AMERICAN JOURNAL of PHYSICS

Using worksheets to solve the Einstein equation

Thomas A. Moore

Citation: American Journal of Physics **84**, 360 (2016); doi: 10.1119/1.4939908
View online: http://dx.doi.org/10.1119/1.4939908
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/84/5?ver=pdfcov
Published by the American Association of Physics Teachers

Using worksheets to solve the Einstein equation

Thomas A. Moore[a)]
Department of Physics and Astronomy, Pomona College, Claremont, California 91711

(Received 17 August 2015; accepted 31 December 2015)

This article describes how one can use worksheets to guide undergraduate students through the process of finding solutions to specific cases of the Einstein equation of general relativity. The worksheets provide expressions for a metric's Christoffel symbols and Ricci tensor components for fairly general metrics. Students can use a worksheet to adapt these expressions to specific cases where symmetry or other considerations constrain the metric components' dependencies, and then use the worksheet's results to reduce the Einstein equation to a set of simpler differential equations that they can solve. This article illustrates the process for both a diagonal metric and a metric with one off-diagonal element. *© 2016 American Association of Physics Teachers.*
[http://dx.doi.org/10.1119/1.4939908]

I. INTRODUCTION

We have argued elsewhere[1] that the recent vitality of general relativity research programs and new teaching resources make teaching general relativity to undergraduates more valuable and easier than ever before. Recent research developments in cosmology, the physics of black holes, and Gravity Probe B, the promise of gravitational wave observatories such as the Laser Interferometer Gravitational Wave Observatory (LIGO) and Virgo, and the practical applications of general relativity in the context of the Global Positioning System all have excited student interest in the field and provide new opportunities for involving students in research. Resources including new textbooks[2–7] and computer software[8–11] also make general relativity more accessible to undergraduates than in even the recent past.

Even so, the mathematics of general relativity can present daunting challenges for undergraduates. One possibility for reducing this complexity is to use computer algebra tools to do the heavy lifting. For example, author James Hartle has provided Mathematica notebooks[8] to support his well-regarded undergraduate textbook[2] that allow students to quickly calculate Christoffel symbols and components of the Riemann, Ricci, and Einstein tensors for known metrics. Similar tools are available elsewhere online.[9]

Such tools have many advantages, allowing undergraduates to quickly move past tedious calculations to interesting applications that decades ago would have required too much work to pursue. However, Mathematica and similar tools themselves have steep initial learning curves, and even with proficiency do not necessarily help students understand the process of *solving* the Einstein equations. For example, Hartle's notebooks allow students to easily verify that certain *known* metrics satisfy the Einstein equation, but do not allow one to solve for an *unknown* metric.

The core pedagogical problem is that we tell students that one can solve the Einstein equation in a given physical situation for the spacetime metric, but then typically give them solutions that *other* people have found. We have found it valuable for students to work through the entire process a few times for themselves, starting with a realistic physical situation, first developing a plausible trial metric and then using the Einstein equation to solve for unknown parts of that trial solution (while satisfying appropriate boundary conditions). In this process, students gain a much deeper appreciation of what actually goes into finding such solutions, and ultimately a greater confidence in their understanding of how the theory works. Going through the process therefore delivers a priceless experience of empowerment.

The main reason that students rarely get this experience in an undergraduate general relativity course is that the calculations required to solve the Einstein equation by hand involve literally hundreds of steps, each of which needs to be done correctly for the solution to make sense. The tediousness of this process, combined with the almost vanishing probability of doing it correctly, makes solving the Einstein equation by hand impractical in the context of a typical undergraduate course. "Toy" problems involving fewer dimensions are not available because nontrivial solutions to the empty-space Einstein equation do not exist in fewer than three spatial dimensions.

To make the calculation practical, one would like a tool that automates enough of the most tedious (and least physically interesting) parts to make success more probable. Such a tool must also have a shallow learning curve, be something the students can trust (because they know how to check its results if necessary, and have practiced doing just that), be general enough to have a wide range of applications, and be inexpensive and easy to deploy.

In this article, we describe just such a tool: a paper worksheet that automates the most tedious and error-prone aspects of solving the Einstein equation. This method extends and further develops an approach that Rindler outlined in an appendix to his textbook *Relativity: Special, General, and Cosmological*.[12] The worksheet allows undergraduates to experience actually solving the Einstein equation in a realistic context without experiencing the tool as a black box or getting hopelessly bogged down in the details. Moreover, the two versions of the worksheet described in this article allow one to handle virtually any metric likely to be encountered in a general relativity course. These worksheets complement the strengths of the computer tools mentioned above, because they are more transparent and allow students to handle metrics with unknown components (even if they don't quite deliver results at the push of a button).

II. THE EINSTEIN EQUATION

The classic form of the Einstein equation of general relativity is

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi G T^{\mu\nu}, \tag{1}$$

where $G^{\mu\nu}$ is the Einstein tensor (which expresses something about the curvature of spacetime), Λ is the so-called cosmological constant, $g^{\mu\nu}$ is the inverse metric tensor, $T^{\mu\nu}$ is the stress-energy tensor (which expresses the density of energy and momentum), and G is the universal gravitational constant (here we choose units where c is 1 but G is not). The Einstein tensor components are complicated nonlinear functions of the spacetime metric components $g_{\mu\nu}$ and their first and second derivatives. Since $g_{\mu\nu}$, $G^{\mu\nu}$, and $T^{\mu\nu}$ are all 4×4 symmetric tensors with ten independent components, the general Einstein equation amounts to ten coupled nonlinear differential equations in the ten independent components of $g_{\mu\nu}$. However, the very nature of the Einstein equation ensures that only six of these equations are actually independent, allowing one the freedom to choose the four coordinates arbitrarily.

This form of the Einstein equation is the simplest conceptually, because it isolates all the terms relating to the curvature of spacetime on the left and everything related to the density of energy and momentum on the right. Moreover, the most straightforward argument for how to construct the Einstein tensor starts with this version of the equation. *Solving* the Einstein equation, however, is easier when we use the mathematically equivalent version

$$R^{\mu\nu} = 8\pi G\left(T^{\mu\nu}_{\text{all}} - \frac{1}{2} g^{\mu\nu} T_{\text{all}}\right), \tag{2}$$

where $R^{\mu\nu}$ is the Ricci tensor, $T^{\mu\nu}_{\text{all}} \equiv T^{\mu\nu} + T^{\mu\nu}_{\text{vac}}$, with $T^{\mu\nu}_{\text{vac}} \equiv \Lambda g^{\mu\nu}/8\pi G$ the effective stress-energy tensor for the vacuum, and $T_{\text{all}} \equiv g_{\mu\nu} T^{\mu\nu}_{\text{all}}$. The reason it is simpler to solve Eq. (2) compared to Eq. (1) is because the Ricci tensor is a significantly simpler function of the metric components $g_{\mu\nu}$ than the Einstein tensor is. It is well worth putting the complexity on the right side of the equation, particularly because we are often interested in vacuum solutions of the Einstein equation with negligible vacuum energy. In this case, the equation

$$R^{\mu\nu} = 0, \tag{3}$$

is *much* easier to solve than the corresponding equation $G^{\mu\nu} = 0$. But even when we are interested in situations where the stress-energy is nonzero (as in the case of solving for the cosmological metric), Eq. (2) is still usually the easier form to work with.

