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Gravitational field of charged gyratons

Valeri P Frolov[1] **and Andrei Zelnikov**[1,2]

[1] Theoretical Physics Institute, University of Alberta, Edmonton, Alberta, T6G 2J1, Canada
[2] Lebedev Physics Institute, Leninsky prospect 53, 119 991, Moscow, Russia

E-mail: frolov@phys.ualberta.ca and zelnikov@phys.ualberta.ca

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Abstract
We study relativistic gyratons which carry an electric charge. The Einstein–Maxwell equations in arbitrary dimensions are solved exactly in the case of a charged gyraton propagating in an asymptotically flat metric.

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1. Introduction

Possible creation of mini black holes in the scattering of highly ultra-relativistic particles has recently attracted a lot of attention. For such a process the energy of particles in their centre of mass must be of the order of or higher than the fundamental (Planckian) mass M_*. In the 'standard' quantum gravity this energy is very high, 10^{28} eV. This makes it difficult to expect that this, theoretically interesting and important, process will be observed in the near future. The fundamental mass scale might be much smaller if recently proposed models with large extra dimensions are valid. The fundamental energy which is often discussed in these models is of the order of a few TeV. This opens an appealing possibility of mini black hole creation in the near future collider and cosmic ray experiments.

In order to estimate the cross section of the mini black hole production one needs to know the gravitational field generated by an ultra-relativistic particle. To calculate this field it is sufficient to boost the Schwarzschild metric keeping the energy fixed and take the limit when the boost parameter $\gamma = 1/\sqrt{1 - v^2/c^2}$ becomes very large (the so-called Penrose limit). The corresponding metric in the four-dimensional spacetime was obtained by Aichelburg and Sexl [1]. It can easily be generalized to the case of a spacetime with arbitrary number of dimensions $D = n + 2$ where it has the form

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}\boldsymbol{x}^2 + \Phi(u, \boldsymbol{x})\,\mathrm{d}u^2, \tag{1}$$

$$\Phi(u, \boldsymbol{x}) = \kappa\sqrt{2}E\delta(u) f(r), \tag{2}$$

$$f(r) = \frac{g_n}{r^{n-2}}, \qquad g_n = \Gamma\left(\frac{n-2}{2}\right) \bigg/ (4\pi^{n/2}). \tag{3}$$

Here $d\boldsymbol{x}^2$ is the metric in the n-dimensional Euclidean space R^n, $r^2 = \boldsymbol{x}^2$, and $\kappa = 16\pi G$. In the four-dimensional spacetime $f(r) = -(1/2\pi)\ln r$.

The constant E which enters (2) is the energy. One can easily smear the δ-like profile of the energy distribution in (2) by substituting a smooth function $\varepsilon(u)$ instead of $E\delta(u)$. In what follows, we assume that the function $\varepsilon(u)$ (and similar functions which appear later) vanishes outside a finite interval $u \in [u_1, u_2]$. For this smooth distribution, $\varepsilon(u)$ has the meaning of the energy density, so that the total energy of such an object moving with the velocity of light is

$$E = \int_{u_1}^{u_2} du\, \varepsilon(u).$$

For example, such a gravitational field is produced by a pulse of light which has the finite energy E and the finite duration (in time u) $L = u_2 - u_1$.

Studies of the gravitational field of beams and pulses of light have a long history. Tolman [2] found a solution in the linearized approximation. Peres [3, 4] and Bonnor [5] obtained exact solutions of the Einstein equations for a pencil of light. These solutions belong to the class of pp-waves. The pp-wave solutions in four dimensions are described in detail in [6]. The four-dimensional pp-waves created by an ultrarelativistic charge or a spinning null fluid have been found by Bonnor [7, 8] (see also [9]). The higher-dimensional generalization of these solutions to the case where the beam of radiation carries angular momentum has been found recently [10, 11]. Such solutions correspond to a pulsed beam of radiation with negligible radius of cross section, finite duration L in time, and which has finite both energy E and angular momentum J. An ultra-relativistic source with these properties was called a gyraton.

In the present paper, we continue studying the properties of gyratons. Namely, we obtain a solution for the electrically charged gyraton and describe its properties.

