A_{\mu\nu\lambda} &= -A_{\nu\lambda\mu} \\ &= A_{\lambda\mu\nu} \\ &= -A_{\mu\nu\lambda} \\ &= 0. \end{aligned} \tag{3.25}$$

Thus, we need only investigate in detail those Green's functions whose Taylor coefficients of second order or higher are needed. These are:

(1) $\Gamma_{\mu\nu}^{(2,0,0)}$. This was done in Section 2.

(2) $\Gamma_{\mu\nu}^{(1,0,0)}$. This is the same algebra as the preceding case.

(3) $\Gamma_{\mu\nu}^{(0,0,2)}$. Here we need only investigate the terms of second order in k.

$$\begin{aligned}\Gamma_{\mu\nu,\lambda\rho}^{(0,0,2)}(k; p_1, p_2) = {} & a g_{\mu\nu} g_{\lambda\rho} k^2 + b k^2 (g_{\mu\lambda} g_{\nu\rho} + g_{\nu\lambda} g_{\mu\rho}) \\ & + c k_\mu k_\nu g_{\lambda\rho} + d k_\lambda k_\rho g_{\mu\nu} \\ & + e(k_\mu k_\lambda g_{\rho\nu} + k_\nu k_\lambda g_{\rho\mu}) \\ & + f(k_\mu k_\rho g_{\lambda\nu} + k_\nu k_\rho g_{\lambda\mu}),\end{aligned} \tag{3.26}$$

where we have restored the suppressed dependence on tensor indices, but not on field-labeling indices. By Eq. (3.17)

$$k^2 g_{\lambda\rho} k_\nu (a + c) + k^2 g_{\nu\rho} k_\lambda (b + e) + k^2 g_{\nu\lambda} k_\rho (b + f) + k_\nu k_\rho k_\lambda (d + e + f) = 0. \tag{3.27}$$

Hence,

$$\begin{aligned} e &= f = -b \\ d &= 2b \\ c &= -a. \end{aligned} \tag{3.28}$$

By Eq. (3.18)

$$g_{\lambda\rho} k^2 (4a + 2b + c) + k_\lambda k_\rho (c + 4d + 2e + 2f) = 0. \tag{3.29}$$

Combining this with the preceding equation

$$3a + 2b = -a + 4b = 0. \tag{3.30}$$

Hence,

$$a = b = 0.$$

(4) $\Gamma_{\mu\nu}^{(0,0,1)}$. Here we need to go to third order.

$$\Gamma_{\mu\nu,\lambda}^{(0,0,1)}(k; k) = a k^2 (k_\mu g_{\nu\lambda} + k_\nu g_{\mu\lambda}) + b k^2 k_\lambda g_{\mu\nu} + c k_\mu k_\nu k_\lambda . \tag{3.31}$$

There are no second-order terms with the proper transformation properties. By Eq. (3.17)

$$a k^4 g_{\nu\lambda} + (b + c)\, k^2 k_\nu k_\lambda = 0. \tag{3.32}$$

Hence,

$$\begin{aligned} a &= 0, \\ c &= -b. \end{aligned} \tag{3.33}$$

By Eq. (3.18),

$$(4b + c)\, k^2 k_\lambda = 0. \tag{3.34}$$

Hence,

$$b = c = 0. \tag{3.35}$$

This solves the homogeneous problem and completes the proof.

4. A Sample Calculation

In this section we shall verify the theorems of Section 2 to lowest nontrivial order in perturbation theory. We shall choose a cutoff such that the meson propagator is

$$D(k^2) = \frac{1}{k^2 - \mu^2} - \frac{c_1^2}{k^2 - M_1^2} - \frac{c_2^2}{k^2 - M_2^2}, \tag{4.1}$$

where

$$c_1^2 + c_2^2 = 1, \tag{4.2}$$

and

$$c_1^2 M_1^2 + c_2^2 M_2^2 = \mu^2. \tag{4.3}$$

Such a cutoff can be effected by introducing, in addition to the scalar field φ, two regulator fields [2], φ_1 and φ_2, and choosing the initial Lagrangian (in the sense of Section 2.3) to be

$$\begin{aligned}\mathscr{L} = {} & \tfrac{1}{2}(\partial_\mu\varphi\, \partial^\mu\varphi - \mu^2\varphi^2) - \tfrac{1}{2}(\partial_\mu\varphi_1\, \partial^\mu\varphi_1 - M_1^2\varphi_1^2) \\ & - \tfrac{1}{2}(\partial_\mu\varphi_2\, \partial^\mu\varphi_2 - M_2^2\varphi_2^2) - \lambda\Phi^4,\end{aligned} \tag{4.4}$$

where

$$\Phi = \varphi + c_1\varphi_1 + c_2\varphi_2 . \tag{4.5}$$

The renormalization counter-terms change the Lagrangian:

$$\mathscr{L} \to \mathscr{L} + \tfrac{1}{2}a\, \partial_\mu\Phi\, \partial^\mu\Phi - \tfrac{1}{2}b\Phi^2 - c\Phi^4 \tag{4.6}$$

where a, b, and c are cutoff-dependent. By the usual interpretation, these counter-terms have the following physical meanings:

$$Z = 1 + a, \tag{4.7}$$

$$\mu_0^2 = Z^{-1}(\mu^2 + b), \tag{4.8}$$

and

$$\lambda_0 = Z^{-2}(\lambda + c). \tag{4.9}$$

According to our theorems, if we define the energy-momentum tensor

$$\Theta_{\mu\nu} = \partial_\mu\varphi\,\partial_\nu\varphi - \partial_\mu\varphi_1\,\partial_\nu\varphi_1 - \partial_\mu\varphi_2\,\partial_\nu\varphi_2 + a\,\partial_\mu\Phi\,\partial_\nu\Phi - g_{\mu\nu}\mathscr{L} - \tfrac{1}{6}(\partial_\mu\partial_\nu - g_{\mu\nu}\Box^2)(\varphi^2 - \varphi_1{}^2 - \varphi_2{}^2 + a\Phi^2),$$

then this object has finite matrix elements. Also,

$$\Theta_\mu{}^\mu = \mu^2\varphi^2 - M_1{}^2\varphi_1{}^2 - M_2{}^2\varphi_2{}^2 + b\Phi^2. \tag{4.11}$$

We will now attempt to verify these assertions to first order in λ.

Before we can do this, we must calculate the renormalization constants to first order in λ. To this order, a and c vanish; b is determined by the condition that it must cancel the first-order self-energy diagram (Fig. 1) at the point zero. Thus,

$$b = -12i\lambda \int \frac{d^4q}{(2\pi)^4}\left(\frac{1}{q^2-\mu^2} - \frac{c_1{}^2}{q^2-M_1{}^2} - \frac{c_2{}^2}{q^2-M_2{}^2}\right). \tag{4.12}$$

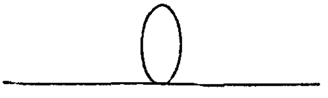

FIG. 1. The mass renormalization counter term to order λ.

