Q. Jl R. astr. Soc. (1985) **26**, 127–136

Understanding the Solutions of Einstein's Equations*

M.A.H. MacCallum

Theoretical Astronomy Unit, School of Mathematical Sciences, Queen Mary College, Mile End Road, London, E1 4NS

(Received 1984 September 17)

SUMMARY

The study of solutions of Einstein's equations of general relativity is of long standing, but has experienced a recent resurgence. New methods have been used, and new results obtained, for problems concerned with general classes of solutions, with approximate solutions and with exact solutions. Here a number of these developments are reviewed. They concern the identification and classification of solutions, the methods of finding new solutions, the properties of the space of solutions and general properties of classes of solutions, and the physical and astrophysical implications of particular solutions.

1 INTRODUCTION

Connoisseurs of this type of festive occasion will recognize my title as one of those which one selects when one is not quite sure what one is going to talk about, but wants to sound intriguing. After committing myself to it, I realized it also sounds as if my enthusiasm ran away with me, since the title suggests a technical character to the talk which would be inappropriate for such an occasion. One might even think that the organizers, by allowing me to choose such a title, had fallen into an error akin to that of an orchestral conductor who violates the rule 'Never encourage the brass, even by so much as a glance'. So I decided, having had time to repent of my choice, to mitigate its effects by working on the principle of the authors of *1066 and All That* (1) who included only those two dates in English history that everybody actually remembers: I prepared the talk by putting in only those points which I could actually remember. (For similar reasons, this written version follows the actual lecture quite closely, rather than expanding the technical details.)

In a sense, all work using general relativity is concerned with understanding properties of the solutions of Einstein's equations. However, what I want to talk about is the progress that has been made in understanding (using rigorous mathematics) either the behaviour of the space of all solutions or some subspace of it, or the properties of particular exact solutions, and in particular to describe results obtained by methods now in use among practitioners in the art of finding new solutions. I arrived at a series of four headings: the identification and classification of solutions; how to find solutions; the space of all solutions and general properties; and how to extract the physics. Before discussing these I want to sketch in a little of the background.

*Invited lecture at the McVittie conference, Canterbury, 1984 June 2.

The study of exact solutions began very early in the history of general relativity. Karl Schwarzschild found his famous solution, the first of those now identified as models of black holes, within a couple of months of Einstein's announcement of the basic equations of general relativity, despite being at the time in the German army on the Russian front (where he died a short time later). Quite a few more solutions, including most of those which appear in introductory texts, were found in the next decade or so.

Looking at the bibliography of the recent monograph on exact solutions of which I was a co-author (**2**), I noticed that among the 700 or so references cited, I could find only 30 published between our new octogenarian's paper of 1933 (**3**) on a point particle in an expanding universe, and his textbook of 1956 (**4**) (one of these, incidentally, being a famous paper (**5**) by the preceding speaker). I would not go so far as to say that McVittie's diversion to other matters was the cause of this dearth, another obvious reason being that the expanding relativistic cosmologies, with zero cosmological constant, led, if Hubble's distances to galaxies were correct, to an age of the Universe less than the geologically-known age of the Earth, a discrepancy only removed in 1952. But it is an interesting coincidence.

In 1956 I was at school. However the impression of the influence of Professor McVittie's text given to me by senior colleagues is borne out by the large number of citations of it (and its various revisions) which I found in the *Science Citation Index*. To me one of its merits is the coupling of an insistence on careful mathematics with a constant and detailed reference to observational data.

First let us consider the problems that face us. Newton's equation for the gravitational field, as Professor Bondi reminded us, is

$$\nabla^2\phi = 4\pi G\varrho$$

a partial differential equation with one dependent variable and three independent variables in which the coordinates are fixed up to constant linear transformations and (after expansion into partial derivatives with respect to the coordinates) there are just four terms. By contrast, Einstein's equations are a set of 10 equations in 10 dependent quantities (the components of a space-time metric) with four independent variables, in which the coordinates are completely arbitrary and which, when written out in terms of coordinate components of a metric, contain, according to Alan Held, 13 280 terms. We know that Newton's equations are difficult enough to deal with, so it is not surprising that we have even more trouble with Einstein's.

