The causal structure of two-dimensional spacetimes

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Abstract. We investigate the causal structure of $(1 + 1)$-dimensional spacetimes. For two sets of field equations we show that at least locally any spacetime is a solution for an appropriate choice of the matter fields. For the theories under consideration we investigate how smoothness of their black hole solutions affects time orientation. We show that if an analogue to Hawking's area theorem holds in two spacetime dimensions, it must actually state that the size of a black hole never *increases*, contrary to what happens in four dimensions. Finally, we discuss the applicability of the Penrose and Hawking singularity theorems to two spacetime dimensions.

1. Introduction

Relativistic theories of gravitation in two spacetime dimensions provide an interesting theoretical laboratory for understanding issues relevant to quantum gravity [1–10]. Such theories reduce the complexity of $(3 + 1)$-dimensional general relativity significantly, thereby offering much hope for obtaining significant insights into its quantization, as well as an understanding of the issues associated with short-distance problems, topology change, singularities and the cosmological constant problem. Recent work has revealed interesting relationships between $(1 + 1)$-dimensional gravitational theories and conformal field theory [9], the Liouville model [2–4], random lattice models [10], and sigma models [11–14].

Although Einstein's field equations are trivial in two spacetime dimensions, there exist a variety of $(1 + 1)$-dimensional generally covariant theories of gravitation [2–8] some of which have non-trivial dynamical structure. A close analogue of the Einstein equations is given by [15, 16]

$$R - \Lambda = 8\pi GT \tag{1}$$

along with the conservation equation

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{2}$$

where R is the Ricci scalar, Λ is a cosmological constant, T is the trace of the stress–energy tensor, G is Newton's constant and we have taken the speed of light to be unity. These equations can be derived from a local action principle [17] by incorporating an auxiliary field whose classical evolution does not affect the gravity/matter system above. In the absence of matter this system reduces to the vacuum field equation used in the Liouville model [2–4]. The classical aspects of this theory

of gravity have been examined in some detail [15, 16, 18] and it has been shown that it has a remarkable similarity to four-dimensional general relativity in many of its features. These features include a Newtonian limit, Robertson–Walker cosmological solutions, interior solutions, gravitational waves and the gravitational collapse of dust into a black hole with an event horizon structure which is the same as that of the four-dimensional Schwarzschild solution. Indeed, the field equations of this theory follow from a dimensional reduction of Einstein's equations in a certain limit [19]; in this sense they form a $(1 + 1)$-dimensional version of general relativity.

These classical features are so closely analogous to $(3 + 1)$-dimensional general relativity that one might hope its quantization would bear a similar resemblance to $(3 + 1)$-dimensional quantum gravity. The semiclassical properties of this theory [17, 20, 21] do indeed yield interesting effects such as Hawking radiation and black hole condensation. These properties are intimately connected with the non-trivial event horizon structures which can form in the theory in a manner quite similar to their $(3 + 1)$-dimensional general relativistic counterparts.

Much more recently it has been shown that other $(1 + 1)$-dimensional theories of gravity which arise in the context of non-critical string theory can also yield a non-trivial event horizon structure. A spacetime exhibiting such features was recently discovered as a solution to a scale-invariant higher-derivative theory of gravity [22], and was later found to be a solution to $c = 1$ Liouville gravity [23] as well as to a non-critical string theory in two spacetime dimensions [24, 25]. This latter result suggests the possiblity of using string-theoretic technology to examine the formation and properties of black holes in more realistic cases.

The field equations associated with this theory are

$$e^{-2\phi}(R_{ab} + 2\nabla_a\nabla_b\phi) = 8\pi G T_{ab} \tag{3}$$

$$R - 4(\nabla\phi)^2 + 4\nabla^2\phi + J + c = 0. \tag{4}$$

where a stress–energy tensor T_{ab} and source J for the dilaton field ϕ have been included. For $J = 0 = T_{ab}$, these equations reduce to those of non-critical $(1 + 1)$-dimensional string theory in the absence of a tachyon field [24]. The black hole metric which follows from (3) and (4) in this case is unique; it is asymptotically flat, and may be matched to a solution for collapsing dust provided appropriate surface stresses are included, where the source J may be understood to arise from the tachyon sector [26]. The quantum properties of the above metric are similar to those found in [18, 21] (see also [27–29]).

In this paper, we investigate the causal structure of solutions to these theories. We begin by motivating the various solutions in section 2. In section 3 we ask whether an arbitrary spacetime can be considered to be a solution to the field equations for an appropriate choice of the matter fields, and we provide a partial answer. In section 4 we examine some black hole solutions and look at their causal structure in detail. We investigate how smoothness of solutions affects their time orientation. We mention that if an analogue to Hawking's area theorem [30] holds in two spacetime dimensions, it must actually state that the size of a black hole never *increases*, which is exactly the opposite of what happens in four dimensions, and we note the difficulty in interpreting the 'size' of a black hole. In section 5 we discuss the applicability of the Penrose and Hawking singularity theorems to two spacetime dimensions, focusing on Penrose's 1965 theorem [30, 31] which predicts that a spacetime containing a closed trapped surface must be singular. We explain the difficulty in defining a closed

trapped surface in two dimensions and show that the energy condition in Penrose's theorem is trivially true for all $(1 + 1)$-dimensional spacetimes. A spacetime with a black hole which contains singularities of an unexpected type is used as an illustrative example. Finally, we summarize our results in a concluding section.

2. Background and motivation

Throughout this paper we will be examining many different spacetimes, so here we introduce each of them and describe the context in which they arise. In general our conventions follow those of Hawking and Ellis [30].

By a *spacetime* (M, g_{ab}) we shall mean a Hausdorff C^∞ manifold M (without boundary) of dimension $\geqslant 2$ with a non-degenerate Lorentzian metric g_{ab}, that is a metric of signature $(-, +, \cdots, +)$. We define a vector v^a to be *timelike, null* or *spacelike* if $g_{ab} v^a v^b$ is negative, zero or positive, respectively. We also assume that the spacetime is time-orientable, i.e. that there exists a continuous timelike vector field on M.

