

# Schwarzschild and Kerr Solutions of Einstein's Field Equation — an introduction —

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## Abstract

Starting from Newton's gravitational theory, we give a general introduction into the spherically symmetric solution of Einstein's vacuum field equation, the Schwarzschild(-Droste) solution, and into one specific stationary axially symmetric solution, the Kerr solution. The Schwarzschild solution is unique and its metric can be interpreted as the exterior gravitational field of a spherically symmetric mass. The Kerr solution is only unique if the multipole moments of its mass and its angular momentum take on prescribed values. Its metric can be interpreted as the exterior gravitational field of a suitably rotating mass distribution. Both solutions describe objects exhibiting an *event horizon*, a frontier of no return. The corresponding notion of a black hole is explained to some extent. Eventually, we present some generalizations of the Kerr solution.

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# 1 Prelude<sup>1</sup>

In Sec.1.1, we provide some background material on Newton’s theory of gravity and, in Sec.1.2, on the flat and gravity-free Minkowski space of special relativity theory. Both theories were superseded by Einstein’s gravitational theory, general relativity. In Sec.1.3, we supply some machinery for formulating Einstein’s field equation without and with the cosmological constant.

## 1.1 Newtonian gravity

Newton’s gravitational theory is described—in particular tidal gravitational forces—and applied to a spherically symmetric body (a “star”).

*Gravity exists in all bodies universally and is proportional to the quantity of matter in each [...] If two globes gravitate towards each other, and their matter is homogeneous on all sides in regions that are equally distant from their centers, then the weight of either globe towards the other will be inversely as the square of the distance between the centers.*

Isaac Newton[136] (1687)

The gravitational force of a point-like mass  $m_2$  on a similar one of mass  $m_1$  is given by Newton’s attraction law,

$$\mathbf{F}_{2\rightarrow 1} = -G \frac{m_1 m_2}{|\mathbf{r}|^2} \frac{\mathbf{r}}{|\mathbf{r}|}, \quad (1)$$

where  $G$  is Newton’s gravitational constant (CODATA 2010),

$$G \stackrel{\text{SI}}{=} 6.67384(80) \times 10^{-11} \frac{(\text{m/s})^4}{\text{N}}.$$

The vector  $\mathbf{r} := \mathbf{r}_1 - \mathbf{r}_2$  points from  $m_2$  to  $m_1$ , see the Fig. 1.

According to *actio = reactio* (Newton’s 3rd law), we have  $\mathbf{F}_{2\rightarrow 1} = -\mathbf{F}_{1\rightarrow 2}$ . Thus a complete symmetry exists of the gravitational interaction of the two masses onto each other. Let us now distinguish the mass  $m_2$  as field-generating active gravitational mass and  $m_1$  as (point-like) passive test-mass. Accordingly, we introduce a hypothetical *gravitational field* as describing the force per unit mass ( $m_2 \leftrightarrow M$ ,  $m_1 \leftrightarrow m$ ):

$$\mathbf{f} := \frac{\mathbf{F}}{m} = -\frac{GM}{|\mathbf{r}|^2} \frac{\mathbf{r}}{|\mathbf{r}|}. \quad (2)$$

With this definition, the force acting on the test-mass  $m$  is equal to *field*

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<sup>1</sup>Parts of Secs.1 & 2 are adapted from our presentation[83] in Falcke et al.[56].

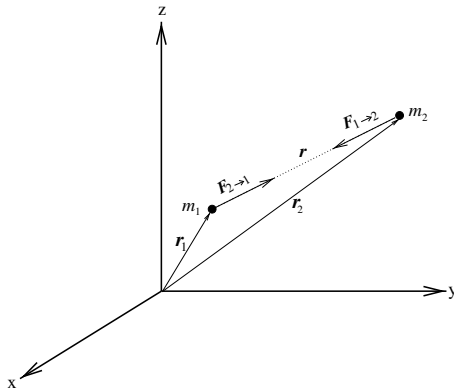


Figure 1: Two mass points  $m_1$  and  $m_2$  attracting each other in 3-dimensional space, Cartesian coordinates  $x$ ,  $y$ ,  $z$ .

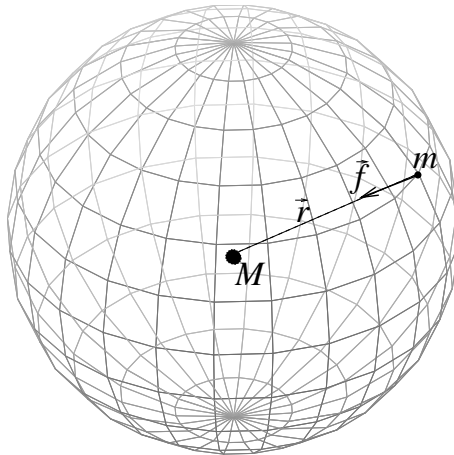


Figure 2: The “source”  $M$  attracts the test mass  $m$ .

*strength*  $\times$  *gravitational charge* (mass) or  $\mathbf{F}_{M \rightarrow m} = m \mathbf{f}$ , in analogy to electrodynamics. The active gravitational mass  $M$  is thought to emanate a gravitational field which is always directed to the center of  $M$  and has the same magnitude on every sphere with  $M$  as center, see Fig. 2. Let us now investigate the properties of the gravitational field (2). Obviously, there exists a potential

$$\phi = -G \frac{M}{|\mathbf{r}|}, \quad \mathbf{f} = -\nabla \phi. \quad (3)$$

Accordingly, the gravitational field is curl-free:  $\nabla \times \mathbf{f} = 0$ .

By assumption it is clear that the source of the gravitational field is the mass  $M$ . We find, indeed,

$$\nabla \cdot \mathbf{f} = -4\pi G M \delta^3(\mathbf{r}), \quad (4)$$

where  $\delta^3(\mathbf{r})$  is the 3-dimensional (3d) delta function. By means of the *Laplace operator*  $\Delta := \nabla \cdot \nabla$ , we infer for the gravitational potential

$$\Delta \phi = 4\pi G M \delta^3(\mathbf{r}). \quad (5)$$

The term  $M \delta^3(\mathbf{r})$  may be viewed as the mass density of a point mass. Eq.(5) is a 2nd order linear partial differential equation for  $\phi$ . Thus the gravitational potential generated by several point masses is simply the linear superposition of the respective single potentials. Hence we can generalize the *Poisson equation* (5) straightforwardly to a continuous matter distribution  $\rho(\mathbf{r})$ :

$$\Delta \phi = 4\pi G \rho. \quad (6)$$

This equation interrelates the source  $\rho$  of the gravitational field with the gravitational potential  $\phi$  and thus completes the quasi-field theoretical description of Newton's gravitational theory.

We speak here of *quasi*-field theoretical because the field  $\phi$  as such represents a convenient concept. However, it has no *dynamical* properties, no genuine degrees of freedom. The Newtonian gravitational theory is an *action at a distance* theory (also called *mass-interaction theory*). When we remove the source, the field vanishes instantaneously. Newton himself was very unhappy about this consequence. Therefore, he emphasized the preliminary and purely descriptive character of his theory. But before we liberate the gravitational field from this constraint by equipping it with its own degrees of freedom within the framework of general relativity theory, we turn to some properties of the Newtonian theory.

A very peculiar fact characteristic to the gravitational field is that the acceleration of a freely falling test-body does not depend on the mass of

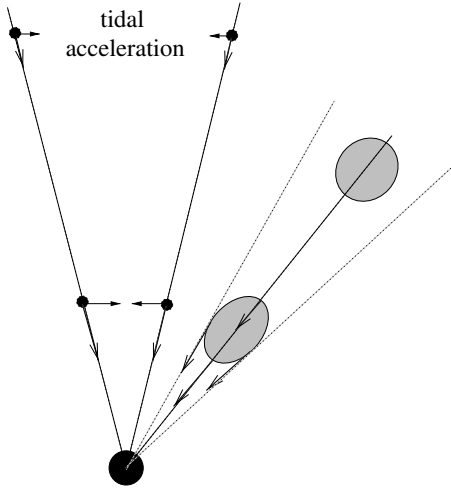


Figure 3: Tidal forces emerging between two freely falling particles and deforming a spherical body.

this body but only on its position within the gravitational field. This comes about because of the equality (in suitable units) of the gravitational and the inertial mass:

$$\overset{\text{inertial}}{m} \ddot{\mathbf{r}} = \mathbf{F} = \overset{\text{grav}}{m} \mathbf{f}. \quad (7)$$

This equality has been well tested since Galileo's time by means of pendulum and other experiments with an ever increasing accuracy, see Will [189].

In order to allow for a more detailed description of the structure of a gravitational field, we introduce the concept of *tidal force*. This can be best illustrated by means of Fig. 3. In a spherically symmetric gravitational field, for example, two test-masses will fall radially towards the center and thereby get closer and closer. Similarly, a spherical drop of water is deformed to an ellipsoidal shape because the gravitational force at its bottom is bigger than at its top, which has a greater distance to the source. If the distance between two freely falling test masses is relatively small, we can derive an explicit expression for their relative acceleration by means of a Taylor expansion. Consider two mass points with position vectors  $\mathbf{r}$  and  $\mathbf{r} + \delta\mathbf{r}$ , with  $|\delta\mathbf{r}| \ll 1$ . Then the relative acceleration reads

$$\delta\mathbf{a} = [\mathbf{f}(\mathbf{r} + \delta\mathbf{r}) - \mathbf{f}(\mathbf{r})] = \delta\mathbf{r} \cdot (\nabla\mathbf{f}). \quad (8)$$

We may rewrite this according to (the sign is conventional,  $\partial/\partial x^a =: \partial_a$ ,  $x^1 = x$ ,  $x^2 = y$ ,  $x^3 = z$ )

$$K_{ab} := -(\nabla\mathbf{f})_{ab} = -\partial_a f_b, \quad a, b = 1, 2, 3. \quad (9)$$

We call  $K_{ab}$  the *tidal force* matrix. The vanishing curl of the gravitational field is equivalent to its symmetry,  $K_{ab} = K_{ba}$ . Furthermore,  $K_{ab} = \partial_a \partial_b \phi$ . Thus, the Poisson equation becomes,

$$\sum_{a=1}^3 K_{aa} = \text{trace } K = 4\pi G \rho. \quad (10)$$

Accordingly, in vacuum  $K_{ab}$  is trace-free.

Let us now investigate the gravitational potential of a homogeneous *star* with constant mass density  $\rho_\odot$  and total mass  $M_\odot = (4/3)\pi R_\odot^3 \rho_\odot$ . For our Sun, the radius is  $R_\odot = 6.9598 \times 10^8$  m and the total mass is  $M = 1.989 \times 10^{30}$  kg.

Outside the sun (in the idealized picture we are using here), we have vacuum. Accordingly,  $\rho(\mathbf{r}) = 0$  for  $|\mathbf{r}| > R_\odot$ . Then the Poisson equation reduces to the *Laplace equation*

$$\Delta \phi = 0, \quad \text{for } r > R_\odot. \quad (11)$$

In 3d polar coordinates, the  $r$ -dependent part of the Laplacian has the form  $(1/r^2) \partial_r (r^2 \partial_r)$ . Thus (11) has the solution

$$\phi = \frac{\alpha}{r} + \beta, \quad (12)$$

where  $\alpha$  and  $\beta$  are integration constants. Requiring that the potential tends to zero as  $r$  goes to infinity, we get  $\beta = 0$ . The integration constant  $\alpha$  will be determined from the requirement that the force should change smoothly as we cross the star's surface, that is, the interior and exterior potential and their first derivatives have to be matched continuously at  $r = R_\odot$ .

Inside the star we have to solve

$$\Delta \phi = 4\pi G \rho_\odot, \quad \text{for } r \leq R_\odot. \quad (13)$$

We find

$$\phi = \frac{2}{3} \pi G \rho_\odot r^2 + \frac{C_1}{r} + C_2, \quad (14)$$

with integration constants  $C_1$  and  $C_2$ . We demand that the potential in the center  $r = 0$  has a finite value, say  $\phi_0$ . This requires  $C_1=0$ . Thus

$$\phi = \frac{2}{3} \pi G \rho_\odot r^2 + \phi_0 = \frac{G M(r)}{2r} + \phi_0, \quad (15)$$

where we introduced the *mass function*  $M(r) = (4/3)\pi r^3 \rho_\odot$  which measures the total mass inside a sphere of radius  $r$ .

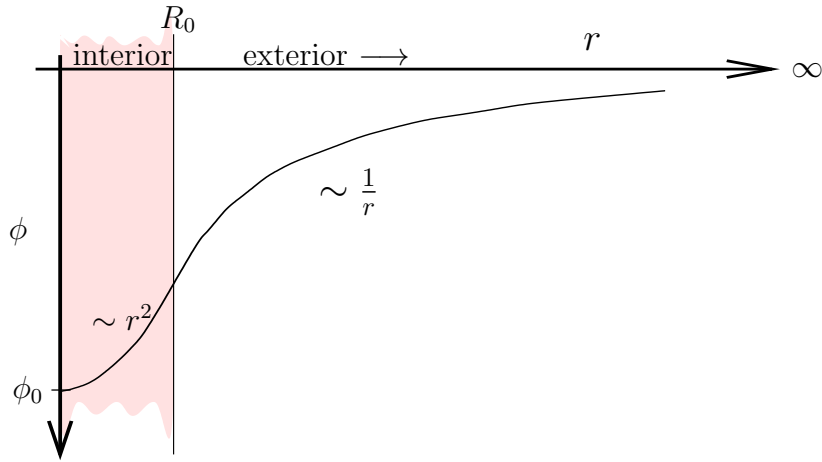


Figure 4: Newtonian potential of a homogeneous star.

Continuous matching of  $\phi$  and its first derivatives at  $r = R_\odot$  finally yields:

$$\phi(\mathbf{r}) = \begin{cases} -G \frac{M_\odot}{|\mathbf{r}|} & \text{for } |\mathbf{r}| \geq R_\odot, \\ G \frac{M_\odot}{2R_\odot^3} |\mathbf{r}|^2 - \frac{3GM_\odot}{2R_\odot} & \text{for } |\mathbf{r}| < R_\odot. \end{cases} \quad (16)$$

The slope of this curve indicates the magnitude of the gravitational force, the curvature (2nd derivative) the magnitude of the tidal force (or acceleration).



## 1.2 Minkowski space

*When, in a physical experiment, gravity can be safely neglected, we seem to live in the flat Minkowski space of special relativity theory. We introduce the metric of the Minkowski space and rewrite it in terms of so-called null coordinates, that is, we use light rays for a parametrization of Minkowski space.*

*Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.*

Hermann Minkowski (1908)

It was Minkowski who welded space and time together into spacetime, thereby abandoning the observer-independent meaning of spatial and temporal distances. Instead, the spatio-temporal distance, the line element,

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

is distinguished as the invariant measure of spacetime. The Poincaré (or inhomogeneous Lorentz) transformations form the invariance group of this spacetime metric. The principle of the constancy of the speed of light is embodied in the equation  $ds^2 = 0$ . Suppressing one spatial dimension, the solutions of this equation can be regarded as a double cone. This light cone visualizes the paths of all possible light rays arriving at or emitted from the cone's apex. Picturing the light cone structure, and thereby the causal properties of spacetime, will be our method for analyzing the meaning of the Schwarzschild and the Kerr solution.