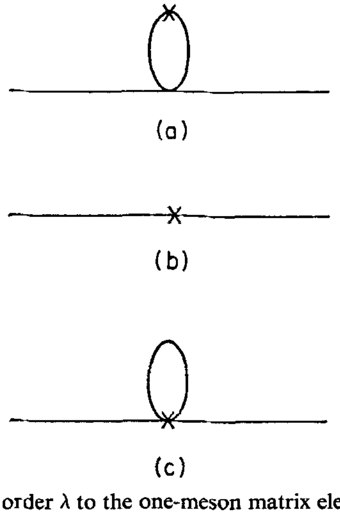

FIG. 2. The corrections of order λ to the one-meson matrix element of the energy momentum tensor. The tensor itself is represented by a cross, and carries off momentum k. Internal momenta are oriented such that they go around the loop clockwise.

FIG. 2(a). The first-order correction to the zeroth-order tensor.

FIG. 2(b) AND 2(c). The zeroth-order matrix elements of two first-order terms in the tensor.

We may now calculate the one-particle matrix element of (4.10). The relevant Feynman diagrams are shown in Fig. 2. Figure 2a shows the contribution from the first correction to the zeroth-order parts, Fig. 2b that from the $b\Phi^2$ term and Fig. 2c that from the $\lambda\Phi^4$ term. (If we had normal ordered the Lagrangian, the last two terms would not have been present; however, in this case, Eq. (4.11) would not have been true). The last two terms cancel; thus, the two-particle matrix element is given by

$$M_{\mu\nu} = 24i\lambda \int \frac{d^4q}{(2\pi)^4} \left\{ \frac{q_\mu(q+k)_\nu - \frac{1}{2}g_{\mu\nu}[q\cdot(q+k) - \mu^2] + \frac{1}{6}(k_\mu k_\nu - g_{\mu\nu}k^2)}{(q^2-\mu^2)[q\cdot(q+k)-\mu^2]} \right.$$
$$\left. - c_1^2(\mu^2 \to M_1^2) - c_2^2(\mu^2 \to M_2^2) \right\}, \tag{4.13}$$

where the last two symbols represent integrands identical to the first, but with μ replaced by M_1 and M_2, respectively. Note that to this order, the amplitude depends only on the total momentum transferred, k. Therefore, with no loss of generality, we can take the external lines to be on the mass shell, and the trace identity takes the simple form

$$g^{\mu\nu}M_{\mu\nu} = M, \tag{4.14}$$

where M is the matrix element of (4.11).

Now,

$$g^{\mu\nu}M_{\mu\nu} = 24i\lambda \int \frac{d^4q}{(2\pi)^4} \left\{ \frac{-q\cdot(q+k) + 2\mu^2 - \frac{1}{3}k^2}{(q^2-\mu^2)[(q+k^2)-\mu^2]} \right.$$
$$\left. - c_1^2(\mu^2 \to M_1^2) - c_2^2(\mu^2 \to M_2^2) \right\}. \tag{4.15}$$

But, by Eq. (4.11),

$$b = -6i\lambda \int \frac{d^4q}{(2\pi)^4} \left\{ \frac{(q^2-\mu^2) + [(q+k)^2-\mu^2]}{(q^2-\mu^2)[(q+k)^2-\mu^2]} \right.$$
$$\left. - c_1^2(\mu^2 \to M_1^2) - c_2^2(\mu^2 \to M_2^2) \right\}. \tag{4.16}$$

Thus,

$$g^{\mu\nu}M_{\mu\nu} = 2b + 24i\lambda \int \frac{d^4q}{(2\pi)^4} \left\{ \frac{\mu^2}{(q^2-\mu^2)[(q+k)^2-\mu^2]} \right.$$
$$\left. - c_1^2(\mu^2 \to M_1^2) - c_2^2(\mu^2 \to M_2^2) \right\}. \tag{4.17}$$

This is precisely M. Thus, we have explicitly verified Eq. (4.11).

If we evaluate (4.17) at the point zero, we find that

$$M(0) = 2b - 2\mu^2(\partial b/\partial \mu^2) - 2M_1^2(\partial b/\partial M_1^2) - 2M_2^2(\partial b/\partial M_2^2). \quad (4.18)$$

This is zero, since b has the dimensions of mass squared, and μ^2, M_1^2, and M_2^2 are the only parameters that are not dimensionless. Thus, if we define the convergent integral

$$F(k^2/\mu^2) = \frac{\partial}{\partial k^2} \int \frac{d^4q}{(2\pi)^4} \frac{\mu^2}{(q^2 - \mu^2)[(q + k)^2 - \mu^2]} \quad (4.19)$$

then,

$$M(k^2) = \int_0^{k^2} dx[F(x/\mu^2) - c_1^2F(x/M_1^2) - c_2^2F(x/M_2^2)]. \quad (4.20)$$

As the regulator masses go to infinity, this becomes

$$M(k^2) = \int_0^{k^2} dx[F(x/\mu^2) - F(0)]. \quad (4.21)$$

This establishes the finiteness of the trace.

By Lorentz invariance, all that remains is to establish the finiteness of (4.12) for $\mu \neq \nu$. In this case,

$$M_{\mu\nu} = k_\mu k_\nu \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} [\tfrac{1}{6} - x(1 - x)]\{[q^2 - \mu^2 + k^2x(1 - x)]^{-2}$$
$$- c_1^2(\mu^2 \to M_1^2) - c_2^2(\mu^2 \to M_2^2)\}, \quad (4.22)$$

where we have introduced a Feynman parameter and shifted the integration vector in the standard way. Since

$$\int_0^1 dx[\tfrac{1}{6} - x(1 - x)] = 0. \quad (4.23)$$

the integral vanishes at the point zero. The finiteness of (4.22) then follows by the same arguments as were used for the trace.

An interesting feature[11] of this calculation is that the contribution of the regulator fields to the trace, Eq. (4.11), does not go to zero as the regulator masses go to infinity. This is because the terms involving the regulator fields are themselves

[11] We have not here examined the interesting question of whether or not it is possible to express the trace of the energy momentum tensor in the renormalized theory purely in terms of the renormalized physical fields with no reference to the regulator fields. We are indebted to K. Wilson for a discussion of the properties of the trace of the energy momentum tensor in a renormalized field theory.

multiplied by the regulator masses. Thus, the very equation with which we began our analysis, in Section 2,

$$\Theta_\mu{}^\mu = \mu_0{}^2\varphi^2 \tag{4.24}$$

is in no sense an acceptable equation in the renormalized theory. This does not affect our proof, of course, since what we need for the proof is not this equation but its generalization in the presence of the regulator fields, Eq. (4.11).