We define the Riemann tensor, Ricci tensor and Ricci scalar as

$$R^a{}_{bcd} = \frac{\partial \Gamma^a{}_{bd}}{\partial x^c} - \frac{\partial \Gamma^a{}_{bc}}{\partial x^d} + \Gamma^a{}_{cf} \Gamma^f{}_{bd} - \Gamma^a{}_{df} \Gamma^f{}_{bc} \tag{5}$$

$$R_{bd} = R^a{}_{bad} \tag{6}$$

and

$$R = R^a{}_a = R^a{}_{bad} g^{bd} \tag{7}$$

respectively.

For our purposes it will be useful to write the static metric in the form

$$ds^2 = -\alpha(x)\, dt^2 + \frac{dx^2}{\alpha(x)}. \tag{8}$$

Such a choice of coordinates is always possible, at least locally. Spacetimes given by static metrics of this form fall into four distinct categories distinguished by the sign of α at large $|x|$:

$$\lim_{|x| \to \infty} \mathrm{sgn}(\alpha(x)) = \begin{cases} +1 & \text{case A} \\ -1 & \text{case B} \\ \mathrm{sgn}(x) & \text{case C} \\ \text{no limit} & \text{case D.} \end{cases} \tag{9}$$

Case A is the spacetime one would expect to arise from the endpoint of gravitational collapse of a distribution of $(1+1)$-dimensional matter. Before collapse the signature of the metric is everywhere $(-, +)$, but afterward certain regions of spacetime develop event horizons. In this case α must have an even number of roots (some pairs of which may be degenerate) We shall cite an example of this below. Physical $(1 + 1)$-dimensional observers (i.e. those abiding in a spacetime of signature $(-, +)$, where t is timelike) may be located at regions of large x, but will ultimately be unable to

receive signals from observers at large $-x$ since all such signals must cross the event horizon once collapse has occurred. The second case is what one might expect in a spacetime which has a cosmological constant, as we will illustrate later. Again, α must have an even number of roots. In this situation, observers are located only in regions of $|x| < R_0$ (where R_0 is some constant), and are unable to receive information from more distant regions of their universe. Case C is somewhat unusual in that it has no $(3+1)$-dimensional analogue: spacetime has signature $(-,+)$ for large x and has signature $(+,-)$ for large $-x$; without loss of generality x may be taken to be positive as above. Originally such spacetimes were considered in the context of a higher-derivative theory [22]; more recently they have become of interest in the context of finding solutions to the system (3) and (4) with $J = T_{ab} = 0$ [24, 25]. Finally, case D involves those spacetimes for which α has no definite sign for large $|x|$.

We begin with solutions to the system (1) and (2). The symmetric, continuous and static solutions for a point particle situated at the origin are given by (8) where

$$\alpha(x) = -\tfrac{1}{2}\Lambda x^2 + 2M|x| - C \tag{10}$$

on $\mathbb{R}^2$. M can be interpreted as the mass of the source and C is an arbitrary (but meaningful) constant. At $x = 0$ the metric is continuous but not differentiable for $M \neq 0$. When $\alpha(x) = 0$ the metric is singular, but for $x \neq 0$ these are just coordinate singularities that result from writing the metric in the form (8). For various choices of Λ, M and C the spacetime represents a black hole, a white hole, a naked singularity or other more complicated structures. This spacetime can also be easily extended to multiple point sources. All of this is explored in detail in [16].

Spacetimes with $\Lambda \neq 0$ can have a qualitatively different structure than their $(3+1)$-dimensional counterparts. In $(3+1)$ dimensions the most general static isotropic metric may be written in the form

$$ds^2 = -B(r)\, dt^2 + A(r)\, dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{11}$$

The rr and tt equations of general relativity imply that AB is a positive constant C and that

$$-\Lambda = \frac{1}{2}\frac{d^2\tilde{B}}{dr^2} + \frac{1}{r}\frac{d\tilde{B}}{dr} \tag{12}$$

where $\tilde{B} = B/C$, whereas the $\theta\theta$ and $\phi\phi$ parts of Einstein's equations imply

$$-r^2\Lambda = -1 + r\frac{d\tilde{B}}{dr} + \tilde{B}. \tag{13}$$

As (13) implies (12), we obtain the solution

$$\tilde{B}(r) = \left(1 - \frac{\Lambda}{3}r^2\right) + \frac{c_3}{r} \tag{14}$$

which is the $(3+1)$-dimensional analogue of (10). The metric (14) is of type B. Note that the ratio between the r^2 coefficient and the constant term is forced by (13) to be $-\Lambda/3$, in contrast to the freedom available in choosing C in the $(1+1)$-dimensional

solution (10). This freedom arises because of the lack of angular information in $(1+1)$ dimensions; there is no equation corresponding to (13). Consequently a cosmological event horizon can only arise for $\Lambda > 0$ in $(3+1)$ dimensions, whereas such horizons can appear for either sign of Λ in (10). Hence in $(1+1)$ dimensions, metrics with cosmological event horizons of type A and B are both possible.

Another interesting solution to equations (1) and (2) arises in the symmetric collapse of an initially static distribution of pressureless dust [18]. The metric for the interior of the dust is given by

$$ds^2 = -dt^2 + (1 - bt^2)^2 \, dx^2 \tag{15}$$

in the region $S = \{(t,x) | 0 \leqslant t < 1/\sqrt{b}, |x| <= r\}$, where $x = \pm r$ are the (constant) positions of the edges of the dust in the comoving coordinate system and $b \equiv 2\pi G\rho_0$. The dust collapses from a static condition of uniform density ρ_0 at $t = 0$ to one of infinite density as $t \to 1/\sqrt{b}$. To match this metric to an external vacuum solution we define

$$X(t,x) = x(1 - bt^2) \tag{16}$$

$$T(t,x) = \frac{1}{2br} \tanh^{-1} \left[\frac{2brt}{\sqrt{bt^2 + e^{2b(x^2-r^2)}(1 - bt^2)}} \right]. \tag{17}$$