2. Ansatz for metric and electromagnetic field

Our starting point is the following ansatz for the metric in the $D = n + 2$-dimensional spacetime

$$ds^2 = d\bar{s}^2 + 2(a_u\, du + a_a\, dx^a)\, du. \tag{4}$$

Here

$$d\bar{s}^2 = -2\, du\, dv + d\boldsymbol{x}^2 \tag{5}$$

is a flat D-dimensional metric, and $a_u = a_u(u, x^a)$, $a_a = a_a(u, x^a)$. The spatial part of the metric $d\boldsymbol{x}^2 = \delta_{ab}\, dx^a\, dx^b$ in the n-dimensional space R^n is flat. Here and later the Greek letters $\mu, \nu \ldots$ take values $1, \ldots, D$, while the Roman lower-case letters $a, b, \ldots$ take values $3, \ldots, D$. We denote

$$a_\mu = a_u \delta^u_\mu + a_a \delta^a_\mu.$$

The form of the metric (4) is invariant under the following coordinate transformation

$$v \to v + \lambda(u, \boldsymbol{x}), \qquad a_\mu \to a_\mu - \lambda_{,\mu}. \tag{6}$$

This transformation for a_μ recalls the gauge transformation for the electromagnetic potential. The quantity

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu,$$

which is the gravitational analogue of the electromagnetic strength tensor, is gauge invariant.

The metric (4) admits a parallelly propagating null Killing vector $l = l^\mu \partial_\mu = \partial_v$. It is the most general D-dimensional null Brinkmann metric [12] with flat transverse space.

Sometimes they are called pp-wave metrics [13]. The metric for the gravitational field of a relativistic gyraton with finite energy and internal angular momentum (spin) is of the form (4) [10, 11].

Let us denote

$$l_\mu = -\delta^u_\mu.$$

Then the flat metric $d\bar{s}^2$ and the metric (4) are related as

$$\bar{g}_{\mu\nu} = g_{\mu\nu} - l_\mu a_\nu - l_\nu a_\mu, \qquad \bar{g}^{\mu\nu} = g^{\mu\nu} + l^\mu a^\nu + l^\nu a^\mu + l^\mu l^\nu a^\epsilon a_\epsilon.$$

In the last relation, $\bar{g}^{\mu\nu}$ is the metric inverse to $\bar{g}_{\mu\nu}$. The operations with the indices of the quantities which enter its right-hand side are performed by using the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. It is easy to check that

$$l^\epsilon a_\epsilon = l^\epsilon f_{\epsilon\mu} = 0. \tag{7}$$

For the metric ansatz (4) all local scalar invariants constructed from the Riemann tensor and its derivatives vanish [11]. This property can be proved off-shell using (7) and the fact that the Riemann tensor is aligned with the null Killing vector l_α. The tensor is called aligned with the vector l_α if it can be written as a sum of terms, where each term contains as a factor at least one vector l_α. Vanishing of all scalar invariants constructed from the curvature and its derivatives is important in the proof of absence of quantum and α' corrections to the gravitational shock waves [14, 15]. One can expect that for the same reason the gyraton metric also does not have local quantum corrections.

Now let us consider the electromagnetic field in the spacetime (4). We choose its vector potential A_μ in the form

$$A_\mu = A_u \delta^u_\mu + A_a \delta^a_\mu, \tag{8}$$

where the functions A_u and A_a are independent of the null coordinate v. The electromagnetic gauge transformations

$$A_\mu \to A_\mu - \Lambda_{,\mu}$$

with $\Lambda = \Lambda(u, \boldsymbol{x})$ preserve the form of the potential (8).