5. Scale Invariance and Conformal Invariance

In this section we shall show that the new energy-momentum tensor which we have introduced is a very natural object to consider if one is interested in the transformation properties of a field theory under scale transformations and conformal transformations. This is a viewpoint that has no apparent relation with the criterion of renormalizability; nevertheless, as we shall demonstrate, a connection does exist. We shall also make some remarks about the representation of the generators of conformal transformations which are not relevant to the main body of our work, but which follow naturally from our analysis, and which may be useful in another context.[4]

1. *Scale Transformations*

As in Section 3.2, we assemble all the fields in a field theory into a vector, φ. A scale transformation is a transformation of the form:

$$\tau\colon \boldsymbol{\varphi}(x) \to e^{\mathbf{D}\tau}\boldsymbol{\varphi}(e^\tau x), \tag{5.1}$$

with D some matrix and τ some real number. The set of all such transformations, with fixed $\mathbf{D}$ and arbitrary τ, form a group. The infinitesimal transformation is of the form

$$\delta\boldsymbol{\varphi} = \mathbf{D}\boldsymbol{\varphi} + x^\mu\partial_\mu\boldsymbol{\varphi}. \tag{5.2}$$

The theory is invariant under scale transformations if the change in the Lagrangian induced by (5.2) is a total divergence. This means

$$\begin{aligned}\delta\mathscr{L} &= 4\mathscr{L} + x^\mu\partial_\mu\mathscr{L}\\ &= \partial_\mu(x^\mu\mathscr{L}).\end{aligned} \tag{5.3}$$

In particular, the Lagrangian (3.1) obeys this condition, if we set all nondimensionless coupling constants (including the masses) equal to zero, and if we choose D to be a matrix that multiplies every Bose field by 1 and every Fermi field by 3/2. These numbers are the conventional dimensions of such fields; for this reason $\mathbf{D}$ is sometimes called "the dimension matrix".

If the condition (5.3) is obeyed, we can use the standard formula to get an expression for the conserved current associated with scale transformations:

$$\begin{aligned} J^\mu &= \boldsymbol{\pi}^\mu \cdot \delta\boldsymbol{\varphi} + x^\mu \mathscr{L} \\ &= \boldsymbol{\pi}^\mu \cdot \mathbf{D}\boldsymbol{\varphi} + \boldsymbol{\pi}^\mu \cdot x^\lambda \partial_\lambda \boldsymbol{\varphi} - x^\mu \mathscr{L} \\ &= \boldsymbol{\pi}^\mu \cdot \mathbf{D}\boldsymbol{\varphi} + x_\lambda T_c^{\mu\lambda}, \end{aligned} \tag{5.4}$$

where $T_c^{\mu\lambda}$ is the canonical energy-momentum tensor,

$$T_c^{\mu\lambda} = \boldsymbol{\pi}^\mu \cdot \partial^\lambda \boldsymbol{\varphi} - g^{\mu\lambda} \mathscr{L}. \tag{5.5}$$

We note that if the Lagrangian does not obey (5.3) but instead obeys

$$\delta \mathscr{L} = \partial_\mu (x^\mu \mathscr{L}) + \Delta, \tag{5.6}$$

where Δ is defined by this equation, then the conservation equation,

$$\partial_\mu J^\mu = 0, \tag{5.7}$$

is replaced by

$$\partial_\mu J^\mu = \Delta. \tag{5.8}$$

For the particular case of the Lagrangian (3.1)

$$\Delta = (\mu_0{}^2)^a \, \varphi^a \varphi^a + m_0{}^a \bar{\psi}^a \psi^a + (M_0{}^2)^a \, A_\mu{}^a A^{\mu a} - \beta^{abc} \varphi^a \varphi^b \varphi^c - 3\alpha^a \varphi^a. \tag{5.9}$$

As we shall see, the similarity between this and Eq. (3.13) is no coincidence.

Equation (5.4) is reminiscent of the canonical expression for the currents associated with the angular-momentum tensor,

$$M^{\lambda\mu\nu} = x^\mu T_c^{\lambda\nu} - x^\nu T_c^{\lambda\mu} + \boldsymbol{\pi}^\lambda \cdot \boldsymbol{\Sigma}^{\mu\nu} \boldsymbol{\varphi} \tag{5.10}$$

In the case of angular momentum, it is possible to build a new set of currents,

$$M^{\lambda\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}, \tag{5.11}$$

where $T^{\mu\nu}$ is the symmetric energy-momentum tensor, defined by Eq. (3.11). The currents (5.11) are not the same as the currents (5.10); however, they are also conserved, and upon integration, yield the same total angular momentum. It is natural to ask if something similar can be done for Eq. (5.4)—if it is possible to defined a new energy-momentum tensor $\Theta^{\mu\nu}$, such that

$$J^\mu = x_\lambda \Theta^{\mu\lambda}, \tag{5.12}$$

obeys Eq. (5.8), and such that the space integral of the time component of (5.12) is the same as that of the time component of (5.4).

We shall show that a sufficient condition for this to be possible is that

$$\boldsymbol{\pi}^\mu \cdot \mathbf{D}\boldsymbol{\varphi} + \boldsymbol{\pi}_\lambda \cdot \boldsymbol{\Sigma}^{\mu\nu}\boldsymbol{\varphi} = \partial_\lambda \sigma^{\mu\lambda}. \tag{5.13}$$

where $\sigma^{\mu\lambda}$ is some tensor. Later, we shall show that this condition is also a necessary one, that a scale-invariant theory be conformally invariant. Eq. (5.13) is satisfied by the most general renormalizable interaction, (3.1), with

$$\sigma^{\mu\lambda} = \tfrac{1}{2}\, g^{\mu\lambda}\varphi^a\varphi^a. \tag{5.14}$$

We now turn to the explicit construction of $\Theta^{\mu\nu}$. From the definition of $T^{\mu\nu}$,

$$\begin{aligned} J^\mu &= \boldsymbol{\pi}^\mu \cdot \mathbf{D}\boldsymbol{\varphi} + x_\nu T^{\mu\nu} - \tfrac{1}{2}x_\nu\partial_\lambda(\boldsymbol{\pi}^\lambda \cdot \boldsymbol{\Sigma}^{\mu\nu}\boldsymbol{\varphi} - \boldsymbol{\pi}^\mu \cdot \boldsymbol{\Sigma}^{\lambda\nu}\boldsymbol{\varphi} - \boldsymbol{\pi}^\nu \cdot \boldsymbol{\Sigma}^{\lambda\mu}\boldsymbol{\varphi}) \\ &= \boldsymbol{\pi}^\mu \cdot \mathbf{D}\boldsymbol{\varphi} + x_\nu T^{\nu\mu} + \boldsymbol{\pi}_\lambda \cdot \boldsymbol{\Sigma}^{\mu\lambda}\boldsymbol{\varphi} - \tfrac{1}{2}\partial_\lambda[x_\nu(\boldsymbol{\pi}^\lambda \cdot \boldsymbol{\Sigma}^{\mu\nu}\boldsymbol{\varphi} - \boldsymbol{\pi}^\mu \cdot \boldsymbol{\Sigma}^{\lambda\nu}\boldsymbol{\varphi} - \boldsymbol{\pi}^\nu \cdot \boldsymbol{\Sigma}^{\lambda\mu}\boldsymbol{\varphi})]. \end{aligned} \tag{5.15}$$