This transformation is a one-to-one transformation of S if $br^2 \leqslant \frac{1}{4}$; otherwise there are cases in which (t_1, x) and (t_2, x) map to the same (T, X). However if we restrict ourselves to $t < \frac{1}{2}br$ the transformation will again be one-to-one. Note that this transformation always transforms the boundary $x = \pm r$ in a one-to-one manner. In (T, X) coordinates the metric can be written

$$ds^2 = -B(T, X) \, dT^2 + \frac{dX^2}{1 - 4b^2x^2t^2} \tag{18}$$

where

$$B(T, X) = \frac{[bt^2 + e^{2b(x^2-r^2)}(1 - bt^2) - 4b^2r^2t^2]^2[bt^2 + e^{2b(x^2-r^2)}(1 - bt^2)]}{e^{4b(x^2-r^2)}[1 - 4b^2x^2t^2]} \tag{19}$$

and in which x and t are defined implicitly by (16) and (17). This matches the static flat outside metric

$$ds^2 = -(4br|X| + 1 - 4br^2) \, dT^2 + \frac{dX^2}{4br|X| + 1 - 4br^2} \tag{20}$$

at the edges $x = \pm r$, $X = \pm r(1 - bt^2)$ of the fluid. This represents a black hole if $br^2 > \frac{1}{4}$ which is exactly when the transformation (16) and (17) is not one-to-one. This condition can also be written as $\rho_0 > \frac{1}{8}\pi Gr^2$. The 'Schwarzschild' radius is

$$|X| = r - \frac{1}{4br} \tag{21}$$

so the dust becomes a black hole when $t = \frac{1}{2}br$, which is precisely the point at which the coordinate transformation first fails to be one-to-one.

The system (3) and (4) has similar solutions. The unique vacuum solution is given by the metric (8) with

$$\alpha(x) = 1 - ae^{-Qx} \tag{22}$$

and dilaton field

$$\phi = -\tfrac{1}{2}Qx \tag{23}$$

on $\mathbb{R}^2$ where a and $Q^2 = J + c$ are constants of integration. This solution has been discussed in the context of a scale-invariant higher-derivative theory of gravity [22], $c = 1$ Liouville gravity [23] and a non-critical string theory [24, 25]. This metric lacks the spatial symmetry of (10) and is of type C; indeed the curvature scalar diverges as $x \to -\infty$. If x is replaced with $|x|$ in (22), the solution models a point source [26]. This point source is, in fact, the endpoint of the gravitational collapse of a pressureless dust. The interior region of the collapsing solution is given by

$$ds^2 = -dt^2 + \left(1 - \lambda \tan\left(\frac{Q}{2}t\right)\right)^2 dx^2 \tag{24}$$

$$\phi = \phi_0 - \ln\left(\cos\left(\frac{Q}{2}t\right)\right) \tag{25}$$

where $0 \leqslant t < (2/Q)\tan^{-1}\lambda^{-1}$ and $|x| \leqslant r$. The exterior vacuum solution is

$$ds^2 = -\left(1 - ae^{-Q|X|}\right)dT^2 + \frac{dX^2}{1 - ae^{-Q|X|}} \cdots \tag{26}$$

$$\phi = -\tfrac{1}{2}Q|X| \tag{27}$$

where $0 \leqslant T$ and

$$|X| \leqslant -\frac{2}{Q}\phi_0 + \frac{1}{Q}\ln\left[1 - (1 - ae^{2\phi_0})\tanh^2\left(\frac{Q}{2}T\right)\right]$$

and $Q > 0$ for asymptotic flatness. For $\phi_0 < 0$, the solution (26) and (27) may be C^0-matched to the solution (24) and (25) provided an appropriate surface stress–energy tensor and dilaton current are included [26].

Finally, as an example of a metric of type D, consider

$$ds^2 = -\cos 2\theta \, dt^2 + 2\sin 2\theta \, dt \, dx + \cos 2\theta \, dx^2 \tag{28}$$

on $\mathbb{R}^2$ where $\theta = \theta(x)$. Note that for θ constant this is Minkowski spacetime in coordinates rotated by the angle θ. Thus when θ varies with x it determines the tilting of the light cones throughout the spacetime. This metric can be expressed in the form (8) by transforming to $t' = t - \int \tan 2\theta \, dx$, which gives

$$ds^2 = -\cos 2\theta \, dt'^2 + \frac{dx^2}{\cos 2\theta}. \tag{29}$$

For a wide variety of choices of $\theta(x)$ this metric will have no definite sign for large $|x|$. We shall make use of this metric in the form (28) as it has no coordinate singularities.

3. Do the field equations put a restriction on the spacetime?

In $(3 + 1)$-dimensional general relativity, the field equations can be written

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}. \tag{30}$$

It is clear that given any spacetime (M, g_{ab}) we can use (30) to *define* T_{ab} and then the spacetime will be a solution to these equations for this choice of T_{ab}. Thus any restrictions on the spacetime metric in general relativity are a consequence of requiring the distribution of matter to be physically reasonable, i.e. to be locally causal and to respect either the weak or dominant energy conditions. Whether or not such a property holds for the $(1 + 1)$-dimensional theories of gravity considered here is the subject of the present section.

Consider first the theory based on the equations (1) and (2). Since the Ricci scalar couples to the trace of the stress–energy, it is not so obvious that the same property holds. That is, given a spacetime (M, g_{ab}) does there exist a symmetric tensor field T_{ab} satisfying (1) and (2)? We prove that locally such a tensor field always exists for a sufficiently smooth metric and give an intuitive argument conjecturing that the global result is also true, at least for simply connected manifolds. Note that (1) and (2) represent three equations and that the tensor field T_{ab} has three independent components.

Theorem 1. Let (M, g_{ab}) be a $(1 + 1)$-dimensional spacetime and assume that g_{ab} is C^4 and that $p \in M$. Then there exists a symmetric tensor field T^{ab} defined on a neighbourhood U of p satisfying (1) and (2).