The Maxwell strength tensor for this vector potential is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Being written in the covariant form, it does not depend on the metric and hence it is the same both for the metric (4) and for the background flat metric (5), $\bar{F}_{\mu\nu} = F_{\mu\nu}$. The Maxwell tensors with the upper indices for these spaces are different and are related as follows:

$$\bar{F}^{\mu\nu} = F^{\mu\nu} + l^\mu a_\sigma F^{\sigma\nu} - l^\nu a_\sigma F^{\sigma\mu}, \qquad \bar{F}_\mu{}^\nu = F_\mu{}^\nu + l^\nu a_\sigma F_\mu{}^\sigma.$$

The quantities A_μ and $F_{\mu\nu}$ obey the relations

$$l^\epsilon A_\epsilon = l^\epsilon F_{\epsilon\mu} = 0, \tag{9}$$

which imply that

$$\bar{F}_\mu{}^\alpha \bar{F}_{\alpha\nu} = F_\mu{}^\alpha F_{\alpha\nu}.$$

3. Field equations

The Einstein–Maxwell action in higher dimensions reads

$$S = \frac{1}{\kappa}\int \mathrm{d}^D \boldsymbol{x}\sqrt{|g|}\left[R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right].$$

Here $\kappa = 16\pi G$ and G is the gravitational coupling constant in D-dimensional spacetime. For this form of the action both the metric $g_{\mu\nu}$ and the vector potential A_μ are dimensionless [16].

The stress–energy tensor for the electromagnetic field is

$$T_{\mu\nu} = \frac{1}{\kappa}\left[F_\mu{}^\epsilon F_{\nu\epsilon} - \frac{1}{4}g_{\mu\nu}F_{\epsilon\sigma}F^{\epsilon\sigma}\right]. \tag{10}$$

For the potential (8) it takes the form

$$T_{\mu\nu} = \frac{1}{\kappa}\left[F_\mu{}^\epsilon F_{\nu\epsilon} - \frac{1}{4}g_{\mu\nu}\boldsymbol{F}^2\right] = \frac{1}{\kappa}\left[\bar{F}_\mu{}^\epsilon \bar{F}_{\nu\epsilon} - \frac{1}{4}(\bar{g}_{\mu\nu} + l_\mu a_\nu + l_\nu a_\mu)\boldsymbol{F}^2\right], \tag{11}$$

where

$$\boldsymbol{F}^2 = F_{\epsilon\sigma}F^{\epsilon\sigma} = \bar{F}_{\epsilon\sigma}\bar{F}^{\epsilon\sigma} = F_{ab}F^{ab}.$$

The trace of the stress–energy tensor

$$T^\mu_\mu = \frac{4-D}{4\kappa}\boldsymbol{F}^2 \tag{12}$$

vanishes in four-dimensional spacetime. This is a consequence of the conformal invariance of the four-dimensional Maxwell theory. Since $F_a{}^\epsilon F_{b\epsilon} = F_a{}^c F_{bc} = \bar{F}_a{}^c \bar{F}_{bc}$ and $g_{ab} = \bar{g}_{ab} = \delta_{ab}$, one has the following expression for the partial trace

$$\delta^{ab}T_{ab} = \frac{6-D}{4\kappa}\boldsymbol{F}^2. \tag{13}$$

Our aim now is to find solutions of the system of Einstein–Maxwell equations

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \tfrac{1}{2}\kappa T_{\mu\nu}, \tag{14}$$

$$F_\mu{}^\nu{}_{;\nu} = J_\mu, \tag{15}$$

for the adopted field ansatze (4) and (8). The stress–energy tensor $T_{\mu\nu}$ is given by (10).

Direct calculations [11] show that for the metric (4) the scalar curvature vanishes $R = 0$ and the only nonzero components of the Ricci tensor are

$$R_{ua} = \tfrac{1}{2}f_{ab}{}^{,b}, \qquad R_{uu} = -(a_u)^{;a}_{,a} + \tfrac{1}{4}f_{ab}f^{ab} + \partial_u\left(a_a{}^{,a}\right).$$

Since $R = \delta^{ab}R_{ab} = 0$, by using relations (12) and (13) one obtains

$$\boldsymbol{F}^2 = 0,$$

and hence $F_{ab} = 0$. Thus in a proper gauge the transverse components of the electromagnetic vector potential vanish, $A_a = 0$. Let us denote $\mathcal{A} = A_u$; then the only non-vanishing components of $F_{\mu\nu}$ and $T_{\mu\nu}$ are

$$F_{ua} = -F_{au} = -\mathcal{A}_{,a}, \qquad T_{uu} = \frac{1}{\kappa}(\nabla\mathcal{A})^2, \tag{16}$$

where $(\nabla\mathcal{A})^2 = \delta^{ab}\mathcal{A}_{,a}\mathcal{A}_{,b}$.