### Null coordinates

We first introduce so-called null coordinates. The Minkowski metric (with  $c = 1$ ), in spherical polar coordinates reads

$$ds^2 = -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) = -dt^2 + dr^2 + r^2 d\Omega^2. \quad (17)$$

We define *advanced* and *retarded null coordinates* according to

$$v := t + r, \quad u := t - r, \quad (18)$$

and find

$$ds^2 = -dv du + \frac{1}{4} (v - u)^2 d\Omega^2. \quad (19)$$

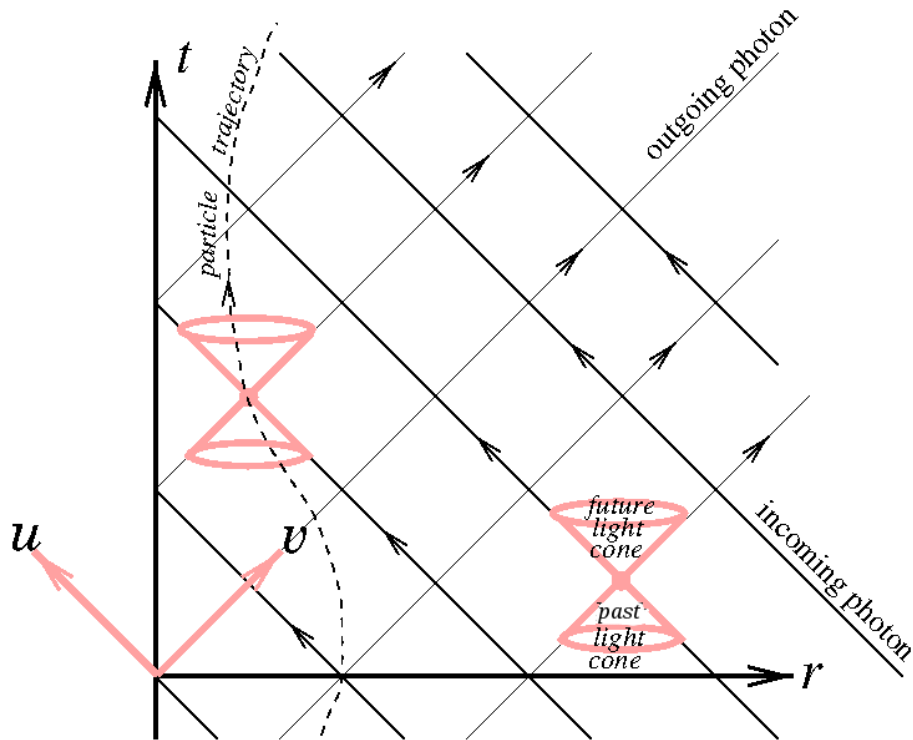


Figure 5: Minkowski spacetime in null coordinates

In Fig. 5 we show the Minkowski spacetime in terms of the new coordinates. Incoming photons, that is, point-like particles with velocity  $\dot{r} = -c = -1$ , move on paths with  $v = \text{const.}$  Correspondingly, we have for outgoing photons  $u = \text{const.}$  The special relativistic wave-equation is solved by any function  $f(u)$  and  $f(v)$ . The surfaces  $f(u) = \text{const.}$  and  $f(v) = \text{const.}$  represent the wavefronts which evolve with the velocity of light. The trajectory of every material particle with  $\dot{r} < c = 1$  has to remain inside the region defined by the surface  $r = t$ . In an  $(r, t)$ -diagram this surface is represented by a cone, the so-called *light cone*. Any point in the *future light cone*  $r = t$  can be reached by a particle or signal with a velocity less than  $c$ . A given spacetime point  $P$  can be reached by a particle or signal from the spacetime region enclosed by the *past light cone*  $r = -t$ .

## Penrose diagram

We can map, following Penrose, the infinitely distant points of spacetime into finite regions by means of a conformal transformation which leaves the light cones intact. Then we can display the whole infinite Minkowski spacetime on a (finite) piece of paper. Accordingly, introduce the new coordinates

$$\tilde{v} := \arctan v, \quad \tilde{u} := \arctan u, \quad \text{for } -\pi/2 \leq (\tilde{v}, \tilde{u}) \leq +\pi/2. \quad (20)$$

Then the metric reads

$$ds^2 = \frac{1}{\cos^2 \tilde{v}} \frac{1}{\cos^2 \tilde{u}} \left[ -d\tilde{v} d\tilde{u} + \frac{1}{4} \sin^2 (\tilde{v} - \tilde{u}) d\Omega^2 \right]. \quad (21)$$

We can go back to time- and space-like coordinates by means of the transformation

$$\tilde{t} := \tilde{v} + \tilde{u}, \quad \tilde{r} := \tilde{v} - \tilde{u}, \quad (22)$$

see (18). Then the metric reads,

$$ds^2 = \frac{-d\tilde{t}^2 + d\tilde{r}^2 + \sin^2 \tilde{r} d\Omega^2}{4 \cos^2 \frac{\tilde{t} + \tilde{r}}{2} \cos^2 \frac{\tilde{t} - \tilde{r}}{2}}, \quad (23)$$

that is, up to the function in the denominator, it appears as a flat metric. Such a metric is called conformally flat (it is conformal to a static Einstein cosmos). The back-transformation to our good old Minkowski coordinates reads

$$t = \frac{1}{2} \left( \tan \frac{\tilde{t} + \tilde{r}}{2} + \tan \frac{\tilde{t} - \tilde{r}}{2} \right), \quad (24)$$

$$r = \frac{1}{2} \left( \tan \frac{\tilde{t} + \tilde{r}}{2} - \tan \frac{\tilde{t} - \tilde{r}}{2} \right). \quad (25)$$

Our new coordinates  $\tilde{t}, \tilde{r}$  extend only over a finite range of values, as can be seen from (24) and (25). Thus, in the Penrose diagram of a Minkowski spacetime, see Fig. 6, we can depict the whole Minkowski spacetime, with a coordinate singularity along  $\tilde{r} = 0$ . All trajectories of uniformly moving particles (with velocity smaller than  $c$ ) emerge from one single point, past infinity  $I^-$ , and all will eventually arrive at the one single point  $I^+$ , namely at future infinity. All incoming photons have their origin on the segment  $\mathcal{I}^-$  (script  $I^-$  or “scri minus”), light-like past-infinity, and will run into the coordinate singularity on the  $\tilde{t}$ -axis. All outgoing photons arise from the coordinate singularity and cease on the line  $\mathcal{I}^+$ , light-like future infinity (“scri plus”). The entire spacelike infinity is mapped into the single point  $I^0$ . For later reference we collect these notions in a table:

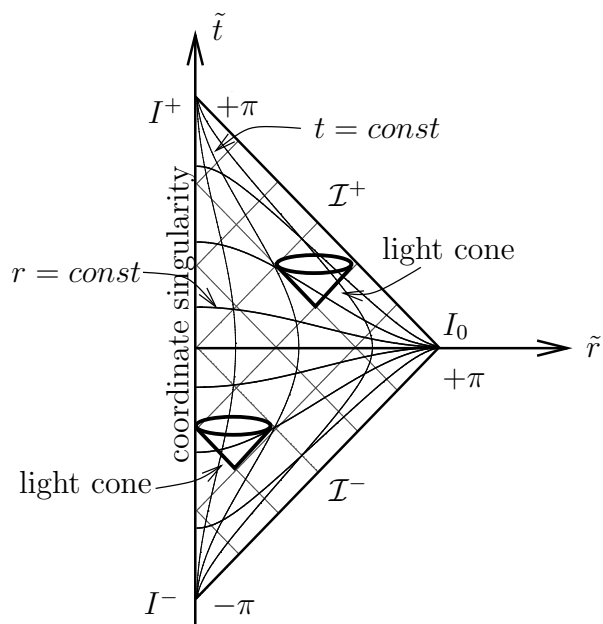


Figure 6: Penrose diagram of Minkowski spacetime.

**Table 1. The different infinities in Penrose diagrams**

$I^-$	timelike past infinity	origin of all particles
$I^+$	timelike future infinity	destination of all particles
$I_0$	spacelike infinity	inaccessible for all particles
$\mathcal{I}^-$	lightlike past infinity	origin of all light rays
$\mathcal{I}^+$	lightlike future infinity	destination of all light rays

Now, we have a really compact picture of the the Minkowski space. Next, we would like to proceed along similar lines in order to obtain an analogous picture for the Schwarzschild spacetime.

### 1.3 Einstein's field equation

We display our notations and conventions for the differential geometric tools used to formulate Einstein's field equation.

We assume that our readers know at least the rudiments of general relativity (GR) as represented, for instance, in Einstein's *Meaning of Relativity*[50], which we still recommend as a gentle introduction into GR. More advanced readers may then want to turn to Rindler[165] and/or to Landau-Lifshitz[106].

We assume a 4d Riemannian spacetime with (Minkowski-)Lorentz signature  $(-+++)$ , see Misner, Thorne, and Wheeler[127]. Thus, the metric field, in arbitrary holonomic coordinates  $x^\mu$ , with  $\mu = 0, 1, 2, 3$ , reads

$$\mathbf{g} \equiv ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu. \quad (26)$$

By partial differentiation of the metric, we can calculate the Christoffel symbols (Levi-Civita connection)

$$\Gamma^\mu{}_{\alpha\beta} := \frac{1}{2} g^{\mu\gamma} (\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}). \quad (27)$$

This empowers us to determine the geodesics (curves of extremal length) of the Riemannian spacetime:

$$\frac{D^2 x^\alpha}{D\tau^2} := \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0. \quad (28)$$

This equation can be read as a vanishing of the 4d covariant acceleration. If we define the 4-velocity  $u^\alpha := dx^\alpha/d\tau$ , then the geodesics can be rewritten as

$$\frac{Du^\alpha}{D\tau} = \frac{du^\alpha}{d\tau} + \Gamma^\alpha{}_{\mu\nu} u^\mu u^\nu = 0. \quad (29)$$

In a neighborhood of any given point in spacetime we can introduce Riemannian normal coordinates, which are such that the Christoffels vanish at that point. In order to find a tensorial measure of the gravitational field, we have to go one differentiation order higher. By partial differentiation of the Christoffels, we find the Riemann curvature tensor<sup>2</sup>

$$R^\mu{}_{\nu\alpha\beta} := 2 (\partial_{[\alpha} \Gamma^\mu{}_{|\nu|\beta]} + \Gamma^\mu{}_{\sigma[\alpha} \Gamma^\sigma{}_{|\nu|\beta]}) . \quad (30)$$

---

<sup>2</sup>Always symmetrizing of indices is denoted by parentheses,  $(\alpha\beta) := \{\alpha\beta + \beta\alpha\}/2!$ , antisymmetrization by brackets  $[\alpha\beta] := \{\alpha\beta - \beta\alpha\}/2!$ , with corresponding generalizations  $(\alpha\beta\gamma) := \{+\alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta + \dots\}/3!$ , etc.; indices standing between two vertical strokes  $| |$  are excluded from the (anti)symmetrization process, see Schouten[170].

The curvature is doubly antisymmetric, its two index pairs commute, and its totally antisymmetric piece vanishes:

$$R_{(\mu\nu)\alpha\beta} = 0, \quad R_{\mu\nu(\alpha\beta)} = 0; \quad R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}; \quad R_{[\mu\nu\alpha\beta]} = 0. \quad (31)$$

If we define collective indices  $A, B, \dots = 1, \dots, 6$  for the antisymmetric index pairs according to the rule  $\{01, 02, 03; 23, 31, 12\} \longrightarrow \{1, 2, 3; 4, 5, 6\}$ , then the algebraic symmetries of (31) can be rephrased as

$$R_{AB} = R_{BA}, \quad \text{trace}(R_{AB}) = 0. \quad (32)$$

Thus, in 4d the curvature can be represented as a trace-free symmetric  $6 \times 6$ -matrix. Hence it has 20 independent components.

With the curvature tensor, we found a tensorial measure for the gravitational field. Freely falling particles move along geodesics of Riemannian spacetime. What about the tidal accelerations between two freely falling particles? Let the ‘‘infinitesimal’’ vector  $n^\alpha$  describe the distance between two particles moving on adjacent geodesics. A standard calculation[127], linear to the order of  $n$ , yields the *geodesic deviation equation*

$$\frac{D^2 n^\alpha}{D\tau^2} = u^\beta u^\gamma R^\alpha{}_{\beta\gamma\delta} n^\delta. \quad (33)$$

This equation describes the relative acceleration of neighboring particles, similar as (8) and (9) in the Newtonian case. The role of the tidal matrix  $K_{ab}$  is taken over by  $\mathcal{K}^\alpha{}_\delta := u^\beta u^\gamma R^\alpha{}_{\beta\gamma\delta}$ .

By contraction of the curvature, we can define the 2nd rank Ricci tensor  $R_{\mu\nu}$  and the curvature scalar  $R$ , respectively:

$$R_{\mu\nu} := R^\alpha{}_{\mu\alpha\nu}, \quad R := g^{\mu\nu} R_{\mu\nu}. \quad (34)$$

For convenience, we can also introduce the Einstein tensor  $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ . The curvature with its 20 independent components can be irreducibly decomposed into smaller pieces according to  $20 = 10 + 9 + 1$ . The Weyl curvature tensor  $C_{\alpha\beta\gamma\delta}$  is trace-free and has 10 independent components, whereas the trace-free Ricci tensor has 9 components and the curvature scalar just 1.

Now we have all the tools for displaying Einstein’s field equation. With  $G$  as Newton’s gravitational constant and  $c$  as velocity of light, we define Einstein’s gravitational constant  $\kappa := 8\pi G/c^4$ . Then, the *Einstein field equation* with cosmological constant  $\Lambda$  reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \quad (35)$$

The source on the right-hand side is the energy-momentum tensor of matter. The *vacuum* field equation, without cosmological constant, simply reduces to  $R_{\mu\nu} = 0$ . Mostly this equation will keep us busy in this article. A vanishing Ricci tensor implies that only the Weyl curvature  $C_{\alpha\beta\gamma\delta} \neq 0$ . Accordingly, the vacuum field in GR (without  $\Lambda$ ) is represented by the Weyl tensor.

Eq.(35) represents a generalization of the Poisson equation (10). There, the contraction of the tidal matrix is proportional to the mass density; in GR, the contraction of the curvature tensor is proportional to the energy-momentum tensor.

The physical mass is denoted by  $M$ . Usually, we use the *mass parameter*,  $m := \frac{GM}{c^2}$ . The Schwarzschild radius reads  $r_S := 2m = \frac{2GM}{c^2}$ . Usually we put  $c = 1$  and  $G = 1$ . We make explicitly use of  $G$  and  $c$  as soon as we stress analogies to Newtonian gravity or allude to observational data.