Proof. Since $(1 + 1)$-dimensional spacetimes are locally conformally flat, we can find a neighbourhood N of p and a conformal factor e^{σ} where $\sigma = \sigma(t, x)$ is C^4, such that in N the metric can be written

$$ds^2 = e^{\sigma}(-dt^2 + dx^2). \tag{31}$$

In this coordinate system equation (1) becomes

$$e^{-\sigma}\left(\frac{\partial^2 \sigma}{\partial x^2} - \frac{\partial^2 \sigma}{\partial t^2}\right) - \Lambda = 8\pi G e^{\sigma}(-T^{tt} + T^{xx}) \tag{32}$$

and equation (2) becomes

$$2\frac{\partial T^{tt}}{\partial t} + 3\frac{\partial \sigma}{\partial t}T^{tt} + 2\frac{\partial T^{tx}}{\partial x} + 4\frac{\partial \sigma}{\partial x}T^{tx} + \frac{\partial \sigma}{\partial t}T^{xx} = 0 \tag{33}$$

$$2\frac{\partial T^{xx}}{\partial x} + 3\frac{\partial \sigma}{\partial x}T^{xx} + 2\frac{\partial T^{tx}}{\partial t} + 4\frac{\partial \sigma}{\partial t}T^{tx} + \frac{\partial \sigma}{\partial x}T^{tt} = 0. \tag{34}$$

We now must show that this system of three equations in three unknowns has a solution in a neighbourhood of p. Let $F = T^{tt} + T^{xx}$, $G = 2T^{tx}$ and $\phi = -2\sigma$. Then using (32), equations (33) and (34) become

$$\frac{\partial F}{\partial t} + \frac{\partial G}{\partial x} = \frac{\partial \phi}{\partial t}F + \frac{\partial \phi}{\partial x}G + M_1 \tag{35}$$

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial t} = \frac{\partial \phi}{\partial x}F + \frac{\partial \phi}{\partial t}G + M_2 \tag{36}$$

where M_1 and M_2 are expressions involving ϕ and its derivatives up to third order and Λ, and thus are C^1. Now define $\tilde{F} = F + G$ and $\tilde{G} = F - G$. Then (35) + (36) becomes

$$\frac{\partial \tilde{F}}{\partial t} + \frac{\partial \tilde{F}}{\partial x} = \left(\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\right)\tilde{F} + M_1 + M_2 \tag{37}$$

and (35) − (36) becomes

$$\frac{\partial \tilde{G}}{\partial t} - \frac{\partial \tilde{G}}{\partial x} = \left(\frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial x}\right)\tilde{G} + M_1 - M_2. \tag{38}$$

Equations (35) and (36) have a solution if and only if equations (37) and (38) do. Theorem 2.1 of [33] ch 4 shows that (37) and (38) will have solutions in a neighbourhood U of p. (In fact, the theorem shows that for any C^1 Cauchy data defined on a C^1 non-characteristic initial curve D in N, there exists a unique C^1 solution to (37) and (38) in a neighbourhood of D.)   □

For the system based on (3) and (4) it is clear that (3) can be used to define T_{ab}. What is not so clear is whether given a spacetime (M, g_{ab}) there exists a scalar field ϕ satisfying (4). We prove that locally such a scalar field always exists for a sufficiently smooth metric.

Theorem 2. Let (M, g_{ab}) be a $(1 + 1)$-dimensional spacetime and assume that g_{ab} is C^2 and that $p \in M$. Then there exists a scalar field ϕ defined on a neighbourhood U of p satisfying (4).

Proof. As before, we can find a neighbourhood N of p and a conformal factor e^σ where σ is C^2, such that in N the metric can be written

$$ds^2 = e^\sigma(-dt^2 + dx^2). \tag{39}$$

In this coordinate system equation (4) becomes

$$-\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} + \left(\frac{\partial \phi}{\partial t}\right)^2 - \left(\frac{\partial \phi}{\partial x}\right)^2 = \tfrac{1}{4}(R_{tt} - R_{xx} - (J + c)e^\phi). \tag{40}$$

Define new coordinates t' and x' by $t' = t + x$ and $x' = t - x$. Then (40) can be written

$$\frac{\partial^2 \phi}{\partial x' \partial t'} = \frac{\partial \phi}{\partial t'}\frac{\partial \phi}{\partial x'} + \frac{1}{16}(-R_{tt} + R_{xx} + (J + c)e^\phi). \tag{41}$$

Theorem 7.1 of [33, ch 4] shows that (41) will have C^2 solutions in a neighbourhood U of p since the right-hand side is a continuous function of $\partial\phi/\partial t'$, $\partial\phi/\partial x'$, t' and x' and also satisfies a Lipschitz condition in $\partial\phi/\partial x'$ and $\partial\phi/\partial t'$. Since the solution will be C^2, it will also satisfy (40). (In fact, the theorem shows that for any appropriate Cauchy data defined on a C^1 non-characteristic initial curve D in N, there exists a unique solution to (41) in a neighbourhood of D.)   □

We conjecture that these results can be extended to any simply connected region S of a spacetime and that theorem 1 is valid for C^3 metrics. It may be possible to construct proofs of these conjectures by investigating the field equations without using conformal coordinates or by using a 'quilting' argument that patches together conformal neighbourhoods using the Cauchy property of the solutions. Starting with a point $p \in S$ we find a region on which a solution exists, by the above theorem. Then along the boundary of this region we apply the theorem repeatedly and use the parenthetical remarks at the end of theorems 1 and 2 to extend the solution to all of the spacetime S.

In the rest of the paper we are not concerned with specific field equations and deal with arbitrary spacetimes, many of which are known solutions to the theories discussed earlier. If our conjectures are true then every spacetime we discuss below is a solution, since each connected component of these spacetimes is simply connected.

4. The causal structure of $(1 + 1)$-dimensional black holes

Penrose diagrams are invaluable tools in the investigation of the causal structure of a spacetime. The causal structure of a spacetime depends only on its conformal structure, since metrics that are related by a conformal factor have the same light cones. As all $(1 + 1)$-dimensional metrics are conformally flat, it is straightforward to choose coordinates so that

$$g = e^{-2\sigma} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{42}$$

for any particular timelike or spacelike region. However it is not, in general, possible to carry out such a transformation for the entire two-dimensional space, as any event horizons are located at $|\sigma| = \infty$. In contrast to this, writing the metric in the form (8) is especially useful in that it clearly illustrates the full event horizon structure of the spacetime in a manner similar to the $(3+1)$-dimensional Schwarzchild metric. In general $\alpha(x)$ will take on both positive and negative values, corresponding respectively to timelike and spacelike regions. Points at which the metric changes signature are given by $\alpha(x) = 0$; these are coordinate singularities and locate the event horizons.

