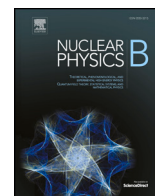




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Introduction of the G_2 -Ricci flow: Geometric implications for spontaneous symmetry breaking and gauge boson massesRichard Pinčák^{a, ,*}, Alexander Pigazzini^b, Michal Pudlák^a, Erik Bartoš^c^a Institute of Experimental Physics, Slovak Academy of Sciences, Watsonova 47, 043 53 Košice, Slovak Republic^b Mathematical and Physical Science Foundation, 4200 Slagelse, Denmark^c Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 845 11 Bratislava, Slovak Republic

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ABSTRACT

This work introduces the G_2 -Ricci flow on seven-dimensional manifolds with non-zero torsion and explores its physical implications. By extending the Ricci flow to manifolds with G_2 structures, we study the evolution of solitonic solutions and their role in spontaneous symmetry breaking in gauge theories. In particular, this model proposes that the masses of the W and Z bosons are determined not by an external scalar field, as in the Higgs mechanism, but by the intrinsic geometric torsion of the manifold. Furthermore, a possible connection between the geometry of extra dimensions and the curvature of our spacetime is explored, with implications for the experimentally observed positive cosmological constant. This approach provides an innovative interpretation of fundamental interactions in theoretical physics, opening new possibilities for studying extra dimensions and the geometry of G_2 -manifolds.

1. Introduction

The Ricci flow has been a powerful tool in differential geometry, most notably used in the proof of the Poincaré conjecture by Perelman [1]. The Ricci flow is a geometric evolution equation that smooths out the metric of a manifold, guiding it toward its canonical geometric structure. Originally introduced by Hamilton [2], the Ricci flow has since found applications in various areas of mathematics and physics, including high-energy theoretical physics and geometry.

Extending the concept of Ricci flow to manifolds with special holonomy groups, the G_2 -Ricci flow naturally arises in the study of 7-dimensional manifolds endowed with G_2 -structures. These structures are not only mathematically interesting due to their unique holonomy properties, but also important in theoretical physics, particularly in string theory, where G_2 -manifolds play a central role in compactifications [3]. Witten [4], highlighted the importance of G_2 -structures in string theory compactifications, showing how they preserve a portion of supersymmetry, making them critical in models of particle physics that emerge from string theory.

A G_2 -structure on a 7-dimensional manifold is characterized by a 3-form φ , which reduces the structure group to the exceptional Lie group G_2 . When φ is both closed and co-closed, the structure is torsion-free, and the associated metric is Ricci-flat [5]. However, in many physical contexts, such as sigma models and gauge theories, it is natural to consider G_2 -structures with non-zero torsion, which complicates the evolution of these structures under the G_2 -Ricci flow [6]. Bryant [7], contributed significantly by providing

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examples of G_2 -structures with torsion and exploring their geometric properties, which has been influential in understanding how torsion affects the evolution under the G_2 -Ricci flow.

Previously, the discrete gravitational field and spacetime with canonical defect was considered [8]. In the Pigazzini-Pinčák braneworld scenario, there exists a class of manifolds in the context of D-branes using partially negative-dimensional product, so-called PNDP manifolds (PNDP), which have virtual dimensions. The topological defects can emerge from non-orientability and they can be related to the appearance of curvature at low-energy scales.

In this paper, we investigate the G_2 -Ricci flow on 7-dimensional manifolds with G_2 -structures, focusing particularly on solutions involving torsion. We rigorously analyze the conditions for existence, regularity, and convergence of solutions. Additionally, we explore how the solitonic solutions derived from the G_2 -Ricci flow influence spontaneous symmetry breaking in gauge theories. These solitons induce symmetry breaking, leading to the generation of massive gauge bosons, introducing a novel geometric mechanism that competes with the Higgs mechanism.

One of the fundamental problems in theoretical physics is to understand the origin of the masses of gauge bosons and the role of geometry in the fundamental interactions. In the Standard Model, the mass of the W and Z bosons arises through the Higgs mechanism, but this requires the ad hoc introduction of a scalar potential. Our approach proposes an alternative based on the geometric torsion of G_2 -manifolds, which could offer a more natural explanation of the spontaneous electroweak symmetry breaking. This raises several fundamental questions: Is it possible to derive symmetry breaking directly from geometry? What are the properties of solitonic solutions of the G_2 Ricci flow? What are the physical consequences of a purely geometric mechanism?

This provides a deeper understanding of the interplay between geometry and high-energy physics [9].

2. Preliminaries

2.1. G_2 -structures on 7-dimensional manifolds

A 7-dimensional differentiable manifold M is said to possess a G_2 -structure if there exists a 3-form φ that reduces the structure group of M to G_2 , a subgroup of $SO(7)$. This form φ not only defines a Riemannian metric g but also determines a corresponding 4-form $*\varphi$ via relations that characterize the G_2 group. Specifically, in a local orthonormal basis $\{e^1, \dots, e^7\}$ of the cotangent bundle, the 3-form φ is expressed as

$$\varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}, \quad (1)$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$. This 3-form φ directly determines a Riemannian metric g_φ on M through a relation between φ and the metric coefficients.

A G_2 -structure is torsion-free if φ is both closed and co-closed, satisfying the conditions

$$d\varphi = 0 \quad \text{and} \quad d*\varphi = 0, \quad (2)$$

where $*\varphi$ is the Hodge dual of φ . When these conditions are met, the manifold M_7 has a torsion-free G_2 -structure, implying that the connection associated with the metric is compatible with the G_2 -structure and the manifold admits a parallel G_2 -form.

2.2. Function spaces

To study the G_2 -Ricci flow, we utilize Sobolev spaces $H^{k,\alpha}(M)$, where k denotes the degree of differentiability and α represents the Hölder exponent. For initial conditions, the 3-form φ is assumed to belong to $H^{k,\alpha}(M)$ for $k \geq 2$, ensuring sufficient regularity to apply elliptic theory.

G_2 -manifolds and Sobolev spaces

A G_2 -structure is said to belong to the Sobolev space $H^{k,\alpha}(M)$ if the components of φ and the associated metric g_φ exhibit appropriate regularity, meaning that derivatives up to order k are Hölder continuous with exponent α . This ensures that the manifold is compatible with the analysis required for the G_2 -Ricci flow.

2.3. Definition of the G_2 -Ricci flow

The G_2 -Ricci flow is defined by the following equation

$$\frac{\partial \varphi}{\partial t} = \Delta_d \varphi + \mathcal{L}_X \varphi + \text{Ric}_\lrcorner * \varphi + T(\varphi), \quad (3)$$

where

- Δ_d is the Hodge-de Rham Laplacian, a second-order elliptic operator that acting on the 3-form φ .
- $\mathcal{L}_X \varphi$ is the Lie derivative of φ along a vector field X . It is first-order operator.
- $\text{Ric}_\lrcorner * \varphi$ is the contraction of the Ricci tensor with the 4-form $*\varphi$.
- $T(\varphi)$ represents the torsion of the G_2 -structure, which measures the deviations from the torsion-free condition.

2.4. Contraction of the Ricci tensor with $*\varphi$

The contraction $\text{Ric}_\perp * \varphi$ is given by

$$(\text{Ric}_\perp * \varphi)_{ijk} = R_{[i}^m (* \varphi)_{|m|jk]}, \quad (4)$$

where $*\varphi$ is the Hodge dual of φ , and the brackets denote cyclic summation over indices. This term accounts for how the Ricci curvature of the manifold interacts with the evolving G_2 -structure during the flow.

The contraction $R_{[i}^m (* \varphi)_{|m|jk]}$ ensures that the resulting tensor is a well-defined 3-form. The operation removes one index via contraction with the Ricci tensor R_i^m , while the symmetrization maintains compatibility with the antisymmetry of $*\varphi$. This guarantees that $\text{Ric}_\perp * \varphi$ is consistent with the structure of differential forms in G_2 -geometry.

3. Problem formulation

We consider a 7-dimensional compact manifold M with an initial G_2 -structure $\varphi(0) \in H^{k,\alpha}(M)$, with $k \geq 2$. Our goal is to study the evolution of the G_2 -structure under the G_2 -Ricci flow, described by the equation

$$\frac{\partial \varphi}{\partial t} = \Delta_d \varphi + \mathcal{L}_X \varphi + \text{Ric}_\perp * \varphi + T(\varphi). \quad (5)$$

The main goal is to determine the evolution of $\varphi(t)$ under the G_2 -Ricci flow, establish the existence and uniqueness of short-time solutions, and study the long-term behavior. In particular, we are interested in understanding under which conditions the flow

1. remains smooth and convergent in the long term,
2. converges to a stationary solution (soliton) depending on the evolution of the torsion $T(\varphi)$.