In practice, faced with this intractable set of equations, we do the obvious thing – cheat. The ways of cheating are many and various (see (**2**) for some of them) but they are all just ways of reducing the problem to more specialized cases which can be handled. Typically, this is done by assuming either that the solution has some symmetry, or simple initial and boundary conditions, or that there is some algebraic simplification (i.e. that terms which are present in the general case happen to vanish identically).

2 CLASSIFYING AND IDENTIFYING SOLUTIONS

It may seem perverse to start discussing the classification of solutions before saying anything about finding solutions; the reason for doing so is that we have to overcome the problem of the arbitrariness of the coordinates, which means that a given solution can appear in an infinite number of different disguises. Although this is in practice limited by humans' inability to solve the equations at all without the help of some simplifying choice of appropriate coordinates (e.g. spherical polars for a spherically symmetric solution), it still leads to a major problem, one which it is helpful to resolve before trying to find new solutions.

There is in fact a workable answer, based on ideas of Cartan and others. It is systematically to choose basis vectors ('axes') in directions fixed by invariant properties of the solutions themselves, such as eigenvectors of the energy-momentum tensor, and then compare quantities calculated using these axes as a basis for vectors and tensors. The result is to reduce the problem to a set of algebraic, rather than differential, equations which two solutions must obey if they are really the same solution in different disguises. For details see, e.g. (**6–8**). The quantities which have to be calculated for this comparison are components of the curvature and its derivatives, and are in fact invariants of the spaces, so that they are directly related to (local) physical properties.

Moreover, the whole method is amenable to computer calculation, and a set of programs to do just this, using the algebraic manipulation system SHEEP (**9**), has been written by J.Åman, G.Joly, myself and others. For some details see (**8, 10**). My standard example of an application is a family of solutions given by Melnick & Tabensky (**11**), which were shown by Bonnor and myself (**12**) to admit symmetries additional to those assumed by the original authors and hence to be equivalent to known metrics – including some studied by one G.C.McVittie!

The next stage of development using these methods is the construction of a database of known solutions, with which any supposed new solution could be compared, and it is in this that I, in collaboration with R.A.d'Inverno, J.Åman and others, am now engaged.

3 FINDING SPECIFIC NEW SOLUTIONS

As I said already, finding solutions is done by intelligent, and to some extent systematic, cheating. I do not propose to describe all possibilities now; it would take far too long. All I will do is give two particular examples (not entirely at random).

The first is the very classical problem of spherically symmetric perfect fluid solutions, the natural simple models for the structure of relativistic stars. The subclass whose properties have recently been further elucidated are those in which the fluid expands (or contracts) without shear. Stephani (**13**) has shown that the system of equations, having been reduced to a single equation

$$L_{,xx} = L^2 F(x)$$

where $L = L(t, x)$, can be studied by Lie's method of analysis of the invariance groups of differential equations in order to find all cases in which the equation (on choice of the arbitrary function F) is integrable. Remarkably, the same set of solutions, in a different coordinate choice, was found in another paper by considering the cases in which the general equations can be integrated in terms of elliptic functions or their specializations. In that same paper, the known occurrences of these solutions in the literature were listed, and their identification with the standard forms given. This *tour de force* is of course by Professor McVittie (**14**); it completes his 1933 work, and I understand his methods for solving the equations have applications to some problems in quantum theory, now being investigated in collaboration with Professors Chisholm and Burt.

The second development has also been in solutions depending on two variables but these are solutions in which the two other variables disappear completely from the metric (i.e. the symmetry generators commute) unlike the spherical case where (known) functions of angular coordinates are unavoidable. The metrics concerned are either stationary and axisymmetric or 'cylindrically symmetric'. The latter name covers the space-times in which both ignorable coordinates are space coordinates, but they are not necessarily truly cylindrical since it is not necessarily the case that one of the ignorable coordinates is angular (with a finite period) and parametrizes rotation about an axis. Moreover, 'cylindrically symmetric' spaces may take two forms, depending on whether the metric of the two-dimensional surfaces of symmetry is varying primarily in time or in space (more precisely, on the nature of the derivative of the gradient of the determinant of this metric); these are interpreted respectively as cosmological models and as examples of gravitational waves.