The last term is the divergence of an antisymmetric tensor; therefore, we may discard it without affecting the conservation of the current or the space integral of its time component. Thus, we redefine the current to be

$$\begin{aligned} J^\mu &= x_\nu T^{\mu\nu} + \boldsymbol{\pi}^\mu \cdot \mathbf{D}\boldsymbol{\varphi} + \boldsymbol{\pi}_\lambda \cdot \boldsymbol{\Sigma}^{\mu\lambda}\boldsymbol{\varphi} \\ &= x_\nu T^{\mu\nu} + \partial_\nu \sigma^{\mu\nu}. \end{aligned} \tag{5.16}$$

Let us write $\sigma^{\mu\nu}$ as the sum of a symmetric and an antisymmetric tensor:

$$\sigma^{\mu\nu} = \sigma_+^{\mu\nu} + \sigma_-^{\mu\nu}. \tag{5.17}$$

By the same reasoning as before, we may drop the contribution of the antisymmetric part to Eq. (5.16), and redefine the current as

$$J^\mu = x_\nu T^{\mu\nu} + \partial_\nu \sigma_+^{\mu\nu}. \tag{5.18}$$

We now define

$$\Theta^{\mu\nu} = T^{\mu\nu} + \tfrac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu}, \tag{5.19}$$

where

$$X^{\lambda\rho\mu\nu} = g^{\lambda\rho}\sigma_+^{\mu\nu} - g^{\lambda\mu}\sigma_+^{\rho\nu} - g^{\lambda\nu}\sigma_+^{\mu\rho} + g^{\mu\nu}\sigma_+^{\lambda\rho} - \tfrac{1}{3}g^{\lambda\rho}g^{\mu\nu}\sigma_{+\alpha}^{\alpha} + \tfrac{1}{3}g^{\lambda\mu}g^{\rho\nu}\sigma_{+\alpha}^{\alpha}\,. \tag{5.20}$$

The added term is symmetric and divergenceless; it makes no contribution to the total four-momentum, nor to the total angular-momentum tensor. From the definition of $\Theta^{\mu\nu}$,

$$J^\mu = x_\nu \theta^{\mu\nu} + \partial_\nu \sigma_+^{\mu\nu} - \tfrac{1}{2}\partial_\lambda \partial_\rho (x_\nu X^{\lambda\rho\mu\nu}) + \tfrac{1}{2}(g_{\lambda\nu}\partial_\rho + g_{\rho\nu}\partial_\lambda)\, X^{\lambda\rho\mu\nu}. \tag{5.21}$$

The third term is the divergence of an antisymmetric tensor; therefore, we may drop it, and redefine the current as

$$\begin{aligned} J^\mu &= x_\nu \Theta^{\mu\nu} + \partial_\nu \sigma_+^{\mu\nu} + \tfrac{1}{2}(g_{\lambda\nu}\partial_\rho + g_{\rho\nu}\partial_\lambda)\, X^{\lambda\rho\mu\nu} \\ &= x_\nu \Theta^{\mu\nu}, \end{aligned} \tag{5.22}$$

because the second and third terms cancel. This is the desired result.

The energy-momentum $\Theta^{\mu\nu}$, (5.19), becomes our energy-momentum tensor (3.12) when $\sigma^{\mu\nu}$ is taken to be (5.14). (More properly, it is our tensor if we replace φ^a by $\varphi^a + \epsilon^a$. However, we always have the freedom to make such a change of variables for scalar fields.) Our energy-momentum tensor was selected to have a "soft" trace—one that involves operators whose Green's functions have low degrees of divergence, and whose counter-terms can therefore be controlled by the Ward identities. These "soft" operators are precisely those terms in the Lagrangian which break scale invariance. However, by Eqs. (5.22) and (5.8),

$$\partial^\mu J_\mu = \Delta = \Theta_\mu{}^\mu. \tag{5.23}$$

This explains why we have arrived in this section at the same energy-momentum tensor we found, by apparently quite different arguments, in Section 3.

2. *Conformal Transformations*

The conformal group is a group of transformations on Minkowski space, defined as the connected part of the smallest group containing the Poincaré group and the inversion in a unit hyperboloid. It is a fifteen-parameter Lie group, isomorphic to the noncompact classical group $SO(4, 2)$. Ten of the fifteen group generators are the usual Poincaré generators; the other five consist of the generator of infinitesimal scale transformations

$$\delta x^\mu = -x^\mu, \tag{5.24}$$

and four generators obtained by inverting infinitesimal translations,

$$\delta_c{}^\nu x^\mu = -2x^\mu x^\nu + g^{\mu\nu} x^2. \tag{5.25}$$

The four transformations (5.25) are called infinitesimal conformal transformations; if a dynamical theory is invariant under such transformations, we say it is conformally invariant. It is easy to see that the commutator of an infinitesimal conformal transformation and an infinitesimal translation is a linear combination of infinitesimal Lorentz transformations and the infinitesimal scale transformation; thus, a Poincaré-invariant theory which is also conformally invariant is necessarily scale-invariant.

We wish to represent these transformations as linear transformations on a set of fields. Since the origin of coordinates is left unchanged, fields at the origin must go into fields at the origin. Thus,

$$\delta\boldsymbol{\varphi}(0) = \mathbf{D}\boldsymbol{\varphi}(0), \tag{5.26}$$

and

$$\delta_c{}^\nu\boldsymbol{\varphi}(0) = \mathbf{K}^\nu\boldsymbol{\varphi}(0), \tag{5.27}$$

with $\mathbf{D}$ and the $\mathbf{K}$'s some matrices. The algebra of the conformal group implies that

$$[\mathbf{D}, \mathbf{K}^\nu] = \mathbf{K}^\nu, \tag{5.28}$$

or, equivalently,

$$e^{+\mathbf{D}\tau}\mathbf{K}^\nu e^{-\mathbf{D}\tau} = e^\tau\mathbf{K}^\nu. \tag{5.29}$$

Thus, the $\mathbf{K}$'s must be nilpotent. This does not necessarily mean they vanish, since there is no reason to assume they are diagonalizable; however, we will restrict ourselves, in the sequel, to the case in which they do vanish,

$$\mathbf{K}^\nu = 0. \tag{5.30}$$

The transformation properties of fields at arbitrary points are now determined by the known commutators of (5.24) and (5.25) with translations:

$$\delta\boldsymbol{\varphi} = \mathbf{D}\boldsymbol{\varphi} + x^\mu\partial_\mu\boldsymbol{\varphi}, \tag{5.31}$$

and

$$\delta_c{}^\mu\boldsymbol{\varphi} = (2x^\mu x^\nu - g^{\mu\nu}x^2)\,\partial_\nu\boldsymbol{\varphi} + 2x^\mu\mathbf{D}\boldsymbol{\varphi} + 2x_\nu\boldsymbol{\Sigma}^{\mu\nu}\boldsymbol{\varphi}. \tag{5.32}$$

The first of these is just Eq. (5.2) again.