We want to show that there are two possible scenarios for the long-term behavior:

1. *Ricci Solitons*: If the torsion $T(\varphi)$ decreases sufficiently over time, the solution $\varphi(t)$ converges to a Ricci soliton.
2. *Mixed Solitons*: If the torsion $T(\varphi)$ stabilizes at a finite value, the solution $\varphi(t)$ converges to a mixed soliton, with non-zero residual torsion.

We assume that the initial shape $\varphi(0)$ belongs to a Sobolev space $H^{k,\alpha}(M)$ with $k \geq 2$, which guarantees that the manifold and the associated metric have sufficient regularity to apply the results of elliptic theory. This regularity is necessary to guarantee the existence of regular local solutions for the parabolic flow.

The G_2 -Ricci flow problem is formulated as follows: given an initial G_2 -structure $\varphi(0)$, study the time evolution of the $\varphi(t)$ structure according to the flow equation and determine the conditions that guarantee the convergence of the solution to a stationary configuration. Possible stationary configurations can include

- *Ricci solitons*: If the torsion tends to zero, the final solution will be a Ricci soliton, which satisfies the torsion-free flow equation.
- *Mixed Solitons*: If the torsion stabilizes at a finite value, the final solution will be a mixed soliton, which includes a residual torsion.

The key questions we will address are

1. *Short-Term Existence and Uniqueness*: Verify that there exists a unique and regular solution for the G_2 -Ricci flow for short times, using the DeTurck trick to obtain a strictly parabolic shape.
2. *Long-Term Behavior*: Study the evolution of the torsion $T(\varphi)$. If $T(\varphi)$ decreases, we expect the solution to converge to a Ricci soliton. Alternatively, if the torsion stabilizes at a non-zero value, the solution will converge to a mixed soliton.
3. *Associated Energy*: Define an energy functional that controls the evolution of the flow and that allows us to study the stability of the solutions in the long term.

The theorem that we will prove in the next section, is formulated as follows

- *Short-Term Existence and Uniqueness*: There exists a time $T > 0$ such that the G_2 -Ricci flow has a unique and regular solution for $t \in [0, T)$, assuming that $\varphi(0) \in H^{k,\alpha}(M)$.
- *Long-Term Convergence (Ricci Soliton)*: If the torsion $T(\varphi(t)) \rightarrow 0$ as $t \rightarrow \infty$, then the solution $\varphi(t)$ converges to a Ricci soliton.
- *Long-Term Convergence (Mixed Soliton)*: If the torsion $T(\varphi(t))$ stabilizes at a finite value T_∞ , then the solution $\varphi(t)$ converges to a mixed soliton with non-zero torsion.

This formulation will allow us to explore how the torsion affects the geometric evolution of the manifold and to understand the asymptotic behavior of the flow.

4. Main theorem

Theorem 1. Let M be a compact, 7-dimensional Riemannian manifold equipped with an initial G_2 -structure $\varphi(0)$ that belongs to the Sobolev space $H^{k,\alpha}(M)$ for $k \geq 2$. Consider the G_2 -Ricci flow defined by

$$\frac{\partial \varphi}{\partial t} = \Delta_d \varphi + \mathcal{L}_X \varphi + \text{Ric} \lrcorner \varphi + T(\varphi), \quad (6)$$

where Δ_d is the Hodge-de Rham Laplacian acting on the 3-form φ , $\mathcal{L}_X \varphi$ is the Lie derivative along a smooth vector field X , $\text{Ric} \lrcorner \varphi$ denotes the contraction of the Ricci tensor with the 4-form φ , and $T(\varphi)$ is a torsion term.

1. **Short-Time Existence and Uniqueness:** There exists a finite time $T > 0$ such that the G_2 -Ricci flow has a unique smooth solution $\varphi(t)$ for $t \in [0, T)$, assuming that the initial data $\varphi(0)$ satisfies the appropriate regularity conditions (smoothness and finite Sobolev norm).
2. **Long-Time Convergence:** If the torsion term $T(\varphi)$ diminishes sufficiently over time, i.e., $\|T(\varphi(t))\|_{C^0} \rightarrow 0$ as $t \rightarrow \infty$, then the solution $\varphi(t)$ exists for all $t \geq 0$ and converges to a Ricci soliton as $t \rightarrow \infty$.
3. **Long-Time Convergence for Mixed Soliton:** If the torsion term $T(\varphi)$ stabilizes at a finite value, i.e.,

$$\|T(\varphi(t)) - T_\infty\|_{L^\infty} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (7)$$

then the solution $\varphi(t)$ exists for all $t \geq 0$ and converges to a mixed soliton φ_∞ , where the torsion does not vanish.

Proof. (Short-Term Existence and Uniqueness): We begin by establishing the short-time existence and uniqueness of the solution to the G_2 -Ricci flow on the compact manifold M .

The G_2 -Ricci flow is governed by the partial differential equation

$$\frac{\partial \varphi}{\partial t} = \Delta_d \varphi + \mathcal{L}_X \varphi + \text{Ric} \lrcorner \varphi + T(\varphi), \quad (8)$$

where

- Δ_d is the Hodge-de Rham Laplacian, an elliptic second-order differential operator acting on differential forms.
- \mathcal{L}_X is the Lie derivative with respect to a smooth vector field X , a first-order differential operator.
- $\text{Ric} \lrcorner \varphi$ involves the Ricci tensor and is also a first-order differential operator.
- $T(\varphi)$ is a torsion term, considered as a lower-order perturbation.

Since M is compact, the parabolic nature of the equation ensures the applicability of standard short-term existence results from the theory of parabolic PDEs. In fact, the term $\Delta_d \varphi$ is a second-order elliptic operator, which guarantees strict parabolicity if the torsion term $T(\varphi)$ is considered a suitable lower-order perturbation. We must verify that the term $T(\varphi)$ does not negatively influence the parabolicity.

$$T(\varphi) = d\varphi, \quad (9)$$

since it is an external derivative, it is a first-order operator that does not alter the global parabolicity, since derivatives of lower order than $\Delta_d \varphi$ cannot destroy the parabolic structure.

Now, we modify the G_2 -Ricci flow equation using DeTurck's trick to transform the flow into a strictly parabolic form. Consider adding a vector field W to the equation, such that

$$\frac{\partial \varphi}{\partial t} = \Delta_d \varphi + \mathcal{L}_W \varphi + Q(\varphi), \quad (10)$$

where W is chosen appropriately to ensure parabolicity, and $Q(\varphi)$ collects the lower-order terms $\mathcal{L}_X \varphi$, $\text{Ric} \lrcorner \varphi$, and $T(\varphi)$. The choice of W is designed to absorb the first-order terms and create a strictly parabolic flow.

The parabolic PDE theory guarantees the short-time existence and uniqueness of solutions for parabolic flows on compact manifolds [2]. In particular, for the G_2 -Ricci flow, we apply the standard theory for quasilinear parabolic systems [10]. Since M is compact, the Sobolev embedding theorems ensure that all Sobolev norms remain finite, and the solution $\varphi(t)$ exists on a short-time interval $[0, T)$ for some $T > 0$.

Moreover, by elliptic regularity, the short-time solution $\varphi(t)$ is smooth, given that the initial data $\varphi(0)$ is smooth and belongs to the Sobolev space $H^{k,\alpha}(M)$. The uniqueness follows from standard results for parabolic PDEs: if two solutions existed, their difference would satisfy a linear parabolic equation with zero initial data, leading to uniqueness by the maximum principle.

Thus, we conclude that there exists a unique smooth solution $\varphi(t)$ for $t \in [0, T)$. \square

Proof. (Long-Term Convergence): Now, we aim to show that the solution $\varphi(t)$ can be extended for all $t \geq 0$ and converges to a Ricci soliton as $t \rightarrow \infty$, under the assumption that the torsion term $T(\varphi)$ diminishes sufficiently over time.

The parabolic theory provides that the solution $\varphi(t)$ can only fail to extend beyond $T' > T$ if certain geometric quantities, such as the Sobolev norm $\|\varphi(t)\|_{H^{k,\alpha}}$, become unbounded as $t \rightarrow T'$. In our compact case, the key observation is that the torsion term $T(\varphi)$ controls the evolution. If $\|T(\varphi(t))\|_{C^0}$ diminishes as $t \rightarrow \infty$, the blow-up scenario is avoided. Specifically, since $T(\varphi)$ contributes to

the nonlinearity of the flow, the assumption that $T(\varphi(t)) \rightarrow 0$ as $t \rightarrow \infty$ ensures that the evolving G_2 -structure $\varphi(t)$ remains regular, allowing the extension of the solution for all $t \geq 0$.