## 2 The Schwarzschild metric (1916)

*Spatial spherical symmetry is assumed and a corresponding exact solution for Einstein's theory searched for. After a historical outline (Sec.2.1), we apply the equivalence principle to a freely falling particle and try to implement that on top of the Minkowskian line element. In this way, we heuristically arrive at the Schwarzschild metric (Sec.2.2). In Sec.2.3, we display the Schwarzschild metric in six different classical coordinate systems. We outline the concept of a Schwarzschild black hole in Sec.2.4. In Secs.2.5 and 2.6, we construct the Penrose diagram for the Schwarzschild(-Kruskal) spacetime. We add electric charge to the Schwarzschild solution in Sec.2.7. The interior Schwarzschild metric, with matter, is addressed in Sec.2.8.*

*It is quite a wonderful thing that from such an abstract idea the explanation of the Mercury anomaly emerges so inevitably.*

Karl Schwarzschild[171] (1915)

### 2.1 Historical remarks

*The genesis of the Schwarzschild solution (1915/16) is described. In particular, we show that Droste, a bit later than Schwarzschild, arrived at the Schwarzschild metric independently. He put the Schwarzschild solution into that form in which we use it today.*

The first exact solution of Einstein's field equation was born in hospital. Unfortunately, the circumstances were more tragic than joyful. The astronomer Karl Schwarzschild joined the German army right at the beginning of World War I and served in Belgium, France, and Russia. At the end of the year 1915, he was admitted to hospital with an acute skin disease. There, not far from the Russian front, enduring the distant gunfire, he found time to "stroll through the land of ideas" of Einstein's theory, as he puts it in a letter to Einstein<sup>3</sup> dated 22 December 1915. According to this letter, Schwarzschild started out from the approximate solution in Einstein's "perihelion paper", published November 25th. Since presumably letters from Berlin to the Russian front took a few days, Schwarzschild[172] found the solution within about a fortnight. Fortunately, the premature field equation of the "perihelion paper" is correct in the vacuum case treated by Schwarzschild.

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<sup>3</sup> The letters from and to Einstein can be found in Einstein's Collected Works[51], see also Schwarzschild's Collected Works[171].



In February 1916, Schwarzschild[173] submitted the spherically symmetric solution with matter—the “interior Schwarzschild solution”—now based on Einstein’s final field equation. In March 1916, he was sent home where he passed away on 11 May 1916.

The field equation used by Schwarzschild requires  $\det g = -1$ . To fulfill this condition, he uses modified polar coordinates (Schwarzschild’s original notation used),

$$x_1 = \frac{r^3}{2}, \quad x_2 = -\cos\theta, \quad x_3 = \phi, \quad x_4 = t.$$

The spherically symmetric ansatz then reads

$$ds^2 = f_4 dx_4^2 - f_1 dx_1^2 - f_2 \frac{dx_2^2}{1-x_2^2} - f_3 dx_3^2 (1-x_2^2),$$

where  $f_1$  to  $f_4$  are functions of  $x_1$  only. The solution turns out to be

$$f_1 = \frac{1}{R^4} \frac{1}{1-\alpha/R}, \quad f_2 = f_3 = R^2, \quad f_4 = 1 - \alpha/R, \quad R = (r^3 + \alpha^3)^{1/3}.$$

In this article, as well as in his letter to Einstein, he eventually returns to the usual spherical polar coordinates,

$$ds^2 = (1 - \alpha/R) dt^2 - \frac{dR^2}{1 - \alpha/R} - R^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad R = (r^3 + \alpha^3)^{1/3}.$$

This looks like the Schwarzschild metric we are familiar with. One should note, however, that the singularity at  $R = \alpha$  is (as we know today) a coordinate singularity, it corresponds to  $r = 0$ . In the early discussion the meaning of such a singularity was rather obscure. Flamm[60] in his 1916 article on embedding constant time slices of the Schwarzschild metric into Euclidean space mentions “the oddity that a point mass has an finite circumference of  $2\pi\alpha$ ”.

In 1917, Weyl[188] talks of the “inside” and “outside” of the point mass and states that “in nature, evidently, only that piece of the solution is realized which does not touch the singular sphere.” In Hilbert’s[86] opinion, the singularity  $R = \alpha$  indicates the illusiveness of the concept of a pointlike mass; a point mass is just the limiting case of a spherically symmetric mass distribution. Illuminating the interior of “Schwarzschild’s sphere” took quite a while and it was the discovery of new coordinates which brought first elucidations. Lanczos[104], in 1922, clearly speaks out that singularities of the metric components do not necessarily have physical significance since they may vanish in appropriate coordinates. However, it took another 38 years to find

a maximally extended fully regular coordinate system for the Schwarzschild metric. We will become acquainted with these Kruskal/Szekeres coordinates in Sec.2.5.

Schwarzschild’s solution, published in the widely read minutes of the Prussian Academy, communicated by Einstein himself, nearly instantly triggered further investigations of the gravitational field of a point mass. Already in March 1916, Reissner[163], a civil engineer by education, published a generalization of the Schwarzschild metric, including an electrical charge; this was later completed by Weyl[188] and by Nordström[140]. Today it is called Reissner-Nordström solution.

Nevertheless, one should not ignore the Dutch twin of Schwarzschild’s solution. On 27 May 1916, Droste[47] communicated his results on “the field of a single centre in Einstein’s theory of gravitation, and the motion of a particle in that field” to the Dutch Academy of Sciences. He presents a very clear and easy to read derivation of the metric and gives a quite comprehensive analysis of the motion of a point particle. Since 1913, he had been working on general relativity under the supervision of Lorentz at Leiden University. *Published in Dutch*, Droste’s results are fairly unknown today. Einstein, probably informed by his close friend Ehrenfest, rather appreciated Droste’s work, praising the graceful mathematical style. Weyl[188] also cites Droste, but in Hilbert’s[86] second communication the reference is not found. Einstein, Hilbert, and Weyl always allude to “Schwarzschild’s solution”.

After Droste took his PhD in 1916, he worked as school teacher and eventually became professor for mathematics in Leiden. He never resumed his work on Einstein’s theory and his name faded from the relativistic memoirs. In Leiden, people like Lorentz, de Sitter, Nordström, or Fokker learned about the gravitational field of a point mass primarily from Droste’s work. Thus, the name “Schwarzschild–Droste solution” would be quite justified from a historical point of view.

The importance of the Schwarzschild metric is made evident by the Birkhoff[16] theorem<sup>4</sup>: For vanishing cosmological constant, the unique spherically symmetric vacuum spacetime is the Schwarzschild solution, which can be expressed most conveniently in Schwarzschild coordinates, see Table 3, entry 1. Thus, a spherically symmetric body is static (outside the horizon). In particular, it cannot emit gravitational radiation. Moreover, the asymptotic Minkowskian behavior of the Schwarzschild solution is dictated by the solution itself, it is *not* imposed from the outside.

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<sup>4</sup>The “Birkhoff” theorem was discovered by Jebsen[95], Birkhoff[16], and Alexandrow[4]. For more details on Jebsen, see Johansen & Ravndal[96]. The objections of Ehlers & Krasiński[48] appear to us as nitpicking.

## 2.2 Approaching the Schwarzschild metric

We start from an ansatz for the metric of an accelerated motion in the radial direction and combine it, in the sense of the equivalence principle, with the free-fall velocity of a particle in a Newtonian gravitational field. In this way, we find a curved metric that, after a coordinate transformation, turns out to be the Schwarzschild metric.

Einstein, in his 1907 *Jahrbuch* article[49], suggests the generalization of the relativity principle to arbitrarily accelerated reference frames.

A plausible notion of a (local) rest frame in general relativity is a frame where the coordinate time is equal to the proper time (for an observer spatially at rest, of course). For a purely radial motion, the following metric would be an obvious ansatz, see also Visser[185]:

$$ds^2 = -dt^2 + [dr + f(r) dt]^2 + r^2 d\Omega^2, \quad \text{with} \quad d\Omega^2 := d\theta^2 + \sin^2 \theta d\phi^2.$$

For  $d\phi = 0$ ,  $d\theta = 0$ , and  $dr/dt = -f(r)$ , we have  $ds^2 = -dt^2$ . Thereby,  $-f(r)$  is identified as a kind of “radial infall velocity”. Note also that constant time-slices,  $dt = 0$ , are Euclidean.

In Newtonian gravity, a particle falling from infinity towards the origin picks up a velocity

$$\frac{dr}{dt} = v = -\sqrt{2\Phi(r)} = -\sqrt{\frac{2GM}{r}} \iff \frac{1}{2}mv^2(r) = m\Phi(r) = m\frac{GM}{r}. \quad (36)$$

Here,  $\Phi$  is the absolute value of the Newtonian potential of a spherical body with mass  $M$ .

Hence, in some Newtonian limit, we demand  $f(r) \rightarrow \sqrt{2\Phi}$ . This leads to the metric

$$ds^2 = -dt^2 + (dr + \sqrt{2\psi} dt)^2 + r^2 d\Omega^2, \quad (37)$$

where we allow for an arbitrary potential  $\psi = \psi(r)$ . This metric generates curvature. The calculations can be conveniently done even by hand. The Ricci tensor reads

$$R_0^0 = R_1^1 = \frac{1}{r} \partial_r \partial_r (r \psi) = 0, \quad R_2^2 = R_3^3 = \frac{2\partial_r(r \psi)}{r^2} = 0.$$

The equations  $R_0^0 = 0 = R_1^1$  are mere integrability conditions of the  $R_2^2 = 0 = R_3^3$  relations. Hence,  $r\psi$  is determined by its first order approximation alone and reads

$$\psi = \frac{\alpha}{r},$$

with  $\alpha$  as an unknown constant so far. By construction, we have

$$\frac{dr}{dt} = -\sqrt{2\psi} = -\sqrt{\frac{2\alpha}{r}} \stackrel{!}{=} -\sqrt{\frac{2GM}{r}} \implies \alpha = GM =: m.$$

The metric (37), expanding the parenthesis and collecting the terms in front of  $dt^2$ , reads

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + 2\sqrt{\frac{2GM}{r}} dt dr + dr^2 + r^2 d\Omega^2. \quad (38)$$

Using different methods, this metric was derived by Gullstrand[70] in May 1921. Gullstrand claimed to have found a new spherically symmetric solution of Einstein's field equation. In his opinion<sup>5</sup>, this showed the ambiguity of Einstein's field equation. However, the metric is of the form

$$ds^2 = -A dt^2 + 2B dt dr + dr^2 + r^2 d\Omega^2, \quad A := 1 - \frac{2GM}{r}, \quad B := \sqrt{\frac{2GM}{r}},$$

and can be diagonalized by completing the square via

$$ds^2 = -A \left(dt - \frac{B}{A} dr\right)^2 + \left(1 + \frac{B^2}{A}\right) dr^2 + r^2 d\Omega^2.$$

Introducing a new time coordinate,

$$dt_S := dt - \frac{B}{A} dr$$

or, explicitly,  $t_S = t - \left(2r\sqrt{\frac{2GM}{r}} - 4GM \operatorname{Artanh}\sqrt{\frac{2GM}{r}}\right)$ , we arrive at ( $A$  and  $B$  re-substituted)

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt_S^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

In contrast to what Gullstrand was aiming at, he “just” rederived the Schwarzschild metric.

Later, applying a coordinate transformation to the Schwarzschild metric, Painlevé[147] obtained the metric (38) independently and presented his result in October 1921. His aim was to demonstrate the vacuity of  $ds^2$  by

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<sup>5</sup>Gullstrand, who was a member of the Nobel committee, was responsible that Einstein did not get his Nobel prize for relativity theory. He thought that GR is untenable.

showing that an exact solution does not determine the physical geometry and is therefore meaningless. In a letter (Dec. 7<sup>th</sup> 1921) to Painlevé, Einstein stresses on the contrary *the meaninglessness of the coordinates!* In the words of Einstein himself (our translation): “... merely results obtained by eliminating the coordinate dependence can claim an objective meaning.”

In the subsequent section, we will meet the Schwarzschild metric in many different coordinate systems. All of them have their merits and their shortcomings.

Using Gullstrand-Painlevé coordinates for the Schwarzschild metric does not change the physics, of course. However, as a coordinate system it is what Gustav Mie[126] calls a *sensible* coordinate system. In contrast to many other coordinate systems, the physics looks quite like we are used to. As an example, we analyze the motion of a radial infalling particle in Schwarzschild and *Gullstrand-Painlevé coordinates*.

The equations of motion for point particles in general relativity are obtained via the geodesic equation (28). It can be shown that this equation is equivalent to the solution of the variational principle  $\delta \int_{x^\alpha} ds^2 = \delta \int \dot{x}^\alpha \dot{x}^\beta g_{\alpha\beta} d\tau^2$ . We choose the proper time  $\tau$  for the parametrization of the curve, the dot denotes the derivative with respect to  $\tau$ . In the present context, we are only interested in the velocity of particles along ingoing geodesics (“freely falling particles”). For time-like geodesics we have  $-1 = \frac{ds^2}{d\tau^2}$ . This allows the algebraic determination of  $\dot{r}$  provided we know  $\dot{t}$ . Since we consider static metrics here,  $t$  is a cyclic variable and  $\left(\frac{\partial}{\partial \dot{t}} \frac{ds^2}{d\tau^2}\right) = K = \text{const.}$  The constant is determined by the boundary condition  $\dot{r} = 0$  for  $r \rightarrow \infty$ . The calculation yields:

**Table 2.** Velocities in different coordinate systems<sup>6</sup>

		Schwarzschild	Gullstrand-Painlevé
particles	coordinate velocity $\frac{dr}{dt}$	$\pm \left(1 - \frac{2GM}{r}\right) \sqrt{\frac{2GM}{r}}$	$-\sqrt{\frac{2GM}{r}}$
	proper velocity $\frac{dr}{d\tau}$	$\pm \sqrt{\frac{2GM}{r}}$	$\pm \sqrt{\frac{2GM}{r}}$
light rays	coordinate velocity $\frac{dr}{dt}$	$\pm \left(1 - \frac{2GM}{r}\right)$	$\pm 1 - \sqrt{\frac{2GM}{r}}$

<sup>6</sup>The velocities of outgoing particles are valid only for the boundary condition specified. The *coordinate* velocity for outgoing particles in GP coordinates does not fit in our table and is thus suppressed.

The difference between the coordinate systems appears in the first line of Table 2: In Gullstrand-Painlevé coordinates, the coordinate velocity of a freely infalling particle increases smoothly towards the center. Nothing special happens at  $r = 2GM$ . From a given position, the particle will plunge into the center in a finite time. Even numerically this looks quite Newtonian. In contrast, the velocity with respect to Schwarzschild coordinates approaches zero as the particle approaches  $r = 2GM$ . Hence, the particle apparently will not be able to go further than  $r = 2GM$ .

For the Gullstrand-Painlevé metric for incoming light the radial coordinate velocity is always larger in magnitude than  $-1$ , at  $r = 2GM$  it is  $-2$ , for outgoing rays it vanishes at  $r = 2GM$  and is negative for  $r < 2GM$ .