For the record, the definition of the Ricci tensor is

$$R_{\mu\nu} \equiv \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\alpha\gamma}\Gamma^\gamma_{\mu\nu} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\mu\alpha}, \tag{4}$$

where $\partial_\alpha \equiv \partial/\partial x^\alpha$ and where $\Gamma^\alpha_{\beta\mu}$ is a Christoffel symbol, defined to be

$$\Gamma^\alpha_{\beta\mu} \equiv \frac{1}{2} g^{\alpha\sigma}(\partial_\beta g_{\mu\sigma} + \partial_\mu g_{\sigma\beta} - \partial_\sigma g_{\beta\mu}), \tag{5}$$

and the usual rules of index manipulations apply. Because each of the indices can range over the four coordinate values, we see that the expression for a Christoffel symbol could involve up to 12 terms and that for a Ricci component could involve hundreds of distinct terms. This is why solving the Einstein equation can be so daunting.

III. THE DIAGONAL METRIC WORKSHEET

Consider a general diagonal metric of the form

$$ds^2 = -A(dx^0)^2 + B(dx^1)^2 + C(dx^2)^2 + D(dx^3)^2, \tag{6}$$

where dx^0 is an arbitrary time coordinate, dx^1, dx^2, and dx^3 are arbitrary spatial coordinates, and A, B, C and D are arbitrary functions of any or all of the coordinates. The Diagonal Metric Worksheet (available online[13]) provides a complete list of the Christoffel symbols and Ricci tensor components for such a metric in terms of the functions A, B, C, and D and their derivatives, using a compact notation where

$$A_0 \equiv \frac{\partial A}{\partial x^0}, \quad B_{12} \equiv \frac{\partial^2 B}{\partial x^1 \partial x^2}, \quad \text{and so on.} \tag{7}$$

To use the worksheet, one first crosses out terms that are zero for the particular metric of interest, and then writes (in the space provided above each term) what each nonzero term becomes for the metric of interest. One then gathers simplified nonzero terms in a space provided at the bottom.

For example, suppose that we are looking for a time-independent and spherically symmetric solution of the empty-space Einstein equation $R_{\mu\nu} = 0$ for a metric tensor based on a time-coordinate $x^0 \equiv t$, a radial coordinate $x^1 \equiv r$, and angular coordinates $x^2 \equiv \theta$ and $x^3 \equiv \phi$. "Spherical symmetry" means that in the spacetime of interest, one can define a concentric set of two-dimensional spatial surfaces that have the same intrinsic geometry as that of a sphere. We can therefore define each such surface to correspond to a constant value of the coordinate r and define the angular coordinates to correspond to the usual angular coordinates θ and ϕ for a spherical surface. In such a case, we can choose the r, θ, and ϕ coordinates so that the metric for such a surface of constant r is the same as the usual metric $ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ for the surface of a sphere. If we do this, our more specific version of the general diagonal metric reduces to the form

$$A = A(r), \ \ B = B(r), \ \ C = r^2, \ \text{ and } D = r^2 \sin^2\theta. \tag{8}$$

Figure 1 illustrates what someone might write on the worksheet to determine the expression for $R_{00} \equiv R_{tt}$ for the metric of interest. Note that since we are supposing that the metric does not depend on the time coordinate t or the angular coordinates (except for D, which depends on $\theta \equiv x^2$), we can cross out all terms involving derivatives with 0, 2, and 3 subscripts (except for D_2). This leaves only a handful of terms, which the user has simplified and gathered at the bottom.

We strongly recommend that students be asked to verify by hand at least *some* elements of the worksheet, such as a few Christoffel symbols and perhaps one off-diagonal element of the Ricci tensor. This will help convince students that the tool is not a black box but rather a convenient summary that they could (in principle) verify, given enough time and patience. The Appendix to this article illustrates such calculations.

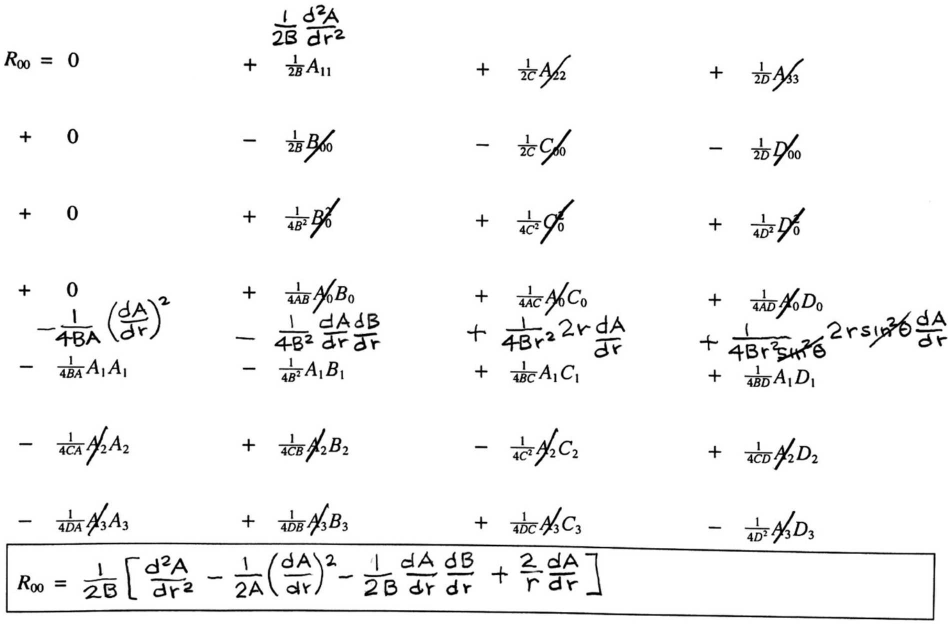

Fig. 1. An example of what someone using the Diagonal Metric Worksheet might write for the specific case of the metric considered in Eq. (8).