The spacetime in (8) with $\alpha = \alpha(x)$ may be maximally extended by carrying out the Kruskal–Szekeres transformation:

$$uv = \exp\left[\int^x \frac{dz}{|\alpha(z)|} \right] \tag{43}$$

$$\frac{u}{v} = -\operatorname{sgn}(\alpha)\exp(t) \tag{44}$$

which yields

$$ds^2 = 4\alpha(uv)\frac{du\,dv}{uv} \tag{45}$$

where $\alpha(uv)$ is implicitly defined via equation (43). As in the $(3 + 1)$-dimensional case, the (u, v) space is a double cover of the (t, x) space and the horizons are at $uv = 0$.

Using the metric in the form (8) makes it easy to identify the generic structure of curvature singularities for static metrics. From (5) and (6) it may be shown that both the Riemann and Ricci tensors are uniquely determinable in terms of the Ricci scalar in two spacetime dimensions. Since $R = -\mathrm{d}^2\alpha/\mathrm{d}x^2$, all curvature singularities will manifest themselves in terms of divergences in the second derivative of $\alpha(x)$. This will not necessarily occur at a point where the metric diverges. Consider the $(1+1)$-dimensional analogue of a spherically symmetric black hole, a black hole for which $\alpha = \alpha(|x|)$. This will correspond to a spacetime of type A or B. The curvature is

$$R = -\alpha''(r) - 2\delta(x)\alpha'(0) \tag{46}$$

where $r = |x|$ and the prime denotes $\mathrm{d}/\mathrm{d}r$. If $\alpha'(0) \neq 0$ then there will be a delta-function singularity in the curvature. If $\alpha''(r)$ and $\alpha(0)$ are finite over the entire range of r $(0 \leqslant r < \infty)$ this will be the only singularity in the curvature.
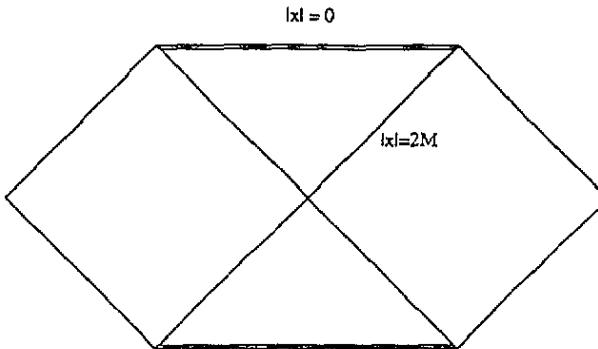

Figure 1. Penrose diagram for the $(1+1)$-dimensional spherically symmetric black hole.

Using the coordinates given in (45), the Penrose diagram for such a black hole may be easily constructed. The result is given in figure 1. It is qualitatively the same as the diagram for the $(3+1)$-dimensional Schwarzchild case, except that each point on the diagram represents a 0-sphere instead of a 2-sphere. Since a 0-sphere consists of two points, an alternative representation of the entire spacetime $-\infty < x < \infty$ may be given by taking two copies of figure 1, one for $x > 0$ and the other for $x < 0$, and joining them at each of the lines at $|x| = 0$ at the respective top and bottom of each copy leading to a singular curvature at the junction. A simple visualization is both sides of the paper. Metrics describing such spacetimes are given by (10) with $\Lambda = 0$ and by (26).

The Penrose diagram for the collapsing fluid discussed in section 2 equations (15) and (20) is given in figure 2. Each point in this diagram represents a 0-sphere. As above, this can be represented by two copies of the figure joined along $|x| = 0$.

Note that the time orientation on each copy of these diagrams is the same, and so closed timelike curves are not present in the spacetimes described by 1 and 2. Indeed, for the collapsing fluid observers on either side of it can travel through the fluid to verify before collapse that they have the same time orientation; smoothness of the metric implies that this is maintained after collapse. The static black holes (10) and (26) model the endpoint of such a collapse and so are taken to have the
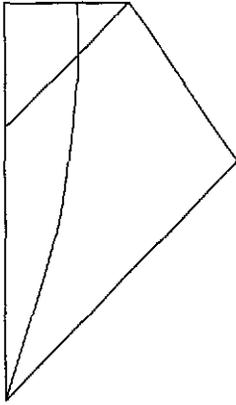
Figure 2. Penrose diagram for a collapsing fluid.
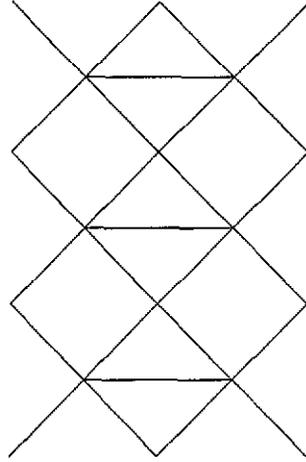

Figure 3. An unfolded Penrose diagram.

same time orientation for positive and negative x outside the horizon. One might consider taking multiple copies of figure 1 and joining them in sequence at $|x| = 0$. However such joinings at $|x| = 0$ (which are not at coordinate singularities but are at delta-function curvature singularities) would make the entire manifold non-Hausdorff since each point on each individual copy represents a 0-sphere [21].

Black hole spacetimes of type C have a different structure. The Penrose diagram is still qualitatively the same as figure 1, except that each point on the diagram represents only one point in the spacetime (as opposed to being a 0-sphere). Hence there is now a 1:1 mapping between points in spacetime and points on the diagram, instead of a 2:1 mapping as before. If no curvature singularities are present, it is possible to extend this diagram by making multiple copies of figure 1 and joining them in sequence along the horizontal lines as in figure 3. Again, instead of each point representing a 0-sphere, there would also be a 1:1 mapping between points in the (extended) spacetime and points on the diagram.