Taking the mere numerical values is misleading. Contemplate for incoming light

$$\frac{\left(\frac{dr}{dt}\right)_{\text{particle}}}{\left(\frac{dr}{dt}\right)_{\text{light}}} = \frac{1}{1 + \sqrt{\frac{r}{2GM}}} \leq 1.$$

So the particle is always slower than light, however it approaches the velocity of light when approaching  $r = 0$ .

The Gullstrand-Painlevé form of the metric is regular at the surface  $r = 2GM$ . This shows that it is not any kind of barrier, but this observation was not made until much later, see Eisenstaedt [52].

### 2.3 Six classical representations of the Schwarzschild metric

*As we mentioned, a coordinate system should be chosen according to its convenience for describing a certain situation. In the following table (Table 3), we collect six widely used forms of the Schwarzschild metric.*

<b>Table 3.</b> Schwarzschild metric in various coordinates	coordinate transformation	characteristic properties
<b>Schwarzschild</b> $ds^2 = - \left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 d\Omega^2 .$ $(t, r, \theta, \phi)$	<hr/>	area of spheres $r = \text{const}$ is the “Euclidean” $4\pi r^2$
<b>Isotropic</b> $ds^2 = - \left(\frac{1 - m/2\bar{r}}{1 + m/2\bar{r}}\right)^2 dt^2 + \left(1 + \frac{m}{2\bar{r}}\right)^4 (d\bar{r}^2 + \bar{r}^2 d\Omega^2)$ $(t, \bar{r}, \theta, \phi)$	$r = \bar{r} \left(1 + \frac{m}{2\bar{r}}\right)^2 .$	constant-curvature time slices
<b>Eddington-Finkelstein</b> $ds^2 = - \left(1 - \frac{2m}{r}\right) dv^2 + 2dv dr + r^2 d\Omega^2 .$ $(u, r, \theta, \phi)$	$v = t + r + 2m \ln \left  \frac{r}{2m} - 1 \right $	ingoing light rays: $dv = 0$
<b>Kerr-Schild</b> $ds^2 = (\eta_{\alpha\beta} + 2ml_{\alpha}l_{\beta}) dx^{\alpha} dx^{\beta} ; l_{\alpha} = \frac{1}{\sqrt{r}} \left(1, \frac{x}{\sqrt{r}}, \frac{y}{\sqrt{r}}, \frac{z}{\sqrt{r}}\right)$ $(\bar{t}, x, y, z)$	$\bar{t} = v - r$ $r^2 = x^2 + y^2 + z^2$	“Cartesian” coordinates
<b>Lemaître</b> $ds^2 = -dT^2 + \frac{2m}{r} dR^2 + r^2 d\Omega^2 , \quad r = \left[\frac{2\sqrt{2m}}{3}(R - T)\right]^{\frac{3}{2}}$ $(T, R, \theta, \phi)$	$dT = dt + \sqrt{\frac{2m}{r}} \frac{1}{1 - \frac{2m}{r}} dr$ $dR = dt + \sqrt{\frac{r}{2m}} \frac{1}{1 - \frac{2m}{r}} dr$	infalling particles: $dR = 0$
<b>Gullstrand-Painlevé</b> $ds^2 = - \left(1 - \frac{2m}{r}\right) d\tilde{t}^2 + 2\sqrt{\frac{2m}{r}} d\tilde{t} dr + dr^2 + r^2 d\Omega^2$ $(\tilde{t}, r, \theta, \phi)$	$dt = d\tilde{t} - \frac{dr}{\sqrt{\frac{r}{2m} - \sqrt{\frac{2m}{r}}}}$	infalling particles: $dr = -\sqrt{\frac{2m}{r}} d\tilde{t}$

## 2.4 The concept of a Schwarzschild black hole

*We first draw a simple picture of a black hole. The event horizon and the stationary limit emerge as characteristic features. These are subsequently defined in a more mathematical way.*

In 1783 John Michell communicated his thoughts *on the means of discovering the Distance, magnitude, etc. of the fixed stars, in consequence of the diminution of the velocity of their light ...* [125] to the Royal Society in London. In the context of Newton’s particle theory of light, he calculated that sufficiently massive stars exhibit a gravitational attraction to such vast an amount that even light could not escape. A few years later (1796) Pierre-Simon Laplace published similar ideas.

In modern notation, we may reconstruct the arguments as follows. We throw a mass  $m$  from the surface of the Earth, assuming that there were no air, in upward direction with an initial velocity  $v$ . It will always fall back, unless its initial velocity reaches a sufficiently high value  $v_{\text{escape}}$  providing the mass with such a kinetic energy that it can overpower the gravitational attraction of the Earth. Energy conservation yields then immediately the formula

$$v_{\text{escape}} = \sqrt{\frac{2GM_{\oplus}}{R_{\oplus}}},$$

where  $G$  is Newton’s gravitational constant and  $M$  and  $R_{\oplus}$  the mass and the radius of the spherically conceived Earth, respectively.

For the Earth we find  $v_{\text{escape}} \approx 11.2 \text{ km/s}$ . If we now compress the Earth appreciably (thought experiment!) until the escape velocity coincides with the speed of light  $v_{\text{escape}} = c$ , its compressed “Schwarzschild” radius becomes  $r_{\oplus} = 2GM_{\oplus}/c^2 \approx 1 \text{ cm}$ . For the Sun, with its mass  $M_{\odot}$ , we have<sup>7</sup>

$$r_{\odot} = \frac{2GM_{\odot}}{c^2} \approx 3 \text{ km}.$$

At any smaller radius the light will be confined to the corresponding body. This is an intuitive picture of a spherically symmetric invisible “black hole”.<sup>8</sup>

It is very intriguing to see how far-sighted Michell anticipated the status of today’s observational black hole physics:

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<sup>7</sup>For the sake of clarity, we display here the speed of light  $c$  explicitly.

<sup>8</sup>The phrase “black hole” is commonly associated with Wheeler (1968). It appears definitely earlier in the literature: In the January 1964 edition of the *Science News Letter* the journalist Ann Ewing entitled her report at the meeting of the American Association for the Advancement of Science in Cleveland “*Black Holes*” in space. And if you have a look into an arbitrary English language dictionary published before ca. 1970, you will learn that “black hole” refers to a notorious dungeon in Calcutta (now Kolkata) in the 18th century, apparently a place of no return ...



*If there should really exist in nature bodies, whose density is not less than that of the sun, and whose diameters are more than 500 times the diameter of the sun, since their light could not arrive at us; [...] we could have no information from sight; yet, if any other luminous bodies should happen to revolve about them we might still perhaps from the motions of these revolving bodies infer the existence of the central ones with some degree of probability . . .*

This could be a verdict on the current observations of the black hole Sgr A\* (“Sagittarius A-star”) in the center of our Milky Way—and this is *not* a thought experiment—for a popular account, see Sanders[168]. Sgr A\* has a mass of about  $4 \times 10^6 M_{\odot}$ . Thus its Schwarzschild radius is far from being minute, it is about  $3 \times 4 \times 10^6$  km or about 17 solar radii.



Figure 7: Not quite seriously: “Schwarzschild” (left) versus “Kerr” (right)

A cautionary remark has to be made, though, see Penrose[150]. In Newtonian gravity  $c$  has no absolute meaning like in special relativity. It is conceivable that the speed of light in strong Newtonian gravitational fields could be larger than  $c$ . Consequently, the Michell type argument becomes only pertinent if  $c$  is the maximal speed for all phenomena like in the Minkowski space of special relativity, or, if gravity is involved, in the Riemannian space of GR.

Let us follow the way of visualizing the black hole concept by means of everyday physics a bit further: We explore the Schwarzschild and, later in Sec.3.4, the Kerr spacetime by boat. Schwarzschild spacetime is mimicked by a hole in a lake in which the surrounding water plunges simply radially without whirling around (Monticello Dam, California). The water flowing towards the hole will drag our boat to the center. Our boat may move around quite freely as long as the current is weak.

However, at some distance from the hole, the current becomes so strong that our boat, engines working at their maximum power, merely can keep its position. This is the *stationary limit*. In the case of our circularly symmetric water hole the stationary limit forms a ring. Bad for the boat: The stationary

limit is also the ring of no return. At best, the boat remains at its position, it never will escape. Any millimeter across the stationary limit will doom the boat, it will be inevitably sucked into the throat. Accordingly, the stationary limit coincides in this spherically symmetric case with the so-called *event horizon*.

### Event horizon

In 1958, Finkelstein[59] characterized the surface  $r = 2m$  as a “semi-permeable membrane” in spacetime, that is, a surface which can be crossed only in one direction. As soon as our boat has passed the event horizon, it can never come back. This property can be formulated in an invariant way: The light cones at each point of the surface have to nestle tangentially to the membrane. In 1964, Penrose[149] termed the null cone which divides observable from unobservable regions an *event horizon*. Mathematically speaking, the event horizon is characterized by having *tangent vectors* which are *light-like* or null at all points. Therefore, the event horizon is a *null hypersurface*. This is what is meant by a *trapped surface*[29], see Fig. 8 and Fig. 9, left image: a compact, spacelike, 2-dimensional submanifold with the property that outgoing future-directed light rays converge in both directions everywhere on the submanifold. All these characterizations quite intuitively show up in the Penrose-Kruskal diagram to be discussed later.

In view of the preceding paragraph, we define a black hole as a region of spacetime separated from infinity by an event horizon, see Carroll[29] and Brill[22].

Observational evidence in favor of black holes was reviewed by Narayan and McClintock[129].

### Killing horizon

The stationary limit surface is rendered more precise in the notion of a *Killing horizon*. A particle at rest (with respect to the infinity of an asymptotically flat, stationary spacetime) is to be required to follow the trajectories of the timelike Killing vector<sup>9</sup>. However, if we have a Killing vector  $K$  describing

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<sup>9</sup> Using the definitions of the covariant derivative and of the Christoffel symbols, we can derive the following equation for an arbitrary vector  $K$ ,

$$K^\alpha \partial_\alpha g_{\mu\nu} = 2\nabla_{(\mu} K_{\nu)} - 2g_{\alpha(\mu} \partial_{\nu)} K^\alpha. \quad (39)$$

Assuming  $K^\alpha$  and  $g_{\mu\nu}$  to be constant in time, demands  $\nabla_{(\mu} K_{\nu)} = 0$ . Hence  $K$  has to be a Killing vector. In this coordinate system, we have  $K^\alpha K_\alpha = g_{00}$ . Although  $K$  acts as time translation, it is not necessarily timelike!

a stationary spacetime, then at some points  $K$  may become lightlike, that is  $K^\mu K_\mu = 0$ . If all these points build up a hypersurface  $\Sigma$ , then this null hypersurface is called a Killing horizon. Apparently, this notion is of a local character, in contrast to the definition of an event horizon, the definition of which refers to events in the future, it is of a nonlocal character, see Fig. 8.

As we will see for the Schwarzschild black hole, see Fig. 9, outside the black hole the Killing vector is timelike, that is,  $K^\mu K_\mu < 0$ , on the Killing horizon it becomes null  $K^\mu K_\mu = 0$  (by definition of the horizon), and inside it becomes spacelike  $K^\mu K_\mu > 0$ .

In the Schwarzschild case it will turn out that the event horizon and Killing horizon coincide, in the Kerr case they separate.

### Surface gravity

From the definition of the Killing horizon it can be shown[29] that the quantity

$$\kappa^2 := -\frac{1}{2}(\nabla_\mu K_\nu)(\nabla^\mu K^\nu) |_\Sigma \quad (40)$$

is constant on the Killing horizon and positive. The quantity  $\kappa$  is called *surface gravity*. In simple cases, it has the interpretation of an acceleration or gravitational force per unit mass on the horizon. In the Schwarzschild spacetime it takes the value  $\kappa = 1/4m$ , which is the acceleration of a particle with unit mass as seen from infinity, compare with the Newtonian “field strength” (2) for  $r = 2m$ :

$$f = \frac{GM}{r^2} = \frac{m}{(2m)^2} = \frac{1}{4m}. \quad (41)$$

In general, there is no such simple interpretation.

### Infinite redshift

Another property associated with the surface  $K^\mu K_\nu = 0$  is the infinite redshift. In view of the relation for the general relativistic time delay,

$$\tau_0(\vec{x}_B) = \frac{\sqrt{g_{tt}(\vec{x}_B)}}{\sqrt{g_{tt}(\vec{x}_A)}} \tau_0(\vec{x}_A).$$

$g_{tt} \rightarrow 0$  can be interpreted as follows. Consider  $\tau_0(\vec{x}_B)$  the time measured by a clock B resting well away from the Killing horizon, whereas clock A with  $\tau_0(\vec{x}_A)$  is nearly at the Killing horizon. If  $g_{tt}(\vec{x}_A) \rightarrow 0$  we get  $\tau_0(\vec{x}_B) \rightarrow \infty$ . From the point of view of clock B, clock A’s last signal, right before A hits

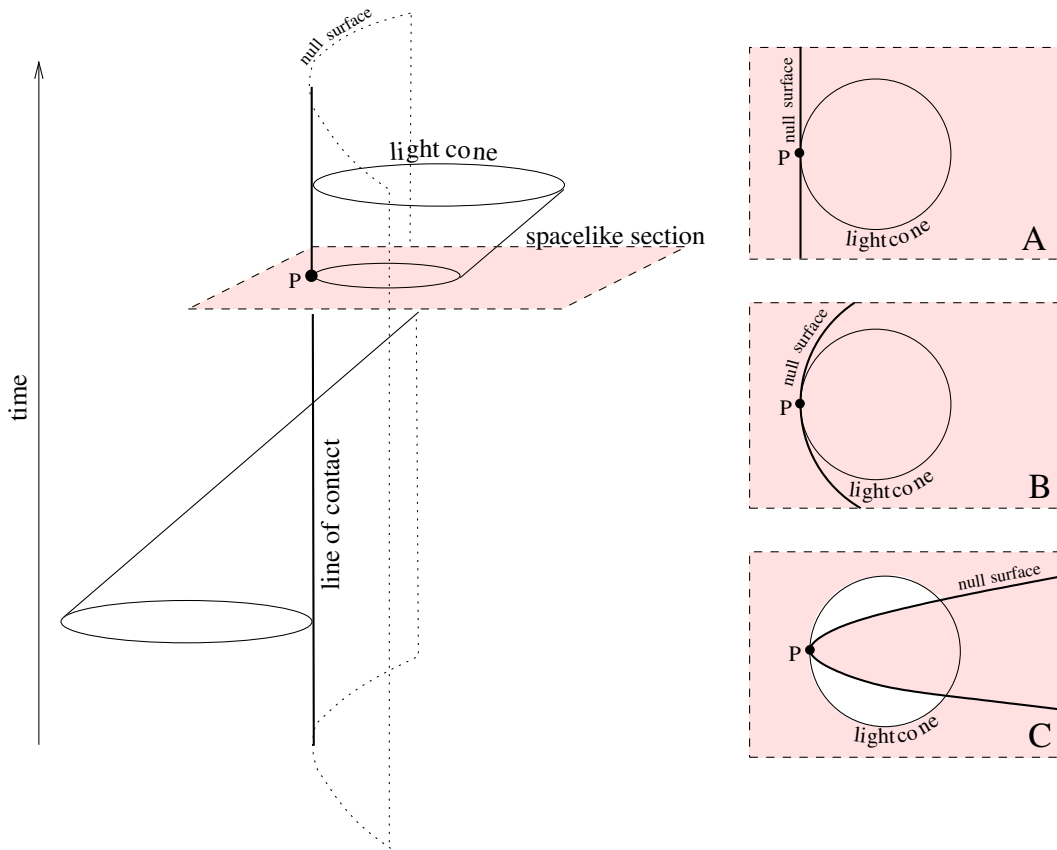


Figure 8: A null hypersurface is not necessarily an event horizon: Imagine a light cone that touches a hypersurface along the line of contact. Thus, the light cone is tangent as well as normal (in a spacetime sense) to the surface. Consequently, all such surfaces are null hypersurfaces. In the cases A and B, the light cone is entirely trapped inside the surface. Case A suggests that the surface does not close in a finite region, therefore the enclosed volume is not compact. Case B represents a (part of a) circle, which encloses all tangential light cones, and this forms an (black hole) event horizon. In case C, the light cone intersects the hypersurface. The white domain is outside the null surface but inside the light cone and, thus, reachable from within the enclosed domain.

the Killing horizon, will not reach B in a finite time, that is, never. To put it a little bit different: Signals sent with respect to A with constant frequency arrive increasingly delayed at B; for B the frequency approaches zero. This is called infinite redshift.