IV. THE SCHWARZSCHILD SOLUTION

How might we use this tool to actually solve the Einstein equation? The general process of solving the Einstein equation is similar to that for solving almost any complicated differential equation. You first develop a plausible trial solution (with perhaps some free parameters and/or functions) that is consistent with the situation's symmetry and/or the solution's known behavior in some limit. You then substitute this trial solution into the equation to be solved, and from that derive mathematical conditions that the free parameters or functions must satisfy to solve the differential equation. If you can find a set of parameters or functions consistent with those conditions, then you have successfully solved the equation. If you cannot satisfy the conditions, then you try a different trial solution until you find a solution that has enough freedom to allow all of the conditions to be satisfied.

As an example, let's use the Diagonal Metric Worksheet to derive the well-known Schwarzschild solution of the Einstein equation. The metric of Eq. (8) is actually a plausible guess for the spacetime metric in the empty space surrounding a static (non-rotating), spherically symmetric object. Because the *object* is spherically symmetric and independent of time, we would plausibly expect the *spacetime* should also be spherically symmetric and independent of time (with a suitable definition of space and time coordinates). The metric of Eq. (8) exhibits spherical symmetry and time-independence, while still having some remaining freedom in the undefined functions $A(r)$ and $B(r)$, which we can adjust to satisfy the Einstein equation. It thus represents a reasonable trial metric for this kind of situation.[14]

If one goes through the same process illustrated in Fig. 1 to evaluate all the components of the Ricci tensor for this trial metric, one finds that

$$R_{tt} \equiv R_{00} = \frac{1}{2B}\left[+\frac{d^2A}{dr^2} - \frac{1}{2A}\left(\frac{dA}{dr}\right)^2 - \frac{1}{2B}\frac{dA}{dr}\frac{dB}{dr} + \frac{2}{r}\frac{dA}{dr}\right], \tag{9}$$

$$R_{rr} \equiv R_{11} = \frac{1}{2A}\left[-\frac{d^2A}{dr^2} + \frac{1}{2A}\left(\frac{dA}{dr}\right)^2 + \frac{1}{2B}\frac{dA}{dr}\frac{dB}{dr} + \frac{2A}{Br}\frac{dB}{dr}\right], \tag{10}$$

$$R_{\theta\theta} \equiv R_{22} = -\frac{r}{2AB}\frac{dA}{dr} + \frac{r}{2B^2}\frac{dB}{dr} + 1 - \frac{1}{B}, \tag{11}$$

and that $R_{\phi\phi} \equiv R_{33} = \sin^2\theta R_{\theta\theta}$, with the off-diagonal components of the Ricci tensor all identically zero. In empty space, the Einstein equation requires that $R_{\mu\nu} = 0$. If we assume that $A \neq 0$ and $B \neq 0$, this means that we must have

$$0 = 2BR_{tt} + 2AR_{rr} = \frac{2}{r}\frac{dA}{dr} + \frac{2A}{Br}\frac{dB}{dr}, \tag{12}$$

in order to solve the Einstein equation. If we additionally assume that $r \neq 0$, this expression reduces to

$$0 = B\frac{dA}{dr} + A\frac{dB}{dr}, \tag{13}$$

so that

$$\frac{d}{dr}(AB) = 0 \quad \Rightarrow \quad AB = \text{const.} \tag{14}$$

We also know that very far from a gravitating object, its gravitational field becomes negligible, so in our case we would like the metric at infinity to reduce to the flat-space metric (in spherical coordinates), which is $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 d\phi^2)$. This means that AB will be 1 at infinity, and since we have just established that AB is a constant, we must have $AB = 1$ at *all* values of r if we are to simultaneously satisfy the Einstein equation and this large-r limit.

If we substitute $B = 1/A$ and Eq. (13) in the form

$$\frac{dB}{dr} = -\frac{B}{A}\frac{dA}{dr} \tag{15}$$

into Eq. (11), and require that $R_{\theta\theta} = 0$ (because we are in empty space), we find that we also must have

$$0 = -r\frac{dA}{dr} + 1 - A \quad \Rightarrow \quad 1 = r\frac{dA}{dr} + A = \frac{d}{dr}(rA), \tag{16}$$

to satisfy the Einstein equation in empty space. Integrating both sides of Eq. (16) with respect to r yields

$$r = rA + K \quad \Rightarrow \quad A = 1 - \frac{K}{r}, \tag{17}$$

where K is a constant of integration. Since $AB = 1$, it immediately follows that

$$B = \frac{1}{A} = \left(1 - \frac{K}{r}\right)^{-1}. \tag{18}$$

One can now apply the geodesic equation to a particle falling from rest at large r to establish that we must have $K = 2GM$ (where M is object's mass and G is the universal gravitational constant) if our solution is to be consistent with the Newtonian limit. The details for how one determines K can be found in virtually any textbook on general relativity[15] (though the Diagonal Metric Worksheet is handy for quickly evaluating the Christoffel symbol that one needs to complete the argument). With this identification, we have arrived at the Schwarzschild solution.

Though using the Diagonal Metric Worksheet to evaluate the Ricci tensor components in this case requires care and diligence, you can see that it involves only algebra and fairly simple calculus. Once we have these components in hand, solving the Einstein equation is a matter of algebra and straightforward calculus. The entirety of this calculation is therefore well within the capability of an (appropriately guided) upper-level undergraduate physics major. Of course, the same could be said *in principle* of a process that involves calculating the Christoffel symbols and Ricci tensor components from their basic definitions. Evaluating these components is the part of the calculation that is the least interesting (and least physical) task, but it is also where one is most likely to make careless but consequential errors. Using the worksheet thus allows students to focus on parts of the calculation that will maximize their physical insight.

One can also use the Diagonal Metric Worksheet in conjunction with the Einstein equation to prove Birkhoff's theorem,[16] to prove that the only plane-symmetric solution to the Einstein equation is flat spacetime,[17] to find the metric in the vacuum around an infinite straight cosmic string,[18] to determine the metric for a homogeneous and isotropic universe,[19] and even to calculate the second-order corrections to the linearized theory of gravitation that are necessary to determine the energy carried by a gravitational wave.[20] The Diagonal Metric Worksheet can also facilitate the calculation of the Christoffel symbols needed to determine the implications of given metrics (without the learning curves associated with computer software), making a number of other applications more practical for undergraduates.