In constructing Penrose diagrams for $(1 + 1)$-dimensional metrics, it is important to note that the criterion for asymptotic flatness is slightly more general than in higher dimensions. It is sufficient to require that $\alpha(x) \rightarrow K|x| + C$ for large $|x|$, since a Rindler transformation may then be applied locally to obtain a flat metric. Taking $\alpha = \ln(\cosh(Kx)) + C$, for example, satisfies this criterion; it has no curvature singularities and its Penrose diagram is of the type given in figure 3. In general such metrics are solutions of the field equations only for physically unreasonable $(1 + 1)$-dimensional stress–energy tensors.

Diagrams for the cosmological cases may also be easily constructed. Consider the metric (10) with $M = 0$. For $C = -1$ this yields either the metric for de Sitter space if $\Lambda > 0$:

$$\mathrm{d}s^2 = -\cos^2\left(\sqrt{\tfrac{1}{2}|\Lambda|}y\right)\mathrm{d}t^2 + \mathrm{d}y^2 \tag{47}$$

(using the transformation $\sin(\sqrt{\tfrac{1}{2}|\Lambda|}y) = \sqrt{\tfrac{1}{2}|\Lambda|}x$) or anti-de Sitter space if $\Lambda < 0$:

$$\mathrm{d}s^2 = -\cosh^2\left(\sqrt{\tfrac{1}{2}|\Lambda|}y\right)\mathrm{d}t^2 + \mathrm{d}y^2 \tag{48}$$

(using the transformation $\sinh(\sqrt{\frac{1}{2}|\Lambda|}y) = \sqrt{\frac{1}{2}|\Lambda|}x$). The Penrose diagrams for these cases are the same as in the $(3+1)$-dimensional case. However, for $C = 1$ an alternate version of anti-de Sitter space is possible with the metric

$$ds^2 = -(\tfrac{1}{2}|\Lambda|x^2 - 1)\,dt^2 + \frac{dx^2}{\frac{1}{2}|\Lambda|x^2 - 1} \tag{49}$$

which may be written as

$$ds^2 = -\sinh^2\left(\sqrt{\tfrac{1}{2}|\Lambda|}y\right)dt^2 + dy^2 \tag{50}$$

using the transformation $\cosh(\sqrt{\frac{1}{2}|\Lambda|}y) = \sqrt{\frac{1}{2}|\Lambda|}x$. For large y the spacetime described by (50) is the same as that described by (48). However there is an event horizon at $y = 0$ ($x = \sqrt{2/|\Lambda|}$) which is absent in the usual anti-de Sitter case.

We close this section by making some general comments on the thermodynamics of $(1+1)$-dimensional black holes. It is straightforward to show using either naive Wick-rotation arguments [16, 26] or a more formal quantum-field-theoretic treatment [20, 21] that the temperature T of a black hole is given by

$$T = M/2\pi \tag{51}$$

where M is the mass parameter. One can then *define* the entropy S of the black hole via the thermodynamic relation [16]

$$dM = T\,dS \tag{52}$$

since one can relate the mass parameter to the energy for both of the theories given by (1), (2) and (3), (4). In the former case one can appeal to the Newtonian limit of the theory [15] and in the latter case one can compute the ADM mass [25]. Hence the entropy varies logarithmically with the mass parameter

$$S \sim \ln(M/M_0) \tag{53}$$

where M_0 is a constant of integration which appears as a fundamental mass scale in the theory; its origin presumably lies within a fully quantized version of the $(1+1)$-dimensional gravitation theories discussed here. A more detailed investigation of the general thermodynamics given by (52) and (53) may be found in [21].

Relating this definition of entropy to an area parameter associated with the black hole is somewhat more problematic. In $(3+1)$ dimensions Hawking's area theorem says that the area of a closed trapped surface will never decrease. The association of an entropy with the area of the horizon then implies that the entropy of a black hole will never decrease in any physical process. In $(1+1)$ dimensions the 'area' of a closed trapped surface is meaningless because the horizon is a zero-dimensional surface. However it may be true that the *volume* of a black hole (that is the geodesic length enclosed by the horizon) has a similar property. If the metric is C^0 within the horizon (i.e. if the horizon encloses only delta-function singularities) then the geodesic length between the horizons is well defined and is given by

$$\ell = \int_a^b ds \tag{54}$$

where a and b are any two opposing points on the horizon's worldlines. For the static black hole given by (10) with $\Lambda = 0$ we obtain

$$\ell = \int_{-1/2M}^{1/2M} \frac{dx}{\sqrt{1 - 2M|x|}} = 2 \int_0^{1/2M} \frac{dx}{\sqrt{1 - 2Mx}} = \frac{2}{M}. \tag{55}$$

Also, the volume of the string-theoretic black hole (26) is given by

$$\ell = \int_{-\ln 2/2M}^{\ln 2/2M} \frac{dx}{\sqrt{2e^{2Mx} - 1}} = 2 \int_0^{\ln 2/2M} \frac{dx}{\sqrt{2e^{2Mx} - 1}} = \frac{\pi}{2M}. \tag{56}$$

Thus for these cases we find that in $(1 + 1)$ dimensions the volume of a black hole *decreases* as matter is added, as follows from the following dimensional arguments. Suppose we have a static black hole in the form (8). Then dimensionally we must have $\alpha = \alpha(Mx)$. Let the horizons occur at x_L and x_R. Then we have that

$$\ell = \int_{x_L}^{x_R} \frac{dx}{\sqrt{\alpha(Mx)}} = \frac{1}{M} \int_{y_L}^{y_R} \frac{dy}{\sqrt{\alpha(y)}} \tag{57}$$

where $y = Mx$.

So it seems to be a general property of static $(1+1)$-dimensional black holes that their volume decreases as their mass (and entropy) increases. A $(1+1)$-dimensional analogue of the area law would then involve demonstrating that in any physical process the volume of the black hole never increases (and hence the entropy never decreases, since it would vary as $S \sim -\ln(V/V_0)$ [16]). Note that for spacetimes of type C there will be a region of signature $(+, -)$ which is not enclosed by two regions of signature $(-, +)$, yielding an infinite geodesic length for such a region and a breakdown of the entropy/length relation. Such objects more closely resemble cosmological event horizons than black holes, since it is difficult to see how they could arise as the endpoint of gravitational collapse of some distribution of matter [26].