Let us work out these ideas for the Schwarzschild solution and let us take “photons” in spacetime instead of boats on a lake.

## 2.5 Using light rays as coordinate lines

*Schwarzschild coordinates exhibit a coordinate singularity at  $r = 2m$ . This obstructs the discussion of the event horizon considerably. As we have seen, light rays penetrate the horizon without difficulty. This suggests to use light rays as coordinate lines. Therefore we introduce in- and outgoing Eddington-Finkelstein coordinates. By combining both, we arrive at Kruskal-Szekeres coordinates, which provide a regular coordinate system for the whole Schwarzschild spacetime.*

### Eddington-Finkelstein coordinates

In relativity, light rays, the quasi-classical trajectories of photons, are null geodesics. In special relativity, this is quite obvious, since in Minkowski space the geodesics are straight lines and “null” just means  $v = c$ . A more rigorous argument involves the solution of the Maxwell equations for the vacuum and the subsequent determination of the normals to the wave surface (rays) which turn out to be null geodesics. This remains valid in general relativity. Null geodesics can be easily obtained by integrating the equation  $0 = ds$ . We find for the Schwarzschild metric, specializing to radial light rays with  $d\phi = 0 = d\theta$ ,

$$t = \pm \left( r + 2m \ln \left| \frac{r}{2m} - 1 \right| \right) + const. \quad (42)$$

If we denote with  $r_0$  the solution of the equation  $r + 2m \ln \left| \frac{r}{2m} - 1 \right| = 0$ , we have for the  $t$ -coordinate of the light ray  $t(r_0) =: v$ . Hence, if  $r = r_0$ , we can use  $v$  to label light rays. In view of this, we introduce  $v$  and  $u$

$$v := t + r + 2m \ln \left| \frac{r}{2m} - 1 \right|, \quad (43)$$

$$u := t - r - 2m \ln \left| \frac{r}{2m} - 1 \right|. \quad (44)$$

Then ingoing null geodesics are described by  $v = const$ , outgoing ones by  $u = const$ , see Fig. 9. We define *ingoing Eddington-Finkelstein coordinates*

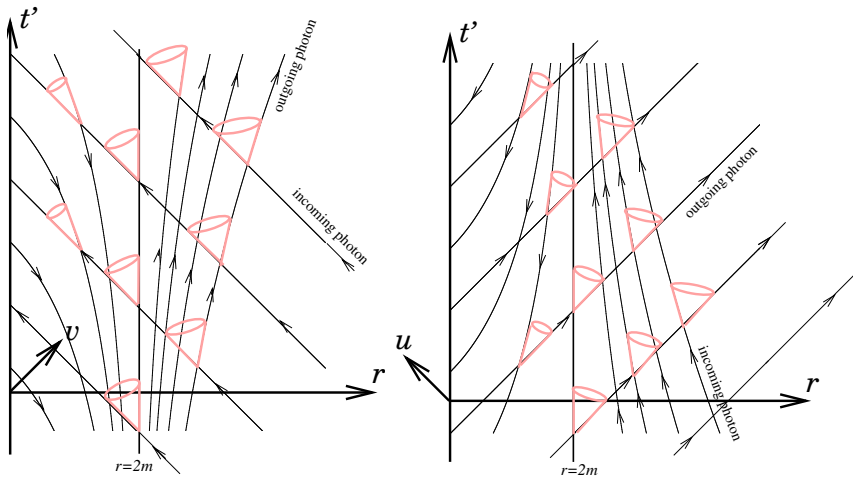


Figure 9: In- and outgoing Eddington-Finkelstein coordinates (where we introduce  $t'$  with  $v = t' + r$ ,  $u = t' - r$ ). The arrows indicate the direction of the original Schwarzschild coordinate time (and thereby the direction of the Killing vector  $\partial_t$ ). The *left* figure illustrates a *black hole*: All incoming photons traverse the event horizon and terminate in the singularity. The *right* figure illustrates a *white hole*: All outgoing photons emerge from the singularity, cross the horizon, and propagate out to infinity.

by replacing the “Schwarzschild time”  $t$  by  $v$ . In these coordinates  $(v, r, \theta, \phi)$ , the metric becomes

$$ds^2 = - \left( 1 - \frac{2m}{r} \right) dv^2 + 2dv dr + r^2 d\Omega^2. \quad (45)$$

For radial null geodesics  $ds^2 = d\theta = d\phi = 0$ , we find two solutions of (45), namely  $v = \text{const}$  and  $v = 4m \ln|r/2m - 1| + 2r + \text{const}$ . The first one describes infalling photons, i.e.,  $t$  increases if  $r$  approaches 0. At  $r = 2m$ , there is no singular behavior any longer for incoming photons. Ingoing Eddington-Finkelstein coordinates are particularly useful in order to describe the gravitational collapse. Analogously, for outgoing null geodesics take  $(u, r, \theta, \phi)$  as new coordinates. In these *outgoing Eddington-Finkelstein coordinates* the metric reads

$$ds^2 = - \left( 1 - \frac{2m}{r} \right) du^2 - 2du dr + r^2 d\Omega^2. \quad (46)$$

Outgoing light rays are now described by  $u = \text{const}$ , ingoing light rays by  $u = -(4m \ln|r/2m - 1| + 2r) + \text{const}$ . In these coordinates, the hypersurface

$r = 2m$  (the “horizon”) can be recognized as a null hypersurface (its normal is null or lightlike) and as a semi-permeable membrane.

### Kruskal-Szekeres coordinates

Next we try to combine the advantages of in- and outgoing Eddington-Finkelstein coordinates in the hope to obtain a fully regular coordinate system of the Schwarzschild spacetime. Therefore we assume coordinates  $(u, v, \theta, \phi)$ . Some (computer) algebra yields the corresponding representation of the metric:

$$ds^2 = - \left( 1 - \frac{2m}{r(u, v)} \right) du dv + r^2(u, v) d\Omega^2. \quad (47)$$

Unfortunately, we still have a coordinate singularity at  $r = 2m$ . We can get rid of it by reparametrizing the surfaces  $u = \text{const}$  and  $v = \text{const}$  via

$$\tilde{v} = \exp\left(\frac{v}{4m}\right), \quad \tilde{u} = -\exp\left(-\frac{u}{4m}\right). \quad (48)$$

In these coordinates, the metric reads ( $r = r(\tilde{u}, \tilde{v})$  is implicitly given by (48) and (44), (43),  $r_S = 2m$ )

$$ds^2 = -\frac{4r_S^3}{r(\tilde{u}, \tilde{v})} \exp\left(-\frac{r(\tilde{u}, \tilde{v})}{2m}\right) d\tilde{v} d\tilde{u} + r^2(\tilde{u}, \tilde{v}) d\Omega^2. \quad (49)$$

Again, we go back from  $\tilde{u}$  and  $\tilde{v}$  to time- and space-like coordinates:

$$\tilde{t} := \frac{1}{2} (\tilde{v} + \tilde{u}), \quad \tilde{r} := \frac{1}{2} (\tilde{v} - \tilde{u}). \quad (50)$$

In terms of the original Schwarzschild coordinates we have

$$\tilde{r} = \sqrt{\left| \frac{r}{2m} - 1 \right|} \exp\left(\frac{r}{4m}\right) \cosh \frac{t}{4m}, \quad (51)$$

$$\tilde{t} = \sqrt{\left| \frac{r}{2m} - 1 \right|} \exp\left(\frac{r}{4m}\right) \sinh \frac{t}{4m}. \quad (52)$$

The Schwarzschild metric

$$ds^2 = \frac{4r_S^3}{r} \exp\left(-\frac{r}{2m}\right) (-d\tilde{t}^2 + d\tilde{r}^2) + r^2 d\Omega^2, \quad (53)$$

in these *Kruskal-Szekeres* coordinates  $(\tilde{t}, \tilde{r}, \theta, \phi)$ , behaves regularly at the gravitational radius  $r = 2m$ . If we substitute (53) into the Einstein equation

(via computer algebra), then we see that it is a solution for all  $r > 0$ . Eqs.(51) and (52) yield

$$\tilde{r}^2 - \tilde{t}^2 = \left| \frac{r}{2m} - 1 \right| \exp\left(\frac{r}{2m}\right). \quad (54)$$

Thus, the transformation is valid only for regions with  $|\tilde{r}| > \tilde{t}$ . However, we can find a set of transformations which cover the entire  $(\tilde{t}, \tilde{r})$ -space. They are valid in different domains, indicated here by I, II, III, and IV, to be explained below:

$$(I) \quad \begin{cases} \tilde{t} = \sqrt{\frac{r}{2m} - 1} \exp\left(\frac{r}{4m}\right) \sinh \frac{t}{4m} \\ \tilde{r} = \sqrt{\frac{r}{2m} - 1} \exp\left(\frac{r}{4m}\right) \cosh \frac{t}{4m} \end{cases} \quad (55)$$

$$(II) \quad \begin{cases} \tilde{t} = \sqrt{1 - \frac{r}{2m}} \exp\left(\frac{r}{4m}\right) \cosh \frac{t}{4m} \\ \tilde{r} = \sqrt{1 - \frac{r}{2m}} \exp\left(\frac{r}{4m}\right) \sinh \frac{t}{4m} \end{cases} \quad (56)$$

$$(III) \quad \begin{cases} \tilde{t} = -\sqrt{\frac{r}{2m} - 1} \exp\left(\frac{r}{4m}\right) \sinh \frac{t}{4m} \\ \tilde{r} = -\sqrt{\frac{r}{2m} - 1} \exp\left(\frac{r}{4m}\right) \cosh \frac{t}{4m} \end{cases} \quad (57)$$

$$(IV) \quad \begin{cases} \tilde{t} = -\sqrt{1 - \frac{r}{2m}} \exp\left(\frac{r}{4m}\right) \cosh \frac{t}{4m} \\ \tilde{r} = -\sqrt{1 - \frac{r}{2m}} \exp\left(\frac{r}{4m}\right) \sinh \frac{t}{4m} \end{cases} \quad (58)$$

The inverse transformation is given by

$$\left(\frac{r}{2m} - 1\right) \exp\left(\frac{r}{2m}\right) = \tilde{r}^2 - \tilde{t}^2, \quad (59)$$

$$\frac{t}{4m} = \begin{cases} \text{Artanh } \tilde{t}/\tilde{r}, & \text{for (I) and (III),} \\ \text{Artanh } \tilde{r}/\tilde{t}, & \text{for (II) and (IV).} \end{cases} \quad (60)$$

The Kruskal-Szekeres coordinates  $(\tilde{t}, \tilde{r}, \theta, \phi)$  cover the entire spacetime, see Fig. 10. By means of the transformation equations we recognize that we need two Schwarzschild coordinate systems in order to cover the same domain. Regions (I) and (III) both correspond each to an asymptotically flat universe with  $r > 2m$ . Regions (II) and (IV) represent two regions with  $r < 2m$ . Since  $\tilde{t}$  is a time coordinate, we see that the regions are time reversed with respect to each other. Within these regions, real physical singularities (corresponding to  $r = 0$ ) occur along the curves  $\tilde{t}^2 - \tilde{r}^2 = 1$ . From the form of the metric we can infer that radial light-like geodesics (and therewith the light cones  $ds = 0$ ) are lines with slope 1. This makes the discussion of the causal structure particularly simple.



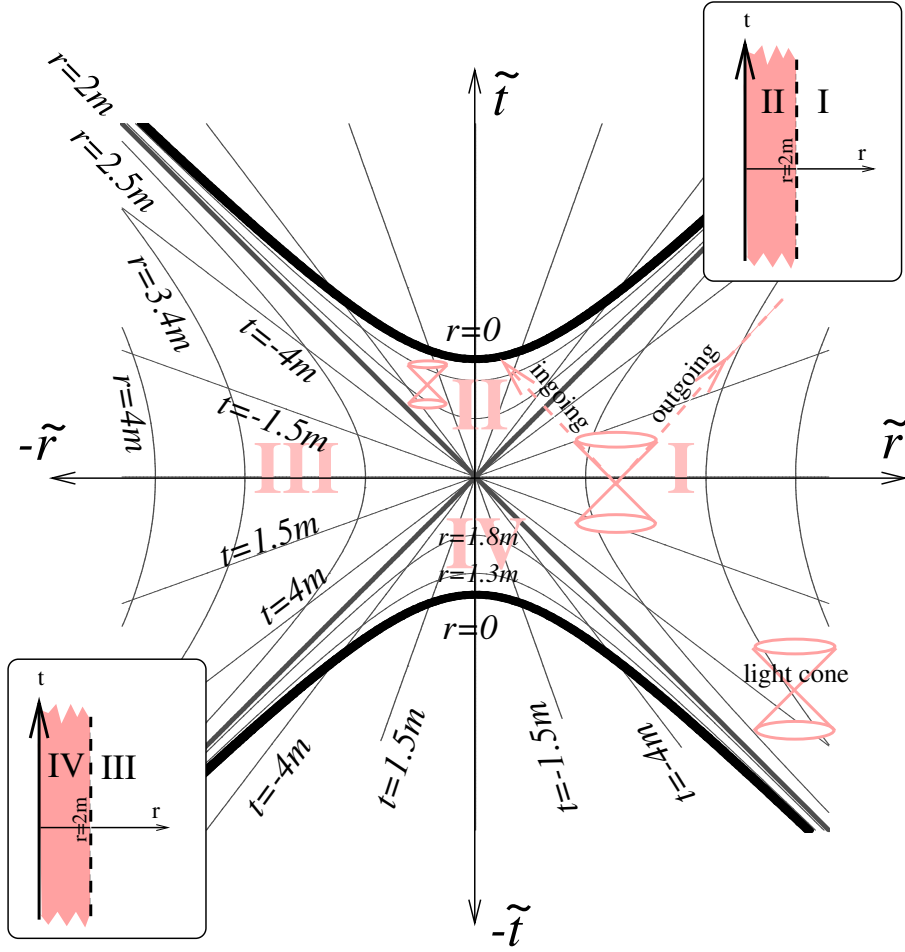


Figure 10: Kruskal-Szekeres diagram of the Schwarzschild spacetime.