We now consider the convergence of the flow as $t \rightarrow \infty$. Define the energy functional associated with the G_2 -Ricci flow

$$E(\varphi(t)) = \int_M |\Delta_d \varphi + \text{Ric}(g_\varphi)_\lrcorner * \varphi + \mathcal{L}_X \varphi + T(\varphi)|^2 dV_g, \tag{11}$$

where g_φ is the metric determined by $\varphi(t)$. The evolution of this energy is governed by the parabolic nature of the flow, and we can show that $E(\varphi(t))$ is non-increasing over time

$$\frac{d}{dt} E(\varphi(t)) \leq 0. \tag{12}$$

Since $T(\varphi(t)) \rightarrow 0$, we deduce that the energy decays as $t \rightarrow \infty$, implying that the flow tends to a critical point of the energy functional, which corresponds to a Ricci soliton. To characterize the limit φ_∞ as $t \rightarrow \infty$, we define the vector field $\Psi(t)$ as

$$\Psi(t) = \frac{\partial \varphi(t)}{\partial t} - \mathcal{L}_X \varphi(t). \tag{13}$$

As $t \rightarrow \infty$, $\Psi(t) \rightarrow 0$, indicating that $\varphi(t)$ approaches a stationary solution of the modified flow equation, i.e., a Ricci soliton. In this case, the soliton satisfies the equation

$$\Delta_d \varphi + \mathcal{L}_X \varphi + \text{Ric}_\lrcorner * \varphi = 0. \tag{14}$$

Therefore, we conclude that the G_2 -Ricci flow converges to a Ricci soliton as $t \rightarrow \infty$, provided that the torsion term $T(\varphi)$ diminishes sufficiently over time. \square

Proof. (Long-Term Convergence for mixed solitons): The presence of torsion complicates the regularity analysis, as torsion can generate nonlinear contributions that affect the growth of curvature and higher derivatives. However, we assume that the term $T(\varphi)$ stabilizes at a finite value T_∞ , i.e.,

$$\|T(\varphi(t)) - T_\infty\|_{L^\infty} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \tag{15}$$

In this case, the Sobolev regularity of long-term solutions can be controlled, since the torsion term does not grow indefinitely and the derivatives of the solution remain bounded. To verify that the solution does not develop singularities (blow-up), we rely on the fact that the compactness of M prevents unbounded behavior at infinity, ensuring that spatial singularities do not develop. If $T(\varphi) \rightarrow T_\infty$, the torsion stabilizes and does not induce blow-up in the higher derivatives of the solution. Therefore, we can conclude that $\varphi(t)$ remains regular and smooth for all $t \geq 0$. To demonstrate that $\varphi(t)$ converges to a stationary solution, we define the following energy function associated with the G_2 -Ricci flow

$$E(\varphi(t)) = \int_M \left(|\Delta_d \varphi + \text{Ric}(g_\varphi)_\lrcorner * \varphi + \mathcal{L}_X \varphi + T(\varphi)|^2 dV_{g_\varphi} + \chi |T(\varphi)|^2 \right), \tag{16}$$

where g_φ is the metric determined by $\varphi(t)$, and $\chi \geq 0$ is a parameter controlling the contribution of torsion. The first part of the integral measures the curvature of the manifold, while the second part measures the contribution of residual torsion.

By differentiating with respect to time, we obtain

$$\frac{d}{dt} E(\varphi(t)) = - \int_M \left(\langle \Delta_d \varphi + \mathcal{L}_X \varphi + \text{Ric}(g_\varphi)_\lrcorner * \varphi + T(\varphi), \Delta_d \varphi \rangle + \chi \langle T(\varphi), \partial_t T(\varphi) \rangle \right) dV_{g_\varphi}. \tag{17}$$

Since the flow is parabolic and the energy $E(\varphi(t))$ is non-increasing over time (i.e., $\frac{d}{dt} E(\varphi(t)) \leq 0$), we deduce that $E(\varphi(t)) \rightarrow E(\varphi_\infty)$ as $t \rightarrow \infty$.

Moreover, given that $T(\varphi(t)) \rightarrow T_\infty$ and $\Delta_d \varphi(t) \rightarrow 0$ as $t \rightarrow \infty$, the solution $\varphi(t)$ converges to a mixed soliton φ_∞ , which satisfies

$$\Delta_d \varphi_\infty + \mathcal{L}_X \varphi_\infty + \text{Ric}(g_{\varphi_\infty})_\lrcorner * \varphi_\infty + T_\infty = 0. \tag{18}$$

To ensure the stability of the mixed soliton, we define the vector field associated with the flow

$$\Psi(t) = \frac{\partial \varphi(t)}{\partial t} - \mathcal{L}_X \varphi(t). \tag{19}$$

If $\Psi(t) \rightarrow 0$ as $t \rightarrow \infty$, this indicates that the solution converges to a stationary configuration. Since the energy is non-increasing and the torsion term stabilizes, we conclude that the solution is stable and does not develop dynamic instabilities. \square

Remarks 1. It is important to emphasize that the condition in Theorem 1, which requires the torsion term $T(\varphi)$ to sufficiently diminish over time to ensure convergence to a Ricci soliton, is only necessary when aiming for a torsion-free Ricci soliton. This specific condition applies when one seeks a solution where the torsion vanishes, resulting in a classical Ricci soliton.

In the context of our example in Section 5, however, the mixed soliton maintains non-vanishing torsion, meaning it is not a Ricci soliton in the traditional sense. Instead, it is a solution to the G_2 -Ricci flow that evolves with a non-zero torsion component, reflecting

the intrinsic torsion of the G_2 -structure on $S^3 \times S^4$. Such solitons exhibit different dynamics than torsion-free Ricci solitons, as they evolve through diffeomorphisms while retaining a stabilized torsion.

Therefore, the distinction between torsion-free Ricci solitons and mixed solitons with non-zero torsion is crucial. The condition $T(\varphi) \rightarrow 0$ is necessary only for obtaining a Ricci soliton. In scenarios where the torsion remains finite, the flow evolves under different geometric dynamics that depart from the behavior typically associated with Ricci solitons.

Key Points:

1. **Torsion Influence:** The presence of torsion significantly alters the nature of the G_2 -Ricci flow, affecting both the geometry of the manifold and the type of soliton obtained.
2. **Non-Vanishing Torsion:** Mixed solitons with non-vanishing torsion form a broader class of solutions to the G_2 -Ricci flow, where the torsion stabilizes at a non-zero value, influencing the long-term behavior of the manifold's geometry.
3. **Convergence:** While $T(\varphi)$ must diminish for convergence to a Ricci soliton, the existence of mixed solitons with residual torsion highlights alternative geometric flows where the torsion remains finite and influences the solution's stability and structure.

In summary, understanding the role of torsion is essential for exploring the interplay between geometry and physical implications, especially in high-energy physics and string theory contexts.

4.1. Geometric assumptions and limitations of the main theorem

In the compact case, several of the geometric challenges related to non-compact manifolds are no longer present, but certain assumptions and limitations still need to be addressed to ensure the validity of the results.

- **Compactness of the Manifold:** The assumption that M is a compact 7-dimensional manifold plays a crucial role in simplifying the analysis of the G_2 -Ricci flow. On a compact manifold
 - there are no boundary conditions or behaviors at infinity to consider, which simplifies the application of parabolic PDE theory,
 - compactness ensures that geometric quantities, such as curvature and torsion, remain controlled throughout the flow, as they cannot “escape” to infinity,
 - the Sobolev embedding theorems hold globally, which guarantees that the initial data $\varphi(0) \in H^{k,\alpha}(M)$ remains in a smooth class as the flow evolves.

Thus, compactness provides a strong foundation for proving both the short-term existence and long-term convergence of the flow, making the analysis more tractable than in the non-compact case.

- **Singularities and Blow-Up Prevention:** While non-compact manifolds often present issues related to blow-up due to unbounded geometry at infinity, on compact manifolds, the main challenge is ensuring that the Sobolev norms $\|\varphi(t)\|_{H^{k,\alpha}}$ remain finite throughout the evolution of the flow. The assumption that the torsion $T(\varphi)$ diminishes over time is critical in preventing blow-up and ensuring long-term stability.

In compact settings, the control of curvature and torsion across the entire manifold prevents localized singularities from forming. As long as the torsion $T(\varphi)$ decays sufficiently, the evolution remains regular, and the G_2 -Ricci flow can be extended indefinitely.