The remarkable discovery is that within the class of solutions just described there are methods of finding 'new solutions for old', that is to say, of deriving new solutions (whose matter content can be vacuum, electromagnetic fields, neutrino or massless scalar fields or the so-called 'stiff' fluids in which the speed of sound is the same as that of light) from known solutions. In fact there are a variety of different ways of generating such new metrics: Backlund transformations, the inverse scattering and soliton methods, the homogeneous Hilbert problem (giving the answer as a complex integral), and transformation groups of generating functions. All these are known in several other areas of application in which similar systems of non-linear partial differential equations arise, but one of the interesting aspects of the cases arising in relativity is the extent to which the interconnection of the different methods has been clarified. For full reviews of the methods, see the forthcoming book (**15**).

Many of the results obtained by these methods are of interest, and I would like to mention a few here. The most obvious is that it becomes possible to write an infinite number of papers giving new solutions of the Einstein equations. Fortunately, once this was appreciated, it immediately went out of favour; the aim now is not to give just any solution, but to find some with interesting properties. For example, it is possible to model complex situations such as a cosmological model with a multiplicity of gravitational waves. However, it is still not completely known how to construct solutions to fit prior physical specifications, and this aspect needs further investigation.

Secondly, it became possible to interrelate known solutions (see, e.g. (**2**), chapter 30, and the work of Kitchingham reported in my contribution to (**15**)). Thus one can now, for example, show how the well-known Kantowski–Sachs and Taub–NUT solutions are related.

Finally, one very nice, and perhaps surprising, result, is that in the stationary axisymmetric case (where the ellipticity of the equations enables one to prove analyticity properties) the previous aspect achieves a remarkable generalization: one can prove that all vacuum solutions, providing they have non-singular points on their axes, are obtainable, by transformations using the generating techniques, from the flat space of special relativity, Minkowski space (**49**). Thus for this case one gets everything from nothing!

4 GENERAL PROPERTIES OF SOLUTIONS

In recent years there have been studies of the complete system of Einstein equations which have aimed at characterizing the space of all solutions. (The work I mention here can be found in (**16–18**) and references therein.) These have shown that in general the space has the structure of an infinite-dimensional differentiable manifold which can be locally represented as the space of four functions of three variables. Moreover, the Einstein equations act on this space as a Hamiltonian system. The two approaches to reduction to the essential degrees of freedom of this space, that used by Fischer, Marsden and Moncrief and that of York, have recently been shown to be equivalent (**18**), in that they are related by canonical transformations; this is a nice result in that the York decomposition is in some ways the more intelligible (it works with the conformal structure of three-dimensional sections of space-time) but had been thought to be non-canonical.

The more alarming aspect of these studies is that (at least for compact spaces, to which many of the results are confined for technical reasons) the space of solutions loses its general structure just exactly at the solutions with symmetry. Instead the space, as represented in the standard way in the space of four functions of three variables, becomes 'cusped' and small variations are only permitted if they satisfy a quadratic condition first written down by Taub (once a colleague of McVittie's at Urbana). The consequence is that at these solutions with symmetry not all solutions of linearized perturbation theory are genuine tangents to families of solutions; they are spurious unless they obey the Taub condition. This in particular means that we must be careful when considering perturbations of 'closed' cosmological models, for example when studying galaxy formation in such models. Marsden has shown similar results for other Hamiltonian systems of equations with constraints.

More recently, Ashtekar (at the conference in Oxford in 1984 March) has announced the result that the spin coefficients of a null tetrad form a local (infinite-dimensional) coordinate system for the space of solutions, though I think only for the asymptotically flat spaces: details of this result will be interesting, as it may be useful in the identification problem mentioned in Section 2.