We shall now investigate the conditions such that a given Lagrangian describes a conformally invariant theory. By the preceding equations,

$$\begin{aligned}\delta_c{}^\mu\mathscr{L} = {} & (2x^\mu x^\nu - g^{\mu\nu}x^2)\,\partial_\nu\mathscr{L} + 2\boldsymbol{\pi}_\lambda\cdot(g^{\mu\lambda}x^\nu + g^{\nu\lambda}x^\mu - g^{\mu\nu}x^\lambda)\,\partial_\nu\boldsymbol{\varphi}\\ & + 2x^\mu\frac{\partial\mathscr{L}}{\partial\boldsymbol{\varphi}}\cdot\mathbf{D}\boldsymbol{\varphi} + 2x^\mu\boldsymbol{\pi}^\nu\cdot\mathbf{D}\partial_\nu\boldsymbol{\varphi}\\ & + 2\boldsymbol{\pi}^\mu\cdot\mathbf{D}\boldsymbol{\varphi} + 2x_\nu\frac{\partial\mathscr{L}}{\partial\boldsymbol{\varphi}}\cdot\boldsymbol{\Sigma}^{\mu\nu}\boldsymbol{\varphi}\\ & + 2x_\nu\boldsymbol{\pi}^\lambda\cdot\boldsymbol{\Sigma}^{\mu\nu}\partial_\lambda\boldsymbol{\varphi} + 2\boldsymbol{\pi}_\lambda\cdot\boldsymbol{\Sigma}^{\mu\nu}\boldsymbol{\varphi}.\end{aligned} \tag{5.33}$$

Now, as a consequence of Lorentz invariance,

$$\frac{\partial \mathscr{L}}{\partial \boldsymbol{\varphi}} \cdot \boldsymbol{\Sigma}^{\mu\nu}\boldsymbol{\varphi} + \boldsymbol{\pi}^{\lambda} \cdot \boldsymbol{\Sigma}^{\mu\nu}\partial_{\lambda}\boldsymbol{\varphi} - \boldsymbol{\pi}^{\nu} \cdot \partial^{\mu}\boldsymbol{\varphi} + \boldsymbol{\pi}^{\mu} \cdot \partial^{\nu}\boldsymbol{\varphi} = 0. \tag{5.34}$$

Likewise, since every conformally invariant theory must be scale-invariant, we must have, as a consequence of Eq. (5.3),

$$\frac{\partial \mathscr{L}}{\partial \boldsymbol{\varphi}} \cdot \mathbf{D}\boldsymbol{\varphi} + \boldsymbol{\pi}^{\lambda} \cdot \mathbf{D}\partial_{\lambda}\boldsymbol{\varphi} + \boldsymbol{\pi}^{\lambda} \cdot \partial_{\lambda}\boldsymbol{\varphi} = 4\mathscr{L}. \tag{5.35}$$

Therefore,

$$\begin{aligned} \delta_c{}^{\mu}\mathscr{L} &= (2x^{\mu}x^{\nu} - g^{\mu\nu}x^2)\,\partial_{\nu}\mathscr{L} + 4x^{\mu}\mathscr{L} + 2\boldsymbol{\pi}^{\mu} \cdot \mathbf{D}\boldsymbol{\varphi} + 2\boldsymbol{\pi}_{\nu} \cdot \boldsymbol{\Sigma}^{\mu\nu}\boldsymbol{\varphi} \\ &= \partial_{\nu}[(2x^{\mu}x^{\nu} - g^{\mu\nu}x^2)\mathscr{L}] + 2\boldsymbol{\pi}^{\mu} \cdot \mathbf{D}\boldsymbol{\varphi} + 2\boldsymbol{\pi}_{\nu} \cdot \boldsymbol{\Sigma}^{\mu\nu}\boldsymbol{\varphi}. \end{aligned} \tag{5.35}$$

Thus, for conformal invariance, two independent conditions must hold: Eq. (5.3), which expresses scale-invariance, and

$$\boldsymbol{\pi}^{\mu} \cdot \mathbf{D}\boldsymbol{\varphi} + \boldsymbol{\pi}_{\nu} \cdot \boldsymbol{\Sigma}^{\mu\nu}\boldsymbol{\varphi} = \partial_{\mu}\sigma^{\mu\nu}. \tag{5.36}$$

where $\sigma^{\mu\nu}$ is some tensor.[12]

This is the same as Eq. (5.13). This should be no surprise, for Eq. (5.13) is the condition that the scale current can be written as

$$J^{\mu} = x_{\nu}\Theta^{\mu\nu}. \tag{5.37}$$

If this can be done, scale invariance implies that

$$\Theta_{\mu}{}^{\mu} = 0. \tag{5.38}$$

But if this is true, then we can define four other conserved currents

$$\partial_{\nu}J^{\mu\nu} = 0, \tag{5.39}$$

[12] A slightly less general form of this condition has been given by Mack and Salam and also by Gross and Wess (footnote 4). As we have remarked, the condition holds for all renormalizable theories; in addition it is valid for all theories of fields with spin 0, $\frac{1}{2}$, and 1, if the kinetic energy is of standard form, and if there are no derivative couplings. A simple example of a theory for which it does not hold is given by the theory of a vector-scalar mixture described by a single vector field, B_{μ}, for which the Lagrangian contains a term proportional to $(\partial^{\mu}B_{\mu})^2$. We emphasize that $\sigma_{\mu\nu}(x)$ must be a local function of the fields; otherwise the standard boundary conditions in Hamilton's principle are not sufficient to ensure that adding a divergence to the Lagrangian does not change the equations of motion.

where

$$J^{\mu\nu} = (2x^\mu x^\lambda - g^{\mu\lambda}x^2)\, \Theta^\nu{}_\lambda. \tag{5.40}$$

It is easy to check that these four currents are the conserved currents associated with invariance under the four conformal transformations.