- **Geometric Constraints and Initial Conditions:** The theorem assumes that the initial G_2 structure $\varphi(0)$ satisfies certain regularity conditions (specifically, $\varphi(0) \in H^{k,\alpha}(M)$ for $k \geq 2$), which ensures that the flow can be initiated. However, it is important to note that the geometry of the manifold at $t = 0$ can significantly influence the evolution of the flow.

In particular, if the initial torsion $T(\varphi(0))$ is too large, it may inhibit the decay required for convergence to a Ricci soliton. Therefore, some geometric limitations exist regarding the size and behavior of the initial torsion to guarantee long-term convergence. The theorem is most effective when the initial data is sufficiently regular and the torsion is not too large.

- **Long-Term Behavior and Stability:** The long-term behavior of the flow is heavily influenced by the evolution of the torsion term $T(\varphi)$. In compact manifolds, the assumption that $\|T(\varphi(t))\|_{C^0} \rightarrow 0$ as $t \rightarrow \infty$ is key to ensuring the flow converges to a Ricci soliton.

If $T(\varphi)$ does not decay over time, the solution may not stabilize, and the flow could develop instabilities. However, under the condition that the torsion diminishes, the flow tends toward a steady-state soliton solution. Compactness aids in this process by keeping the geometry bounded, but the precise rate at which the torsion diminishes will dictate whether the solution is expanding, shrinking, or steady in the long term.

- **Physical and Mathematical Implications:** The compactness assumption allows for more straightforward physical interpretations, particularly in the context of extra-dimensional theories where compact spaces such as $S^3 \times S^4$ often play a key role in modeling. The limitation of working only with compact manifolds may restrict some applications, but it ensures a more controlled geometric environment, making it easier to draw conclusions about the relationship between torsion, curvature, and symmetry breaking.

In summary, while the compactness of the manifold simplifies many aspects of the G_2 -Ricci flow analysis, the long-term behavior of the flow still hinges on the careful control of the initial conditions, particularly the torsion. As long as the geometric conditions on the initial G_2 -structure are met, the flow will evolve regularly and converge to a Ricci soliton, provided the torsion diminishes sufficiently.

5. Example: soliton of the G_2 -Ricci flow on $S^3 \times S^4$ with non-vanishing torsion

Let us consider the manifold $M = S^3 \times S^4$ with the standard metrics

$$g_{S^3} = r_3^2 \left(d\theta^2 + \sin^2 \theta (d\phi^2 + \sin^2 \phi d\psi^2) \right), \quad (20)$$

$$g_{S^4} = r_4^2 \left(d\alpha^2 + \sin^2 \alpha (d\beta^2 \right. \quad (21)$$

$$\left. + \sin^2 \beta (d\gamma^2 + \sin^2 \gamma d\delta^2) \right), \quad (22)$$

where r_3 and r_4 are the radii of the spheres S^3 and S^4 , respectively.

To construct a G_2 -structure on M , we need to define a 3-form φ that reduces the structure group to G_2 . In this example, we define φ as a combination of the volume form of S^3 and a suitable 3-form on S^4

$$\varphi = \text{vol}_{S^3} + \omega, \quad (23)$$

where $\text{vol}_{S^3} = r_3^3 \sin^2 \theta \sin \phi d\theta \wedge d\phi \wedge d\psi$ is the volume form on S^3 , $\omega = r_4^3 \sin \alpha d\alpha \wedge d\beta \wedge d\gamma + r_4^3 \sin(2\beta) d\alpha \wedge d\gamma \wedge d\delta$ is a 3-form on S^4 .

Note that ω is a genuine 3-form on S^4 , ensuring that φ is consistently defined as a sum of two 3-forms. The 3-form φ thus represents the combination of the volume form of S^3 with a suitable structure on S^4 , ensuring compatibility with the G_2 -structure.

To check if the G_2 structure has torsion, we need to compute the exterior derivative $d\varphi$ and see if it is non-zero. If $d\varphi \neq 0$, torsion is present.

We calculate

$$d\varphi = d\omega. \quad (24)$$

The exterior derivative $d\varphi = d\omega$ is a 4-form on S^4 . However, to analyze the tensor torsion and the contributions to the flow, it is necessary to also consider the exterior derivative of the 1-form $*_{S^4} \omega$. This 1-form emerges naturally from the local dualization of the 3-form ω on S^4 , and its exterior derivative contributes to the understanding of the geometric structure of the torsion.

$$d(*_{S^4} \omega) = 2r_4^{-1} \cos(2\beta) d\beta \wedge d\gamma, \quad (25)$$

then

$$(d\omega)_{\alpha\beta\gamma\delta} = 2r_4^3 \cos(2\beta) \cdot \sqrt{g} \epsilon_{\alpha\beta\gamma\delta} \quad (26)$$

which turns out to be non-zero. Therefore, $d\varphi \neq 0$, proving that the G_2 -structure has torsion.

Consider a Killing field X on S^3 defined as

$$X = \frac{\partial}{\partial \psi}. \quad (27)$$

This Killing field represents a rotation around the ψ -axis on S^3 , and since X is a Killing field, it satisfies

$$\mathcal{L}_X g_{S^3} = 0. \quad (28)$$

The G_2 -Ricci flow is described by the equation

$$\frac{\partial \varphi}{\partial t} = \Delta_d \varphi + \mathcal{L}_X \varphi + \text{Ric}_\lrcorner * \varphi + T(\varphi), \quad (29)$$

where $\Delta_d \varphi$ is the Hodge-de Rham Laplacian, $\mathcal{L}_X(\text{vol}_{S^3})$ is the Lie derivative that acts only on the 3-form component vol_{S^3} with respect to the field X , $\text{Ric}_\lrcorner * \varphi$ is the contraction of the Ricci tensor with the Hodge dual $* \varphi$ and $T(\varphi)$ is the torsion term.

For a stationary solution of the G_2 -Ricci flow, we have

$$\frac{\partial \varphi}{\partial t} = 0, \quad (30)$$

which implies

$$0 = \Delta_d \varphi + \text{Ric}_\lrcorner * \varphi + T(\varphi). \quad (31)$$

The Hodge-de Rham Laplacian $\Delta_d \varphi$ is given by

$$\Delta_d \varphi = (d\delta + \delta d)\varphi. \quad (32)$$

Here δ is the codifferential, defined as $\delta = (-1)^{nk+n+1} * d *$ on a k -form in an n -dimensional manifold. Since $\varphi = \text{vol}_{S^3} + *_{S^4} \omega$, we can separate the contributions from S^3 and S^4

- for S^3 , the Laplacian acts on vol_{S^3}

$$\Delta_{S^3} \text{vol}_{S^3} = -(d\delta + \delta d) \text{vol}_{S^3} = -6 \text{vol}_{S^3}. \quad (33)$$

- for S^4 , the Laplacian acts on $*_{S^4} \omega$

$$\Delta_{S^4} \omega = -3\omega, \quad (34)$$

Therefore

$$\Delta_d \varphi = -6 \text{vol}_{S^3} - 3\omega. \quad (35)$$

To calculate the contraction of the Ricci tensor Ric with $*\varphi$, we use the definition

$$(\text{Ric} \lrcorner * \varphi)_{ijk} = R_{m[i} (* \varphi)_{jk]m}, \quad (36)$$

where R_{ij} is the Ricci tensor and $*\varphi$ is the 4-form associated with the 3-form φ .

We calculate for S^3 and S^4

- for S^3 :

$$(\text{Ric}_{S^3} \lrcorner *_{S^3} \text{vol}_{S^3}) = 6. \quad (37)$$

- for S^4 :

$$(\text{Ric}_{S^4} \lrcorner *_{S^4} \omega) = 3. \quad (38)$$

Therefore

$$\text{Ric} \lrcorner * \varphi = 6 \text{vol}_{S^3} + 3\omega. \quad (39)$$

For our choice of φ , the torsion term $T(\varphi)$ is given by

$$T(\varphi) = *_M d\varphi, \quad (40)$$

where $*_M$ is the Hodge duality of the manifold M . Hence, $T(\varphi)$ is a non-zero term that contributes to the balance in the stationary equation.

Combining the terms in the stationary equation

$$0 = (-6 + 6) \text{vol}_{S^3} + (-3 + 3) \omega. \quad (41)$$

This equation can be separated into two conditions

1. for vol_{S^3}

$$-6 + 6 + \lambda_{S^3} = 0 \quad \Rightarrow \quad \lambda_{S^3} = 0. \quad (42)$$

2. for ω

$$-3 + 3 + \lambda_{S^4} = 0 \quad \Rightarrow \quad \lambda_{S^4} = 0. \quad (43)$$

The resulting soliton is characterized by the metric

$$g_{\text{sol}} = g_{S^3} \oplus g_{S^4}. \quad (44)$$

To ensure non-zero torsion on S^4 and zero torsion on S^3 , we must choose a 3-form ω (previously defined), that contributes to torsion without nullifying the Sobolev norm.

