Modeling some two-dimensional relativistic phenomena using an educational interactive graphics software[a)]

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This paper describes several relativistic phenomena in two spatial dimensions that can be modeled using the COLLISION program of Spacetime Software. These include the familiar aberration, the Doppler effect, the headlight effect, and the invariance of the speed of light in vacuum, in addition to the rather unfamiliar effects like the dragging of light in a moving medium, reflection at moving mirrors, Wigner rotation of noncommuting boosts, and relativistic rotation of shrinking and expanding rods. All these phenomena are exhibited by tracings of composite computer printouts of the COLLISION movie. It is concluded that an interactive educational graphics software with pleasing visuals can have considerable investigative power packed within it.

I. INTRODUCTION

In this paper we describe some relativistic phenomena in two spatial dimensions which can be modeled using the COLLISION program of Spacetime Software.[1] Spacetime Software mainly consists of two interactive graphics utilities developed by Edwin F. Taylor to help students understand relativity better. These programs enable the students to simulate, set up, solve, and investigate a variety of problems, posers, and paradoxes in relativity.[2] The results of the simulation appear on the screen of the monitor as pictures, movies, graphs, tables, and perspective plots.

About a year ago we received a gift set of the software from E. F. Taylor. Enclosed with the software were the User's and Teacher's manuals and a set of projects worked out by students at MIT during the last 2 years using this software. The manuals were very helpful, and learning to use these programs took only a few days.

After working out Taylor's projects we began designing some on our own. The animations and movies were very enticing, and what started as a pastime proved to be an absorbing affair. By the time we finished writing a handful of projects we realized that we were seeing well-known phenomena from perspectives that we missed earlier.

For instance, we learned that the ray surface in a moving medium[3] can come from velocity addition, the relativistic rotation of a moving rod[4] from particle aberration, the reflection of light at a moving mirror[5] from inverse Compton scattering, the Wigner rotation due to noncommutative boosts[6] from velocity addition, etc.

An inspection of the literature cited above shows that some of these tidbits may be somewhat original, but most of them can be found in textbooks and journals. But the fact remains that we learned most of them not while scouring the library for books and journals but while playing with the PC, punching its buttons, and watching the movies. The literature survey came later on.[7] Clearly, our experience shows that an interactive educational graphics software with pleasing visuals can combine instruction with enjoyment and pack within it a lot of investigative potential.

II. THE COLLISION PROGRAM

Spacetime Software consists mainly of two programs: SPACETIME and COLLISION. SPACETIME deals with Lorentz transformations in one spatial dimension.[1] COLLISION, on the other hand, is a two-dimensional program. It is built around the conservation of energy–momentum in relativistic collisions, creations, transformations, annihilations, and decays in two spatial dimensions. The first display of this program is a Table.[1] There is provision in the Table for entering the mass, energy, momentum, and angle of motion of up to two input particles A, B and up to three output particles C, D, and E.

The student enters into the Table what he knows and wants about the input and output particles and asks the computer to complete the Table. It does so by solving the collision and displays messages on the screen as it goes about its task. After solving the collision one can ask the computer to play the collision as a movie. (This, without doubt, is the most charming part of Spacetime Software.) Massless particles such as photons appear as open circles whose size increases with the energy of the photon (e.g., see Figs. 1 and 2). Massive particles appear as dark circles (see Fig. 3) whose size increases with the mass of the particle. The student can command the movie to go forward, backward, halt, step, restart, etc. The movie adjusts its timings in such a way that, in every frame, the collision takes place at $t = 0$.

The key feature of the COLLISION program is its ability to transform quickly the solved collision to any other frame moving with any desired speeds β_x and β_y with respect to the original frame. After performing each transformation, the movie displays the collision as seen in the new frame. Each of these displays can be filed and called back at will. Since the original collision is in two spatial dimensions, the ability to jump to other frames facilitates the modeling of many two-dimensional relativistic phenomena using the COLLISION program.

III. ALL ABOUT LIGHT

The propagation of light in vacuum is surely the kingpin of relativity. When a source sitting at the origin emits a flash of light, the wave front is an expanding sphere centered on the origin. When the same flash is emitted by a moving source, the wave front is again an expanding sphere centered on the site of the emission. The speed of the source does not add to the speed of light in vacuum, as was first postulated by Einstein. Although the speed of light is invariant, other features of light such as its frequency, intensity, and angular distribution do depend on whether the source is a moving source or a static one in the given frame.

All these aspects of the propagation of light can be modeled using the COLLISION program. Figure 1 is the tracing of a composite printout generated by the COLLISION movie. It shows the "still" picture of 12 identical photons distributed uniformly on a circle[8] at $t = 10$. All these photons were emitted at $t = 0$ by a source of light sitting at the origin of the lab frame. We describe briefly how to get such pictures using the COLLISION program because all the printouts in this paper are obtained by a similar procedure.

Since the COLLISION program has provision for a maximum number of three output particles (C, D, E), the 12 photons in Fig. 1 are obtained in four passes through the printer on the same paper. The three photons, say, 1, 5, 9 are obtained by setting up the table to simulate the decay of a massive particle (A) at rest into three identical photons (C,D,E) at the desired angles 0°, 120°, and 240°. We ask the computer to complete the table and play the movie. The movie starts at $t = -10$ and shows the particle A sitting at the origin till $t = 0$. It then explodes and the three decay photons are emitted in the three chosen directions. These photons travel radially out until $t = 10$. At this instant the screen would show the three photons labeled 1, 5, 9 at the positions shown in Fig. 1. We take a printout of this display. We repeat the procedure, making necessary changes in the table to shoot the photons 2, 6, 10 and take a printout of this movie display at $t = 10$ on the same paper that contained the prints of the photons 1, 5, 9. In this way we get Fig. 1 in four passes.

