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Spacetime singularities as sources of a conserved gravitational flux

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Abstract. Spacetime singularities are regarded as sources or sinks of a conserved gravitational flux. The flux lines are viewed as curves belonging, not to the spacetime manifold $\mathcal{M}$, but to the tangent fibre bundle over $\mathcal{M}$. These lines are determined by timelike and null geodesics referred to a suitable non-affine parameter. With the help of this approach to the study of singularities a simple proof is given of Penrose's theorem on the occurrence of a singularity in a star undergoing asymmetric gravitational collapse.

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1. Introduction

In classical electrodynamics a most useful way of visualizing an electromagnetic field is by drawing the electric and magnetic lines of force. They show not only the direction of the vector fields at any given point in space, but also provide a quantitative measure of the fluxes with the help of the related concept of tubes of force. On the other hand in general relativity no equivalent procedure has been developed so far. This is a consequence of the non-linearity of the Einstein field equations which implies that, even in the absence of external sources, the gravitational field itself acts as its own source. Furthermore the principle of superposition is no longer valid, so that we cannot add the fields created by the individual separate sources. However in general relativity there are families of curves which give a beautiful visualization of the gravitational field, namely, the geodesic lines. Besides, these curves provide a geometrical definition of the Riemann curvature tensor by means of the geodesic deviation equation. Unfortunately geodesics do not qualify as flux lines due to the following two reasons: (a) an infinite number of geodesics, either timelike, spacelike or null, pass through every point in the spacetime manifold; (b) even if we restricted ourselves to a single congruence of geodesics, the vector field defined by the unit tangent vectors to these lines would not be divergence-free. This means that the associated flux is not conserved in empty spacetime and, therefore, we cannot give a meaning to the flux tubes.

In this paper we show that the first difficulty may be removed provided we consider the flux lines as curves belonging, not to the spacetime manifold $\mathcal{M}$, but to the tangent fibre bundle $T(\mathcal{M})$. This is possible because there is an intrinsic horizontal vector field defined in $T(\mathcal{M})$ by the connection. The second difficulty may also be removed when geodesics are considered as sequences of events in spacetime. This fact allows us to introduce a non-affine parameter so that the new tangent vector is divergence-free. In section 2 we perform the corresponding calculations in detail and identify the sources of the gravitational flux with spacetime singularities. In section 3 we work out two simple examples of geodesic congruences where the fluxes and sources may be explicitly determined. Finally, in section 4, we apply the law of conservation of gravitational flux to prove Penrose's singularity theorem.

2. Construction of a divergence-free vector field in $T(\mathcal{M})$

Let us consider the flux lines as curves belonging to the tangent fibre bundle $T(\mathcal{M})$ over the spacetime manifold $\mathcal{M}$. The connection $\Gamma^{\rho}_{\mu\nu}(\rho, \mu, \nu = 0, 1, 2, 3)$ defines a horizontal subspace H_u in the tangent space to the bundle $T(\mathcal{M})$ at the point $u = (p, \boldsymbol{k})$, where $p \in \mathcal{M}$ and $\boldsymbol{k} \in T_p(\mathcal{M})$. In terms of a local coordinate basis $\{x^{\mu}, k^{\nu}\}$ of $T(\mathcal{M})$, the horizontal subspace H_u has the associated basis of vectors

$$\left\{\frac{\partial}{\partial x^{\mu}} - k^{\nu}\Gamma^{\rho}_{\mu\nu}\frac{\partial}{\partial k^{\rho}}\right\} \tag{1}$$

from which one obtains the dual basis of 1-forms

$$\{\mathrm{d}x^{\mu}\}. \tag{2}$$

Since this basis is isomorphic to the corresponding one on $\mathcal{M}$, it follows that in both manifolds the volume and surface integrals have the same measure-valued forms.

The vector $\boldsymbol{k}$ itself corresponds to a horizontal vector field $\bar{\boldsymbol{k}}$ defined in $T(\mathcal{M})$ by the connection (see [1] section 2.9). In terms of the basis given by (1) this vector field has the expression

$$\bar{\boldsymbol{k}} = k^{\mu}\left(\frac{\partial}{\partial x^{\mu}} - k^{\nu}\Gamma^{\rho}_{\mu\nu}\frac{\partial}{\partial k^{\rho}}\right). \tag{3}$$

The integral curve of $\bar{\boldsymbol{k}}$ through a point $u_0 = (p_0, \boldsymbol{k}_0)$ of $T(\mathcal{M})$ is the horizontal lift of the geodesic in $\mathcal{M}$ with tangent vector $\boldsymbol{k}_0$ at p_0. Therefore the vector field $\bar{\boldsymbol{k}}$ represents all geodesics on $\mathcal{M}$. Each congruence of geodesics corresponds to a different cross section of the tangent bundle $T(\mathcal{M})$. The first difficulty mentioned in the introduction is thus removed since these curves are non-intersecting everywhere in $T(\mathcal{M})$.