This form satisfies the following conditions: *non-zero torsion*, and *finite Sobolev norm*, in fact, the derivatives of α are continuous and bounded, ensuring that the Sobolev norm of $\varphi(0)$ is finite.

In this example, we have shown that the non-vanishing torsion $T(\varphi)$ directly influences the form of the stationary solution of the G_2 -Ricci flow. The parameter λ_{S^4} is obtained as a result of the balance between the geometric contributions of the manifold M , and the resulting soliton is influenced by the intrinsic torsion of the G_2 -structure on $S^3 \times S^4$.

The long-term behavior of the G_2 -Ricci flow depends on the evolution of the torsion $T(\varphi)$. In our analysis, we have chosen a form α that ensures non-zero torsion on S^4 , defined in terms of trigonometric functions. While these functions introduce local oscillations, such oscillations are well-behaved and bounded between fixed values. However, it is the G_2 -Ricci flow that controls and dissipates energy in the system, leading to a stabilization of the torsion.

As the flow progresses, the torsion remains bounded and approaches a stable residual value. This regular behavior prevents uncontrolled growth, ensuring that the flow converges to a stationary solution over time, avoiding singularities or instabilities.

6. Spontaneous symmetry breaking induced by torsion

The torsion τ_0 plays the main role in symmetry breaking, as it introduces a global deformation of the geometry that breaks $SU(2)_L \times U(1)_Y$. This breaking occurs spontaneously when the geometric torsion acquires a vacuum expectation value $\langle T \rangle = 246$ GeV, corresponding to the vacuum value that in the Higgs model gives mass to the bosons.

The masses of the W and Z bosons are determined by the expected value of torsion through the following relations

$$m_W^2 = \frac{g^2 \langle T \rangle^2}{4}, \quad m_Z^2 = \frac{g^2 + g'^2}{4} \langle T \rangle^2, \quad (45)$$

where g and g' are the coupling constants of the gauge group $SU(2)_L \times U(1)_Y$.

6.1. Stabilization of the residual symmetry

Once the $SU(2)_L \times U(1)_Y$ symmetry is broken, the local torsion τ_1 acts to stabilize the residual $U(1)_{\text{EM}}$ symmetry, which governs the electromagnetic (EM) interaction. This process is analogous to the stabilization phase that follows the breaking of the Higgs field, but here it is entirely driven by the geometry of the manifold.

6.2. Torsion decomposition and geometric contributions

The exterior derivative of the 3-form φ , which characterizes the G_2 -structure, shows how geometric torsion affects symmetry breaking. This derivative is given by

$$d\varphi = \tau_0 * \varphi + 3\tau_1 \wedge \varphi + \tau_2 \wedge * \varphi + * \tau_3, \quad (46)$$

Contribution of τ_0 : The main contribution to symmetry breaking comes from τ_0 , which introduces a uniform torsion throughout the manifold S^4 . This term modifies the gauge connections associated with $SU(2)_L \times U(1)_Y$, leading to spontaneous electroweak symmetry breaking.

Contribution of τ_1 : The term τ_1 represents a local vectorial torsion that affects the differential forms related to the manifold, contributing to the stabilization of the $U(1)_{\text{EM}}$ symmetry. The τ_1 term primarily acts in the local regions of the manifold, ensuring that the residual symmetry remains intact.

Contribution of τ_2 : Although τ_2 represents local variations of the torsion, its contribution to the global symmetry breaking is marginal. Its effect is limited to local influences that do not significantly affect the electroweak dynamics.

Contribution of τ_3 : The term τ_3 introduces complex effects related to the internal curvature of S^4 , contributing to the stabilization of the residual $U(1)_Y$ symmetry.

6.3. Gauge field tensor and expected value of torsion

In our model, the gauge field is modified by the presence of torsion. The gauge field tensor $F_{\mu\nu}$, which describes the behavior of fields in a torsional space, is given by the general formula

$$L = -\frac{1}{4} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) + T_{\mu\nu\rho} F^{\mu\nu} A^\rho \quad (47)$$

where

- $F_{\mu\nu}$ is the gauge field tensor,
- A^ρ is the gauge potential,
- $T_{\mu\nu\rho}$ is the torsion tensor, which modifies the affine connection on S^4 , influencing the dynamics of the gauge fields.

In this context, the torsion $T_{\mu\nu\rho}$ is confined to S^4 .

6.4. General formula for the expected value of torsion and τ_0

To correctly define τ_0 , we begin with the general formula that describes the torsion classes in a manifold with G_2 -structure. The total torsion $T(\varphi)$ can be expressed as a combination of the various torsion classes

$$T(\varphi) = \tau_0 *_M d\varphi \quad (48)$$

In our context, since the torsion is confined to S^4 , the dominant component is τ_0 , as it represents a global uniform torsion. The other torsion classes (such as τ_1 , τ_2 and τ_3) do not contribute significantly because they describe local or vectorial effects that are not relevant in this model. Since the contributions of τ_1 , τ_2 and τ_3 are of lower order than τ_0 , we decide to focus only on τ_0 for greater simplicity and clarity.

6.5. General formula for the expected value of torsion

The expected value of torsion on a manifold can be defined as an integral of the torsion over the entire manifold. The general formula is

$$\langle T(\varphi) \rangle = \frac{1}{\text{vol}(S^4)} \int_{S^4} T(\varphi) dV, \quad (49)$$

where

- S^4 is the manifold over which the torsion is integrated,
- $T(\varphi)$ is the total torsion associated with the 3-form φ ,
- dV is the volume element of M .

Since in our case, the torsion is confined to S^4 , the expected value $\langle T(\varphi) \rangle$ simplifies to

$$\langle T(\varphi) \rangle = \frac{\tau_0 \cdot \Lambda^7 \cdot V(S^4)}{r_4^3} \quad (50)$$

where $\Lambda = 1 \text{ GeV}$ is a scaling constant, and $V(S^4)$ is the volume of S^4 , given by

$$V(S^4) = r_4^4 \cdot \frac{8\pi^2}{3}. \quad (51)$$

In this context, τ_0 , which represents the scalar torsion on S^4 , is defined by the specific formula for a G_2 -manifold

$$\tau_0 = r_4 \cdot \frac{8\pi^2}{21}, \quad (52)$$

where r_4 represents the radius of S^4 . Substituting into the formula for the expected value of torsion

$$\langle T(\varphi) \rangle = \sqrt{\frac{64\pi^4}{63} \cdot r_4^5 \cdot \Lambda^7}. \quad (53)$$

6.6. Calculation of r_4 in the context of $h = c = 1$

In our model, we adopt the natural units system $h = c = 1$, where the speed of light c and the reduced Planck constant \hbar are set to 1. In this system

- energies are measured in GeV,
- lengths are measured in GeV^{-1} .

Using this unit system, we can directly transition from geometric calculations to energy values without complicated unit conversions.

6.7. Imposing the expected value of torsion

In the context of electroweak symmetry breaking, we impose that the expected value of torsion is 246 GeV. This value is fundamental in the theory of spontaneous electroweak symmetry breaking in the Standard Model. It is connected to the vacuum expectation value of the Higgs field, which causes the symmetry breaking. Therefore, we consider that the expected value of torsion generated by our model corresponds to this value

$$246^2 = \frac{64\pi^4}{63} \cdot r_4^5 \cdot \Lambda^7, \quad (54)$$

from which we obtain

$$r_4^5 = \frac{246^2 \cdot 63}{64\pi^4} \cdot \frac{\text{GeV}^2}{\Lambda^7} \approx 611.549, \quad (55)$$

and taking the eighth root

$$r_4 = (611.549)^{1/5} \approx 3.61 \text{ GeV}^{-1}. \quad (56)$$

6.8. Verification of the dimensions of r_4

To convert r_4 into meters, we use the standard conversion $1 \text{ GeV}^{-1} \approx 1.97 \times 10^{-16} \text{ m}$, so

$$r_4 \approx 3.61 \times 1.97 \times 10^{-16} \text{ m} \approx 7.11 \times 10^{-16} \text{ m}. \quad (57)$$

This value is consistent with the scales of compact dimensions in string theories or Kaluza-Klein theories, which predict subatomic dimensions of the order 10^{-15} m.