5. Comments on singularity theorems in two dimensions

As mentioned in the previous section, the structure of curvature singularities is easily analysed using the metric in the form (8). Extending the well known singularity theorems [30, 31] to two dimensions is somewhat more problematic. Consider Penrose's 1965 theorem, which states:

Theorem 3. (Penrose 1965) A four-dimensional spacetime (M, g_{ab}) cannot be null geodesically complete if
    (i) $R_{ab} K^a K^b \geqslant 0$ for all null vectors K^a;
    (ii) there is a non-compact Cauchy surface in M; and
    (iii) there is a closed trapped surface in M.

A proof of this theorem is given in [30, 31].
    A spacetime is said to be *null* (respectively *timelike, spacelike*) *geodesically complete* if all null (respectively timelike, spacelike) geodesics can be extended to arbitrary

affine parameter values. A spacetime is usually said to be *singular* if it is not geodesically complete.

The first two conditions are easily generalized to $(1+1)$ dimensions. Condition 1 is called the null energy condition and implies in higher dimensions that the expansion of congruences of null geodesics monotonically decreases along the geodesics. This condition is trivially true for all $(1+1)$-dimensional spacetimes because the identity $R_{ab} = \frac{1}{2}g_{ab}R$ implies that for all null vectors K^a

$$R_{ab}K^aK^b = \frac{1}{2}g_{ab}RK^aK^b = 0. \tag{58}$$

A *Cauchy surface* is a spacelike hypersurface which every inextendible non-spacelike curve intersects exactly once, a concept which may also be extended to $(1+1)$ dimensions. If a spacetime admits a Cauchy surface, one can predict the state of the spacetime at any time in the past or future if one knows the relevant data on the surface. See [30] for more discussion of these definitions.

Although these definitions make sense for spacetimes of arbitrary dimension ($\geqslant$ 2), the following definition only applies to spacetimes of dimension 3 or greater. Let (M, g_{ab}) be an n-dimensional spacetime. A *closed trapped surface* S is a C^2 compact spacelike $(n-2)$ surface without boundary such that the two families of null geodesics orthogonal to S are converging at S, that is $_1\hat{\chi}_{ab}g^{ab}$ and $_2\hat{\chi}_{ab}g^{ab}$ are negative, where $_1\hat{\chi}_{ab}$ and $_2\hat{\chi}_{ab}$ are the two null second fundamental forms of S. Intuitively this definition is saying that the gravitational field is so strong at S that even light cannot escape. Extending this definition to two spacetime dimensions is difficult in that the closed trapped surface would have to be zero dimensional, most likely consisting of two distinct points. We have been unable to find a rigorous definition of this concept in two dimensions. Part of the problem is that the idea is a local one—it only depends on the properties of the spacetime near S. But, as shown in section 4, in two dimensions there exist black holes for which spacetime is flat over large regions, so locally the event horizon (which is a likely candidate for the closed trapped surface) has no distinguishing properties. Also, when a $(1+1)$-dimensional black hole contains a singularity, the spacetime is often disconnected and part of the event horizon is in one half and part in the other, complicating definitions based on the 'volume' enclosed by the surface (although this may not be a serious problem for delta-function type singularities). And, the 'area' of the surface is no help since the surface is zero dimensional.

As an attempt to see if some form of Penrose's theorem applies in two spacetime dimensions, we have been investigating the causal structure of various $(1+1)$-dimensional spacetimes containing a surface satisfying the intuitive idea behind the definition of a closed trapped surface, and also containing a Cauchy surface. If such a spacetime were found that was non-singular, it would show that either Penrose's theorem was false in two dimensions, or that the conditions need to be strengthened. Recall that we need not worry about the null energy condition, as it is always true is two dimensions.

Consider the static spacetime defined by the metric (28). Although this metric is smooth for smooth choices of $\theta(x)$, the spacetime is null geodesically incomplete whenever θ is non-constant and $\cos 2\theta = 0$ for some x. This is seen as follows. Let $x^a(\lambda) = (t(\lambda), x(\lambda))$ be a geodesic. Then the geodesic equations

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}\lambda^2} + \Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\frac{\mathrm{d}x^c}{\mathrm{d}\lambda} = 0 \tag{59}$$

can be written

$$\frac{d^2 t}{d\lambda^2} - \frac{d\theta}{dx}\left[\sin^2 2\theta \left(\frac{dt}{d\lambda}\right)^2 + 2\cos 2\theta \sin 2\theta \frac{dt}{d\lambda}\frac{dx}{d\lambda} + (\cos^2 2\theta + 1)\left(\frac{dx}{d\lambda}\right)^2\right] = 0$$

(60)

$$\frac{d^2 x}{d\lambda^2} + \frac{d\theta}{dx}\sin 2\theta \left[-\cos 2\theta \left(\frac{dt}{d\lambda}\right)^2 + 2\sin 2\theta \frac{dt}{d\lambda}\frac{dx}{d\lambda} + \cos^2 2\theta \left(\frac{dx}{d\lambda}\right)^2\right] = 0.$$

(61)

Note that the factor in brackets in equation (61) is precisely $g_{ab}(dx/d\lambda)^a(dx/d\lambda)^b$, so for a null geodesic this equation becomes simply $d^2 x/d\lambda^2 = 0$. Thus for non-vertical null geodesics, we may choose $\lambda = x$ as our affine parameter. This justifies the claim earlier that these spacetimes are null geodesically incomplete when $\cos 2\theta = 0$ for some x since the vertical lines through these points are null geodesics and so null geodesics going in the same direction (i.e. 'left' or 'right') cannot cross these lines (by the uniqueness property of geodesics through a specific point with a specific tangent vector). Since x is an affine parameter for these geodesics, they must be incomplete.

Equation (60) can be integrated for null geodesics to give

$$t = \int \tan(\theta(x) \pm \tfrac{1}{4}\pi)\,dx + c$$

(62)

where the choice of sign selects right- or left-moving null geodesics.