## 2.6 Penrose-Kruskal diagram

We represent the Schwarzschild spacetime in a manner analogous to the Penrose diagram of the Minkowski spacetime. To this end, we proceed along the same line as in the Minkowskian case.

First, we switch again to null-coordinates  $v' = \tilde{t} + \tilde{r}$  and  $u' = \tilde{t} - \tilde{r}$  and perform a conformal transformation which maps infinity into the finite (again, by means of the tangent function). Finally we return to a time-like coordinate  $\hat{t}$  and a space-like coordinate  $\hat{r}$ . We perform these transformations all in one according to

$$\tilde{t} + \tilde{r} = \tan \frac{\hat{t} + \hat{r}}{2}, \quad \tilde{t} - \tilde{r} = \tan \frac{\hat{t} - \hat{r}}{2}. \quad (61,62)$$

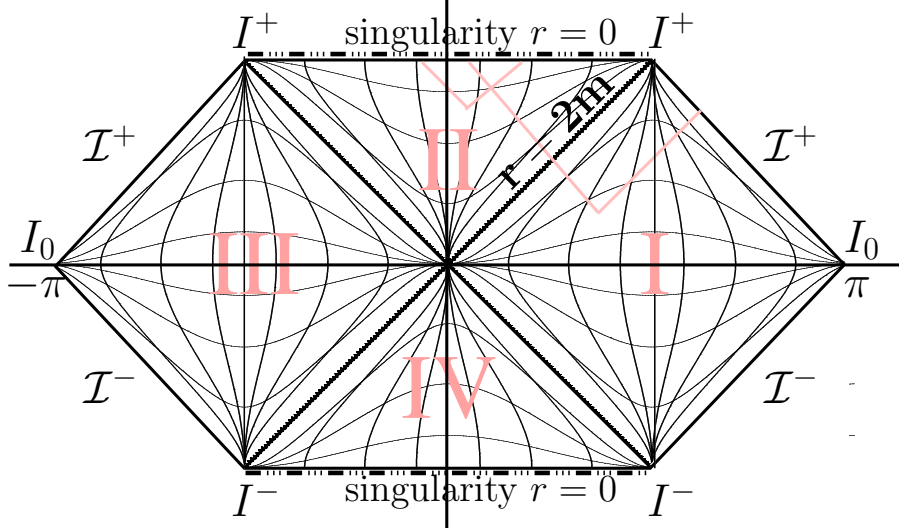


Figure 11: Penrose-Kruskal diagram of the Schwarzschild spacetime. Region II corresponds to a black hole, region IV to a white hole. Regions I and III correspond to two universes.

The Schwarzschild metric then reads

$$ds^2 = \frac{r_S^3}{r(\hat{r}, \hat{t})} \frac{\exp\left(-\frac{r(\hat{r}, \hat{t})}{2m}\right) (-d\hat{t}^2 + d\hat{r}^2)}{\cos^2 \frac{\hat{t} + \hat{r}}{2} \cos^2 \frac{\hat{t} - \hat{r}}{2}} + r^2(\hat{t}, \hat{r}) d\Omega^2, \quad (63)$$

where the function  $r(\hat{t}, \hat{r})$  is implicitly given by

$$\left(\frac{r}{2m} - 1\right) \exp\left(\frac{r}{2m}\right) = \tan \frac{\hat{t} + \hat{r}}{2} \tan \frac{\hat{t} - \hat{r}}{2}. \quad (64)$$

The corresponding Penrose-Kruskal diagram is displayed in Fig. 11. The notations for the different infinities can be extracted from Table 1. In contrast to Minkowski space, light rays and particles may not escape to infinity but enter the black hole (II). Likewise, light rays and particles may not emerge from infinity but from the white hole (IV).

## 2.7 Adding electric charge and the cosmological constant: Reissner Nordström

As mentioned in the historical remarks, soon after Schwarzschild's solution, the first generalizations, including electric charge and the cosmological constant were published. We can be even quicker ... We already calculated the Ricci tensor for the Gullstrand-Painlevé ansatz. If we use the well-known energy-momentum tensor for a point charge[81], the field equation may be written as<sup>10</sup>

$$R_{\mu}{}^{\nu} - \frac{1}{2} R \delta_{\mu}^{\nu} + \Lambda \delta_{\mu}^{\nu} = \kappa \lambda_0 \frac{q^2}{2r^4} \text{diag}(-1, -1, 1, 1). \quad (65)$$

Taking the trace, we find  $R = 4\Lambda$  and arrive at

$$R_2{}^2 = R_3{}^3 = \frac{2\partial_r(r\psi)}{r^2} = \Lambda - \frac{q^2}{r^4}. \quad (66)$$

This equation can be integrated elementarily,

$$2\psi = \frac{1}{3} \Lambda r^2 - \frac{q^2}{r^2} + \frac{2\alpha}{r}. \quad (67)$$

This function also solves the remaining two field equations. The integration constant  $\alpha$  is again the mass  $m$ . Substituted into (37) and transformed to Schwarzschild coordinates ( $f = 1 - 2\psi$ ) the solution reads

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad (68)$$

with

$$f(r) := 1 - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{\Lambda}{3} r^2. \quad (69)$$

A detailed derivation using Schwarzschild coordinates and computer algebra can be found in Puntigam et al.[158]

A discussion of the Reissner-Nordström(-de Sitter) solution can be found in Griffiths & Podolsky[69], for example. We only remark, that we recover the Schwarzschild solution for  $q = 0$  and  $\Lambda = 0$ . The algebraic structure of the solution is identical to the Schwarzschild case. Thus, we find, in general, a singularity at  $r = 0$ . However, a pure cosmological solution,  $m = 0, q = 0$

---

<sup>10</sup>Einstein's gravitational constant is denoted by  $\kappa$ ,  $\lambda_0 = \sqrt{\frac{\epsilon_0}{\mu_0}}$  is the admittance of the vacuum. With  $c = 1$  and  $G = 1$  we have  $\kappa\lambda_0 = 2$ .

and  $\Lambda \neq 0$ , possesses no singularity and no horizon! On the other hand, an electrically charged black hole,  $\Lambda = 0$ , exhibits *two* horizons,

$$f(r) = 0 \quad \Leftrightarrow \quad r_{\pm} = m \pm \sqrt{m^2 - q^2}. \quad (70)$$

In this respect, the charged black hole shows some similarities to a rotating (Kerr) black hole. We will pick up this discussion in Sec.3.4.

## 2.8 The interior Schwarzschild solution and the TOV equation

In the last section we investigated the gravitational field outside a spherically symmetric mass-distribution. Now it is time to have a look inside matter, see Adler, Bazin, and Schiffer [1]. Of course, in a first attempt, we have to make decisive simplifications on the internal structure of a star. We will consider cold catalyzed stellar material during the later phase of its evolution which can be reasonably approximated by a perfect fluid. The typical mass densities are in the range of  $\approx 10^7 \text{ g/cm}^3$  (white dwarfs) or  $\approx 10^{14} \text{ g/cm}^3$  (neutron stars, e.g., pulsars). In this context we assume vanishing angular momentum.

We start again from a static and spherically symmetric metric

$$ds^2 = -e^{A(r)} c^2 dt^2 + e^{B(r)} dr^2 + r^2 d\Omega^2 \quad (71)$$

and the energy-momentum tensor

$$T_{\mu\nu} = \left( \rho + \frac{p}{c^2} \right) u_{\mu} u_{\nu} + p g_{\mu\nu}, \quad (72)$$

where  $\rho = \rho(r)$  is the spherically symmetric mass density and  $p = p(r)$  the pressure (isotropic stress). This has to be supplemented by the equation of state which, for a simple fluid, has the form  $p = p(\rho)$ .

We compute the non-vanishing components of the field equation by means of computer algebra as (here  $\kappa = 8\pi G/c^4$  is Einstein's gravitational constant and  $(\prime) = d/dr$ )

$$-e^B \kappa r^2 c^2 \rho + e^B + B'r - 1 = 0, \quad (73)$$

$$-e^B \kappa p r^2 - e^B + A'r + 1 = 0, \quad (74)$$

$$-4e^B \kappa p r + 2A''r + (A')^2 r - A'B'r + 2A' - 2B' = 0. \quad (75)$$

The  $(\phi, \phi)$ -component turns out to be equivalent to the  $(\theta, \theta)$ -component. For convenience, we define a *mass function*  $m(r)$  according to

$$e^{-B} =: 1 - \frac{2m(r)}{r}. \quad (76)$$

We can differentiate (76) with respect to  $r$  and find, after substituting (73), a differential equation for  $m(r)$  which can be integrated, provided  $\rho(r)$  is assumed to be known

$$m(r) = \int_0^r \frac{\kappa}{2} \rho c^2 \xi^2 d\xi. \quad (77)$$

Differentiating (74) and using all three components of the field equation, we obtain a differential equation for  $A$ :

$$A' = -\frac{2p'}{p + \rho c^2}. \quad (78)$$

We can derive an alternative representation of  $A'$  by substituting (76) into (74). Then, together with (78), we arrive at the *Tolman-Oppenheimer-Volkoff* (TOV) equation

$$p' = -\frac{(\rho c^2 + p)(\mathbf{m} + \kappa p r^3/2)}{r(\mathbf{r} - 2m)}. \quad (79)$$

Terms that survive in the Newtonian limit are emphasized by boldface letters. The system of equations consisting of (77), (78), the TOV equation (79), and the equation of state  $p = p(\rho)$  forms a complete set of equations for the unknown functions  $A(r)$ ,  $\rho(r)$ ,  $p(r)$ , and  $m(r)$ , with

$$ds^2 = -e^{A(r)} c^2 dt^2 + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\Omega^2. \quad (80)$$

These differential equations have to be supplemented by initial conditions.

In the center of the star, there is, of course, no enclosed mass. Hence we demand  $m(0) = 0$ . The density has to be finite at the origin, i.e.  $\rho(0) = \rho_c$ , where  $\rho_c$  is the density of the central region. At the surface of the star, at  $r = R_\odot$ , we have to match matter with vacuum. In vacuum, there is no pressure which requires  $p(R_\odot) = 0$ . Moreover, the mass function should then yield the total mass of the star,  $m(R_\odot) := GM_\odot/c^2$ . Finally, we have to match the components of the metric. Therefore, we have to demand  $\exp[A(r_0)] = 1 - 2m(R_\odot)/R_\odot$ .

Equations (73), (74), (75) and certain regularity conditions which generalize our boundary conditions, that is,

- regularity of the geometry at the origin,
- finiteness of central pressure and density,
- positivity of central pressure and density,

- positivity of pressure and density,
- monotonic decrease of pressure and density,

impose conditions on the functions  $\rho$  and  $p$ . Then, even without the explicit knowledge of the equation of state, the general form of the metric can be determined. For recent work, see Rahman and Visser [162] and the literature given there.

We can obtain a simple solution, if we assume a constant mass density

$$\rho = \rho(r) = \text{const.} \quad (81)$$

One should mention here that  $\rho$  is not the physically observable fluid density, which results from an appropriate projection of the energy-momentum tensor into the reference frame of an observer. Thus, this model is not as unphysical as it may look at the first. However, there are serious but more subtle objections which we will not discuss further in this context.

When  $\rho = \text{const.}$ , we can explicitly write down the mass function (77),

$$m(r) = \frac{r^3}{2\widehat{R}^2}, \quad \text{with } \widehat{R} = \sqrt{\frac{3}{\kappa\rho c^2}}, \quad m_\odot := \frac{R_\odot^3}{2\widehat{R}^2}. \quad (82)$$

This allows immediately to determine one metric function

$$e^B = \frac{1}{1 - \frac{r^2}{\widehat{R}^2}}. \quad (83)$$

The TOV equation (79) factorizes according to

$$\frac{dp}{dr} = -\frac{1}{2}(\rho c^2 + p)(1 + \kappa\widehat{R}^2 p) \frac{r}{\widehat{R}^2 - r^2}. \quad (84)$$

It can be elementarily solved by separation of variables,

$$p(r) = \rho c^2 \frac{\sqrt{\widehat{R}^2 - R_\odot^2} - \sqrt{\widehat{R}^2 - r^2}}{\sqrt{\widehat{R}^2 - r^2} - 3\sqrt{\widehat{R}^2 - R_\odot^2}}. \quad (85)$$

Using (78) as  $A' = -2[\ln(p + \rho c^2)]'$  and continuous matching to the exterior, eventually yields the *interior* & *exterior Schwarzschild solution* for a spherically symmetric body [173]

$$ds^2 = \begin{cases} -\left(\frac{3}{2}\sqrt{1 - \frac{R_\odot^2}{\widehat{R}^2}} - \frac{1}{2}\sqrt{1 - \frac{r^2}{\widehat{R}^2}}\right)^2 c^2 dt^2 + \frac{1}{1 - \frac{r^2}{\widehat{R}^2}} dr^2 + r^2 d\Omega^2, & r \leq R_\odot, \\ -\left(1 - \frac{2m_\odot}{r}\right) c^2 dt^2 + \frac{1}{1 - \frac{2m_\odot}{r}} dr^2 + r^2 d\Omega^2, & r > R_\odot. \end{cases} \quad (86)$$

The solution is only defined for  $R_{\odot} < \widehat{R}$ . For the Sun<sup>11</sup> we have  $M_{\odot} \approx 2 \times 10^{30}$  kg,  $R_{\odot} \approx 7 \times 10^8$  m and accordingly  $\rho_{\odot} \approx 1.4 \times 10^3$  kg/m<sup>3</sup>. This leads to  $\widehat{R} \approx 3 \times 10^{11}$  m, that is, the radius of the star  $R_{\odot}$  is much smaller than  $\widehat{R}$ :  $R_{\odot} < \widehat{R}$ . Hence the square roots in (86) remain real.

The condition  $R_{\odot} < \widehat{R}$  suggests that a sufficiently massive object cannot be stable since no static gravitational field seems possible. This conjecture can be further supported. Even before reaching  $\widehat{R}$ , the central pressure becomes infinite,

$$p(0) \rightarrow \infty \text{ for } R_{\odot} \rightarrow \sqrt{\frac{8}{9}} \widehat{R}, \quad \text{or} \quad m_{\odot} \rightarrow \frac{4}{9} R_{\odot}. \quad (87)$$

If there is no static solution and the situation remains spherically symmetric, we are forced to the conclusion that such a mass distribution must radially collapse; either in an infinite time or to a single point in space. With reasonable simplifications, it was first shown by Oppenheimer and Snyder[145] that the second alternative is true: A very massive object collapses to a black hole. As various singularity theorems show today, this behavior is indeed generic, see Chruściel et al.[38] and Sec.3.10.