In general the vector field $\bar{\boldsymbol{k}}$ is not divergence-free, so that its flux is not conserved outside the sources. However, when geodesics are considered as sequences of events in spacetime, one can describe them in terms of a non-affine parameter. By this token we can replace the field $\bar{\boldsymbol{k}}$ by another field $\bar{\boldsymbol{m}} = f(x)\bar{\boldsymbol{k}}$ and choose $f(x)$ such that $\bar{\boldsymbol{m}}$ be divergence-free.

Since the vector field $\bar{\boldsymbol{k}}$ given by (3) is geodesic, and is parametrized by an affine parameter λ, it satisfies the equations

$$k^{\mu} = \mathrm{d}x^{\mu}/\mathrm{d}\lambda \tag{4}$$

$$k^{\mu}_{;\nu}k^{\nu} = 0. \tag{5}$$

As mentioned earlier, this vector field is not divergence-free in empty spacetime, namely,

$$k^{\mu}_{;\mu} \neq 0. \tag{6}$$

Nevertheless, it is possible to obtain a conserved field m^{μ} by parametrizing the geodesics with a suitable non-affine parameter η. Explicitly one has

$$m^{\mu} = \mathrm{d}x^{\mu}/\mathrm{d}\eta \tag{7}$$

$$m^{\mu}_{;\nu}m^{\nu} = \kappa(\eta)m^{\mu}. \tag{8}$$

From (4) and (7) one obtains

$$m^{\mu}(\eta)=fk^{\mu}(\lambda) \tag{9}$$

where $f=\mathrm{d}\lambda/\mathrm{d}\eta$.

Our purpose now is to find a function f such that $m^{\mu}_{;\mu}=0$. This is equivalent to

$$(\log f)_{,\mu}k^{\mu}=-k^{\mu}_{;\mu}. \tag{10}$$

Writing $a(\lambda)=-k^{\mu}_{;\mu}(\lambda)$, and taking into account (4), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\log f(\lambda)=a(\lambda). \tag{11}$$

The general solution of this equation reads

$$f(\lambda)=K\exp\left[\int a(\lambda)\,\mathrm{d}\lambda\right] \tag{12}$$

where K is the constant of integration. We are always going to set $K>0$ so that the vector fields k^{μ} and m^{μ} point in the same direction. When dealing with non-spacelike geodesics, this is equivalent to choosing the positive sense of the gravitational flux as the one followed by a test mass or photon.

Having determined $f(\lambda)$ we can obtain the function $\kappa(\eta)$ in (8) by combining (5), (9) and (10). The result is

$$\kappa(\eta)=f(\eta)a(\eta). \tag{13}$$

Therefore, the second difficulty has also been removed.

Having constructed a divergence-free vector field $m^{\mu}(\eta)$ defined on the tangent bundle $T(\mathcal{M})$, we can obtain all congruences of geodesics referred to η by taking cross sections of $T(\mathcal{M})$. Note that two field lines lying on the same cross section of $T(\mathcal{M})$ cannot intersect at a non-singular event unless $m^{\mu}=0$. Otherwise the field vector would have two directions at the point of intersection. Therefore the caustics, defined as points where geodesics meet, are either zeros or singularities of the field m^{μ}. According to (12) this situation takes place when $k^{\mu}_{;\mu}=\pm\infty$.

The procedure we have followed to construct the vector field m^{μ} makes use of a coordinate patch in $\mathcal{M}$, so that this field has only been defined locally. However, on choosing a covering of $\mathcal{M}$ by coordinate neighbourhoods, this construction may be extended over the whole congruence.

Since m^{μ} is divergence-free we can build tubes of flux for any non-spacelike congruence of geodesics by considering the field lines which pass through the points of a simple closed curve in spacetime. Let us consider now a segment of a flux tube between two sections Σ_1 and Σ_2. By applying Gauss' theorem and assuming there are no sources or sinks of flux inside the tube, we obtain

$$\Phi_1\equiv\int_{\Sigma_1}m^{\mu}\,\mathrm{d}\Sigma_{\mu}=\int_{\Sigma_2}m^{\mu}\,\mathrm{d}\Sigma_{\mu}\equiv\Phi_2=\text{constant}. \tag{14}$$

Thus, the gravitational flux Φ is constant along any segment of a flux tube containing no sources.