The spaces which are asymptotically flat, and which therefore provide models of those isolated physical systems such as two orbiting bodies for which 'infinity' can be taken to be at a distance small compared with cosmological scales, have been the arena for three interesting recent results on general properties of solutions. (Incidentally, it would be of interest to have more detailed work on the influence of exterior solutions which represent the cosmological environment more realistically.)

The most impressive of these is the proof that the energy of the space, by which is meant the mass (or monopole term in the gravitational field) as measured at infinity, has been shown to be non-negative, and to be zero precisely at flat space alone, so that flat empty space is energetically stable. This result was sought for many years, and eventually proved by Schoen & Yau (**19**), but even more remarkably it has been given a much shorter and more intelligible proof which is in a way simply a dressed-up version of the first-year undergraduate proof of the uniqueness of the solutions of Laplace's equations. The twist is that this proof, due to E.Witten (**20**), uses, in an essential way, spinor fields defined on space-like hypersurfaces and satisfying a special equation (the Witten equation), perhaps the first time spinors have been seen to be more than just a convenient way of rewriting tensorial equations.

The second topic, in which results have also been sought for a long time, is cosmic censorship. The aim, first enunciated by Penrose (**21**), is essentially to prove that singularities in the gravitational field can only be formed if they lie inside an event horizon (like the surface of a black hole) so that they are censored from the view of a distant observer. In fact we now have not just one theorem, but several, due to Krolak (**22–24**) and Newman (**25–27**). These have been proved under different technical assumptions whose relative merits have been the subject of some debate. As a spectator, I got the impression that Newman's conditions, which relate to the strength of the gravitational field, may prove the more useful, but probably both versions will have interesting applications. In any event, both sets of papers represent quite an achievement. Incidentally, Newman had G.C.McVittie as supervisor for his PhD thesis (and myself as an examiner).

The third result, due to Gibbons & Stewart (**28**), resolves a third long-standing debate, in this case about whether freely-falling bodies really radiate. What they proved, by a fairly simple integration of some of the Newman–Penrose (**29**) equations near the null infinity of an empty asymptotically flat space-time, is that such a space-time cannot be periodic and non-radiative. Hence in particular one cannot have an isolated binary orbiting system in periodic motion which does not radiate, even though each body may be in free-fall.

5 EXTRACTING THE PHYSICS

This is of course easily the most difficult part of the process of understanding solutions of the equations, whether these solutions are approximate or exact.

One example is that the problem of observations in cosmological models which have some of the lumpiness of the real universe has been tackled by using the 'Swiss cheese' model, in which spherical pieces of the usual isotropic homogeneous cosmologies are replaced by Schwarzschild black holes (**30, 31**), and by using the McVittie point-mass models (**32, 33**) and these models give quantitatively different results (**33**).

Another approximate calculation which has been put under the microscope recently is the derivation of the quadrupole formula for the emission of gravitational wave energy from an isolated system. This, for long of only theoretical interest, suddenly became observationally relevant with the discovery of the binary pulsar (**34**). Theoreticians knew that the derivation, which paralleled that for the dipole radiation of electromagnetism, was based on methods with very strong physical motivation but without mathematically rigorous proof, and indeed some authors claimed to have found counter-examples (**35**). This argument continues, but the present weight of opinion seems to be that the quadrupole approximation was indeed the correct one to apply to the binary pulsar, and hence the observations really do give a good test of the theory of general relativity (**36**).

The comparison with the binary pulsar data complemented the very extended calculations of Nordtvedt, Thorne, Will, Ni and others on the effects of non-Newtonian theories on the dynamics of the Solar System and gravitational effects on the Earth, which on comparison with observation have shown that general relativity does indeed give correct predictions while most of its competitors do not (see (**37, 38**) for some details). (These calculations are for 'metric' theories; their extension to non-metric theories of the 'metric-affine' type, involving a non-metric connection specifying particle paths, was considered by Coley, a former student of mine, in his PhD work (**39–41**).)

Turning back to exact solutions, there are a number of recent results of interest.