Thus the requirement that the electromagnetic field is consistent with the Einstein equations for the gyraton metric ansatz (4) implies that the vector potential a_μ can be chosen $a_\mu \sim l_\mu$, i.e., to be aligned with the null Killing vector. In this case $F_{\mu\nu}$ and all its covariant derivatives are also aligned with l_μ. Together with the orthogonality conditions (9), these properties can be used to prove that all local scalar invariants constructed from the Riemann tensor, the Maxwell tensor (16) and their covariant derivatives vanish. This property generalizes the analogous property for non-charged gyratons [11] and gravitational shock waves. This property can also be used to prove that the charged gyraton solutions of the Einstein–Maxwell equations are also exact solutions of any other nonlinear electrodynamics and the Einstein equations[3].

Eventually, the Einstein equations reduce to the following two sets of equations in the n-dimensional flat space R^n:

$$(a_u)_{,a}^{;a} - \partial_u\left(a_a{}^{,a}\right) = \tfrac{1}{4} f_{ab} f^{ab} - \tfrac{1}{2}(\nabla \mathcal{A})^2, \tag{17}$$

$$f_{ab}{}^{,b} = 0, \qquad f_{ab} = a_{b,a} - a_{a,b}. \tag{18}$$

We are looking for the field outside the region occupied by the gyraton, where $J_\mu = 0$. The Maxwell equations then reduce to the relation

$$\triangle\, \mathcal{A} = 0. \tag{19}$$

Here $\triangle$ is a flat n-dimensional Laplace operator.

If the gyraton carries an electric charge

$$Q = \frac{1}{\kappa} \int_\Sigma J^\mu \, \mathrm{d}\Sigma_\mu,$$

then by using Stoke's theorem it can be written as

$$Q = \frac{1}{\kappa} \int_{\partial\Sigma} F^{\mu\nu} \, \mathrm{d}\sigma_{\mu\nu}.$$

Thus the electric charge of the gyraton is determined by the total flux of the electric field across the surface $\partial\Sigma$ surrounding it.

Let us choose Σ to be $(D-1)$-dimensional region on the surface $v = \text{const}$ with the boundary $\partial\Sigma$ which consists of the surface of the cylinder $r = \text{const}$, $\partial\Sigma_r$, and two discs of radius r, $\partial\Sigma_1$ and $\partial\Sigma_2$, located at $u = u_1$ and $u = u_2$, respectively. Since $F^{vu} = 0$, the fluxes through the boundary discs vanish. Thus we have

$$Q = \frac{r^{n-1}}{\kappa} \int_{u_1}^{u_2} \mathrm{d}u \int \mathrm{d}\Omega_{n-1} F_{ur}(r, \boldsymbol{n}). \tag{20}$$

Here $\boldsymbol{n}$ is the point on the unit $(n-1)$-dimensional sphere and $\mathrm{d}\Omega_{n-1}$ is its volume element. For the linear distribution of charge one has

$$Q \equiv \int_{u_1}^{u_2} \mathrm{d}u\, \rho(u), \tag{21}$$

where $\rho(u)$ is a linear charge density. By comparing (20) and (21) one can conclude that in the asymptotic region $r \to \infty$ the following relations are valid:

$$F_{ur} \approx \begin{cases} \dfrac{\kappa(n-2) g_n \rho(u)}{r^{n-1}}, & \text{for} \quad n > 2, \\[2ex] \dfrac{\kappa \rho(u)}{2\pi r}, & \text{for} \quad n = 2. \end{cases}$$

[3] Let us denote $\mathcal{F} = F_{\mu\nu} F^{\mu\nu}$ and $\mathcal{K} = \mathrm{e}^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$; then in the nonlinear electrodynamics the Lagrangian for the field A_μ is of the form $L = \mathcal{F} + \mathcal{L}(\mathcal{F}, \mathcal{K})$, where $\frac{\partial \mathcal{L}}{\partial \mathcal{F}}|_{\mathcal{F}=0} = \frac{\partial \mathcal{L}}{\partial \mathcal{K}}|_{\mathcal{K}=0} = 0$. Since for the gyraton ansatz $\mathcal{F} = \mathcal{K} = 0$ the nonlinear term $\mathcal{L}$ in the Lagrangian does not affect the electromagnetic field equations. So the solutions for the nonlinear electrodynamics coincide with the solutions of the standard Maxwell equations. We are grateful to David Kubiznak for drawing our attention to this fact.