V. THE OFF-DIAGONAL METRIC WORKSHEET

We have documented the Diagonal Metric Worksheet and its applications elsewhere.[21] Here, we announce the availability of an extension of this tool for off-diagonal metrics of the form

$$ds^2 = -A(dx^0)^2 + B(dx^1)^2 + C(dx^2)^2 + D(dx^3)^2 + 2F(dx^0)(dx^1), \tag{19}$$

where F is (like A, B, C, and D) an arbitrary function of the coordinates. This metric is sufficiently general to embrace virtually any metric one is likely to encounter in a general relativity course. Like the Diagonal Metric Worksheet, the Off-Diagonal Metric Worksheet (also available online[13]) provides a complete list of the Christoffel symbols and Ricci tensor components for such a metric in terms of the functions A, B, C, D, and F and their derivatives. For simplicity, this worksheet also uses the shorthand that

$$H \equiv AB + F^2, \tag{20}$$

a combination that occurs often as a factor in terms (though we have broken down derivatives of H into derivatives of A, B, and F). The calculations were so complicated that we had to write a custom computer program to do the algebra (Mathematica was ill-suited to this particular task). The source code for this program is also available online.[13]

Figure 2 shows the worksheet page for R_{00} in this particular case. As one can see, the presence of the single off-diagonal term in the metric approximately doubles the number of nonzero terms in the Ricci tensor components, which obviously makes solving the Einstein equation more complicated. Even so, one can still practically use the Off-Diagonal Metric Worksheet to find solutions to the Einstein equation, as Sec. VII illustrates.

VI. WHAT AN OFF-DIAGONAL METRIC MEANS

An obvious application of the Off-Diagonal Metric Worksheet would be to derive the Kerr metric from the Einstein equation in the same way we did the Schwarzschild solution. However, because the Kerr metric depends on two variables (r and θ) instead of one, this calculation is challenging even with the worksheet and is ill-suited as an example in this short paper. Instead, in Sec. VII we will use the Off-Diagonal Metric Worksheet to find an alternative solution to the Einstein equation in the vacuum surrounding a static spherical object, a solution called "rain coordinates"[22] or "global rain coordinates."[23] In this section, we will address why we might be interested in such a solution, and (in the process) illustrate the process for *interpreting* an off-diagonal metric.

Imagine constructing a lattice for measuring the Schwarzschild coordinates of events around a static spherical object. We can (without reference to any coordinate system) determine concentric surfaces of symmetry around our spherical object, and define the radial direction to be perpendicular to such surfaces. Let's then construct our lattice so that one of the girders at each lattice intersection is radial, and the other two are orthogonal and lie on one of these spherical surfaces. If we also put a "t-meter" (a device that displays the coordinate time t) at each lattice intersection, then in principle, we can determine the spacetime coordinates of any event occurring in the lattice by noting the coordinates of the nearest lattice intersection and the coordinate time displayed by the t-meter there. But how do we connect

$R_{00} \; = \; 0$	$+\; \frac{A}{2H} A_{11}$	$+\; \frac{1}{2C} A_{22}$	$+\; \frac{1}{2D} A_{33}$
$+\; 0$	$-\; \frac{A}{2H} B_{00}$	$-\; \frac{1}{2C} C_{00}$	$-\; \frac{1}{2D} D_{00}$
$+\; 0$	$+\; \frac{A^2}{4H^2} B_0^2$	$+\; \frac{1}{4C^2} C_0^2$	$+\; \frac{1}{4D^2} D_0^2$
$+\; 0$	$+\; \frac{AB}{4H^2} A_0 B_0$	$+\; \frac{B}{4HC} A_0 C_0$	$+\; \frac{B}{4HD} A_0 D_0$
$-\; \frac{AB}{4H^2} A_1 A_1$	$-\; \frac{A^2}{4H^2} A_1 B_1$	$+\; \frac{A}{4HC} A_1 C_1$	$+\; \frac{A}{4HD} A_1 D_1$
$-\; \frac{B}{4HC} A_2 A_2$	$+\; \frac{A}{4HC} A_2 B_2$	$-\; \frac{1}{4C^2} A_2 C_2$	$+\; \frac{1}{4CD} A_2 D_2$
$-\; \frac{B}{4HD} A_3 A_3$	$+\; \frac{A}{4HD} A_3 B_3$	$+\; \frac{1}{4DC} A_3 C_3$	$-\; \frac{1}{4D^2} A_3 D_3$
$+\; 0$	$+\; \frac{AF}{4H^2} A_0 B_1$	$-\; \frac{F}{4HC} A_0 C_1$	$-\; \frac{F}{4HD} A_0 D_1$
$+\; 0$	$-\; \frac{AF}{4H^2} A_1 B_0$	$+\; \frac{F}{4HC} A_1 C_0$	$+\; \frac{F}{4HD} A_1 D_0$
$+\; \frac{A}{H} F_{01}$	$-\; \frac{AF}{H^2} F_0 F_1$	$+\; \frac{A}{2HC} F_2 F_2$	$+\; \frac{A}{2HD} F_3 F_3$
$-\; \frac{AB}{2H^2} A_0 F_1$	$-\; \frac{AF}{2H^2} A_1 F_1$	$-\; \frac{F}{2HC} A_2 F_2$	$-\; \frac{F}{2HD} A_3 F_3$
$+\; 0$	$+\; \frac{AF}{2H^2} B_0 F_0$	$+\; \frac{F}{2HC} C_0 F_0$	$+\; \frac{F}{2HD} D_0 F_0$
$+\; 0$	$-\; \frac{A^2}{2H^2} B_1 F_0$	$+\; \frac{A}{2HC} C_1 F_0$	$+\; \frac{F}{2HD} D_1 F_0$

$R_{00} \; =$

Fig. 2. This figure shows the portion of the Off-Diagonal Metric Worksheet that lists the nonzero terms for the R_{00} component of the Ricci tensor for the metric given by Eq. (19). Note that adding just one off-diagonal term to the metric roughly doubles the number of terms in the expression for R_{00} (compare with Fig. 1).

the locations of lattice intersections and t-meters in our lattice to Schwarzschild coordinates?

The *metric* is the only thing that gives physical meaning to coordinates (which are otherwise completely arbitrary). The Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{21}$$

tells us that because $r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ is the metric for a spherical surface of radius r in ordinary angular coordinates θ and ϕ, then a surface of constant r ($dr = 0$) at a specific instant of time t ($dt = 0$) in Schwarzschild spacetime must be one of those spheres of symmetry. So the Schwarzschild θ and ϕ coordinates correspond to the standard angular coordinates on the surface of each sphere on the lattice, and r is a radial coordinate whose value on any spherical lattice surface corresponds to the area of that sphere divided by 4π.

The metric also tells us that a t-meter at rest ($dr = 0$, $d\theta = 0$, $d\phi = 0$) at a lattice intersection registers a proper time $d\tau = \sqrt{-ds^2} = (1 - 2GM/r)^{1/2}\, dt$. Thus, we can construct a Schwarzschild t-meter by modifying an ordinary clock so that it displays on its face a value $(1 - 2GM/r)^{-1/2}$ times the (proper) time τ that the clock actually measures. (Note that clocks at $r \to \infty$ register t directly.)