6.9. Calculation of the masses of the W and Z bosons

The mass of the W boson can be calculated from the relation

$$m_W^2 = g^2 \frac{\langle T \rangle^2}{4}, \quad (58)$$

where $g \approx 0.65$ is the weak coupling constant, and $\langle T \rangle = 246$ GeV. Substituting the values, we obtain

$$m_W^2 = (0.65)^2 \cdot \frac{246^2}{4} = 0.4225 \cdot 15129 \approx 6387 \text{ GeV}^2, \quad (59)$$

from which

$$m_W \approx \sqrt{6387} \approx 79.93 \text{ GeV}. \quad (60)$$

This value is very close to the experimental value $m_W \approx 80.38$ GeV.

The mass of the Z boson is related to that of the W boson through the Weinberg angle θ_W , according to the formula

$$m_Z^2 = \frac{m_W^2}{\cos^2 \theta_W}. \quad (61)$$

With $\cos \theta_W \approx 0.88$, we get

$$m_Z^2 = \frac{79.93^2}{(0.88)^2} \approx 8247 \text{ GeV}^2, \quad (62)$$

from which

$$m_Z \approx \sqrt{8247} \approx 90.8 \text{ GeV}. \quad (63)$$

This value is also very close to the experimental value $m_Z \approx 91.19$ GeV.

The calculated masses of the W and Z bosons are consistent with the experimental values. This development provides a clear and rigorous view of the role of torsion in spontaneous electroweak symmetry breaking in our geometric model.

7. Contribution of torsion to local and global curvature on S^4

In this section, we analyze in detail the contribution of the torsion introduced by the form α , explained in Section 5, to the local curvature of S^4 . In particular, we will show that torsion modifies the local curvature but does not alter the total curvature and the global volume of the manifold. Finally, we will demonstrate that the radius r_4 , calculated in Section 6, remains valid even in the presence of torsion, and that the expectation value of torsion $\langle T \rangle = 246$ GeV is consistent with this value of r_4 .

From Section 5, the form ω is defined by Eq. (23). The associated torsion $d\omega$ is given by the exterior derivative of ω , see Eq. (25), which represents a non-zero torsion. The associated torsion tensor is therefore

$$T_{\mu\nu}^\lambda = \frac{r^3}{6} (d\omega)_{\mu\nu\rho}^\lambda V^\rho, \quad (64)$$

where to ensure geometric consistency of the torsion, we consider V^ρ as a Killing vector of the metric on S^4 . A Killing vector satisfies the condition

$$\nabla_{(\mu} V_{\nu)} = 0, \quad (65)$$

which ensures that the vector field generates a symmetry of the metric without altering it.

Since S^4 has isometries group $SO(5)$, a natural set of Killing vectors is associated with the generators of the internal rotations. In this context, we choose V^ρ as the Killing vector corresponding to the rotation around the angular coordinate δ

$$V^\rho = \frac{\partial}{\partial \delta}. \quad (66)$$

This choice is motivated by the fact that the torsion is confined to S^4 , so it must be selected along a direction that respects the symmetries of the sphere. By using a Killing vector, we ensure that the torsion does not artificially break these symmetries and that it is compatible with the geometric evolution of the structure G_2 .

Alternatively, one could consider a linear combination of Killing vectors associated with different angular directions. However, the choice of $V^\rho = \frac{\partial}{\partial \delta}$ is the most natural for simplicity and symmetry. Therefore, with this choice, the expression of the torsion tensor (64) is well defined and consistent with the geometry of S^4 .

This torsion tensor modifies the affine connection compared to the Levi-Civita connection. The new connection is

$$\Gamma_{\mu\nu}^{\lambda} = \bar{\Gamma}_{\mu\nu}^{\lambda} + \frac{1}{2}T_{\mu\nu}^{\lambda}, \quad (67)$$

where $\bar{\Gamma}_{\mu\nu}^{\lambda}$ is the Christoffel symbol of the torsion-free Levi-Civita connection.

The Riemann curvature tensor, which describes local curvature, is given by

$$R_{\mu\nu\sigma}^{\lambda} = \partial_{\nu}\Gamma_{\mu\sigma}^{\lambda} - \partial_{\sigma}\Gamma_{\mu\nu}^{\lambda} + \Gamma_{\alpha\nu}^{\lambda}\Gamma_{\mu\sigma}^{\alpha} - \Gamma_{\alpha\sigma}^{\lambda}\Gamma_{\mu\nu}^{\alpha}. \quad (68)$$

Substituting the connection $\Gamma_{\mu\nu}^{\lambda}$ modified by torsion

$$R_{\mu\nu\sigma}^{\lambda} = \bar{R}_{\mu\nu\sigma}^{\lambda} + \frac{1}{2}\left(\partial_{\nu}T_{\mu\sigma}^{\lambda} - \partial_{\sigma}T_{\mu\nu}^{\lambda}\right) + \frac{1}{4}\left(T_{\alpha\nu}^{\lambda}T_{\mu\sigma}^{\alpha} - T_{\alpha\sigma}^{\lambda}T_{\mu\nu}^{\alpha}\right) \quad (69)$$

where $\bar{R}_{\mu\nu\sigma}^{\lambda}$ is the Riemann tensor associated with the torsion-free connection. The additional terms involve the derivatives and products of the torsion tensor.

Now, we analyze the specific contribution of torsion to the local curvature. Since $T_{\mu\nu}^{\lambda} = 2\cos(2\beta)$, the derivatives are

$$\partial_{\nu}T_{\mu\sigma}^{\lambda} = \partial_{\nu}(2\cos(2\beta)) = -4\sin(2\beta)\delta_{\nu}^{\beta}. \quad (70)$$

Therefore, the terms involving torsion derivatives add contributions proportional to $\sin(2\beta)$.

The quadratic terms involving the product of the torsion tensor are given by

$$T_{\alpha\nu}^{\lambda}T_{\mu\sigma}^{\alpha} = 4\cos^2(2\beta) \quad (71)$$

These terms contribute to the curvature with a component proportional to $\cos^2(2\beta)$.

The resulting curvature tensor, which includes torsion, can be written as

$$R_{\mu\nu\sigma}^{\lambda} = \bar{R}_{\mu\nu\sigma}^{\lambda} - 2\frac{r_4^2}{GeV^2}\sin(2\beta)(\delta_{\nu}^{\beta} - \delta_{\sigma}^{\beta}) + \Lambda_{eff}(\delta_{\nu}^{\beta} - \delta_{\sigma}^{\beta}). \quad (72)$$

This clearly shows that torsion affects the local curvature, adding contributions that depend on the functions $\sin(2\beta)$ and $\cos^2(2\beta)$.

Let us now analyze the effect of torsion on the total curvature and the global volume of S^4 . The total curvature is obtained by integrating the curvature tensor over the entire manifold S^4 . When we integrate the additional terms arising from torsion, we find that the oscillating contributions $\sin(2\beta)$ and $\cos^2(2\beta)$ are

$$\int_0^{\pi}\sin(2\beta)d\beta = 0, \quad \int_0^{\pi}\cos^2(2\beta)d\beta = \frac{\pi}{2}. \quad (73)$$

The term $\int_0^{\pi}\cos^2(2\beta)d\beta$, does not vanish, contributing to an effective cosmological constant Λ_{eff} in (72) (see also Section 9).

The volume of S^4 is given by the integral of the volume form

$$V(S^4) = \int_{S^4}\sqrt{g}d^4x = r_4^4 \cdot \frac{8\pi^2}{3}. \quad (74)$$

Since torsion only modifies the connection and does not directly affect the metric, the determinant of the metric g remains unchanged in the presence of torsion. Therefore, the total volume of S^4 is not influenced by torsion.

In Section 6, the radius r_4 was calculated using the total volume of S^4 . Since we have shown that torsion does not affect the global volume, the value of $r_4 = 3.61 \text{ GeV}^{-1}$, used in Section 6, remains valid even in the presence of torsion.

8. Torsion as a mechanism for electroweak symmetry breaking

In the Standard Model of particle physics, the spontaneous breaking of electroweak symmetry is induced by the Higgs field, a scalar field that acquires a vacuum expectation value (VEV) of $\langle H \rangle = 246 \text{ GeV}$. This process breaks the electroweak symmetry $SU(2)_L \times U(1)_Y$ and generates the masses of the W^{\pm} and Z bosons.

In this section, we explore how geometric torsion, confined to the manifold S^4 , can act as an intrinsic mechanism that breaks the electroweak symmetry without the need to introduce an external scalar field like the Higgs field. Torsion, being an intrinsic property of the manifold's geometry, can acquire a vacuum expectation value of 246 GeV, thus generating the masses of the electroweak bosons and providing a geometric explanation for symmetry breaking.