Here g_n is defined by (3).

Now we return to the Einstein equations. The combination which enters the left-hand side of (17) is invariant under the transformation (6). One can use this transformation to put $a_a{}^{;a} = 0$. We shall use this 'gauge' choice and denote $a_u = \frac{1}{2}\Phi$ in this 'gauge'. Equations (17)–(18) take the form

$$\triangle\,\Phi = -j, \qquad j = j_f + j_{\mathcal{A}}, \qquad j_f = -\tfrac{1}{2} f_{ab} f^{ab}, \qquad j_{\mathcal{A}} = (\nabla\mathcal{A})^2, \tag{22}$$

$$\triangle\, a_a = 0. \tag{23}$$

The set of equations (19), (22), and (23) determines the metric

$$\mathrm{d}s^2 = \mathrm{d}\bar{s}^2 + \Phi\,\mathrm{d}u^2 + 2a_a\,\mathrm{d}x^a\,\mathrm{d}u$$

and the electromagnetic field $\mathcal{A}$ of a gyraton. (Let us emphasize again that these equations are valid only outside the region occupied by the gyraton.)

Equations (19) and (23) are linear equations in an Euclidean n-dimensional space R^n. The first one is the equation for the gravitomagnetic field a_μ and it formally coincides with the equation for the magnetic field. The second is an electrostatic equation for the electric potential $\mathcal{A}$. The last (gravitoelectric) equation (22) is a linear equation in the Euclidean space for the gravitoelectric potential Φ which can be solved after one finds solutions for f_{ab} and $\mathcal{A}$. Thus for a chosen ansatz for the metric and the electromagnetic fields, the solution of the Einstein–Maxwell equations in D-dimensional spacetime reduces to linear problems in an Euclidean n-dimensional space ($n = D - 2$).

4. Solutions

Let us first discuss the scalar (electrostatic) equation (19). Its general solution for a point-like charge distribution in the n-dimensional space can be written in the form (see, e.g., [17] and references therein)

$$\mathcal{A} = \sum_{l=0}^{\infty}\sum_{q} \frac{\mathcal{Y}^{lq}}{r^{n+2l-2}}. \tag{24}$$

Here

$$\mathcal{Y}^l = C_{c_1\ldots c_l} x^{c_1}\ldots x^{c_l},$$

where $C_{c_1\ldots c_l}$ is a symmetric traceless rank-l tensor. The index q enumerates linearly independent components of coefficients $C_{c_1\ldots c_l}$. It takes the value from 1 to $d_0(n,l)$, where

$$d_0(n,l) = \frac{(l+n-3)!(2l+n-2)}{l!(n-2)!}$$

is the total number of such independent components for given n and l. For the gyraton solution $C_{c_1\ldots c_l}$ are arbitrary functions of u.

In a similar way, a solution of the gravitomagnetic equations (18) for a point-like source can be written as follows (see [17] and references therein):

$$a_a = \sum_{l=1}^{\infty}\sum_{q} \frac{\mathcal{Y}_a^{lq}}{r^{n+2l-2}}. \tag{25}$$

Here

$$\mathcal{Y}_a^l = C_{abc_1\ldots c_{l-1}} x^b x^{c_1}\ldots x^{c_{l-1}},$$

where $C_{abc_1\ldots c_{l-1}}$ is a $(l+1)$th-rank constant tensor which possesses the following properties: it is antisymmetric under interchange of a and b, and it is traceless under contraction of any pair of indices [17]. Again, we use an index q to enumerate different linearly independent vector spherical harmonics. The total number of these harmonics for given l is [17]

$$d_1(n,l) = \frac{l(n+l-2)(n+2l-2)(n+l-3)!}{(n-3)!(l+1)!}.$$

In the gyraton solution the coefficients $C_{abc_1\ldots c_{l-1}}$ are arbitrary functions of the retarded time u. The functions a_a obey the following gauge fixing condition $a_a{}^{;a} = 0$.