Finally, because the Schwarzschild metric is independent of time and is invariant under the transformation $dr \to -dr$, a radially moving light flash takes the same Schwarzschild coordinate time Δt to travel between two lattice intersections separated by a given radial displacement Δr, whether it is going outward or inward. Therefore, we can synchronize lattice clocks on the same radial line but on different spherical surfaces by having a t-meter on one surface send a radial light flash to a t-meter on the other, which then reflects the flash back to the first. The two t-meters are synchronized if the reflecting t-meter registers the coordinate time of the reflection event to be the coordinate time halfway between the emission and reception events as registered by the sending t-meter.

Now, the Schwarzschild solution exhibits well-known difficulties at $r = 2GM$, because $B = g_{rr} = (1 - 2GM/r)^{-1}$ is singular at that radius. This singularity turns out to be an artifact of the choice of coordinate system, for reasons that our previous discussion helps us better understand. Inside $r = 2GM$, no physical object can remain at rest; that is, Eq. (21) tells us that $|dr|$ must be nonzero for the object to have a timelike ($ds^2 < 0$) worldline. So we cannot even begin to construct a lattice with fixed intersections and t-meters inside this radius, meaning that our scheme for assigning coordinates falls apart. The singularity in the $B = g_{rr}$ term of the metric at $r = 2GM$ is one indication that our coordinate system is breaking down there.

One can change coordinates to find non-singular diagonal metrics (such the Kruskal-Szekeres coordinate system[24]) for the same spacetime, but the metrics for such coordinate systems have the disadvantage of not reducing to the flat-space metric at infinity, making them very difficult to interpret. Also, the coordinate transformations to such coordinates are themselves necessarily singular at $r = 2GM$, making many students suspicious that we are actually just brushing the problem under the rug.

The global rain coordinate system gets around this problem by introducing an off-diagonal term g_{tr} $(= g_{rt})$ in the metric. What does this off-diagonal term do? A nonzero term $2g_{rt}\, dr\, dt$ in the metric equation means that the metric is no longer invariant under the transformation $dr \to -dr$. This in turn means that ingoing and outgoing light flashes no longer take the same coordinate time (as defined by the new metric) to travel between two given spherical surfaces. The result is that t-meters synchronized using the Schwarzschild scheme described above will *not* be synchronized according to the new coordinate time we are defining.

We see that introducing the off-diagonal term changes the way we synchronize t-meters on different spherical shells in our lattice. (To say it another way, introducing the off-diagonal term redefines the hypersurfaces in spacetime corresponding to fixed values of the new coordinate time t.) It turns out (as we will see) that we can use our freedom to adjust both t-meter rates on a *given* spherical shell and the relative synchronization of t-meters on *different* shells to align the coordinate times registered by our lattice t-meters so that they agree with the actual clock time τ displayed by clocks that are *falling through the lattice* after being synchronized to clocks at some extremely large radius ($r \approx \infty$) and dropped from rest there. In essence, we are choosing to synchronize our lattice t-meters using these dropped clocks instead of using light flashes.

Better yet, we can *replace* our fixed lattice clocks by an endless "rain" of clocks dropped from $r \approx \infty$. The rain coordinate time of a given event will simply be the time displayed by the nearest falling clock. Because freely falling clocks can exist and display well-defined values even inside $r = 2GM$, this procedure extends our definition of coordinate time meaningfully to *all* nonzero values of r, and suggests that introducing an off-diagonal term in the metric might allow us to sidestep the problem with the Schwarzschild coordinate time and so avoid the problems that make the Schwarzschild metric singular.

The usual treatments[22,23] of global rain coordinates arrive at the global rain metric via a (singular) coordinate transformation from Schwarzschild coordinates. In Sec. VII, we will instead use the Off-Diagonal Metric Worksheet to look for a *direct* solution of the empty-space Einstein equation with an off-diagonal g_{rt} term. Indeed, we will find a whole family of solutions (corresponding to different ways of defining constant-t hypersurfaces in Schwarzschild spacetime), one of which is the non-singular global rain coordinate system described in this section. By deriving this non-singular solution directly from the Einstein equation, we also sidestep the criticism that we are simply masking the Schwarzschild problem by using an equally problematic coordinate transformation.

VII. THE GLOBAL RAIN METRIC AND ITS SIBLINGS

Following the path proposed in Sec. VI, let's consider a trial metric of the form

$$ds^2 = -A\, dt^2 + B\, dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 + 2F\, dt\, dr, \tag{22}$$

where $t \equiv x^0$ is some kind of time coordinate, $r \equiv x^1$ is a radial coordinate, and $\theta \equiv x^2$ and $\phi \equiv x^3$ are the usual angular coordinates. The r, θ, and ϕ coordinates in this metric have the same physical meaning as the corresponding Schwarzschild coordinates, but the time coordinate t does not. Spherical symmetry and time independence again suggest that A, B, and F should be functions of r alone, so we will assume that. We would also like the metric to reduce to the flat-space metric at infinity, so we will also assume that A and B both go to 1 at $r \to \infty$ and F goes to zero in the same limit.

In this case, one can use the Off-Diagonal Metric Worksheet (in the same way as described in Sec. III) to show that

$$R_{tt} = \frac{A}{4H^2}\left[+2H\frac{d^2A}{dr^2} - B\left(\frac{dA}{dr}\right)^2 - A\frac{dA}{dr}\frac{dB}{dr} + \frac{4H}{r}\frac{dA}{dr} - 2F\frac{dA}{dr}\frac{dF}{dr}\right], \tag{23}$$

$$R_{rr} = \frac{B}{4H^2}\left[-2H\frac{d^2A}{dr^2} + B\left(\frac{dA}{dr}\right)^2 + A\frac{dA}{dr}\frac{dB}{dr} + \frac{4AH}{Br}\frac{dB}{dr} + 2F\frac{dA}{dr}\frac{dF}{dr} + \frac{8FH}{Br}\frac{dF}{dr}\right], \tag{24}$$

and

$$R_{\theta\theta} = -\frac{A}{H} + 1 - \frac{r}{H}\frac{dA}{dr} + \frac{ABr}{2H^2}\frac{dA}{dr} + \frac{A^2r}{2H^2}\frac{dB}{dr} + \frac{FAr}{H^2}\frac{dF}{dr}. \tag{25}$$

We also see that $R_{\phi\phi} = \sin^2\theta R_{\theta\theta}$ (as before), $R_{tr} = R_{rt} = -(B/F)R_{tt}$, and that all other components of $R_{\mu\nu}$ are zero. Thus, the vacuum Einstein equation will be satisfied as long as $R_{tt} = R_{rr} = R_{\theta\theta} = 0$.