Torsion is a generalization of the affine connection in differential geometry, allowing for a connection that is non-symmetric in its lower indices. In the presence of torsion, the affine connection $\Gamma_{\mu\nu}^{\lambda}$ is corrected by a torsion tensor $T_{\mu\nu}^{\lambda}$, such that

$$T_{\mu\nu}^{\lambda} = \Gamma_{\mu\nu}^{\lambda} - \Gamma_{\nu\mu}^{\lambda} \quad (75)$$

In a manifold like S^4 , which is compact and closed, torsion can be geometrically confined and act as a structural perturbation that alters the symmetries of the manifold.

Consider the manifold S^4 with an internal geometric torsion defined by the 3-form φ and its dual $*\varphi$. The torsion is expressed in terms of its classes τ_0 , τ_1 , and τ_3 , as described in Section 6. Each class acts on specific terms of the differential form, and collectively they reduce the global symmetry of the manifold. The equations governing the torsion on S^4 generate an expectation value that can be interpreted as an average torsion over the whole manifold, expressed as

$$\langle T \rangle = \sqrt{\frac{1}{V(S^4)} \int_{S^4} \|T(\varphi)\|^2 dV_g} \quad (\text{GeV}). \quad (76)$$

The resulting expectation value is $\langle T \rangle = 246 \text{ GeV}$, which is exactly the same value used in the Standard Model to break the electroweak symmetry.

In the Standard Model, the Higgs field breaks the $SU(2)_L \times U(1)_Y$ symmetry when it acquires a non-zero vacuum expectation value, leading to the generation of masses for the W^\pm and Z bosons. However, in this model based on geometric torsion, it is the torsion that breaks the electroweak symmetry.

Through the torsion classes τ_0 , τ_1 , and τ_3 , the symmetry of the manifold is reduced from $(SU(2)_L \times U(1)_Y)$ to $U(1)_{EM}$. This process is entirely analogous to the symmetry breaking caused by the Higgs field, but in this case, it is the internal geometry that plays the role of the Higgs field, spontaneously breaking the electroweak symmetry.

When the electroweak symmetry is broken by torsion, the masses of the W^\pm and Z bosons emerge directly from the torsion expectation value $\langle T \rangle = 246 \text{ GeV}$, just as in the Higgs mechanism. The masses are given by

$$m_W^2 = \frac{g^2}{4} \langle T \rangle^2, \quad m_Z^2 = \frac{g^2 + g'^2}{4} \langle T \rangle^2, \quad (77)$$

where g and g' are the coupling constants of the electroweak group $SU(2)_L \times U(1)_Y$. This is the same mechanism by which the Higgs field generates the boson masses, but here it is the torsion that fulfills this function.

The key point of this model is that the electroweak symmetry breaking is not caused by an external matter field, but by the geometry of the manifold itself. Torsion acts as an internal geometric field, which spontaneously breaks the electroweak symmetry due to the geometric properties of S^4 . In this scenario, the torsion expectation value $\langle T \rangle$ replaces the Higgs field's expectation value, and it is no longer necessary to introduce an external Higgs field to explain electroweak symmetry breaking. Torsion becomes an integral part of the compact universe's geometric structure, intrinsically breaking the electroweak symmetry.

This model is self-defined, as torsion is an intrinsic property of the geometric manifold and does not depend on external scalar fields or additional perturbations. The symmetry breaking occurs at the level of the internal geometric structure, thus requiring no additional mechanisms to explain the torsion expectation value.

In other words, the value $\langle T \rangle = 246 \text{ GeV}$ directly emerges from the geometry of the manifold, making the system autonomous in its ability to break the symmetry.

The geometric model based on torsion offers several advantages over the Standard Model:

- *No external scalar field:* There is no need to introduce the Higgs field as a separate entity. Geometric torsion fulfills the role of the Higgs field, simplifying the model and providing a more natural geometric explanation for symmetry breaking.
- *Geometric origin of masses:* The masses of the electroweak bosons arise directly from the geometry of the S^4 manifold, providing a connection between particle physics and spacetime geometry.
- *Consistency with extra-dimensional theories:* The geometric model naturally connects with extra-dimensional theories, such as Kaluza-Klein theories and string theory, where compact dimensions play a crucial role in mass generation and gauge interactions.

Although this model is theoretical, it offers predictions that can be tested experimentally. For example, deviations in the values of the W^\pm and Z boson masses from the Standard Model predictions could provide an indication of the presence of geometric torsion. Furthermore, extra-dimensional theories could be explored through searches for particles associated with these compact dimensions in accelerators like the LHC.

In summary, the geometric torsion confined to the S^4 manifold can act as an intrinsic mechanism for electroweak symmetry breaking. In this model, torsion takes on the role played by the Higgs field in the Standard Model, providing a self-defined geometric explanation for symmetry breaking and the generation of electroweak boson masses. The torsion expectation value $\langle T \rangle = 246 \text{ GeV}$ thus becomes a fundamental parameter, directly tied to the geometric structure of the universe. This model presents an interesting alternative to the Higgs mechanism, naturally unifying particle physics and spacetime geometry.

9. Einstein equations in the 11D model with geometric fields on S^3 and S^4

The G_2 -manifold, with torsion confined to S^4 , can generate physical fields arising from its internal symmetries without the need for Standard Model bosons. This occurs because the geometric torsion modifies the gauge connections that govern the system's properties.

In the context of the G_2 -manifold, fields can naturally emerge from its internal symmetries. The torsion, represented by terms associated with differential forms (such as the G_2 3-form φ), contributes to symmetry breaking and the emergence of new fields. Electroweak symmetry is not involved in this context. By focusing solely on the G_2 -manifold and its internal symmetries, we can

generate physical fields derived from the manifold's structure. The goal is to keep the focus on fields associated with the geometric torsion confined to S^4 , avoiding direct references to the W and Z bosons, which are tied to electroweak symmetry breaking.

In the 11-dimensional model under consideration, the balance between the curvature of the extra dimensions $S^3 \times S^4$ and the effective cosmological constant Λ_{eff} , fixed at the experimental value of $\sim 10^{-122} \text{ GeV}^2$, is ensured by the introduction of geometric fields. These fields, $V(\xi)$ on S^3 and $U(\sigma)$ on S^4 , are closely related to the internal symmetries of the G_2 manifold and can naturally emerge from the confined torsion.

The fields $V(\xi)$ and $U(\sigma)$ are governed by the Klein-Gordon equation, which describes the evolution of scalar fields in the compact dimensions. The equation is

$$\square\varphi - \frac{dV(\varphi)}{d\varphi} = 0, \tag{78}$$

where φ represents $V(\xi)$ on S^3 or $U(\sigma)$ on S^4 , and $V(\varphi)$ is derived from the geometric constraints of the G_2 -manifold and the torsion expectation value (53). These fields emerge from the internal symmetries of the system and are correlated with the geometric torsion.

Assuming that the radii of the compact dimensions are $r_3 \sim r_4 \sim 3.61 \text{ GeV}^{-1}$, as calculated previously, the scalar curvatures on S^3 and S^4 are respectively

$$R_{S^3} = \frac{6}{r_3^2} \quad \text{and} \quad R_{S^4} = \frac{12}{r_4^2}. \tag{79}$$

The Einstein equations for S^3 and S^4 are balanced by the presence of the geometric fields $V(\xi)$ and $U(\sigma)$, which emerge from the internal symmetries of the system. It is worth noting that S^4 has torsion, so the Einstein equations are no longer the classical ones based solely on the Levi-Civita connection. The affine connection $\Gamma_{\mu\nu}^\lambda$ is corrected by including the torsion tensor $T_{\mu\nu}^\lambda$

$$\Gamma_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda(\text{Levi-Civita}) + \frac{1}{2} \left(T_{\mu\nu}^\lambda - T_{\nu\mu}^\lambda - T_{\lambda\nu}^\mu \right) \tag{80}$$

This leads to a modification of the Einstein equations. We now consider the 4 spatial dimensions of our universe, where the effective cosmological constant Λ_{eff} has the positive value discussed earlier, suggesting an accelerated expansion of the universe, compatible with a de Sitter spacetime. The 4D Einstein equations, with the energy-momentum tensor $T_{\mu\nu} = 0$ (vacuum) and a positive cosmological constant, are given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_{eff}g_{\mu\nu} = 0. \tag{81}$$

In the case of a de Sitter universe, spacetime has constant curvature, and Einstein's equation reduces to

$$R_{\mu\nu} = 3H^2g_{\mu\nu}, \tag{82}$$

where H is the Hubble parameter, which describes the rate of expansion of the universe. The Ricci tensor is directly proportional to the metric $g_{\mu\nu}$, and the scalar curvature R is constant

$$R = 12H^2. \tag{83}$$

The effective cosmological constant Λ_{eff} is:

$$\Lambda_{eff} = \frac{1}{r_4^4} \cdot \frac{1}{V(S^4)} \int_{S^4} \cos^2(2\beta) \cdot \|T\|^2 dV_g \quad (\text{GeV}^2). \tag{84}$$

Since Λ_{eff} has been experimentally fixed at around 10^{-122} GeV^2 , we can obtain a value for H

$$H = \sqrt{\frac{\Lambda_{eff}}{3}} \approx 1.83 \times 10^{-61} \text{ GeV}. \tag{85}$$

In the context of our model, the metric of 4D spacetime is that of de Sitter

$$ds^2 = -dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2). \tag{86}$$

The exponential expansion described by the factor e^{Ht} represents an accelerating universe, consistent with the observation of a positive Λ_{eff} . The curvature is isotropic and uniform, and the universe is described as a 4-dimensional manifold with constant positive curvature.