An interesting choice of $\theta(x)$ is

$$\theta(x) = \tan^{-1}[2x/(1 + x^2)]$$

(63)

(see figure 4). This has the nice properties that it tends to 0 as $x \rightarrow \pm\infty$ and equals 0 at 0 and $\pm\tfrac{1}{4}\pi$ at ± 1. Thus at ± 1 the light cones are tilting inwards at 45° and no non-spacelike geodesic can leave the region $-1 \leqslant x \leqslant 1$. Also, since the metric is independent of t, the distance between the points $(t, -1)$ and $(t, 1)$ is independent of t. So intuitively it would seem that this is a black hole and that for each t, $(t, -1), (t, 1)$ should be called a 'closed trapped surface', whatever that means in two dimensions. (The independence of t is important, for if we take the portion of Minkowski spacetime with $t > 0$ in the standard (t, x) coordinates and transform to the coordinates (t, x') where $x' = x/t$ it would seem at first glance that the resulting metric has a black hole with horizons at $x' = \pm 1$. But clearly the distance between $(t, x' = -1)$ and $(t, x' = 1)$ is $2t$ so the horizons do not enclose a bounded region.)

This black hole is interesting in that there are no coordinate singularities in the metric. (In fact, it is C^∞ and non-degenerate everywhere.) But according to the earlier discussion, the null geodesics are incomplete. The integral (62) is easy to calculate for this choice of θ and we find that the null geodesics are given by

$$t = \pm x + \frac{4}{1 \mp x} + 4\ln|1 \mp x| + c.$$

(64)

(See figure 5.) Although the geodesics are unbounded as $x \rightarrow \pm 1$, they still have finite affine length since x is an affine parameter. It is curious how different this
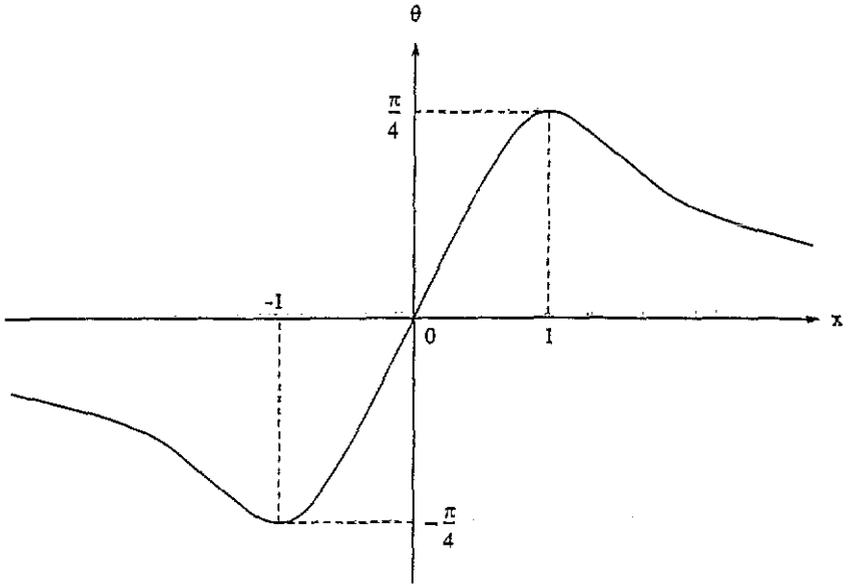
Figure 4. Plot of $\theta(x) = \tan^{-1}[2x/(1+x^2)]$.
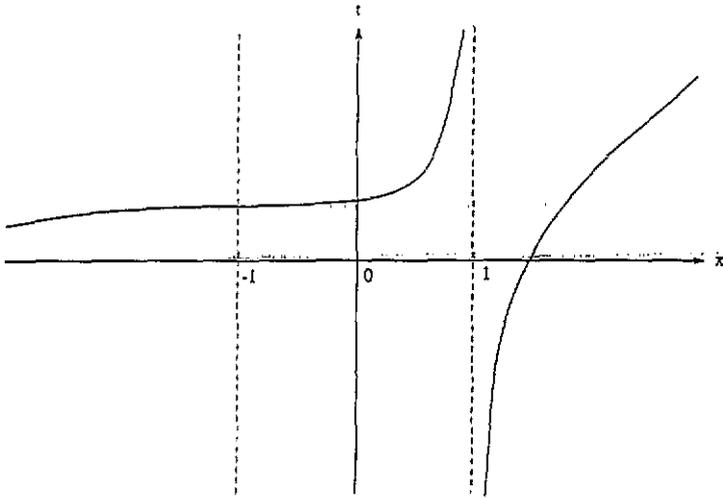

Figure 5. Some right-moving null geodesics for the metric with $\theta(x) = \tan^{-1}[2x/(1+x^2)]$. All null geodesics can be obtained from those in the figure by shifting vertically and/or reflecting in the t-axis.

singular behaviour is from that typically found within black holes: the incomplete null geodesics approach the event horizon on the opposite side of the black hole instead of encountering a singularity at the centre of the black hole, and the metric is C^∞ everywhere. Nevertheless Penrose's theorem is found to hold in this case and in several other cases that we investigated.

Although we have not yet found a counterexample, we speculate that if a reasonable definition of a closed trapped surface in two dimensions is discovered, Penrose's

theorem will be found to be false as stated earlier, but will be true with a stronger energy condition such as the weak energy condition ($R_{ab} K^a K^b \geqslant 0$ for all *nonspacelike* vectors K^a). We suspect that the weak energy condition will be sufficient because all of the examples that we experimented with that were 'close' to violating Penrose's theorem also violated the weak energy condition.

6. Conclusions

Theories of gravitation in two spacetime dimensions possess a wealth of solutions whose causal structure is far from trivial. Many of these have counterparts in $(3 + 1)$-dimensional general relativity, but a number of them have features which are quite distinct from the higher dimensional case. A more complete understanding of the implications of these spacetimes for $(1 + 1)$-dimensional gravity will entail a deeper exploration of the singularities, the entropy/volume law and, ultimately, full quantization.

Acknowledgments

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