Ibañez & Verdaguer (**42**), and Carr & Verdaguer (**43**), have used metrics obtained by the generating techniques mentioned in Section 3 to study the influence of gravitational waves in cosmology, finding solutions that look like shock waves in the early Universe but settle to small perturbations in the far future.

Siklos (**44**) and Wainwright (**45**) have shown that certain spatially homogeneous cosmologies asymptotically approach plane-wave solutions, while if the cosmological constant is non-zero stationary solutions which are regular and asymptotically de Sitter are de Sitter space (**46**). This last result is of relevance to some popular variants of quantum gravity.

As a spin-off from the techniques used on the 'equivalence problem' (Section 2) Karlhede, Lindstrom & Åman (**47**) showed that there is an invariant, in principle measurable, which passes through zero exactly at the horizon of a Schwarzschild black hole. This invariant is composed of first derivatives of the curvature, and its properties mean that although the geometry shows no local singularity one can detect the approach to passing through the horizon and hence avoid it. Since the quantities used in the equivalence problem characterize the space, they must encode all its (local) properties so one might reasonably hope for further insights of this kind.

Goode (**48**) has taken the Szekeres metrics, almost the only known cosmological metrics in which the cheating to obtain a solution does not take the usual form of the imposition of symmetry, and shown that they can be rewritten in such a way that the time-dependent functions involved are the functions governing the standard cosmologies and their linearized perturbations. This means one has in some sense exact perturbations of the usual cosmologies.

Finally I would like to mention another solution obtained by the generating techniques. In the case of stationary axisymmetric solutions, one can start from the static Weyl solutions, which are writable in terms of axisymmetric Newtonian potentials (though these do not directly match the physical content, since for example the Schwarzschild solution is derived from the potential for a rod along the axis). Taking a pair of negative mass rods and applying four distinct transformations, one at each end of each rod, Hoenselaers & Dietz (**15**) have found a solution for a pair of spinning masses. Computing the mass M and angular momentum J for each body by integrals over surfaces surrounding the bodies, one can relate these to the properties of the singularity which is in general required on the axis joining the two bodies. Interpreting this singularity as representing the force necessary to keep the two bodies apart against their gravitational attraction, one finds that while it can be zero with certain choices of parameters, the forces for the spinning and non-spinning cases in general obey

$$\frac{F_{\text{spinning}}}{F_{\text{static}}} - 1 = -\frac{3}{d^2}\left[\frac{J_1}{M_1} + \frac{J_2}{M_2}\right]^2$$

where d is the separation of the bodies. The upshot is that if laboratory equipment could measure the Newtonian constant of gravity to 1 in 10^7 it could measure this gravitational spin-spin interaction, although it is a truly non-linear effect (a linearized approach gives only the J_1J_2 term). This is a fascinating prospect for the future.

6 CONCLUSION

I hope the methods and results I have described in this talk have shown that the study of approximate and exact solutions of Einstein's equations, far from being an outdated and dry subject, can play a very important part in advancing our understanding of gravitational physics and the world around us. Its continued and renewed vitality has much in common with, and owes something to, the characteristics of the man, slightly older than my subject, in whose honour this talk was presented.

REFERENCES

(1) Sellar, W.C. & Yeatman, R.J., 1930. *1066 and All That: a memorable history of England, comprising all the parts you can remember, including 103 good things, 5 bad kings, and 2 genuine dates*, Plymouth Press, London (reprinted by Penguin, London, 1960).

(2) Kramer, D., Stephani, H., MacCallum, M.A.H. & Herlt, E., 1980. *Exact Solutions of Einstein's Field Equations,* Deutscher Verlag der Wissenschaften, Berlin and Cambridge University Press (Russian edition, ed. Vladimirov, Yu.S., Energoisdat, Moscow, 1982).

(3) McVittie, G.C., 1933. The mass-particle in an expanding universe, *Mon. Not. R. astr. Soc.,* **93**, 325.

(4) McVittie, G.C., 1956. *General Relativity and Cosmology,* Chapman & Hall, London.