If we assume that $A \neq 0$ and $B \neq 0$, the Einstein equation requires

$$0 = BR_{tt} + AR_{rr} = \frac{AB}{Hr}\frac{dA}{dr} + \frac{A^2}{Hr}\frac{dB}{dr} + \frac{2AF}{Hr}\frac{dF}{dr}. \tag{26}$$

If we also assume that $H \neq 0$ and $r \neq 0$, then this becomes

$$0 = B\frac{dA}{dr} + A\frac{dB}{dr} + 2F\frac{dF}{dr} = \frac{d}{dr}\left(AB + F^2\right) = \frac{dH}{dr}, \tag{27}$$

implying that H is a constant. Since we are requiring that $A = B = 1$ and $F = 0$ at infinity, we find that H must be 1 at infinity, and therefore $H = 1$ *everywhere* to satisfy the Einstein equation.

Substituting this result into the equation $R_{\theta\theta} = 0$ yields

$$\begin{aligned} 0 &= -A + 1 - r\frac{dA}{dr} + \frac{Ar}{2}\left[B\frac{dA}{dr} + A\frac{dB}{dr} + 2F\frac{dF}{dr}\right] \\ &= -A + 1 - r\frac{dA}{dr} + [0]. \end{aligned} \tag{28}$$

This is the same equation we solved in Eq. (16), so the solution $A = 1 - K/r$ is the same as well. Requiring that the gravitational field corresponds to the Newtonian limit once again results in $K = 2GM$. Substituting this into the condition $H = 1$ implies that

$$1 = H = AB + F^2 = \left(1 - \frac{2GM}{r}\right)B + F^2. \tag{29}$$

At this point, we have exhausted the constraints that the Einstein equation puts on B and F. Even though we have three equations ($R_{tt} = 0, R_{rr} = 0$, and $R_{\theta\theta} = 0$) to solve for the unknowns A, B, and F, the three equations are actually not independent. We can see as follows. If we put $A = 1 - 2GM/r$ back into either $R_{tt} = 0$ or $R_{rr} = 0$, the components become zero no matter what B or F might be. Therefore, the Einstein equation tells us that we are free to choose B and F subject to the constraints that $B \to 1$ and $F \to 0$ at infinity and $H = (1 - 2GM/r)B + F^2 = 1$. This freedom allows us to specify how to synchronize clocks on surfaces with differing r.

If we choose $F = 0$, then we end up with $AB = 1$, meaning that $B = 1/A$, recovering the Schwarzschild solution (as we must). The most straightforward alternative is to choose $B = 1$, which leads to

$$F = \sqrt{\frac{2GM}{r}}, \tag{30}$$

yielding the metric

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2GM}{r}\right)dt^2 + dr^2 + 2\sqrt{\frac{2GM}{r}}\,dt\,dr \\ &+ r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \end{aligned} \tag{31}$$

As desired, this is a metric for Schwarzschild spacetime that is non-singular at all values of $r > 0$ and reduces to the flat-space metric at infinity.

We can see that this is indeed the global rain metric as follows. As many general relativity texts show,[25] the proper radial velocity $dr/d\tau$ for an object dropped from rest at infinity is given by $dr/d\tau = -\sqrt{2GM/r}$. Since the metric definition of the r coordinate is the same in both the Schwarzschild metric [Eq. (21)] and the metric in Eq. (31), and every observer must agree on what time τ the dropped clock's face reads, the same result must apply in the coordinate system of Eq. (31). If we multiply Eq. (31) by -1, divide through by $d\tau^2 = -ds^2$, and substitute $dr/d\tau = -\sqrt{2GM/r}$ and $d\theta/d\tau = d\phi/d\tau = 0$, we see that

$$1 = \left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + 2\left(\frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right) - \frac{2GM}{r}. \tag{32}$$

Applying the quadratic equation yields the solution $dt/d\tau = 1$ (as one can check by direct substitution) and a negative solution that we discard (because our coordinate time can't go backward as τ marches forward). So we see that a falling clock's proper time τ does indeed coincide with the coordinate time t defined by this metric up to an additive constant, which we can set to zero by synchronizing our falling clocks to master clocks at rest at infinity (which also measure coordinate time t) before dropping them. Therefore, this metric does indeed define the global rain coordinate system described in Sec. VI. We note that because $g_{rr} = 1$ in Eq. (31), the spatial part of the metric is the same as that for flat space, meaning that this metric's particular definition of $t =$ constant hypersurfaces means that such hypersurfaces happen to have a spatially flat geometry.

One of the insights we gain by approaching the problem this way is that other potentially interesting solutions might exist as well. For example, another simple solution results if we choose $B = (1 + 2GM/r)$, which, when combined with Eq. (29), implies that

$$F = \frac{2GM}{r}. \tag{33}$$

This choice leads to a coordinate system called advanced Eddington-Finkelstein coordinates,[26] whose metric equation is

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{4GM}{r}\,dt\,dr \\ &+ \left(1 + \frac{2GM}{r}\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \end{aligned} \tag{34}$$

In this coordinate system, one can show that a radially ingoing light flash travels at coordinate speed $dr/dt = -1$, so an observer at any event can determine the coordinate time t by noting the time registered by a clock at essentially infinite r (as carried in by a radially ingoing light signal from that clock) and adding the radial coordinate distance $|\Delta r|$ between that clock and the observer's current position. Since we can carry out this procedure at any radial position (including inside $r = 2GM$), this metric for Schwarzschild spacetime is also well-defined and nonsingular at all r.

We see that solving the Einstein equation directly for such off-diagonal metrics is useful for three reasons. First, it addresses the worry that a singular coordinate transformation simply masks the problem with Schwarzschild coordinates. Second, it helps us better understand what freedoms we have in choosing the time coordinate. And third, it yields a whole family of possible new metrics for Schwarzschild spacetime that are worth exploring.

Besides this application and the Kerr metric, other possible applications of the Off-Diagonal Metric Worksheet might be in the study of rotating universes or to directly solving the Einstein equation for the metric of the vacuum surrounding a rotating object in the large-r limit (without going through the formalism of linearized gravity or gauge transformations).

VIII. CONCLUSION

We have seen that the Diagonal Metric Worksheet and the Off-Diagonal Metric Worksheet provide useful tools that make it practical for upper-level undergraduates to have the experience of solving the Einstein equation in a number of interesting contexts. We believe such experiences are a crucial element in helping students gain "ownership" of the Einstein equation, by helping them see how one can start from the definition of the Christoffel symbols and Ricci tensor and proceed to deriving solutions.

Moreover, both worksheets (particularly the new Off-Diagonal Metric Worksheet) allow one to explore applications of the Einstein equation in situations previously inaccessible to undergraduates and not yet treated in textbooks. We offer these tools to the community in the hope that they will open up a variety of interesting new areas for exploration.