It is now only a matter of punching buttons on the computer to get Fig. 2 from Fig. 1, where Fig. 2 is the Lorentz transformation of Fig. 1 to a rocket frame moving to the right with a $\beta_x = 0.8$ with respect to the lab frame. Each of the four movie displays contained in Fig. 2 would show that in the rocket frame the source of light (particle A) moves to the left from $t = -10$ to $t = 0$ with the speed $\beta_x = -0.8$ and explodes at $t = 0$ releasing the transformed photons. These move out in the new directions shown in Fig. 2.

Figure 2 packs within it the following physics:

(i) *Invariance of the speed of light in vacuum.* The joining of the 12 photons is seen once again to be a circle of the same radius centered on the origin of the rocket frame. This demonstrates the fundamental postulate of relativity.

(ii) *Aberration of light.* Each photon, say photon 10, of Fig. 2 is the aberrated version of the same photon 10 of Fig. 1. The change in the direction of motion can be checked against the aberration formula. (Photon 10, in particular, is falling vertically downward in Fig. 1 and can be used to simulate the well-known stellar aberration.)

(iii) *Doppler effect.* Each photon, say photon 10, of Fig. 2 is also the Doppler-shifted version of the same photon 10 of Fig. 1. The change in the sizes of the photons shows the systematic variation predicted by the Doppler shift formula. (Photon 10, in particular, can simulate the transverse Doppler effect if *carefully* interpreted. The transverse Doppler effect is a purely relativistic effect due to time dilation and has been checked in the laboratory by Ives and Stilwell.)

(iv) *Headlight effect.* Figure 2 shows that the distribution of photons in the rocket frame is tilted forward compared to their distribution in the lab frame (Fig. 1) where

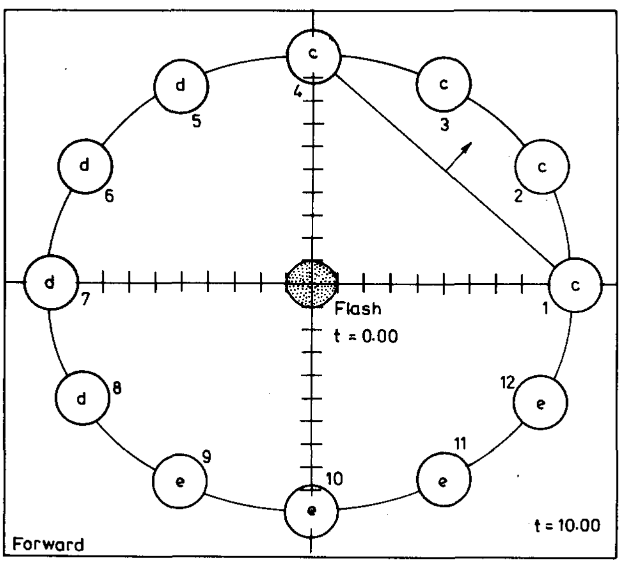

Fig. 1. This figure shows the locus of 12 photons emitted by a static source in vacuum. The wave front is a sphere although it looks squashed owing to a slight difference in the scales along the x and y axes.

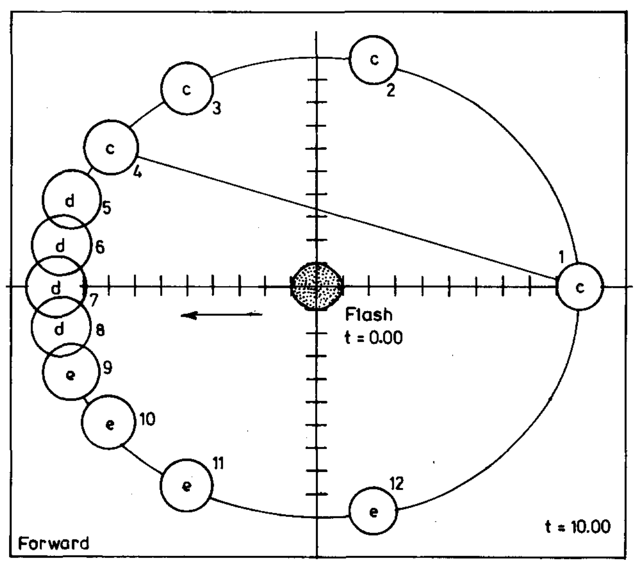

Fig. 2. This is the transform of Fig. 1 to a rocket frame. It exhibits the phenomena of aberration, Doppler effect, headlight effect, invariance of the speed of light in vacuum, and the relativistic rotation of 66 moving rods.

the source was at rest. This bunching of the photons in the direction of motion of the source is the well-known headlight effect. It is a consequence of aberration and contributes partly to the resolution of the "dark night sky paradox" (or Olbers' paradox).[9]

(v) *Relativistic rotation.* Suppose we join photons 1 and 4 of Fig. 1. This simulates an expanding rod oriented at $-45°$ to the x axis and moving normal to itself with a speed $\beta(u) = 1/\sqrt{2}$ in the lab frame. In the rocket frame we take the join of the transformed photons 1 and 4 of Fig. 2. As Fig. 2 is a "still" of photons 1 and 4, the line joining them shows that the 1–4 rod has now undergone a relativistic rotation. (Both the rods considered are expanding while moving forward, but this aspect will be discussed in detail in Sec. VII.) Since Fig. 1 contains 12 photons, it follows that there are as many as 66 rods in Fig. 1, each moving with a different speed in a different direction with a different orientation. Figure 2 contains the relativistic rotation data of all these 66 rods.