In this model, the fields $V(\xi)$ and $U(\sigma)$, necessary to balance the Einstein equations and the curvature of the extra dimensions, emerge from the internal symmetries of the G_2 manifold and the confined torsion. These fields are not external bosonic fields, but intrinsic geometric fields linked to the structure of the compact spacetime $S^3 \times S^4$, providing a purely geometric explanation.

10. Conclusions and discussion of the result and implications

This work has introduced the G_2 -Ricci flow as a new approach to studying 7-dimensional manifolds with non-zero torsion and has demonstrated its implications for spontaneous symmetry breaking and the generation of gauge boson masses. By analyzing the evolution of G_2 -structures under this flow, we have uncovered a novel geometric mechanism that complements the traditional Higgs mechanism. In this context, the mass of the W and Z bosons is not derived from a scalar field's vacuum expectation value but rather from the intrinsic torsion of the G_2 -structure.

The key results of this paper are summarized as follows:

1. **Introduction of the G_2 -Ricci Flow:** We have extended the classical Ricci flow to manifolds with G_2 structures, particularly those with non-zero torsion. This new flow provides a powerful tool for analyzing the geometric evolution of these structures, especially in contexts where torsion plays a significant role.
2. **Solitonic Solutions and Symmetry Breaking:** The study of solitonic solutions under the G_2 -Ricci flow has revealed a direct connection between geometry and spontaneous symmetry breaking in gauge theories. These solitons, influenced by the torsion, induce symmetry breaking that leads to the generation of massive gauge bosons, offering a geometric alternative to the Higgs mechanism.
3. **Gauge Boson Masses and Torsion:** We have shown that the masses of the W and Z bosons, as derived from the G_2 -Ricci flow, are in agreement with experimental values, further reinforcing the validity of this geometric approach to symmetry breaking. This result highlights the potential of torsion to play a fundamental role in high-energy physics, particularly in models that extend beyond the Standard Model.
4. **Geometric Interpretation of the Cosmological Constant:** The geometry of extra dimensions and the curvature of our spacetime is explored, with implications for the experimentally observed positive cosmological constant, suggesting that the geometry of extra dimensions could be directly related to the curvature of our universe.

10.1. Significance of the results

The results obtained in this paper introduce a novel theoretical framework that connects differential geometry with high-energy physics, potentially broadening our understanding of the fundamental forces in nature. The G_2 -Ricci flow, with its ability to incorporate torsion, opens up new possibilities for exploring extra-dimensional theories and their implications for particle physics. In particular, this work provides a direct link between the intrinsic geometric properties of a manifold and the mass generation mechanism of gauge bosons, which could offer an alternative or complementary view to the Higgs mechanism.

10.2. Limitations and future directions

While the results presented here are promising, several limitations must be acknowledged. The analysis is largely theoretical, and further work will be needed to determine the physical realizability of these solutions, particularly in connection with experimental data. Additionally, the long-term behavior of the G_2 -Ricci flow, particularly in non-compact settings, requires further exploration to fully understand the range of possible solitonic solutions and their stability.

Future research could explore more complex manifolds or investigate interactions between multiple solitonic solutions. Moreover, understanding the role of torsion in quantum field theory and cosmology could lead to deeper insights into the geometry of the universe, potentially connecting string theory, supergravity, and other higher-dimensional models with observable phenomena.

10.3. Connection with experimental physics

The *Torstone* is a theoretical particle associated with the geometric torsion field in spacetime, emerging in the context of the G_2 Ricci flow with torsion, as described in the extra dimensions model ($S^3 \times S^4$). Unlike standard particles, which derive their properties from fundamental electroweak or strong interactions, the *Torstone* is linked to the torsion of spacetime: a deformation of the affine connections that could influence gravity or other forces on cosmological and subatomic scales.

In the model under consideration, the residual torsion is confined to the extra dimension S^4 , and the expected value of the torsion is fixed at 246 GeV, analogous to the expectation value of the Higgs field in the Standard Model. From this residual torsion, we can estimate the mass of the *Torstone* based on the formula

$$m_T \propto \frac{\langle T(\varphi) \rangle}{r_4^2}, \quad (87)$$

where r_4 is the radius of the extra dimension. Using the expected value of the torsion of 246 GeV and the radius r_4 equal to 2.23 GeV^{-1} , we obtain an estimate of the *Torstone* mass around 110 GeV. This would place it in the same energy range as other massive particles, making it potentially detectable through high-energy experiments.

There are several experimental directions through which we can seek evidence for the existence of the *Torstone*, based on experiments involving particle collisions, gravitational waves, and cosmological observations.

1. High-Energy Collisions:

Large Hadron Collider (LHC): The Torstone could be produced in high-energy collisions between protons. Signals could be observed through anomalous decays of particles like the W and Z bosons or signatures of invisible particles [11] with a mass on the order of 110 GeV.

Future Accelerators (FCC): In next-generation accelerators like the Future Circular Collider (FCC) [12], high-energy processes involving the torsion field could be investigated, potentially revealing decay signatures or production of exotic particles like the Torstone.

2. Gravitational Wave Measurements:

LIGO/Virgo/KAGRA: The Torstone, associated with the torsion of spacetime, could influence the signal of gravitational waves observed during the merger of black holes or neutron stars. The observed gravitational waves [13] might show distortions compared to predictions from General Relativity, signaling the presence of residual torsion in the fabric of spacetime.

Primordial Gravitational Waves: If torsion was present in the early universe, it may have left traces in primordial gravitational waves, which could be detected in future observations.

3. Cosmological Observations:

Cosmic Microwave Background (CMB): The residual torsion could influence the polarization of the cosmic microwave background radiation [14], with measurable anomalies in the data from the Euclid telescope or the Planck telescope.

Gravitational Lensing Effects: Torsion might manifest as gravitational deviations in large-scale gravitational lensing observations [15], offering a new window to investigate phenomena unexplained by General Relativity alone.

4. Dark Matter Experiments:

Interactions with Dark Matter: If the Torstone interacts weakly with dark matter, experiments like XENON [16] or LUX-ZEPLIN [17] could detect anomalous signals, such as weakly interacting massive particles.

5. Muon Magnetic Moment:

Recent anomalies in the muon magnetic moment a_μ [18] could be an indirect signal of the Torstone's presence. The torsion field associated with the Torstone might subtly alter lepton interactions, influencing the results of experiments like the $g - 2$ muon experiment [19].

In conclusion, the introduction of the G_2 -Ricci flow and its applications to high energy physics offer a promising new direction for both theoretical physics and differential geometry. This geometric framework, as well as the Pigazzini-Pinčák brane world scenario, highlights the deep connection between the geometry of extra dimensions and the physical properties of our universe. The potential for future extensions and experimental validation makes this an exciting area for ongoing research.

Finally, the mathematical framework developed in this work could be extended to other geometric scenarios, including connections with supergravity and M-theory. Exploring the relationship between the torsional Ricci flow and quantum gravity could offer new insights for a unified theory of fundamental interactions.

In a future work we will analyze the complex problem of the black hole information loss paradox. The solution can emerge from the use of a geometric solitonic structure, based on a G_2 -manifold with torsion, analyzed in the present paper. This configuration can preserve information through a stationary solution, preventing the complete evaporation of the black hole and predicting a non-zero residual mass. We will try to show how the geometric torsion can influence both the dynamics of matter and the Hawking radiation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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