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<sup>11</sup>To ascertain the consistency of dimensions and units, we recollect the basic definitions:

$$[G] = \frac{(\text{m/s})^4}{\text{N}} = \frac{\text{m}^3}{\text{kg s}^2}, \quad \kappa = \frac{8\pi G}{c^4}.$$

The mass  $M$  carries the unit kg, the mass *parameter* has the dimension of a length:

$$m := \frac{GM}{c^2}, \quad [m] = \frac{\text{m}^3 \text{kg s}^2}{\text{kg s}^2 \text{m}^2} = \text{m}.$$

The definition of  $m(r)$  in Eq.(77) is consistent. For  $\rho = \text{const}$ , we have

$$m(r) = \frac{\kappa}{2} \rho c^2 \frac{1}{3} r^3 = \frac{G}{c^2} \frac{4}{3} \pi r^3 \rho = \frac{GM(r)}{c^2}.$$

Here  $\rho$  denotes the physical mass density,  $[\rho] = \text{kg/m}^3$ . Thus

$$M(r) := \frac{4}{3} \pi r^3 \rho$$

is the physical mass with the unit kg.

### 3 The Kerr metric (1963)

*After some historical reminiscences (Sec.3.1), we point out how one can arrive at the Kerr metric (Sec.3.2). For that purpose, we derive, in cylindrical coordinates, the four corresponding partial differential equations and explain how this procedure leads to the Kerr metric. In Sec.3.3, we display the Kerr metric in three classical coordinate systems. Thereafter we develop the concept of the Kerr black hole (Sec.3.4). In Secs.3.5 to 3.7, we depict and discuss the geometrical/kinematical properties of the Kerr metric. Subsequently, in Sec.3.8, we turn to the multipole moments of the mass and the angular momentum of the Kerr metric, stressing analogies to electromagnetism. In Sec.3.9, we present the Kerr-Newman solution with electric charge. Eventually, in Sec.3.10, we wonder in which sense the Kerr black hole is distinguished from the other stationary axially symmetric vacuum spacetimes, and, in Sec.3.11, we mention the rotating disk metric of Neugebauer-Meinl as a relevant interior solution with matter.*

*...When I turned to Alfred Schild, who was still sitting in the armchair smoking away, and said “Its rotating!” he was even more excited than I was. I do not remember how we celebrated, but celebrate we did!*

Roy P. Kerr (2009)

#### 3.1 Historical remarks

*The search for axially symmetric solutions of the Einstein equation started in 1917 with static and was extended in 1924 to stationary metrics. It culminated in 1963 with the discovery of the Kerr metric.*

The Schwarzschild solution, as we have seen, describes the gravitational field of a spherically symmetric body. Obviously, most planets, moons, and stars rotate so that spherical symmetry is lost and one spatial direction is distinguished by the 3-dimensional angular momentum vector  $\mathbf{J}$  of the body. Hence the next problem to attack was to search for the gravitational field of a massive rotating body.

When one considers a *static* and axially symmetric situation—this is the case if the body does not carry angular momentum—then one can choose the rotation axis as the  $z$ -axis of a cylindrical polar coordinate system:  $x^1 = z$ ,  $x^2 = \rho$  and  $x^3 = \phi$ . Then static axial symmetry means that the components of the metric  $g_{\mu\nu} = g_{\mu\nu}(z, \rho)$  do not depend on the time  $t$  and the azimuthal



angle  $\phi$  (we have here one timelike and one spacelike Killing vector<sup>12</sup>).

Already in 1917, Weyl[188] started to investigate static axially symmetric vacuum solutions of Einstein's field equation. He took cylindrical coordinates and proposed the following "canonical" form of the static axisymmetric vacuum line element:<sup>13</sup>

$$ds^2 = f dt^2 - \left\{ h(dz^2 + d\rho^2) + \frac{\rho^2 d\phi^2}{f} \right\}; \quad (89)$$

here  $f = f(z, \rho)$  and  $h = h(z, \rho)$  and ( $x^0 = t, x^1 = z, x^2 = \rho, x^3 = \phi$ ). Weyl was led, in analogy to Newton's theory, to a Poisson equation and found thereby a family of static cylindrically symmetric solutions that could be understood as the exterior field of a line distribution of mass along the rotation axis. Similar investigations were undertaken by Levi-Civita[108] (1917/19).

In the year 1918, Lense and Thirring[107] investigated a rotating body. They specified the energy-momentum tensor of a slowly rotating ball of matter of homogeneous density and integrated the Einstein equation in lowest approximation. They found, for a ball rotating around the  $z$ -axis of a spatial Cartesian coordinate system, the linearized Schwarzschild solution in isotropic coordinates, see Table 2, together with two new "gravitomagnetic" correction terms in off-diagonal components of the metric ( $\kappa$  is Einstein's gravitational constant):

$$ds^2 = \underbrace{\left(1 - \frac{2\kappa M}{r}\right) dt^2 - \left(1 + \frac{2\kappa M}{r}\right) (dx^2 + dy^2 + dz^2)}_{\text{linearized Schwarzschild}} - \underbrace{\frac{4\kappa J_z}{r^3} (x dy - y dx) dt}_{\text{gravitomagnetic term}}; \quad (90)$$

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<sup>12</sup>*Remark on Killing vectors:* Consider a point  $P$  of spacetime with coordinates  $x^\alpha$ . We specify a direction  $\xi^\mu$  at  $P$ . If we have a flat Minkowski space, the components  $g_{\mu\nu}$  of the metric, given in Cartesian coordinates, would not change under a motion in the  $\xi$ -direction. However, in a curved spacetime, the  $g_{\mu\nu}$  will change in general. If  $\xi^\mu$  fulfills the Killing equations (see Stephani [179])

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0, \quad (88)$$

with  $\nabla$  as covariant derivative operator, then  $\xi^\mu$  is called a *Killing vector*, and this vector specifies a direction under which the metric does not change. The Schwarzschild metric is static, that is, it has one timelike Killing vector along the time coordinate. Furthermore, it is spherically symmetric and thus has three additional spacelike Killing vectors. In the Weyl case, because of the axial symmetry around the  $z$ -axis, two of those spacelike Killing vectors get lost. Left over in the Weyl case are the two Killing vectors, one timelike <sup>(1)</sup> $\xi^t = \partial_t$  and one spacelike <sup>(2)</sup> $\xi^\phi = \partial_\phi$ .

<sup>13</sup>Weyl used  $\rho \rightarrow r, \phi \rightarrow \vartheta$ .

this is valid for  $\kappa M \ll r$  and  $\kappa J_z \ll r^2$ . This gravitomagnetic effect (“the Lense-Thirring effect”) is typical for GR: in Newton’s theory a rotating rigid ball has the same gravitational field as a non-rotating one. Gravitomagnetism is alien to Newton’s gravitational theory.

In the meantime, the Lense-Thirring effect has been experimentally confirmed by the Gravity Probe B experiment, see Everitt et al.[55]. They took a gyroscope in a satellite falling freely around the (rotating) Earth. The spin axis of the gyroscope pointed to a fixed guide star. Because of the gravitomagnetic term in (90), the gyroscope executed a (very small) *Lense-Thirring precession*.<sup>14</sup> This can be understood as an interaction of the spin of the gyroscope with the spin of the Earth (spin-spin interaction). Since the gyroscope moves along a 4d geodesic of a spacetime curved by the mass of the Earth, an additional *geodetic precession* occurs that has to be experimentally separated from the Lense-Thirring term. The geodetic precession had already been derived earlier by *de Sitter*[43] in 1916.<sup>15</sup>

In spherical polar coordinates we have  $ydx - xdy = r^2 \sin^2\theta d\phi$ . Thus, the gravitomagnetic cross-term in (90) may be rewritten as  $(4\kappa J_z \sin^2\theta/r) dt d\phi$ . A comparison with (89) shows that the canonical Weyl form of the static metric is too narrow for describing rotating bodies.

From 1919 on, there appeared further articles on axisymmetric solutions. Levi-Civita<sup>16</sup> (1919) reacted to Weyl’s article, and Bach[8] (1922) pushed the Lense-Thirring line element to the second order in the approximation.

Then, in 1924, Lanczos[105] extending the Weyl ansatz, started to investigate *stationary*<sup>17</sup> solutions. He found an exact solution for uniformly rotating dust. However, his work was apparently partially overlooked. Later, Akeley[2, 3] (1931), Andress[6] (1930) and, in a more definite form, Lewis[109] (1932) generalized the static Weyl metric to a stationary one by taking into account the gravitomagnetic term of Lense-Thirring. Lewis (1932) wrote, in cylindrical polar coordinates ( $x_1 \rightsquigarrow \rho, x_2 \rightsquigarrow z$ ),

$$ds^2 = f dt^2 - (e^\mu dx_1^2 + e^\nu dx_2^2 + l d\phi^2) - 2mdt d\phi. \quad (91)$$

He found some exact solutions, typically for rotating cylinders, but not for

<sup>14</sup>For related experiments, see Ciufolini et al.[35, 34] and Iorio et al.[89, 92] A recent comprehensive review was given by Will[189]. A textbook presentation may be found in Ohanian & Ruffini[143].

<sup>15</sup>De Sitter had applied it to the Earth-Moon system conceived as a gyroscope precessing around the Sun (the rotation of which can be neglected). This effect can nowadays be measured by Lunar Laser Ranging, see Will[189].

<sup>16</sup>See Ref.[108], note 8 with the subtitle “*Soluzioni binarie di Weyl*”.

<sup>17</sup>Stationary spacetimes are those that admit a time-like Killing vector. Static spacetimes are stationary spacetimes for which this Killing vector is hypersurface orthogonal; physically this implies time reversal invariance and thus the absence of rotation.

rotating balls. It became definitely clear that, in the axially symmetric case, we may have many different exact vacuum solutions, in contrast to the case of spherical symmetry with, according to the Birkhoff theorem, the Schwarzschild solution as being unique.

Not much later, van Stockum[184] (1937) determined the gravitational field of an infinite rotating cylinder of dust particles, thereby recovering the Lanczos solution, inter alia. He fitted one of the interior matter solutions of Lewis to an exterior vacuum solution. Continuing on this line of research, Papapetrou[148] (1953) started from the Andress-Lewis line element, putting it in a slightly different form, suitable for all stationary axisymmetric vacuum solutions:

$$ds^2 = -e^\mu(d\rho^2 + dz^2) - ld\phi^2 - 2m d\phi dt + f dt^2. \quad (92)$$

The functions  $\mu, l, m$ , and  $f$  depend only on  $\rho$  and  $z$ . Papapetrou integrated the field equations and found exact stationary rotating vacuum solutions. However, his solution carried either *mass and no angular momentum* or *angular momentum and no mass*. Thus[148], “this solution is very special and physically of little interest.”

A year later, a new result was published, which gave the problem of finding solutions for a rotating ball a new direction. Petrov[152] (1954), from Kazan, classified algebraically the Einstein vacuum field, that is, the Weyl curvature tensor, according to its eigenvalues and eigenvectors. This information reached the West, in the time of the Cold War, with some delay. A bit later, Pirani[154] (1957) developed a related formalism. It was the Petrov classification and the picking of a suitable class for the gravitational field of an isolated body (Petrov class D, with two double principal null directions) that finally led to the discovery of the Kerr solution during 1963, ten years after the unphysical solutions of Papapetrou.

Accordingly, it turned out to be a formidable task to find an exact solution for a rotating ball and it was only found nearly half a century after the publication of Einstein’s field equation, namely in 1963 by Roy Kerr [98], a New Zealander, who worked at the time in Texas within the research group of Alfred Schild. It is a 2-parameter solution of Einstein’s vacuum field equation with mass  $M$  and rotation (or angular momentum) parameter  $a := J/M$ .

The story of the discovery of the Kerr solution was told by Kerr himself at a conference on the occasion of his 70th birthday [99]. A decisive starting point of Kerr’s investigations was, as mentioned, the Petrov classification. Melia, in his popular book[124] “Cracking the Einstein Code”, which does not contain any mathematical formula—apart from those appearing in two copies of Kerr’s notes and on a blackboard in another figure—has told this

fascinating battle for solving Einstein’s equation, see also the Kerr story in Ferreira[58].

Dautcourt [40] discussed the work of people who were involved in this search for axially symmetric solutions but who were not so fortunate as Kerr. In particular, Dautcourt himself got this problem handed over from Papapetrou in 1959 as a subject for investigation. He used the results of Papapetrou (1953). Dautcourt’s scholarly article is an interesting complement to Melia’s book. In particular, it becomes clear that the (Lanczos-Akeley-Andress-Lewis-)Papapetrou line element (92) was the correct ansatz for the stationary axially symmetric metric and the Kerr metric is a special case therefrom. The Papapetrou approach with the line element (92) was later, after Kerr’s discovery, brought to fruition by Ernst[53] and by Kramer and Neugebauer[100].

### 3.2 Approaching the Kerr metric

*We derive a 2nd order partial differential equation, the Ernst equation, that governs the stationary axially symmetric metrics in Einstein’s theory. Subsequently, we sketch how the Kerr solution emerges as a simple case therefrom.*

#### Papapetrou line element and vacuum field equation

In more modern literature, the Papapetrou line element (92), which describes some rotation around the axis with  $\rho = 0$ , is usually parametrized as follows<sup>18</sup>,

$$ds^2 = f(dt - \omega d\phi)^2 - f^{-1} [e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2] , \quad (93)$$

$$t \in (-\infty, \infty), \quad \rho \in [0, \infty), \quad z \in (-\infty, \infty), \quad \phi \in [0, 2\pi) ;$$

we assume  $f > 0$ . We compute the vacuum field equation of this metric. Nowadays we can do this straightforwardly with the assistance of a computer algebra system. During the 1960s, when this work was mainly done, there were no computer algebra systems around. Hearn[79] released the computer algebra system REDUCE in 1968. Back then, one had to be in command of huge computer resources in order to bring the underlying computer language LISP to work. Today, Reduce can run on every laptop; for other computer algebra systems, see Grabmeier et al.[68] and Wolfram[191].