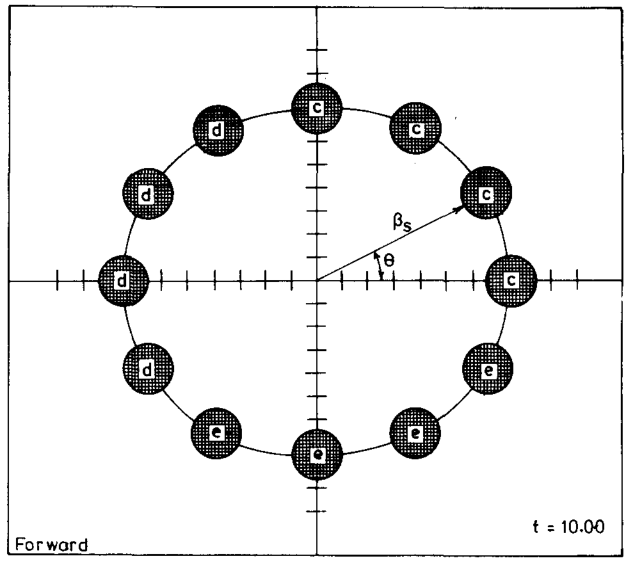

Fig. 3. This figure shows the ray surface of light in a static medium like water. It is a sphere (although it looks a little squashed) expanding with a speed less than the speed of light in vacuum.

IV. RAY SURFACE IN A MOVING MEDIUM

We saw in Sec. III that an expanding sphere in one frame remains an expanding sphere in every other frame provided it expands with the speed of light in vacuum. We now ask what happens to a sphere expanding with a speed less than the speed of light in vacuum. This simulates the wavefront generated by a flash of light emitted by a static source in an isotropic homogeneous transparent medium at rest, like still water. If the medium has a refractive index n ($n = \frac{4}{3}$, say), the speed of the light signals in the lab frame is reduced to $\beta_s = 1/n$ ($= 0.75$ for water).

A printout of this signal velocity (or ray) surface in such a medium is shown in Fig. 3. Figure 3 is obtained from the COLLISION program precisely like Fig. 1 except that the output particles C, D, E, instead of being photons, are now chosen to be massive particles moving out with speeds $\beta_s = 0.75$. These particles simulate the light signals in the medium as far as the kinematical properties of light propagation are concerned.[10] Figure 3 is a sphere expanding with a speed $\beta_s = 0.75$ centered around the flash at the origin.

We now jump to a subluminal rocket frame moving to the right with a speed $\beta_x = 0.6$ and ask the computer to transform the ray surface of Fig. 3 to this frame. In the rocket frame the entire medium, together with the source,[10] is moving to the left with the speed $\beta_x = -0.6$. Isotropy is now reduced to axial symmetry and we no longer expect the ray surface to remain a sphere. This is indeed borne out by Fig. 4, which is the computer-generated printout of the transform of Fig. 3. It shows that the ray surface in a moving medium is an oval[8] that is partially dragged by the medium but still encloses the origin of the flash.

Now, all that the collision transform function could be doing is to apply the particle aberration formula to each of the outgoing particles C, D, and E. Equivalently it is just applying the velocity addition formulas. We therefore conclude that the ray surface in a moving medium should be derivable from the well-known relativistic velocity addition formulas:

$$\beta_s \cos\theta = (\beta_s' \cos\theta' + \beta)/(1 + \beta\beta_s' \cos\theta'), \quad (1)$$

$$\beta_s \sin\theta = (\beta_s' \sin\theta' \sqrt{1-\beta^2})/(1 + \beta\beta_s' \cos\theta'), \quad (2)$$

where the primed quantities refer to the rocket frame (Fig. 4). (We dropped the suffix x over β_x to save writing.) Squaring and adding Eqs. (1) and (2) we note that θ drops out and gives a quadratic equation for β_s'. The solution of this quadratic equation $\beta_s'(\theta')$ is the desired equation of the ray surface:

$$\beta_s' = \left| \frac{\beta(n^2-1)\cos\theta' \pm \{n^2(1-\beta^2)[(1-\beta^2) - \beta^2(n^2-1)\sin^2\theta']\}^{1/2}}{(n^2-\beta^2)\cos^2\theta' + n^2(1-\beta^2)\sin^2\theta'} \right|, \quad (3)$$

where n is the refractive index ($n = 1/\beta_s$) of the medium in its rest frame. It is gratifying to note that the result of this simple derivation tallies exactly with the equation for the ray surface in a moving medium obtained from the electrodynamics of moving media.[11]

The dragged oval of Fig. 4 can be put to some interesting pedagogical uses. For instance, one can use it to predict the results of celebrated experiments like the Fizeau's running water experiment or a Michelson–Morley experiment in a possible ether wind or refractive index $n > 1$. The arms of the Michelson interferometer could have unequal length, and may be arbitrarily oriented relative to the direction of the ether wind. It is easy to check that as $n \to 1$, Eq. (3) goes back to the

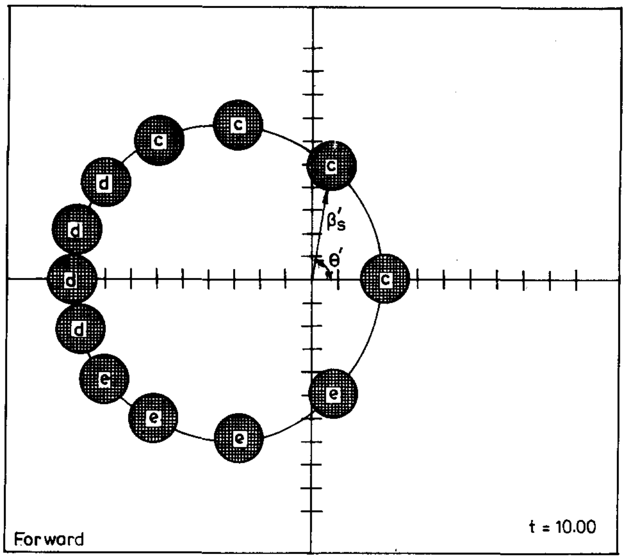

Fig. 4. This is the transform of Fig. 3 to a subluminal rocket frame, in which the medium flows to the left. Notice that the ray surface is actually a dragged oval (although it looks more like a sphere—for an actual sphere in these scales see Fig. 3). The dragging is not complete and the oval still encloses the flash at the origin.

equation of the spherical wavefront in vacuum, as it should, and we get back the null result of the usual Michelson–Morley experiment.