We conclude this section with a final remark[13] about the relevance of our tensor $\Theta^{\mu\nu}$ to conformal transformations. Consider a conformally invariant theory, such as the massless scalar field with a φ^4 self-coupling; i.e., the theory given by (1.1), (1.2), and (1.3) with $\mu_0{}^2 = 0$. Even though conformal transformations are symmetry operations in this theory, one may verify by use of (5.32) that $T^{\mu\nu}$ as given by (1.2) does not transform covariantly under these transformations; i.e.,

$$\delta_c{}^\lambda T^{\mu\nu}(0) \neq 0. \tag{5.41}$$

On the other hand when the variation of $\Theta^{\mu\nu}$, as given by (1.3), is computed from (5.32) one does find a covariant transformation law

$$\delta_c{}^\lambda \Theta^{\mu\nu}(0) = 0. \tag{5.42}$$

6. Gravitational Interactions

1. *Observable Gravitational Effects*

It is only when we consider gravitational interactions that the full physical import of the energy-momentum tensor emerges, for it is the source of gravity, to lowest order in the gravitational coupling. To demand that our $\Theta_{\mu\nu}$ be the source of gravity is to demand that all gravitational effects be finite, to lowest order in the gravitational coupling and to all orders in all other couplings. (Of course, this does not guarantee that all gravitational effects are finite to all orders in the gravitational coupling. To attempt to fulfill this condition would lead us into the thorny problem of the divergences of quantized general relativity.)

We shall begin by assuming that our $\Theta_{\mu\nu}$ is the source of gravity, to lowest order, and explore the consequences of this assumption. Later, we shall show how to alter the gravitational interaction such that this is the case. To avoid notational complexity, we shall restrict ourselves to the case of a single scalar field with quartic self-interaction, discussed in Section 2; the generalization is trivial.

Let us begin with the process

$$a \to b + \text{graviton}, \tag{6.1}$$

[13] This point was developed through conversations with K. Johnson.

with a and b arbitrary states. The matrix element for this process is proportional to

$$\epsilon^{\mu\nu}\langle b \mid \Theta_{\mu\nu} \mid a\rangle = \epsilon^{\mu\nu}\langle b \mid T_{\mu\nu} \mid a\rangle + \tfrac{1}{6}\epsilon^{\mu\nu}(q_\mu q_\nu - g_{\mu\nu}q^2)\langle b \mid \varphi^2 \mid a\rangle, \tag{6.2}$$

where $\epsilon^{\mu\nu}$ is the graviton polarization tensor, and q is the momentum transferred to the graviton. For a free graviton, it is always possible to choose a gauge such that

$$q_\mu \epsilon^{\mu\nu} = 0, \tag{6.3}$$

and

$$g_{\mu\nu}\epsilon^{\mu\nu} = 0. \tag{6.4}$$

Thus, the last term in Eq. (6.2) vanishes, and, for this process, there is no observable difference between using our tensor and using the conventional tensor.

However, the situation is quite different when one considers scattering in an external field. For example, to lowest order, scattering in the field of the sun is also given by Eq. (6.2) with

$$\epsilon^{\mu\nu}(q) = \frac{8\pi GM}{q^2}\,\delta(q^0)[\delta_0{}^\mu\delta_0{}^\nu - \tfrac{1}{2}g^{\mu\nu}], \tag{6.5}$$

where M is the mass of the sun. We thus find that

$$\epsilon^{\mu\nu}\langle b \mid \Theta_{\mu\nu} \mid a\rangle = \epsilon^{\mu\nu}\langle b \mid T_{\mu\nu} \mid a\rangle + \tfrac{2}{3}\pi GM\delta(q_0)\langle b \mid \varphi^2 \mid a\rangle. \tag{6.6}$$

Therefore, in this case, there is an observable effect; we would have obtained a divergent matrix element if we had used the conventional energy-momentum tensor.

2. *The Gravitational Lagrangian*

Conventional gravitation theory [5] is described by an action principle of the form

$$I = \int d^4x\, \sqrt{-g}\, [(16\pi G)^{-1}R + \mathscr{L}_m] \tag{6.7}$$

where R is the curvature scalar, g is the determinant of the metric tensor, and $\mathscr{L}_m$ is the Lagrangian for matter (everything except gravity), with all derivatives replaced by covariant derivatives. In our case

$$\mathscr{L}_m = \tfrac{1}{2}g^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi - \tfrac{1}{2}\mu_0{}^2\varphi^2 - \lambda_0\varphi^4. \tag{6.8}$$

If we vary (6.7), we find the usual Einstein field equations

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -8\pi GT_{\mu\nu}\,, \tag{6.9}$$

where $T_{\mu\nu}$ is the conventional energy-momentum tensor. Thus, (6.7) predicts that the conventional tensor is the source of gravity. We wish to alter (6.7) so that the source of gravity becomes our new tensor, $\Theta_{\mu\nu}$.

One obvious possibility is to add a term to $\mathscr{L}_m$ that does not change the matter field equations in flat space; that is to say, to make a transformation of the form

$$\mathscr{L}_m \to \mathscr{L}_m + \partial_\mu V^\mu, \tag{6.10}$$

where V^μ is some vector. Unfortunately, this has no effect on the gravitational field equations, for the covariant version of this change is

$$\sqrt{-g}\,\mathscr{L}_m \to \sqrt{-g}\,\mathscr{L}_m + \sqrt{-g}\,V^\mu{}_{;\mu}. \tag{6.11}$$

where the semicolon indicates covariant differentiation. However,

$$\sqrt{-g}\,V^\mu{}_{;\mu} = \partial_\mu(\sqrt{-g}\,V^\mu). \tag{6.12}$$

Thus this does not change (6.7).

The proper procedure is to replace (6.7) by the more general form

$$I = \int d^4x\,\sqrt{-g}\,(f(\varphi)R + \mathscr{L}_m). \tag{6.13}$$

where f is a function we will determine shortly. This leads to the field equations

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -\frac{1}{2f}\,[T_{\mu\nu} + 2f_{\mu;\nu} - 2g_{\mu\nu}f^\lambda{}_{;\lambda}]. \tag{6.14}$$

Thus we can obtain the desired lowest-order result if we choose

$$f = \frac{1}{16\pi G} - \frac{1}{12}\,\varphi^2. \tag{6.15}$$

The field equations of matter[14] are also changed; they become

$$\varphi^\mu{}_{;\mu} = -\mu_0{}^2\varphi - 4\lambda_0\varphi^3 - \tfrac{1}{6}R\varphi. \tag{6.16}$$

If we take the trace of Eq. (6.14), and use Eq. (6.16) to evaluate the right-hand side, we find

$$2fR = \mu_0{}^2\varphi^2 - \tfrac{1}{6}R\varphi^2. \tag{6.17}$$

[14] This equation, with zero mass and no self-interaction, occurs in R. Penrose, Proc. Roy. Soc. (London) **284A**, 204 (1965). Not surprisingly, this paper is a study of conformally-invariant equations for massless particles. It also appears in F. Gürsey, Ann. Phys. **24**, 211 (1963), as the equation for $(-9)^{1/8}$.

This implies that

$$R = 8\pi G \mu_0^2 \varphi^2. \tag{6.18}$$

Inserting (6.18) in (6.16) we find

$$\varphi^{\mu}{}_{;\mu} = -\mu_0^2 \varphi - (\lambda_0 + \tfrac{4}{3}\pi G \mu_0^2)\, \varphi^3. \tag{6.19}$$

Equations (6.16) and (6.19) are mathematically equivalent; their physical interpretations, however, are completely different. Equation (6.16) tells us that the mass of a particle depends on the surrounding gravitational field; this is in obvious violation of the principle of equivalence. Equation (6.19) tells us that the only effect of our new gravitation theory is to change, in a universal way, the strength of the quartic self-interaction; this is obviously consistent with the principle of equivalence.