In order to obtain a general solution of the gravitostatic equation (22) it is convenient to write Φ in the form

$$\Phi = \varphi + \psi,$$

where φ is a general solution for a point-like source and ψ is a special solution of the inhomogeneous equation

$$\triangle\,\psi = -j \tag{26}$$

in the absence of a point-like source. A general solution φ can be written in a form similar to (24), while the special solution ψ can be presented as follows:

$$\psi(u,\boldsymbol{x}) = \int \mathrm{d}\boldsymbol{x}'\mathcal{G}_n(\boldsymbol{x},\boldsymbol{x}')\,j(u,\boldsymbol{x}'). \tag{27}$$

Here $\mathcal{G}_n(\boldsymbol{x},\boldsymbol{x}')$ is Green's function for the n-dimensional Laplace operator

$$\triangle\,\mathcal{G}_n(\boldsymbol{x},\boldsymbol{x}') = -\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{28}$$

which can be written in the following explicit form:

$$\mathcal{G}_2(\boldsymbol{x},\boldsymbol{x}') = -\frac{1}{2\pi}\ln|\boldsymbol{x}-\boldsymbol{x}'|, \tag{29}$$

$$\mathcal{G}_n(\boldsymbol{x},\boldsymbol{x}') = \frac{g_n}{|\boldsymbol{x}-\boldsymbol{x}'|^{n-2}}, \qquad n>2, \tag{30}$$

where g_n is given by (3).

5. Higher-dimensional charged gyratons

We consider now special solutions for charged gyratons in a spacetime with the number of dimensions $D > 4$. (The four-dimensional case will be discussed in the next section.) These solutions are singled out by the property that in the harmonic decomposition of the functions φ, a_a and $\mathcal{A}$ only the terms with the lowest multipole momenta are present:

$$\varphi = \frac{\varphi_0}{r^{n-2}}, \qquad \mathcal{A} = \frac{\mathcal{A}_0}{r^{n-2}}, \qquad a_a = \frac{a_{ab}x^b}{r^n}, \tag{31}$$

where φ_0, $\mathcal{A}_0$, and a_{ab} are functions of u. It can be shown [11] that

$$\varphi_0(u) = \kappa\sqrt{2}g_n\varepsilon(u), \qquad a_{ab}(u) = \kappa(n-2)g_n j_{ab}(u), \tag{32}$$

where $\varepsilon(u)$ and $j_{ab}(u)$ are the density of energy and angular momentum, respectively. One can also relate $\mathcal{A}_0$ to the density of the electric charge distribution ρ:

$$\mathcal{A}_0(u) = \kappa g_n\rho(u). \tag{33}$$

Let us denote $b_{ab} = a_{ac}a_{bc}$ and $b = \delta^{ab}b_{ab} = a_{ac}a_{ac}$. Then one has

$$j_f = -\frac{1}{2} f_{ab} f^{ab} = \frac{2b}{r^{2n}} + \frac{n(n-4)b_{ab}x_a x_b}{r^{2n+2}},$$

$$j_{\mathcal{A}} = (\nabla \mathcal{A})^2 = \frac{(n-2)^2 \mathcal{A}_0^2}{r^{2n-2}}.$$

Instead of using (27) and (30), one can determine ψ by directly solving the Laplace equation. We write $\psi = \psi_f + \psi_{\mathcal{A}}$ as a sum of two solutions of (26) with the sources j_f and $j_{\mathcal{A}}$, respectively. By using the relations

$$\triangle\left(\frac{1}{r^{2m}}\right) = \frac{2m(m+1)}{r^{2m+2}},$$

$$\triangle\left(\frac{x_a x_b}{r^{2m}}\right) = 2m(m-2)\frac{x_a x_b}{r^{2m+2}} + 2\frac{\delta_{ab}}{r^{2m}},$$

one can check that

$$\psi_f = \frac{\alpha}{r^{2n-2}} + \frac{\beta_{ab}x_a x_b}{r^{2n}}, \qquad \psi_{\mathcal{A}} = \frac{(n-2)}{2(n-1)}\frac{\mathcal{A}_0^2}{r^{2n-4}}. \tag{34}$$