It can be checked from Eq. (3) that the dragged oval of Fig. 4 encloses the origin of the flash only as long as the speed of the rocket frame (β) is less than the signal speed (β_s) in the lab frame. Let us ask the COLLISION program to transform Fig. 3 to a superluminal rocket frame moving with $\beta > \beta_s$, say, $\beta_x = 0.9$. The result is the printout shown in Fig. 5. An inspection of Fig. 5 shows that the dragging of light by the medium is now overwhelming and the oval no longer encloses the origin of the flash that generated it. As a result, there exists a "Mach" cone in this frame, outside of which no light signal ever reaches and inside which light is doubled up, with two pulses of light in each direction. The situation is somewhat similar to the sound of a bell ringing in a supersonic wind tunnel. It can be checked from Eq. (3) and tallied from Fig. 5 that the half-angle θ_M' of this Mach cone is given by

$$\tan \theta_M' = \sqrt{1-\beta^2}/\sqrt{\beta^2 n^2 - 1}\,. \tag{4}$$

A *careful* analysis would lead us from Eq. (4) of the Mach cone to the equation of the well-known Čerenkov cone by means of a simple Lorentz transformation.[12]

V. REFLECTION OF LIGHT AT A MOVING MIRROR

Let us first model the reflection of light at a static mirror using the COLLISION program. By definition, a mirror can absorb a lot of momentum with little change in its velocity. So we simulate the mirror by a very heavy particle (a) at rest (see Fig. 6). The incident beam of light can be modeled by the input photon (b) and the reflected beam by the output photon (d). We choose the directions of the input and output photons such that the law of reflection $i = r$ is obeyed in this lab frame. But this is nothing but a Compton scattering event of a soft photon by a supermassive target, and the completed COLLISION table would show that there is neither a frequency shift of the photon nor any appreciable recoil of the massive target. We therefore see that the computer printout of this event (Fig. 6) correctly simulates the reflection of light at a static mirror.

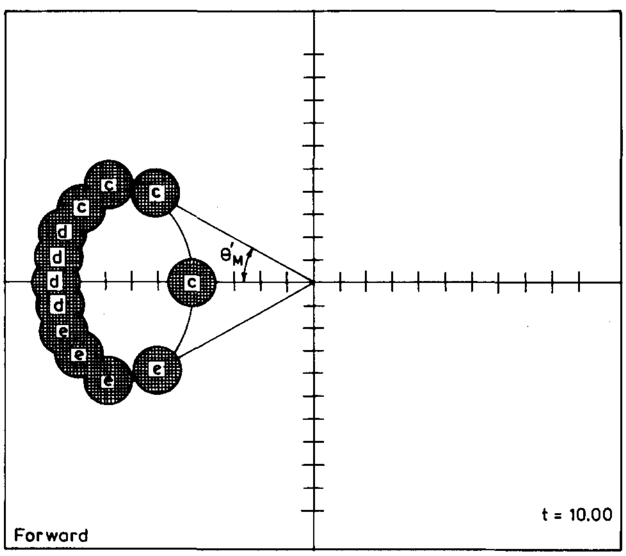

Fig. 5. This is the transform of Fig. 3 to a superluminal rocket frame. Notice that the dragging is overwhelming. The ray surfaces develop an envelope (the Mach cone) outside of which no light ever reaches.

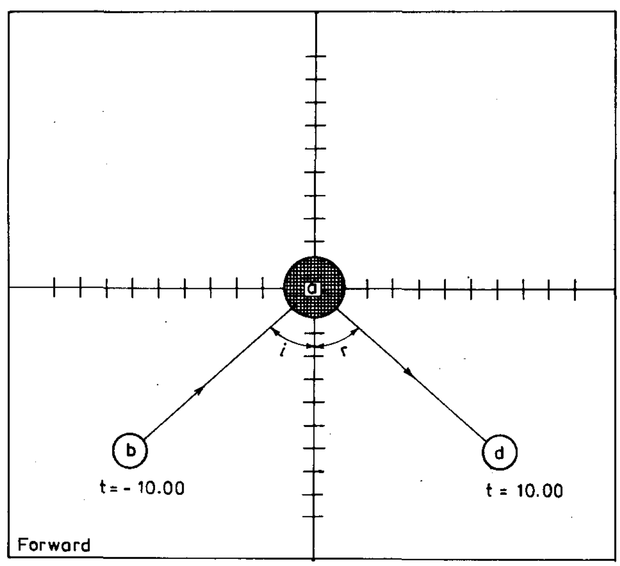

Fig. 6. This is a simulation of the reflection of light at a static mirror viewed as a Compton scattering of a soft photon by a supermassive target.

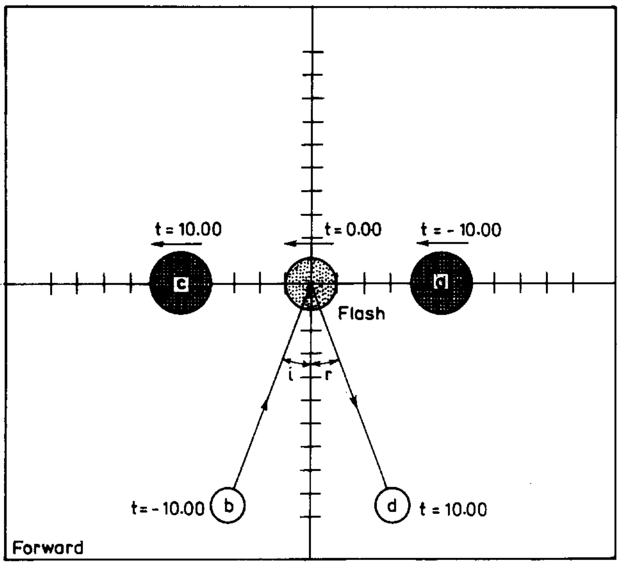

Fig. 7. This is the transform of Fig. 6 to a rocket frame moving with $\beta_x = 0.5$. In this frame the mirror is moving to the left parallel to itself. Note that the law of reflection is still obeyed ($i = r$).