Both interpretations are valid, but the second is more useful for comparing the results of the new theory with that of general relativity. Together with the fact that, in the absence of matter, Eq. (6.14) reduces to Einstein's field equation, it tells us that all the classical tests of general relativity (including the Eötvös experiment) are met by our new theory.

APPENDIX 1. Some Remarks on Vector Fields

The purpose of this appendix is to prove the assertions at the end of Section 3.2, by giving a more careful analysis of the structure of the energy-momentum tensor in the presence of vector fields. For simplicity, we restrict ourselves to the Lagrangian (3.6); the generalization is trivial.

The energy-momentum tensor for this theory is given by

$$\Theta_{\mu\nu} = \mu^2 A_\mu A_\nu - \partial_\mu A\, \partial_\nu A - \tfrac{1}{2} g_{\mu\nu}(\mu^2 A_\lambda A^\lambda - \partial_\lambda A\, \partial^\lambda A) + \cdots, \tag{A1.1}$$

where the triple dots indicate terms in which the fields A_μ and A enter only in the combination $\mathcal{A}_\mu$. Such terms need not concern us here, since diagrams in which they enter have the superficial degrees of divergence used in our proof. On the other hand, the terms displayed explicitly are a possible source of difficulty, because of the bad behaviour of the propagator (3.5).

By Eq. (3.7),

$$\Theta_{\mu\nu} = \mu^2 \mathcal{A}_\mu \mathcal{A}_\nu - \tfrac{1}{2} g_{\mu\nu} \mu^2 \mathcal{A}_\lambda \mathcal{A}_\lambda - \mu[\mathcal{A}_\mu\, \partial_\nu A + \mathcal{A}_\nu\, \partial_\mu A - g_{\mu\nu} \mathcal{A}^\lambda\, \partial_\lambda A] + \cdots. \tag{A1.2}$$

As a consequence of the equations of motion,

$$\partial^\mu \partial_\mu A = 0, \tag{A1.3}$$

and

$$\partial^\mu A_\mu = \partial^\mu \mathcal{A}_\mu = 0. \tag{A1.4}$$

Therefore,

$$\partial^\mu[\mathcal{A}_\mu\, \partial_\nu A + \mathcal{A}_\nu\, \partial_\mu A - g_{\mu\nu}\mathcal{A}^\lambda\, \partial_\lambda A] = 0. \tag{A1.5}$$

Thus, we can drop this term, and redefine the energy-momentum tensor to be

$$\Theta_{\mu\nu} = \mu^2 \mathcal{A}_\mu \mathcal{A}_\nu - \tfrac{1}{2}\mu^2 g_{\mu\nu}\mathcal{A}_\lambda \mathcal{A}^\lambda + \cdots . \tag{A1.6}$$

But this is the same expression as the original energy-momentum tensor, before we introduced the auxiliary fields, except that A_μ has everywhere been replaced by $\mathcal{A}_\mu$. Thus we have justified the counting of divergences in Section 3.

Because of Eq. (A1.3), the terms involving A make no contribution to the matrix elements of (A1.1) between on-the-mass-shell states. Thus, these matrix elements are finite even if we do not introduce the auxiliary field A; the auxiliary field is introduced only to simplify our proof, and does not affect the final result.

APPENDIX 2. Explicit Form of the Ward and Trace Identities

In this appendix we shall give the explicit form of the Ward and trace identities. The expressions we shall derive are valid for any field theory in which Eq. (5.36) is valid, and in which the fundamental fields commute (or anticommute) at equal times. (Note that, in the case of vector fields, the latter condition is true only if we use the formalism explained in the preceding appendix. The components of $\mathcal{A}_\mu$ are kinematically independent; those of A_μ are not.) This class of theories includes, but is larger than, the class of renormalizable theories.

We begin by establishing some notation. As in the body of the paper, we assemble all of our fields into a vector $\boldsymbol{\varphi}$. In the same spirit, we write the n-particle Green's functions, $\boldsymbol{\Gamma}_{\mu\nu}^{(n)}$, $\mathbf{G}^{(n)}$, and $\boldsymbol{\Gamma}^{(n)}$, as tensors. If $\boldsymbol{\varphi}$ has N components, each of these tensors has n indices (in addition to the explicitly displayed Lorentz indices), each of which runs from 1 to N. If M is any $N \times N$ matrix, the expression

$$\mathbf{M}^{(r)}\mathbf{G}^{(n)} \tag{A2.1}$$

is to be read as meaning that $\mathbf{M}$ acts only on the r-th index of $\mathbf{G}^{(n)}$.

According to Eqs. (3.11) and (5.19),

$$\begin{aligned}\Theta^{\mu\nu} = {} & [\boldsymbol{\pi}^\mu \cdot \partial^\nu \boldsymbol{\varphi} - g^{\mu\nu}\mathscr{L} + \tfrac{1}{2}\partial_\lambda(\boldsymbol{\pi}^\lambda \cdot \boldsymbol{\Sigma}^{\mu\nu}\boldsymbol{\varphi})] \\ & - \tfrac{1}{2}\partial_\lambda(\boldsymbol{\pi}^\mu \cdot \boldsymbol{\Sigma}^{\lambda\nu}\boldsymbol{\varphi} + \boldsymbol{\pi}^\nu \boldsymbol{\Sigma}^{\lambda\mu} \cdot \boldsymbol{\varphi}) + \tfrac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu}.\end{aligned} \tag{A2.2}$$

As we explained in Section 2, the off-the-mass-shell matrix elements of this object depend on whether or not we use the equations of motion to explicitly do the indicated differentiations. We shall adopt the convention that we shall do the differentiations only on the terms enclosed in square brackets. Thus, in the position-space definitions of the Green's functions, these derivatives are within the time-ordering symbol; the others are without. This makes the calculation of the Ward identity especially simple: For the first two terms, we have to bring the external derivative inside the time-ordering symbol; this gives us the divergence of the canonical energy-momentum tensor, which is zero. For the third term, we have to bring the internal derivative outside; this makes the total expression (apart from the first two terms) the divergence of an antisymmetric tensor, which has vanishing divergence. Thus we find

$$k^\mu \mathbf{\Gamma}_{\mu\nu}^{(n)}(k; p_1 \cdots p_n) = -i \sum_P (p_1 + k)_\nu \mathbf{G}^{(n)}(p_1 + k, p_2 \cdots p_n) + \tfrac{1}{2} i k^\mu \sum_P \mathbf{\Sigma}_{\mu\nu}^{(1)} \mathbf{G}^{(n)}(p_1 + k, p_2 \cdots p_n). \tag{A2.3}$$

For the trace identity, we lust bring the derivatives on the unbracketed terms inside the time-ordering symbol. The calculation of the necessary commutators can be drastically simplified with the aid of Eq. (5.36). We shall not give the details of the calculation here, but merely state the final result:

$$g^{\mu\nu} \mathbf{\Gamma}_{\mu\nu}^{(n)}(k; p_1 \cdots p_n) = \mathbf{\Gamma}^{(n)}(k; p_1 \cdots p_n) - i \sum_P \mathbf{D}^{(1)} \mathbf{G}^{(n)}(k + p_1, p_2 \cdots p_n). \tag{A2.4}$$

None of this has taken account of the ϵ-dependent terms in the final definition of the finite energy-momentum tensor, Eq. (3.12). However, since for these terms the derivatives are always outside the time ordering symbol, they contribute to neither the Ward nor the trace identities.