ACKNOWLEDGMENTS

The author would like to thank the reviewers for their valuable suggestions for improvement.

APPENDIX: VERIFYING WORKSHEET RESULTS

This appendix will illustrate the calculations that lie behind the Diagonal Metric Worksheet. These are the kind of calculations that students should duplicate for themselves to ensure that the worksheet is not simply a black box. We start with the diagonal metric given in Eq. (6), repeated here for convenience to be

$$ds^2 = -A(dx^0)^2 + B(dx^1)^2 + C(dx^2)^2 + D(dx^3)^2, \tag{A1}$$

where A, B, C, and D can be functions of any or all of the four coordinates x^0, x^1, x^2, and x^3. Suppose we would like to check the Diagonal Metric Worksheet's result for Γ^0_{11}. According to Eq. (5), this should be

$$\Gamma^0_{11} = \frac{1}{2} g^{0\sigma}(\partial_1 g_{1\sigma} + \partial_1 g_{\sigma 1} - \partial_\sigma g_{11}), \tag{A2}$$

with an implied sum over σ. But the metric $g_{\alpha\beta}$ is diagonal by hypothesis, and the components $g^{\alpha\beta}$ of the inverse of a diagonal metric are simply $g^{\alpha\beta} = 1/g_{\alpha\beta}$ for all choices of α and β, so the metric inverse is also diagonal. This in turn means that all terms in the implied sum over σ in Eq. (A2) are zero unless $\sigma = 0$. Therefore, since $g^{00} = 1/g_{00} = -1/A$, $g_{11} \equiv B$, and the off-diagonal metric components are zero, we have

$$\begin{aligned}\Gamma^0_{11} &= \frac{1}{2} g^{00}(\partial_1 g_{10} + \partial_1 g_{01} - \partial_0 g_{11}) \\ &= -\frac{1}{2A}\left(0 + 0 - \frac{\partial B}{\partial x^0}\right) = \frac{1}{2A} B_0,\end{aligned} \tag{A3}$$

using the worksheet's compact notation. This is the result given in the worksheet.

Once one has come to trust the Christoffel symbol results, one can use that part of the worksheet to check the Ricci tensor components. For example, consider the component R_{12}. According to Eq. (4), we have

$$R_{12} \equiv \partial_\alpha \Gamma^\alpha_{12} - \partial_2 \Gamma^\alpha_{1\alpha} + \Gamma^\alpha_{\alpha\gamma}\Gamma^\gamma_{12} - \Gamma^\alpha_{2\sigma}\Gamma^\sigma_{1\alpha}. \tag{A4}$$

If we expand the implied sums in the first two terms of this expression and note that (according to the Diagonal Metric Worksheet) Christoffel symbols of the form Γ^α_{12} are nonzero only if $\alpha = 1$ or 2, we get

$$\begin{aligned}\partial_\alpha \Gamma^\alpha_{12} - \partial_2 \Gamma^\alpha_{1\alpha} = {} & \partial_1 \Gamma^1_{12} + \underline{\partial_2 \Gamma^2_{12}} - \partial_2 \Gamma^0_{10} \\ & - \partial_2 \Gamma^1_{11} - \underline{\partial_2 \Gamma^2_{12}} - \partial_2 \Gamma^3_{13}.\end{aligned} \tag{A5}$$

Note that the underlined terms cancel. Substituting in the worksheet values of the other Christoffel symbols and evaluating the derivatives yields

$$\begin{aligned}\partial_\alpha \Gamma^\alpha_{12} - \partial_2 \Gamma^\alpha_{1\alpha} &= \partial_1\left(\frac{1}{2B}B_2\right) - \partial_2\left(\frac{1}{2A}A_1\right) \\ &\quad - \partial_2\left(\frac{1}{2B}B_1\right) - \partial_2\left(\frac{1}{2D}D_1\right) \\ &= -\frac{1}{2B^2}B_1B_2 + \frac{1}{2B}B_{12} + \frac{1}{2A^2}A_2A_1 \\ &\quad - \frac{1}{2A}A_{21} + \frac{1}{2B^2}B_2B_1 - \frac{1}{2B}B_{21} \\ &\quad + \frac{1}{2D^2}D_2D_1 - \frac{1}{2D}D_{21}.\end{aligned} \tag{A6}$$

Note that the four terms involving the Bs cancel because neither the order of multiplication nor that of partial differentiation matters.

To simplify the third and fourth terms, we first use the Diagonal Metric Worksheet to identify nonzero terms. Christoffel symbols of the form Γ^γ_{12} are nonzero only if $\gamma = 1$ or 2, so the third term becomes

$$\begin{aligned}\Gamma^\alpha_{\alpha\gamma}\Gamma^\gamma_{12} = {} & (\Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31})\Gamma^1_{12} \\ & + (\Gamma^0_{02} + \Gamma^1_{12} + \Gamma^2_{22} + \Gamma^3_{32})\Gamma^2_{12}.\end{aligned} \tag{A7}$$

Of the sixteen terms in the double sum expressed by the fourth term in Eq. (A4), the Diagonal Metric Worksheet tells us that only following six terms are nonzero

$$\begin{aligned}-\Gamma^\alpha_{2\sigma}\Gamma^\sigma_{1\alpha} = {} & -\Gamma^0_{20}\Gamma^0_{10} - \underline{\Gamma^1_{21}\Gamma^1_{11}} - \Gamma^1_{22}\Gamma^2_{11} \\ & - \underline{\Gamma^2_{21}\Gamma^1_{12}} - \underline{\Gamma^2_{22}\Gamma^2_{12}} - \Gamma^3_{23}\Gamma^3_{13}.\end{aligned} \tag{A8}$$

(One can check this by noting that for a diagonal metric, a Christoffel symbol is only nonzero if two of its three indices are the same.) If we recall that Christoffel symbols are symmetric in their lower indices, then we see that the three underlined terms in Eq. (A8) each cancel a corresponding term in Eq. (A7). The remaining terms are

$$\Gamma^{\alpha}_{\alpha\gamma}\Gamma^{\gamma}_{12} - \Gamma^{\alpha}_{2\sigma}\Gamma^{\sigma}_{1\alpha} = (\Gamma^{0}_{01} + \Gamma^{3}_{31})\Gamma^{1}_{12} + (\Gamma^{0}_{02} + \Gamma^{1}_{12} + \Gamma^{3}_{32})\Gamma^{2}_{12} - \Gamma^{0}_{20}\Gamma^{0}_{10} - \Gamma^{1}_{22}\Gamma^{2}_{11} - \Gamma^{3}_{23}\Gamma^{3}_{13}. \tag{A9}$$