Here

$$\alpha = \frac{b}{2(n-1)(n-2)}, \qquad \beta_{ab} = \frac{n-4}{2(n-2)} b_{ab}, \tag{35}$$

6. Four-dimensional charged gyratons

The four-dimensional case is degenerate and requires special consideration. In the lowest order of the harmonic decomposition one has

$$\varphi = -\frac{\kappa}{\pi\sqrt{2}}\varepsilon(u)\ln r, \qquad \mathcal{A} = -\frac{\kappa}{2\pi}\rho(u)\ln r, \qquad a_a = \frac{\kappa j(u)}{2\pi}\frac{\epsilon_{ab}x^b}{r^2}. \tag{36}$$

Simple calculations give

$$j_f = 0, \qquad j_{\mathcal{A}} = \frac{\kappa^2}{4\pi^2}\rho^2(u)\frac{1}{r^2}.$$

To obtain $\psi_{\mathcal{A}}$ we use the relation $(p = p(r))$

$$\triangle p = \frac{\mathrm{d}^2 p}{\mathrm{d}r^2} + \frac{n-1}{r}\frac{\mathrm{d}p}{\mathrm{d}r}.$$

Solving equation $\triangle\psi_{\mathcal{A}} = j_{\mathcal{A}}$, one finds

$$\psi_{\mathcal{A}} = \frac{\kappa^2}{8\pi^2}\rho^2(u)\ln^2(r). \tag{37}$$

To summarize, the metric of the four-dimensional charged gyraton can be written in the form

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2 + \frac{\kappa}{2\pi}j(u)\,\mathrm{d}u\,\mathrm{d}\phi + \Phi\,\mathrm{d}u^2, \tag{38}$$

$$\Phi = -\frac{\kappa\sqrt{2}}{2\pi}\varepsilon(u)\ln r + \frac{\kappa^2}{8\pi^2}\rho^2(u)\ln^2(r). \tag{39}$$

This solution can easily be generalized to the case when the energy density distribution has higher harmonics. It is sufficient to add to Φ the function

$$\varphi' = \sum_{n\neq 0}\frac{b_n}{r^{|n|}}\,\mathrm{e}^{\mathrm{i}n\phi}, \qquad \bar{b}_n(u) = b_{-n}(u). \tag{40}$$

The 4D solution of Einstein–Maxwell equations corresponding to a cylindrical non-rotating charged source moving with the speed of light was found in 1970 by Bonnor [7]. He also obtained [8] the 4D solution for the gravitational field of a neutral spinning null fluid. These solutions outside the source correspond to our formulae for a particular choice of charge and angular momentum distributions.

7. Summary and discussions

In the present paper, we obtained solutions of Einstein–Maxwell equations for the relativistic charged gyratons. These solutions generalize the results of [10, 11] to the case when in addition to the finite energy and angular momentum the gyraton has an electric charge. The parameters specifying the solutions are arbitrary functions of u. The obtained solutions have an important property that all scalar invariants constructed from the Riemann curvature, electric field strength and their covariant derivatives vanish. In the case of shock waves this property was the reason why they do not receive quantum and α' corrections [14, 15]. So one can expect that the gyraton metric also does not have quantum corrections and, hence, it is probably the exact solution of the quantum problem too. We demonstrated that for the charged gyratons the Einstein–Maxwell field equations in D-dimensional spacetime reduce to a set of linear equations in the Euclidean $(D-2)$-dimensional space. This property of gyraton solutions is similar to the properties of charged M-branes in string theory [18][4]. In the absence of the charge the gyraton solutions can be generalized to the case of asymptotically AdS spacetimes [19]. It would be interesting to find a similar generalization for charged gyratons. In the supergravity theory there are other charged objects similar to gyratons, where, for example, the Kalb–Ramond field appears instead of the Maxwell field. It happens that in the Brinkmann metric ansatz the supergravity equations can also be reduced to a system of linear equations and solved exactly [20].

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