APPENDIX 3. Equal-Time Commutation Relations

If $\Theta_{\mu\nu}$ is any energy-momentum tensor, its components can be shown [6, 7] as a consequence of locality and Lorentz-invariance, to obey the following equal-time commutation relations:

$$i[\Theta^{00}(\mathbf{x}, t), \Theta^{00}(\mathbf{y}, t)] = [\Theta^{0j}(\mathbf{x}, t) + \Theta^{0j}(\mathbf{y}, t)]\, \partial_j \delta^{(3)}(\mathbf{x} - \mathbf{y}) - \partial_i^x \partial_j^x \partial_k^y \partial_l^y \tau_1^{ijkl}(\mathbf{x}, \mathbf{y}, t), \tag{A3.1}$$

and

$$i[\Theta^{00}(\mathbf{x}, t), \Theta^{0i}(\mathbf{y}, t)] = [\Theta^{ij}(\mathbf{x}, t) - g^{ij}\Theta^{00}(\mathbf{y}, t)]\, \partial_j \delta^{(3)}(\mathbf{x} - \mathbf{y}) - \partial_k{}^x\, \partial_l{}^x\, \partial_j{}^y [\tau_2^{klij}(\mathbf{x}, y, t) - \tfrac{1}{2}\partial_0 \tau_1^{klij}(\mathbf{x}, \mathbf{y}, t)] \tag{A3.2}$$

where the τ's are objects which have support only at $\mathbf{x} = \mathbf{y}$.

Schwinger [6] has argued on general grounds that τ_1 vanishes, and has shown explicitly that this is the case for a wide class of theories involving fields of spin less than or equal to one, if $\Theta_{\mu\nu}$ is the conventional energy-momentum tensor. (We have called this object $T_{\mu\nu}$ in the body of the text.) On the other hand, Boulware and Deser [7] have shown, as a consequence of the positivity of the metric in Hilbert space, that τ_2 can never vanish.

For the $\lambda\varphi^4$ theory, discussed in Section 2, naive canonical manipulation indicates that, for the conventional tensor, both τ's vanish. (This contradicts the Boulware-Deser result, just as the corresponding calculation in spinor electrodynamics gives the false result that the Schwinger term for the electromagnetic current vanishes. Here, just as in that case, a more rigorous analysis can easily be done for the free theory; it shows that τ_2 does *not* vanish, but is only a c number.)

It is interesting to ask what is the state of the τ's in the same theory, for our new tensor. We present here the result of a naive canonical computation:

$$\tau_1 = 0, \tag{A3.3}$$

and

$$\tau_2^{klij} = -\tfrac{1}{12}(g^{ki}g^{lj} + g^{kj}g^{li} - \tfrac{2}{3}g^{kl}g^{ij})\, \varphi^2(\mathbf{x}, t)\, \delta^{(3)}(\mathbf{x} - \mathbf{y}).$$

Although we have not attempted to verify these equations by any more sophisticated method of calculation, we suspect that their most prominent feature is indeed true, that τ_2 is a q number.

Similar canonical q-number τ's appear in any theory for which our tensor is not the same as the conventional tensor—that is to say, for any theory involving spinless mesons. This is a close parallel to the situation for the canonically-calculated Schwinger terms for the electric current, which appear only in theories with charged spinless fields.

APPENDIX 4. Remarks Added in Proof

The proof of the finiteness of the energy-momentum tensor given in the body of this paper puts great emphasis on the softness of the trace of this tensor. However, our explicit calculation of Sec. 4 shows that this trace is not soft. This has caused K. Symanzik (private communication) to suggest that perhaps the proof of renormalizability is in error and the tensor does not, in fact, have finite matrix elements. Phrased in another way, the terms involving large regulator masses in Eq. (4.11) may cause difficulties in higher orders of perturbation theory, although they cause none in lowest order.

We do not know, at this time, whether this actually happens; however, to be safe, we offer in this appendix an alternative method for constructing the tensor which we believe circumvents these difficulties. The construction is as follows:

(1) Define the naive trace to be that combination of the operators (3.16) whose value at the point zero is determined by the Ward and trace indentities. (Notice that in the specific calculation of Sec. 4, this would lead to a different answer than that we obtained by using regulator fields; the last term in Eq. (4.21) would be missing.)

(2) Define the functions $\Gamma^{(2,0,0)}$ by the trace identities, and fix the coefficient of the extra term in the energy-momentum tensor involving second derivatives of scalar fields by demanding that the term of order k^2 in the expansion of these functions about the point zero be equal to the corresponding term for the corresponding functions constructed for the naive trace. If the naive calculation of the trace (ignoring the regulator fields) given in Sec. 3 were valid, this would fix the coefficient to be 1/6. However, in any event, it does fix the coefficient at some value, and ensures that the trace has its naive value, at least up to this order in the power-series expansion.

(3) Although, in Sec. 3.5, we used the finiteness of the trace to show that $\Gamma^{(0,0,2)}_{\mu\nu\lambda\sigma}$, and $\Gamma^{(0,0,2)}_{\mu\nu,\lambda}$ were finite, we did not need to do this; we could have, instead, used the Ward identities that come from taking the divergence of the *vector* fields.

With these alterations, the arguments for finiteness of the tensor run along the same lines as before, and the need to use the equations of motion to calculate the trace is eliminated.

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References

1. K. Hepp, *Comm. Math. Phys.* **1** (1965), 95. (This paper contains a self-contained and rigorous development of the theory, as well as references to the original work of Bogoliubov and Parasiuk.)
2. S. N. Gupta, *Proc. Phys. Soc.* **A66** (1953), 129.
3. Y. Nambu, *Prog. Theor. Phys. (Kyoto)* **7** (1952), 131.
4. F. J. Belinfante, *Physica (Utrecht)* **7** (1940), 449.
5. We use the notational conventions of R. Adler, M. Bazin, and M. Schiffer, "Introduction to General Relativity," McGraw-Hill, New York, 1965.
6. J. Schwinger, *Phys. Rev.* **130** (1963), 800.
7. D. G. Boulware and S. Deser, *J. Math. Phys.* **8** (1967), 1468.