Substituting in the worksheet values of these components, we find that

$$\Gamma^{\alpha}_{\alpha\gamma}\Gamma^{\gamma}_{12} - \Gamma^{\alpha}_{2\sigma}\Gamma^{\sigma}_{1\alpha} = \frac{1}{4AB}A_1B_2 + \frac{1}{4BD}D_1B_2 + \frac{1}{4AC}A_2C_1 + \underline{\frac{1}{4BC}B_2C_1} + \frac{1}{4DC}D_2C_1 - \frac{1}{4A^2}A_1A_2 - \underline{\frac{1}{4BC}B_2C_1} - \frac{1}{4D^2}D_1D_2. \tag{A10}$$

Note that the underlined terms also happen to cancel (though this is not obvious from the Christoffel symbols). So, gathering the surviving terms in Eqs. (A6) and (A10) yields

$$\begin{aligned} R_{12} &= +\frac{1}{2A^2}A_2A_1 - \frac{1}{2A}A_{21} + \frac{1}{2D^2}D_2D_1 - \frac{1}{2D}D_{21} \\ &\quad + \frac{1}{4AB}A_1B_2 + \frac{1}{4BD}D_1B_2 + \frac{1}{4AC}A_2C_1 \\ &\quad + \frac{1}{4DC}D_2C_1 - \frac{1}{4A^2}A_1A_2 - \frac{1}{4D^2}D_1D_2 \\ &= +\frac{1}{4A^2}A_2A_1 - \frac{1}{2A}A_{21} + \frac{1}{4D^2}D_2D_1 - \frac{1}{2D}D_{21} \\ &\quad + \frac{1}{4AB}A_1B_2 + \frac{1}{4BD}D_1B_2 + \frac{1}{4AC}A_2C_1 \\ &\quad + \frac{1}{4DC}D_2C_1. \end{aligned} \tag{A11}$$

Here, we have again used the fact that multiplication is commutative to combine some terms. If we compare with the worksheet (and note that $A_{21} = A_{12}$ and $D_{21} = D_{12}$), we see that Eq. (A11) delivers exactly what the worksheet says R_{12} should be.

The calculation for, say, R_{12} is the kind of thing that you will want to do at most once in your life (if you can help it). (Note that the calculation for a *diagonal* component of the Ricci tensor will be even more complicated.) But if you *have* done it once, then subsequently using the Diagonal (or Off-Diagonal) Metric Worksheet means that you can solve the Einstein equation for even novel situations without having to do a calculation like those in this appendix *ever again*, while at the same time knowing that you thoroughly understand where its results come from (and that you *could* verify them all if necessary).

[a]Electronic mail: tmoore@pomona.edu

[1]N. Christensen and T. Moore, "Teaching general relativity to undergraduates," Phys. Today **65**(6), 41–47 (2012).
[2]J. B. Hartle, *Gravity* (Addison-Wesley, San Francisco, 2003).
[3]E. F. Taylor and J. A. Wheeler, *Exploring Black Holes: Introduction to General Relativity* (Addison Wesley Longman, San Francisco, 2000).
[4]T. Cheng, *Relativity, Gravitation and Cosmology* (Oxford U.P., Oxford, 2005).
[5]B. F. Schutz, *A First Course in General Relativity*, 2nd ed. (Cambridge U.P., New York, 2009).
[6]R. J. A. Lambourne, *Relativity, Gravitation, and Cosmology* (Cambridge U.P., New York, 2010).
[7]T. A. Moore, *A General Relativity Workbook* (University Science Books, Mill Valley, 2013).
[8]See the links under "Curvature and the Einstein Equation" on the website <http://web.physics.ucsb.edu/~gravitybook/mathematica.html>.
[9]See, for example, the computer tools "GRTensor II" (<http://grtensor.phy.queensu.ca>), "Maxima" (<http://maxima.sourceforge.net>), or "Ricci" (<http://www.math.washington.edu/~lee/Ricci/>). See also the list at <https://en.wikipedia.org/wiki/Tensor_software>, though this may not be up to date.
[10]One can find a variety of general relativity simulation programs (as well as other resources) at <http://www.compadre.org/relativity/search/search.cfm?gs=1374&SS=1384&b=1>.
[11]Programs for simulating orbits around black holes and the effective potential energy for objects falling in Kerr spacetime are available at <http://pages.pomona.edu/~tmoore/grw/resources.html>.
[12]W. Rindler, *Relativity: Special, General, and Cosmological*, 2nd ed. (Oxford U.P., Oxford, 2006), pp. 419–421. Note that although Rindler outlines a method for tabulating Ricci tensor components, he does not explicitly provide the tabulation.
[13]See supplementary material at http://dx.doi.org/10.1119/1.4939908, which includes PDF versions of the Diagonal Metric Worksheet and the Off-Diagonal Metric Worksheet, the Xojo source code, and a PDF listing of the source code. To run the computer code, download the Xojo IDE (free) from <http://xojo.com>. This program extensively uses a Xojo feature called "operator overloading," which allows for the definition of what the $+$, $-$, and $*$ operators do when they appear between variables pointing to instances of the Expression class. Please note that the program's output still required quite a bit of human post-processing to yield the Off-Diagonal Metric Worksheet. Also note that the program uses "G" where this paper and the worksheet use "H" (which we substituted in the post-processing stage because "G" could easily be confused with the universal gravitational constant).
[14]For a somewhat more detailed argument for this metric, see R. P. Gruber *et al.*, "The impossibility of a simple derivation of the Schwarzschild metric," Am. J. Phys. **56**, 265–269 (1988).
[15]For example, Ref. 12, p. 230; M. P. Hobson, G. Efstathiou, and A. N. Lasenby, *General Relativity: An Introduction for Physicists* (Cambridge U.P., Cambridge, 2006), p. 201; H. C. Ohanian and R. Ruffini, *Gravitation and Spacetime*, 2nd ed. (Norton, New York, 1994), p. 395; Ref. 7, pp. 268–269; Ref. 4, p. 90.
[16]Ref. 7, pp. 267–268.
[17]Ref. 7, problem P23.1 (page 276).
[18]Ref. 7, problem P23.6 (page 278).
[19]Ref. 7, pp. 292–302, 306.
[20]Ref. 7, pp. 376–378.
[21]Ref. 7, p. 271ff.
[22]Ref. 3, pp. B-4, B-12 through B-15.
[23]Ref. 7, pp. 180–182.
[24]Kruskal–Szekeres coordinates are discussed in Ref. 7, pp. 182–183, 186–187; Ref. 2, pp. 269–275; and many other general relativity textbooks.
[25]For example, Ref. 2, p. 198; Ref. 15 (Hobson *et al.*), p. 210; Ref. 7, p. 174 and p. 184.
[26]Ref. 15 (Hobson *et al.*), p. 256.