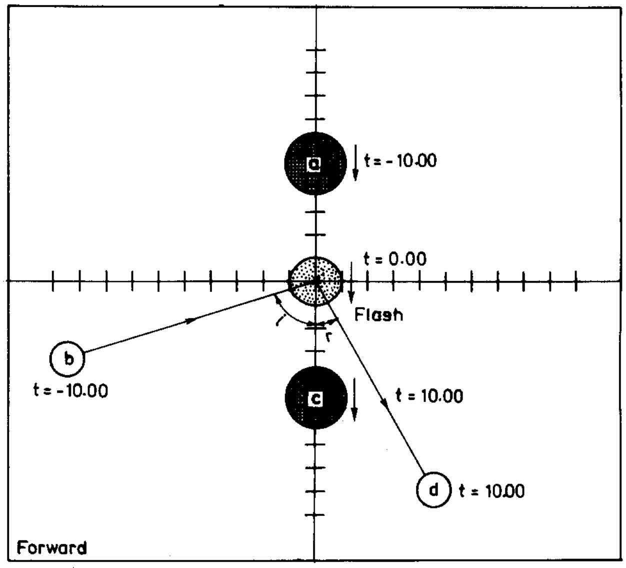

Fig. 8. This is the transform of Fig. 6 to a rocket frame moving with $\beta_y = 0.5$. In this frame the mirror swoops down on the incident photon. The law of reflection is no longer obeyed ($i \neq r$). There is also a blue shift of the incident photon.

We now jump to a rocket frame with, say $\beta_x = 0.5$ with respect to the lab frame. In the rocket frame the mirror (a) would be moving parallel to itself leftward with $\beta_x = -0.5$. The transform of Fig. 6 generated by the computer is shown in Fig. 7. It shows the rather unexpected result that the law of reflection $i = r$ is still obeyed in this rocket frame.[13] The accompanying table (not given here) would show that there is no Doppler shift either.

Let us next jump from the lab frame to a rocket frame moving with a speed $\beta_y = 0.5$ along the y axis of the lab frame. The transformed printout is shown in Fig. 8. It is now seen that in this frame the mirror (a) moves normal to itself and swoops down on the incident photon (b). It is evident that the reflected photon (d) no longer obeys the usual law of reflection. The table accompanying this event (not given here) would show that the reflected photon suffers a considerable blue shift. All these results can be shown to agree quantitatively with the standard formulas for the reflection of light at moving mirrors.[5]

The event shown in Fig. 8 is actually an inverse Compton scattering event that is of some importance in astrophysics. The COLLISION program would show that at near normal incidence on a high-velocity target, the boost in energy of the scattered Compton photon can be tremendous. It is believed that some of the intergalactic x-ray emission could be due to such an escalation of energy of the microwave background photons suffering inverse Compton scattering by energetic cosmic ray particles.[14]

VI. WIGNER ROTATION DUE TO NONCOMMUTING BOOSTS

It is well known that finite spatial rotations do not commute. Therefore, the result of a product of two rotations depends in general on the order in which the rotations are performed. Since a Lorentz boost is a rotation in four-space, the resultant of two successive boosts in general depends on the order in which the boosts are applied. The commutator of two generators of pure boosts is a generator of spatial rotation.[15] It is therefore reasonable that the resultants of two boosts applied in different orders should differ by a spatial rotation. This spatial rotation is generally called the Wigner rotation.[6] The relativistic rotation of moving rods mentioned in Sec. III(v) and discussed in more detail in Sec. VII is an indirect consequence of the Wigner rotation.

Since a product of two boosts is nothing but addition of two three-velocities, it follows that velocity addition is, in general, noncommutative in the theory of relativity (i.e., $\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2 \neq \boldsymbol{\beta}_2 + \boldsymbol{\beta}_1$) unlike in classical physics. The correct application of relativistic velocity addition would show that the result of combining two perpendicular velocities $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ in different orders results in boosts with the same magnitude:

$$|\boldsymbol{\beta}_{12}| = |\boldsymbol{\beta}_{21}| = [\beta_1^2 + \beta_2^2 - \beta_1^2\beta_2^2]^{1/2}, \tag{5}$$

but which are rotated by the Wigner rotation angle ψ given by

$$\tan\psi = \beta_1\beta_2\gamma_1\gamma_2/(\gamma_1 + \gamma_2), \tag{6}$$

where the symbols have their usual meaning.

Figure 9 shows the composite printouts of the addition of two perpendicular velocities $\beta_1 = \beta_2 = 0.9$ in different orders. It can be checked from the printout that the magnitude of the resultant velocity and the Wigner rotation angle ψ agree with Eqs. (5) and (6).

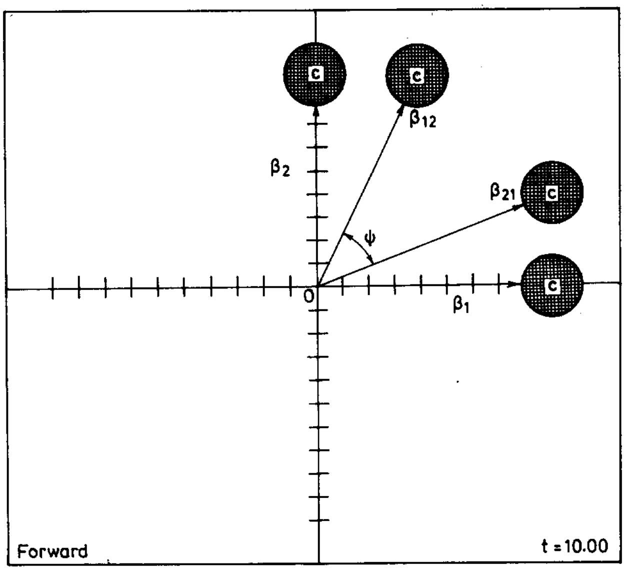

Fig. 9. This figure shows that the addition of two velocities β_1 and β_2 depends on the order of their addition. The resultants β_{12} and β_{21} have the same magnitude but are inclined to one another by the Wigner rotation angle ψ.