The timelike or null flux lines can be extended up to infinity unless they hit a source or sink of gravitational flux. This situation means that the corresponding geodesic ends at a finite value of its affine parameter, so that these sources or sinks must be identified with spacetime singularities (see e.g. [1] section 8.1). On the other hand spacelike geodesics are not associated to flux lines since it is impossible to define for them an invariant sense of flux circulation.

Notice that the reparametrization has eliminated not only the spurious gravitational sources associated to the energy pseudotensor $t_{\mu\nu}$, but also the genuine sources associated to the energy tensor $T_{\mu\nu}$. However, the action of this tensor on the flux lines is determined by the geodesic deviation equation, much in the same manner as the electric and magnetic fields give rise to the Faraday tensions and pressures acting on the electric and magnetic lines of force in electrodynamics

The identification of spacetime singularities with sources of the divergence-free vector field $\bar{m}$, in the tangent bundle $T(\mathcal{M})$, allows us to explore the nature of these singularities by studying the behaviour of the flux in a neighbourhood of the source. One can thus avoid the conceptual trouble introduced by the fact that singularities cannot be regarded as being part of the spacetime manifold.

3. The Schwarzschild and Kerr geometries

To illustrate the procedure described in the preceding section we work out here two simple examples of geodesic congruences. Let us consider first the congruences of ingoing and outgoing timelike radial geodesics in the Schwarzschild geometry. Taking curvature coordinates referred to the affine parameter λ, the tangent vector k^μ reads (see [2] section 25.3)

$$k^\mu(\lambda) \equiv \mathrm{d}x^\mu/\mathrm{d}\lambda = [(1 - r_g r^{-1})^{-1}, \pm(r_g r^{-1})^{1/2}, 0, 0] \tag{15}$$

where we have introduced natural units so that $c = 1$, $G = 1$. The plus sign here refers to the outgoing congruence and the minus sign to the ingoing one.

From (9) we see that m^μ is determined by the function $f(\lambda)$, which in turn is given by (12) with $a(\lambda) = -k^\mu_{;\mu}(\lambda)$. Taking into account that $\sqrt{-g} = r^2 \sin\theta$, we obtain

$$a(\lambda) = -\tfrac{3}{2} r_g^{1/2} r^{-(3/2)} \tag{16}$$

and consequently

$$f(\lambda) = Kr^{-(3/2)}. \tag{17}$$

Combining now (9) and (17) and recalling that $f = \mathrm{d}\lambda/\mathrm{d}\eta$ we arrive at

$$m^\mu(\eta) \equiv \mathrm{d}x^\mu/\mathrm{d}\eta = [Kr^{-(3/2)}(1 - r_g/r)^{-1}, \pm K r_g^{1/2} r^{-2}, 0, 0] \tag{18}$$

showing the presence of a singularity at $r = 0$. Starting from a finite value of r, this singularity is reached after a finite interval of the non-affine parameter η.

Let us proceed now to the evaluation of the flux of the ingoing field. Since (18) does not depend of time it is advantageous to choose the surface Σ in (14) to be a segment of a timelike circular hypersurface of constant radius r and length Δt. Thus

$$\Delta\Phi = -\Delta t \int_{r=\mathrm{constant}} m^1(\eta) r^2 \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\varphi = 4\pi K r_g^{1/2} \Delta t \tag{19}$$

where the surface element has been evaluated according to (2). It is convenient to choose $K = (4\pi)^{-1} r_g^{-(1/2)} M$ in order that the flux per unit time take the simple form $\mathrm{d}\Phi/\mathrm{d}t = M$. This means that there is a sink at $r = 0$ which absorbs M units of flux per second. We recall that $r = 0$ is not a point but a spacelike hypersurface.

Let us consider now the gravitational field of a static spherically symmetric star of radius $r_0 > r_g$. In this case there is no spacetime singularity at $r=0$. Therefore, on crossing the symmetry axis, each ingoing geodesic becomes an outgoing one. In the spacetime manifold $\mathscr{M}$, all geodesics lying on the surface of a cone intersect on the line $r=0$, where the polar coordinates are singular. However in $T(\mathscr{M})$ each geodesic crosses the axis at a different point of the fibre defined by the event ($r=0$, $t=$ constant), so that there is no singularity of the field $\bar{m}$. Hence, the cross section defined by the ingoing congruence is connected with the cross section defined by the outgoing one through the fibres associated to the points of the symmetry axis. By this token the gravitational flux is conserved everywhere in $T(\mathscr{M})$.