(5) Bondi, H., 1947. Spherically symmetrical models in general relativity, *Mon. Not. R. astr. Soc.,* **107**, 410.

(6) Ehlers, J., 1981. Christoffel's work on the equivalence problem for Riemannian spaces and its importance for modern field theories of physics, in *E.B.Christoffel*, eds Butzer, P.L. & Feher, E., Birkhauser Verlag, Basel.

(7) Karlhede, A., 1980. A review of the geometrical equivalence of metrics in general relativity, *Gen. Rel. grav.,* **12**, 693.

(8) MacCallum, M.A.H., 1983. Classifying metrics in theory and practice, in *Unified Field Theories of more than 4 Dimensions, Including Exact Solutions,* eds de Sabbata, V. & Schmutzer, E., World Scientific, Singapore.

(9) Frick, I., 1977. The computer algebra system SHEEP – what it can and cannot do in general relativity, *Stockh. Univ. Inst. theor. Phys. Rep. 77-14.*

(10) MacCallum, M.A.H., 1984. Algebraic computing in general relativity, *Classical General Relativity,* eds Bonnor, W.B., Islam, J.N. & MacCallum, M.A.H., Cambridge University Press.

(11) Melnick, J. & Tabensky, R., 1975. Exact solutions to Einstein field equations, *J. Math. Phys.,* **16**, 958.

(12) Bonnor, W.B. & MacCallum, M.A.H., 1982. The Melnick-Tabensky metrics have high symmetry, *J. Math. Phys.,* **22**, 1039.

(13) Stephani, H., 1983. A new interior solution of Einstein's field equations for a spherically symmetric perfect fluid in shearfree motion, *J. Phys. A,* **16**, 3529.

(14) McVittie, G.C., 1984. Elliptic Functions in spherically symmetric solutions of Einstein's equations, *Ann. Inst. H. Poincaré,* **40**, 235.

(15) Hoenselaers, C. & Dietz, W. (eds), 1984. *Solutions of Einstein's Equations: Techniques and Results (Proceedings, Retzbach, Germany, 1983)*, Lecture Notes in Physics 205, Springer-Verlag, Berlin.

(16) Fischer, A.E. & Marsden, J.E., 1980. The initial value problem and the dynamical formulation of general relativity, in *General Relativity, an Einstein Centenary Survey*, eds Hawking, S.W. & Israel, W., Cambridge University Press.

(17) Isenberg, J. & Marsden, J.E., 1983. A slice theorem for the space of solutions of Einstein's equations, *Phys. Rep.,* **89**, 179.

(18) Isenberg, J. & Marsden, J.E., 1983. The York map is canonical, in *Contributed papers 10th int. Conf. General Relativity and Gravitation*, eds Bertotti, B., de Felice, F. & Pascolini, A., Padova.

(19) Schoen, R. & Yau, S.-T., 1979. Proof of the positive mass theorem II, *Communs Math. Phys.,* **79**, 231.

(20) Witten, E., 1981. A new proof of the positive energy theorem, *Communs Math. Phys.,* **80**, 381.

(21) Penrose, R., 1969. Gravitational collapse: the role of general relativity, in *The Growth Points of Physics,* inaugural Conference of the European Physical Society, Florence, *Riv. Nuovo Cim.,* **1**, 252.

(22) Krolak, A., 1983. A proof of the cosmic censorship hypothesis, *Gen. Rel. grav.,* **15**, 99.

(23) Krolak, A., 1984. Black holes and the strong cosmic censorship, *Gen. Rel. grav.,* **16**, 121.

(24) Krolak, A., 1984. Black holes and the weak cosmic censorship, *Gen. Rel. grav.,* **16**, 365.

(25) Newman, R.P.A.C., 1983. Cosmic censorship and curvature growth, *Gen. Rel. grav.,* **15**, 641.

(26) Newman, R.P.A.C., 1984. A theorem of cosmic censorship: a necessary and sufficient condition for future asymptotic predictability, *Gen. Rel. grav.,* **16**, 175.