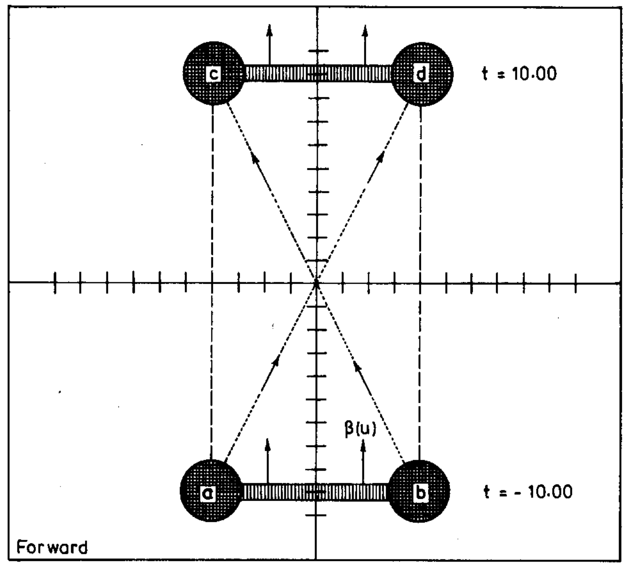

Fig. 10. This figure shows a rod ab that moves perpendicular to itself along the Y axis with the speed $\beta(u) = 0.9$.

VII. RELATIVISTIC ROTATION OF A MOVING ROD

Suppose that a rod lies in the xy plane of the lab frame parallel to the x axis and moves normal to itself along the y axis with a speed $\beta(u)$. We now jump to a rocket frame moving with the speed $\beta(v)$ along the x axis. Relativity predicts the surprising result that the rod is now rotated by an angle ϕ and no longer moves normal to itself. This spatial rotation and aberration of the rod is crucial to the resolution of a number of two-dimensional length contraction paradoxes.[4]

Let us try to simulate this rod using the COLLISION program. Any rod can be simulated by just two particles, say, the end particles of the rod as shown in Fig. 10. But there is no way to show a rod moving parallel to itself and keeping a constant length using the COLLISION program that requires that all its particles (such as a and b) must collide at the origin. The collision shown in Fig. 10 is therefore actually the simulation of a shrinking and expanding "elastic" rod moving perpendicular to itself with a speed $\beta(u) = 0.9$.

We now jump to a rocket frame moving with the speed $\beta_x = 0.9$ and ask the computer to transform Fig. 10 to this frame. The result is Fig. 11 which does show a relativistic rotation by a large angle ϕ. Out of curiosity we checked whether this rotation angle ϕ of the "elastic" rod checks with the known result[4] for the "rigid" rod. It turned out that it *precisely* agrees with the predicted result for the "rigid" rod. Now, all that the COLLISION program could be doing is to apply effectively the velocity addition formulas to particles a and b. It therefore follows that the formula for the relativistic rotation of a rod that is usually derived by Lorentz transforming the equation of motion of the rod can also be derived from the velocity addition formulas. The interested reader can check the general case for himself.[16]

Let us now revert to Sec. III(v) which describes the relativistic rotation of 66 rods. Since we joined the 12 photons by a smooth wave front in Fig. 1, the number 66 in reality tends to infinity, for we can consider any photon, say, photon 3 of Fig. 1. If we now imagine the rod connecting this photon 3 with another photon (not shown) infinitesimally

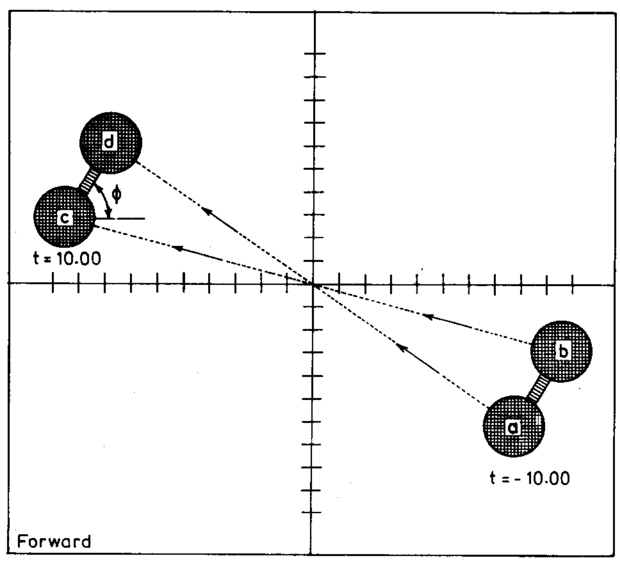

Fig. 11. This is the transform of Fig. 10 to a rocket frame moving to the right with the speed $\beta(v) = 0.9$. Notice that the rod appears tilted by a large angle ϕ.

separated from it on the wave front, the connecting rod goes over to the tangent to the wave surface at the site of photon 3. We now jump to the rocket frame (Fig. 2) and ask where this tangent rod has gone. We see that it suffers a rotation by the same angle that photon 3 suffers aberration. The locus of the photons is still a circle centered on the origin, and tangents at all points of a circle are normal to the corresponding radius vectors. Hence follows the known result that photons and the wave normals associated with their electromagnetic waves aberrate and rotate jointly in vacuum.