The second example we want to discuss refers to the principal null congruences of the Kerr geometry. In Boyer–Lindquist coordinates the vector field $k^\mu(\lambda)$ is given by (see [2] section 33.6)

$$k^\mu(\lambda) = [E(r^2+a^2)\Delta^{-1}, \pm E, 0, Ea\Delta^{-1}] \tag{20}$$

where λ is the affine parameter. Furthermore,

$$\Delta = r^2 + a^2 - 2Mr \tag{21a}$$

$$\sqrt{-g} = (r^2 + a^2 \cos^2\theta)\sin\theta = \rho^2 \sin\theta. \tag{21b}$$

A straightforward calculation gives

$$m^\mu = K\rho^{-2}k^\mu. \tag{22}$$

From the results obtained by Carter [3] we know that the lines of flux of the ingoing congruence come in from $r=\infty$, pass through the equatorial disc $r=0$ towards a 'new universe' with a negative radial coordinate r, travelling out to $r=-\infty$. The sole exception are those lines lying on the equatorial plane which hit the singularity at $\rho^2=0$. The total flux per unit time across the pseudospheres $r=$ constant is equal to $4\pi EK$ since the contribution of the equator is of measure zero. We can again choose the constant K so that $\mathrm{d}\Phi/\mathrm{d}t$ is equal to M. To observers in the 'new universe' this flux is outgoing and therefore they see a source having the mass $-M$.

4. Penrose's singularity theorem

In this last section we want to exhibit the power of our approach by applying the law of conservation of gravitational flux we have discussed in this paper to give a simple proof of Penrose's singularity theorem [4]. There is no loss in generality in assuming the deviation from spherical symmetry is small when the collapsing star crosses its event horizon. Therefore, we can depict its history in a spacetime diagram very similar to the corresponding one in the symmetrical case as shown in figure 1.

Consider now a spacelike hypersurface S bounded by two trapped 2-surfaces T_1 and T_2 lying outside the collapsing star. This hypersurface may be thought of as generated by a continuous sequence of trapped 2-surfaces. We can next build a flux tube whose walls consist of two lightlike hypersurfaces L_1 and L_2 generated by the outgoing null geodesics orthogonal to the 2-surfaces T_1 and T_2. Assuming the local energy density is non-negative, these geodesics will be steadily convergent until they merge. Since we are dealing here with the *outgoing* radial congruence, these geodesics remain confined to a single cross section of $T(\mathscr{M})$. This fact implies that the meeting points must be cusps, i.e., points where the geodesics share the same tangent vector.

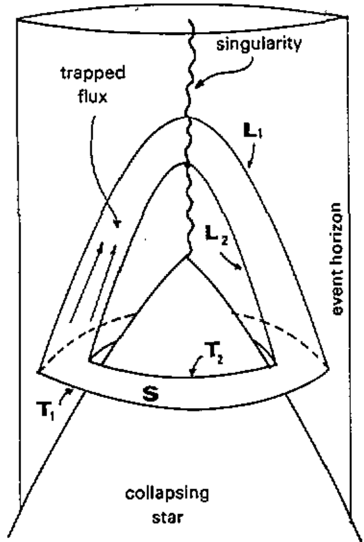

Figure 1. The law of conservation of gravitational flux applied to prove Penrose's singularity theorem. Here the time coordinate is vertical and the space coordinates are horizontal, with one spatial dimension suppressed.

Hence the future directed flux crossing the hypersurface S remains trapped inside the tube. On the other hand this 'trapped flux' must be conserved in empty spacetime. Therefore the result of the collapse must be either the occurrence of a singularity or the formation of a Cauchy horizon joining 'another universe' to the original one in which the star collapsed. In the first case the singularity acts as a sink of the trapped flux, like in the Schwarzschild geometry. In the second case a topological hole develops inside the tube, allowing the trapped flux to escape through it towards the 'new universe'. This second alternative resembles the situation we found with the Kerr geometry. However, there is strong support for the argument that Cauchy horizons are unstable, so that the development of a singularity will be unavoidable in any realistic collapse process (see [5] chapter 12, section 12.3.2).

The proof we have given here of Penrose's theorem illustrates the strong physical insight we can gain with the simple intuitive approach introduced in this paper.

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