(27) Newman, R.P.A.C., 1984. Cosmic censorship, persistent curvature and asymptotic causal pathology, in *Classical General Relativity*, eds Bonnor, W.B., Islam, J.N. & MacCallum, M.A.H., Cambridge University Press.

(28) Gibbons, G.W. & Stewart, J.M., 1984. Absence of asymptotically flat solutions of Einstein's equations which are periodic and empty near infinity, in *Classical General Relativity*, eds Bonnor, W.B., Islam, J.N. & MacCallum, M.A.H., Cambridge University Press.

(29) Newman, E.T. & Penrose, R., 1962. An approach to gravitational radiation by a method of spin coefficients, *J. Math. Phys.*, **3**, 566.

(30) Kantowski, R., 1969. Corrections in the luminosity redshift relations of the homogeneous Friedmann models, *Astrophys. J.*, **155**, 89.

(31) Dyer, C.C. & Roeder, R.C., 1974. Observations in locally inhomogeneous cosmological models, *Astrophys. J.*, **174**, L115.

(32) Newman, R.P.A.C. & McVittie, G.C., 1982. A point particle model universe, *Gen. Rel. grav.*, **14**, 591.

(33) Newman, R.P.A.C., 1979. Singular perturbations of the empty Robertson-Walker cosmologies, *PhD thesis*, University of Kent.

(34) Taylor, J.H., 1975. Discovery of a pulsar in a binary system, *Ann. NY Acad. Sci.*, **262**, 490.

(35) Cooperstock, F.J. & Hobill, D., 1980. Free-fall sources of gravitational radiation and the failure of the quadrupole formula, *Astrophys. J.*, **235**, L55.

(36) Walker, M., 1984. The quadrupole approximation to gravitational radiation, in *Proc. 10th int. Conf. General Relativity and Gravitation*, eds Bertotti, B., de Felice, F. & Pascolini, A., Reidel, Dordrecht.

(37) Hellings, R., 1984. Testing relativity with solar system dynamics, in *Proc. 10th int. Conf. General Relativity and Gravitation*, eds Bertotti, B., de Felice, F. & Pascolini, A., Reidel, Dordrecht.

(38) Will, C.M., 1981. *Theory and Experiment in Gravitational Physics*, Cambridge University Press.

(39) Coley, A.A., 1983. Analysis of non-metric theories of gravity I: Electromagnetism, *Phys. Rev. D*, **27**, 728.

(40) Coley, A.A., 1984. Analysis of non-metric theories of gravity II: the weak equivalence principle, *Phys. Rev. D*, **28**, 1829.

(41) Coley, A.A., 1984. Analysis of non-metric theories of gravity III: summary of the analysis and its application to theories in the literature, *Phys. Rev. D*, **28**, 1844.

(42) Ibañez, J. & Verdaguer, E., 1983. Soliton collision in general relativity, *Phys. Rev. Lett.*, **51**, 1313.

(43) Carr, B.J. & Verdaguer, E., 1983. Soliton solutions and cosmological gravitational waves, *Phys. Rev. D*, **28**, 2995.

(44) Siklos, S.T.C., 1980. Field equations for spatially-homogeneous space-times, *Phys. Lett. A*, **76**, 19.

(45) Wainwright, J., 1983. A spatially homogeneous cosmological model with plane-wave singularity, *Phys. Lett. A*, **99**, 301.

(46) Boucher, W., 1984. Cosmic no-hair theorems, in *Classical General Relativity*, eds Bonnor, W.B., Islam, J.N. & MacCallum, M.A.H., Cambridge University Press.

(47) Karlhede, A., Lindstrom, U. & Åman, J.E., 1982. A note on a local effect at the Schwarzschild sphere, *Gen. Rel. grav.*, **14**, 569.

(48) Goode, S.W., 1983. Spatially inhomogeneous cosmologies and their relation with the Friedman–Robertson–Walker models, *PhD thesis*, University of Waterloo.

(49) Hauser, I. & Ernst, F.J., 1981. On a Geroch conjecture, *J. Math. Phys.*, **22**, 1057.