The situation is drastically different in a medium, as can be seen by an inspection of Figs. 3 and 4. The ray surface is now a dragged oval and no longer a sphere. The tangent at an arbitrary point on the oval is in general not normal to the radius vector from the origin. It therefore follows that one has to distinguish the wave normal velocity from the ray velocity in a moving medium. We have already seen that the wave normals can be tracked by the tangent rods while the rays can be simulated by radially moving particles. This bears out the well-known "pedal" relationship between the ray surface and the wave normal surface. Further, we have shown that the ray surface in a moving medium can be derived from the particle aberration formula. It then can be surmised that the wave normal surface should be derivable from the relativistic rotation formula.

It is considerations such as these that led de Broglie to associate waves with particles moving with speeds less than that of light in a relativistically invariant manner, and led to the birth of quantum mechanics.[17]

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[a] This paper is dedicated to the memory of the late Professor H. N. Bose.

[1] Edwin F. Taylor, *Spacetime Software, User's Manual and Teacher's Manual,* available from Spacetime Software, 20 Davis Road, Belmont, MA 02178. The software is available for both Macintosh and PC. The PC version is used in this paper. See the following papers by Edwin F. Taylor, "Learning from computers about physics teaching," Am. J. Phys. **56**, 975–980 (1988); "Comparison of different uses of computers in teaching physics," Plenary lecture, reprinted in *Proceedings of the International Conference on Trends in Physics Education*, Sophia University, Tokyo, 24–29 August 1986, edited by Koichi Shimoda and Tae Ryu, pp. 109–124. Reprinted in Chin. J. Phys. Educ. and in Phys. Educ. **22**, 201 (1987); "Computer graphics utilities in special relativity," Proceedings of the 1986 IBM Advanced Education Project Conference, San Diego, CA, 5–8 April, 1986, pp. 111–141; "Spacetime Software: Computer graphics utilities in special relativity," Am. J. Phys. **57**, 508–514 (1989).

[2] For a short bibliography of several of the paradoxes in relativity see G. P. Sastry, "Is length contraction really paradoxical?" Am. J. Phys. **55**, 943–946 (1987).

[3] For a derivation of the equation of the ray surface in a moving medium by a Lorentz transformation of the wave surface in the static medium, see C. Møller, *The Theory of Relativity* (Clarendon, Oxford, 1972), pp. 56–66. For a derivation from the Maxwell's equations in moving media, see T. H. O'Dell, "The electrodynamics of magneto-electric media," in *Selected Topics in Solid State Physics*, edited by E. P. Wohlfarth (North-Holland, Amsterdam, 1970). For a proof that the group velocity transforms like the particle velocity, see José Mario Saca, "Addition of group velocities," Am. J. Phys. **53**, 173–174 (1985). See also the references cited in Ref. 12. The derivation we give here of the ray surface comes straight from the velocity addition.

[4] For the rotation of a moving rod and the paradoxical situations its ignorance leads to, see R. Shaw, "Length contraction paradox," Am. J. Phys. **30**, 72 (1962). See also Bryon L. Coulter, "Relativistic motion in a plane," Am. J. Phys. **48**, 633–638 (1980); Charles A. Bordner, Jr., "Comment on Relativistic motion in a plane," Am. J. Phys. **51**, 74–75 (1983) and other citations in Ref. 2 above. The result that the relativistic rotation of a shrinking and expanding rod can be derived from velocity addition and that it coincides with the rotation of a "rigid" rod were first shown up by the Spacetime Software on the PC and can perhaps be found nowhere else.

[5] This problem played a big role in the birth of special relativity and its correct solution dates back to Einstein's 1905 paper. For the relativistic formulas for reflection at moving mirrors from aberration phenomenon, see, e.g., V. A. Ugarov, *Special Theory of Relativity* (Mir, Moscow, 1979), pp. 272–276. The connection between reflection and scattering was discussed earlier by Jørgen Olsen, "Reflection and elastic scattering," Am. J. Phys. **47**, 1094–1095 (1979). It is interesting to note that an intriguing forward reflection effect we saw in our simulations on the monitor screen had been deduced earlier from the wave theory by Bernard M. Jaffe, "Forward reflection of light by a moving mirror," Am. J. Phys. **41**, 577–578 (1973).

[6] The simulation of this effect was suggested by Table I of the article: Ari Ben-Menahem, "Wigner's rotation revisited," Am. J. Phys. **53**, 62–66 (1985). For a variety of general formulas on the result of the composition of two boosts from matrices, spinors, etc., see e.g., A. C. Hirshfeld and F. Metzger, "A simple formula for combining rotations and Lorentz boosts," Am. J. Phys. **54**, 550–552 (1986); C. B. van Wyk, "Rotation associated with the product of two Lorentz transformations," Am. J. Phys. **52**, 853–854 (1984); James T. Cushing, "Vector Lorentz transformations," Am. J. Phys. **35**, 858–862 (1967) and citations in these papers.

[7] However, the textbook by Wolfgang Rindler, *Introduction to Special Relativity* (Clarendon, Oxford, 1982) was lying on our desk throughout last year and our indebtedness to this slim but splendid volume may be more than we are conscious of. Problems 13–15 of Chap. III, p. 53, Sec. 24 on p. 67, and Sec. 33 on p. 94 have a bearing on the topics discussed in this paper. In view of the variety of topics discussed in this paper, our citations are merely representative and by no means exhaustive.

[8] The circle appears a little squashed in the printout because the scales on its two axes are slightly different. Similarly, the circle in Fig. 3 also looks a little squashed, while the oval of Fig. 4 looks more like a circle. Hence a little care is required in measuring distances and angles from the printout. It must be remembered, however, that the Tables accompanying the movie displays contain numerical output up to seven figures and any exact comparison with analytical formulas is best done on the Tables, which are not given here.

[9] See, e.g., Ø. Grøn, "Expansion of the universe and a hierarchical structure of the universe as solutions of the dark night sky paradox," Am. J. Phys. **46**, 923–927 (1978).

[10] In this section we shall be simulating only the "shape" of the ray surface and not the intensity distribution of light, or its Doppler effect in moving media. The shape of the wave and ray surfaces are well known to be independent of the motion of the source because they depend only on the speed of light to which the speed of the source does not add, as is true in any wave theory.

[11] See O'Dell's book cited in Ref. 3 above.

[12] The jump to Čerenkov effect needs a little explanation. Figure 5 shows the "Mach" cone emitted by a "flash" at the origin. If we now imagine a source "sitting" at the origin of the superluminal rocket frame and emitting light continuously, the ray surfaces keep leaving the source one by one and travel backward filling the "Mach" cone. If we now jump back to the lab frame, the source that was "sitting" at the origin of the rocket frame appears to move forward with a superluminal speed, while the water comes to rest. Thus we simulate a particle moving above the

Čerenkov threshold in a static medium. Due to the jump in the frames and the accompanying length contraction, the cone angle gets narrowed down. Further, the particle will now carry its "Mach" cone convectively forward (much as a moth carries its wings along). The complementary cone of this Mach cone is the Čerenkov cone, whose semivertical angle can be checked against the computer printout. For further material on the Čerenkov cone in a moving medium, see R. T. Compton, Jr. "The time dependent Green's function for electromagnetic waves in moving simple media," J. Math. Phys. 7, 2145–2152 (1966). See also G. P. Sastry, "Coulomb's law in a moving medium—a review exercise in advanced undergraduate electromagnetism," Am. J. Phys. **46**, 554–559 (1978). For an interesting paradox associated with the Čerenkov cone, see G. M. Volkoff, "Electric field of a charge moving in a medium," Am. J. Phys. **31**, 601–605 (1963) and G. P. Sastry, "Gauss's law paradox and the melting Čerenkov cone," Am. J. Phys. **44**, 707–708 (1976). For the important role played by the ray surface as against the wave normal surface in Čerenkov effect, see G. P. Sastry and K. Kumar, "Čerenkov ray cones in crystalline media," Proc. R. Soc. Lond. Ser. A **411**, 35–47 (1987).

[13] It is interesting to note that this result is *a priori* assumed, often without mention, in practically all the conventional treatments of the Michelson–Morley experiment. See, e.g., Robert Resnick, *Introduction to Special Relativity* (Wiley Eastern, New Delhi, 1972), p. 22, Fig. 1–7.

[14] See Steven Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), p. 527.

[15] See, e.g., J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), p. 541.

[16] Suppose we want to simulate an expanding rod by its radially moving particles. Let the rod move normal to itself with a speed $\beta(u)$ along a direction making an angle ξ with the X axis of the lab frame. A typical particle of the rod whose radius vector to the origin makes an angle η with the X axis moves radially with the speed $\beta(u)\sec(\eta-\xi)$. It can then be deduced rigorously from either the velocity addition formulas or the Lorentz transformation formulas that the relativistic rotation of such an elastic rod in the rocket frame turns out to be exactly the same as that of a rigid rod, with the parameter η dropping out from the final expression, as it should!

[17] See Møller's textbook or Rindler's textbook cited in Refs. 3 and 7.

Computer simulation of ergodicity and mixing in dynamical systems

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This article discusses the microcomputer simulation of dynamical systems that have a few degrees of freedom but which illustrate the properties associated with ergodicity and mixing shared by more realistic many-particle systems. The models discussed are a stadium-shaped billiard table model, and a particle moving in a regular array of hard disks or two-dimensional Lennard–Jones particles. The examples show the need for a statistical description and are very useful for pedagogical purposes.

I. INTRODUCTION

Introductory teaching of statistical mechanics describes this science as having the special function of providing reasonable methods for treating the behavior of many-particle systems. The statistical averaging process is introduced to take into account our lack of knowledge of (or interest in) the exact dynamical state of the system, and not as a consequence of any objective aspect of the physical phenomena.

In order to apply the averaging process some fundamental hypotheses must be accepted. They have a reasonable character but must nevertheless be regarded as postulates, as for example:

(a) the hypothesis of equal *a priori* probabilities in the phase space for a system in equilibrium;

(b) the hypothesis that an isolated system left to itself will ultimately attain a final equilibrium situation in which it is likely to be found in any one of its accessible states.

Many thermodynamic systems can be described using simple dynamical models, for example, hard spheres or particles interacting through simple potentials. Students very soon are confused in understanding the relationships between the dynamical description with its reversible and deterministic laws of evolution and the thermodynamic description with its law of monotonic increase of entropy.

A goal of the teaching of classical mechanics is still to make students conscious of the complete predictability of systems based on Newton's laws of motion. Some phenomena, like the roll of the dice, are said to be random only because it is difficult (but not impossible) to gather and to process a sufficient amount of information, but there is no reason to doubt that a precise predictability could be achieved in principle.

A pedagogical problem in justifying or establishing a statistical mechanics for classical dynamical models may be to demonstrate that they may exhibit phase space trajectories (generated by $f=ma$) which are ergodic and mixing. Mathematical demonstrations of these properties are far from easy and an "experimental" demonstration by computer model simulation can be a useful teaching strategy at an introductory level.

In this paper we will report some examples of computer simulation programs whose main objective is to show that deterministic equations can yield solutions strongly depending on the initial conditions of the systems, and for them only a predictability horizon[1] is definable. The reported examples are simple dynamical systems with few degrees of freedom but connected with simple models of the structure of matter. For these systems it will be shown that the instability of motions with respect to initial conditions only gives the possibility of a statistical description.

The first example is the billiard table system that enables us to introduce easily some concepts connected with the meaning of mixing and ergodic behavior. These concepts are successively applied to the study of hard-disk and Lennard–Jones particle systems.