Because of its efficiency, we will use Schrüfer’s Reduce-package EXCALC, which was built for manipulating expressions within the calculus of exterior

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<sup>18</sup>See Ernst [53], Buchdahl[23], de Felice & Clarke[57], Quevedo [160], O’Neill[144], Stephani et al.[180], Eq.(19.21), Griffiths & Podolský[69], and Sternberg[181].

forms. For that purpose, we reformulate the metric (93) in terms of an orthonormal coframe of four 1-forms  $\vartheta^\alpha = e_i^\alpha dx^i$ , with the unknown functions  $f = f(\rho, z)$ ,  $\omega = \omega(\rho, z)$ , and  $\gamma = \gamma(\rho, z)$ , namely

$$\vartheta^{\hat{0}} = f^{\frac{1}{2}} (dt - \omega d\phi) = e_i^{\hat{0}} dx^i = f^{\frac{1}{2}} (dx^0 - \omega dx^3), \quad (94)$$

$$\vartheta^{\hat{1}} = f^{-\frac{1}{2}} e^\gamma d\rho = e_i^{\hat{1}} dx^i = f^{-\frac{1}{2}} e^\gamma dx^1, \quad (95)$$

$$\vartheta^{\hat{2}} = f^{-\frac{1}{2}} e^\gamma dz = e_i^{\hat{2}} dx^i = f^{-\frac{1}{2}} e^\gamma dx^2, \quad (96)$$

$$\vartheta^{\hat{3}} = f^{-\frac{1}{2}} \rho d\phi = e_i^{\hat{3}} dx^i = f^{-\frac{1}{2}} \rho dx^3. \quad (97)$$

Because of the orthonormality of the coframe  $\vartheta^\alpha$ , we have

$$ds^2 \equiv g = +\vartheta^{\hat{0}} \otimes \vartheta^{\hat{0}} - \vartheta^{\hat{1}} \otimes \vartheta^{\hat{1}} - \vartheta^{\hat{2}} \otimes \vartheta^{\hat{2}} - \vartheta^{\hat{3}} \otimes \vartheta^{\hat{3}}. \quad (98)$$

Eqs.(94) to (98) are equivalent to (93). The corresponding computer code, as input for Reduce-Excalc, reads as follows:

```
pform f=0, omega=0, gamma=0 $
fdomain f=f(rho,z), omega=omega(rho,z), gamma=gamma(rho,z);

coframe o(0) = sqrt(f) * (d t - omega * d phi),
o(1) = sqrt(f)**(-1) * exp(gamma) * d rho,
o(2) = sqrt(f)**(-1) * exp(gamma) * d z,
o(3) = sqrt(f)**(-1) * rho * d phi
with signature (1,-1,-1,-1);
```

Isn't that simple enough? From this data, the Einstein equation is calculated, with the Einstein tensor  $G^\mu_\nu$ . The complete, fairly trivial program is documented in the Appendix. Note in particular that we used a L<sup>A</sup>T<sub>E</sub>X interface allowing us to output the expressions directly in L<sup>A</sup>T<sub>E</sub>X. This computer output—without changing anything of the formulas—after some post-editing for display purposes, reads as follows:

$$G^0_0 := (4 \cdot \partial_{\rho,\rho} f \cdot f \cdot \rho^2 - 5 \cdot \partial_\rho f^2 \cdot \rho^2 + 4 \cdot \partial_\rho f \cdot f \cdot \rho + 4 \cdot \partial_{z,z} f \cdot f \cdot \rho^2 - 5 \cdot \partial_z f^2 \cdot \rho^2 - 4 \cdot \partial_{\rho,\rho} \gamma \cdot f^2 \cdot \rho^2 - 4 \cdot \partial_{z,z} \gamma \cdot f^2 \cdot \rho^2 + 3 \cdot \partial_\rho \omega^2 \cdot f^4 + 3 \cdot \partial_z \omega^2 \cdot f^4) / (4 \cdot e^{2\gamma} \cdot f \cdot \rho^2), \quad (99)$$

$$G^3_0 := f \cdot (2 \cdot \partial_\rho f \cdot \partial_\rho \omega \cdot \rho + 2 \cdot \partial_z f \cdot \partial_z \omega \cdot \rho + \partial_{\rho,\rho} \omega \cdot f \cdot \rho - \partial_\rho \omega \cdot f + \partial_{z,z} \omega \cdot f \cdot \rho) / (2 \cdot e^{2\gamma} \cdot \rho^2), \quad (100)$$

$$G^1_1 := (\partial_\rho f^2 \cdot \rho^2 - \partial_z f^2 \cdot \rho^2 - 4 \cdot \partial_\rho \gamma \cdot f^2 \cdot \rho - \partial_\rho \omega^2 \cdot f^4 + \partial_z \omega^2 \cdot f^4) / (4 \cdot e^{2\gamma} \cdot f \cdot \rho^2), \quad (101)$$

$$G^1_2 := (\partial_\rho f \cdot \partial_z f \cdot \rho^2 - 2 \cdot \partial_z \gamma \cdot f^2 \cdot \rho - \partial_\rho \omega \cdot \partial_z \omega \cdot f^4) / (2 \cdot e^{2\gamma} \cdot f \cdot \rho^2) \quad (102)$$

$$G^3_3 := (-\partial_\rho f^2 \cdot \rho^2 - \partial_z f^2 \cdot \rho^2 - 4 \cdot \partial_{\rho,\rho} \gamma \cdot f^2 \cdot \rho^2 - 4 \cdot \partial_{z,z} \gamma \cdot f^2 \cdot \rho^2 - \partial_\rho \omega^2 \cdot f^4 - \partial_z \omega^2 \cdot f^4) / (4 \cdot e^{2\gamma} \cdot f \cdot \rho^2). \quad (103)$$

This calculation of the Einstein tensor by machine did not require more than about 15 minutes, including the programming and the typing in; for sample programs, see Socorro et al.[176] and Stauffer et al.[178].

Inspecting these equations, it becomes immediately clear that the numerator of (100) does not depend on  $\gamma$ . In order to get a better overview, we abbreviate the partial derivatives of Reduce  $\partial_\rho f$  by subscripts,  $f_\rho$ , and drop the superfluous multiplication dots of Reduce. We find

$$G^3_0 = 0 \rightarrow 0 = f(\omega_{\rho\rho} + \omega_{zz} - \frac{1}{\rho}\omega_\rho) + 2(f_\rho\omega_\rho + f_z\omega_z). \quad (104)$$

Moreover, by subtracting (103) from (99) we find another equation free of  $\gamma$ :

$$G^0_0 - G^3_3 = 0 \rightarrow 0 = f(f_{\rho\rho} + \frac{1}{\rho}f_\rho + f_{zz}) - f_\rho^2 - f_z^2 + \frac{f^4}{\rho^2}(\omega_\rho^2 + \omega_z^2). \quad (105)$$

Left over are the equations (101) and (102), which can be resolved with respect to the first derivatives of  $\gamma$ :

$$G^1_1 = 0 \rightarrow \gamma_\rho = \frac{\rho}{4f^2}(f_\rho^2 - f_z^2) + \frac{f^2}{4\rho}(\omega_z^2 - \omega_\rho^2), \quad (106)$$

$$G^1_2 = 0 \rightarrow \gamma_z = \frac{\rho}{2f^2}f_\rho f_z - \frac{f^2}{2\rho}\omega_\rho\omega_z. \quad (107)$$

Collected, we have these four equations determining the stationary ax-symmetric vacuum metric:

$$0 = f(f_{\rho\rho} + \frac{1}{\rho}f_\rho + f_{zz}) - f_\rho^2 - f_z^2 + \frac{f^4}{\rho^2}(\omega_\rho^2 + \omega_z^2), \quad (108)$$

$$0 = f(\omega_{\rho\rho} + \omega_{zz} - \frac{1}{\rho}\omega_\rho) + 2(f_\rho\omega_\rho + f_z\omega_z), \quad (109)$$

$$\gamma_\rho = \frac{\rho}{4f^2}(f_\rho^2 - f_z^2) + \frac{f^2}{4\rho}(\omega_z^2 - \omega_\rho^2), \quad (110)$$

$$\gamma_z = \frac{\rho}{2f^2}f_\rho f_z - \frac{f^2}{2\rho}\omega_\rho\omega_z. \quad (111)$$

Let us underline how effortless—under computer assistance—we arrived at these four partial differential equations (PDEs) for determining stationary axially symmetric solutions of Einstein's field equation.

### Ernst equation (1968)

It is one step ahead, before we arrive at a still more convincing form of these PDEs. After some attempts, one recognizes that (109) can be written as

$$\left(\frac{f^2}{\rho}\omega_\rho\right)_\rho + \left(\frac{f^2}{\rho}\omega_z\right)_z = 0. \quad (112)$$

With the ansatz ( $\Omega = \Omega(\rho, z)$ ),

$$\Omega_z = \frac{f^2}{\rho}\omega_\rho, \quad \Omega_\rho = -\frac{f^2}{\rho}\omega_z, \quad (113)$$

Eq.(112) is identically fulfilled. We substitute (113) into (108):

$$f(f_{\rho\rho} + \frac{1}{\rho}f_\rho + f_{zz}) - f_\rho^2 - f_z^2 + \Omega_\rho^2 + \Omega_z^2 = 0. \quad (114)$$

Since (109) is already exploited, we can find  $\Omega$  by differentiating the  $\Omega$ 's in (113) with respect to  $z$  and  $\rho$ , respectively, and by adding the emergent expressions ( $\omega_{\rho z} = \omega_{z\rho}$ ):

$$f(\Omega_{\rho\rho} + \frac{1}{\rho}\Omega_\rho + \Omega_{zz}) - 2f_\rho\Omega_\rho - 2f_z\Omega_z = 0. \quad (115)$$

Eqs.(108) and (115) can be put straightforwardly into a vector analytical form, if we recall that our functions do not depend on the angle  $\phi$ :<sup>19</sup>

$$f\Delta f - (\nabla f) \cdot \nabla f + (\nabla \Omega) \cdot \nabla \Omega = 0, \quad (116)$$

$$f\Delta \Omega - 2(\nabla f) \cdot \nabla \Omega = 0. \quad (117)$$

The last equation can also be written as  $\nabla \cdot (f^{-2}\nabla \Omega) = 0$ . Eqs.(116) and (117) liberate ourselves from the cylindrical coordinates, that is, this expression is now put in form independent of the specific 3d coordinates. With the potential ( $i^2 = -1$ )

$$\mathcal{E} := f + i\Omega, \quad (118)$$

which was found by Ernst[53] and Kramer & Neugebauer[100], we find the Ernst equation[53]

$$(\text{Re } \mathcal{E}) \Delta \mathcal{E} = \nabla \mathcal{E} \cdot \nabla \mathcal{E}, \quad (119)$$

or, in components,

$$(\text{Re } \mathcal{E}) \left[ \frac{\partial^2 \mathcal{E}}{\partial z^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \mathcal{E}}{\partial \rho} \right) \right] = \left( \frac{\partial \mathcal{E}}{\partial z} \right)^2 + \left( \frac{\partial \mathcal{E}}{\partial \rho} \right)^2. \quad (120)$$

The ‘‘Re’’ denotes the real part of a complex quantity. Under stationary axial symmetry—the corresponding metric is displayed in (93)—the Ernst equation (119), together with Eqs.(118, 113, 110, 111), are equivalent to the vacuum Einstein field equation.

<sup>19</sup>In cylindrical coordinates, we have for a vector  $\mathbf{V}$  and a scalar  $s$  the following formulas, see Jackson[94]:

$$\begin{aligned} \nabla \cdot \mathbf{V} &= \frac{1}{\rho} \partial_\rho(\rho V_1) + \partial_z V_2 + \frac{1}{\rho} \partial_\phi V_3, & \nabla^2 s &\equiv \Delta s = \frac{1}{\rho} \partial_\rho(\rho \partial_\rho s) + \partial_z^2 s + \frac{1}{\rho^2} \partial_\phi^2 s, \\ \nabla s &= \mathbf{e}_1 \partial_\rho s + \mathbf{e}_2 \partial_z s + \mathbf{e}_3 \frac{1}{\rho} \partial_\phi s, & \nabla s \cdot \nabla s &= (\partial_\rho s)^2 + (\partial_z s)^2 + \frac{1}{\rho^2} (\partial_\phi s)^2. \end{aligned}$$

## From Ernst back to Kerr

This reduces the problem of axial symmetry to the solution of the second order PDE (119). This method, which came along only five years after Kerr's publication, led to many new exact solutions, amongst them the Kerr solution (1963) as one of the simplest cases. We are only going to sketch how one arrives at the Kerr solution eventually. We follow here closely Buchdahl[23].

One introduces a new complex potential  $\xi$  by

$$\mathcal{E} =: \frac{\xi - 1}{\xi + 1}. \quad (121)$$

Then the Ernst equation becomes

$$(\xi\bar{\xi} - 1)\Delta\xi = 2\bar{\xi}\nabla\xi \cdot \nabla\xi, \quad (122)$$

where the overline denotes complex conjugation. If one has a solution of this equation, we can determine the functions  $f$ ,  $\omega$  and  $\gamma$  by

$$f = \operatorname{Re}\frac{\xi - 1}{\xi + 1}, \quad (123)$$

$$\omega_\rho = -2\rho \frac{\operatorname{Im}[(\bar{\xi} + 1)^2\xi_z]}{(\xi\bar{\xi} - 1)^2}, \quad \omega_z = 2\rho \frac{\operatorname{Im}[(\bar{\xi} + 1)^2\xi_\rho]}{(\xi\bar{\xi} - 1)^2}, \quad (124)$$

$$\gamma_\rho = \rho \frac{\xi_\rho\bar{\xi}_\rho - \xi_z\bar{\xi}_z}{(\xi\bar{\xi} - 1)^2}, \quad \gamma_z = 2\rho \frac{\operatorname{Re}(\xi_\rho\bar{\xi}_z)}{(\xi\bar{\xi} - 1)^2}. \quad (125)$$

For rotating bodies, *spherical prolate coordinates*  $x, y$ , with a constant  $k$ , are much more adapted:

$$\rho = k(x^2 - 1)^{\frac{1}{2}}(1 - y^2)^{\frac{1}{2}}, \quad z = kxy. \quad (126)$$

It turns out that one simple potential solving the Ernst equation, with the constants  $p$  and  $q$ , is

$$\xi = px - iqy \quad \text{with} \quad p^2 + q^2 = 1; \quad (127)$$

it leads to the Kerr metric. For this purpose, one has to introduce the redefined constants  $m := k/p$  (mass) and  $a := kq/p$  (angular momentum per mass) and to execute subsequently the transformations  $px = (\tilde{\rho}/m) - 1$  and  $qy = (a/m)\cos\theta$  to the new coordinates  $\tilde{\rho}$  and  $\theta$ . Then one arrives at the Kerr metric in Boyer-Lindquist coordinates, which is displayed in the table on the next page. For more detail, compare, for instance, the books of Buchdahl[23], Islam[93], Heusler[85], Meinel et al.[123], or Griffiths et al.[69].



By similar techniques, a Kerr solution with a topological defect was found by Bergamini et al.[14].

Incidentally, in the context of the Ernst equation, Geroch made the following interesting conjecture: A subset of all stationary axially symmetric vacuum space-times, including all of its asymptotically flat members, that is, in particular the Kerr solution, can be obtained from Minkowski space by transformations generated by an infinite-dimensional Lie group. This conjecture was “proved” by Hauser and Ernst[76], see also Ref.[75] However, the proof contained a mistake that was subsequently corrected in Ref.[77]

Starting from 4d ellipsoidal coordinates, Dadhich[39] gave a heuristic derivation of the Kerr metric by requiring, amongst other things, that light propagation should be influenced by gravity.

### 3.3 Three classical representations of the Kerr metric

We collected these three classical versions of the Kerr metric in Table 4, see also Visser[186]. Three more coordinate systems should at least be mentioned:

- *Pretorius & Israel*[157] double null coordinates:  
very convenient to tackle the initial value problem
- *Doran*[46] coordinates:  
Gullstrand-Painlevé like; useful in analog gravity
- *Debever/Plebański/Demiański*[155] coordinates:  
components of the metric are rational polynomials; convenient for (computer assisted) calculations

**Table 4. Kerr metric: the three classical representations**

<b>Kerr-Schild</b>	$(t, x, y, z)$	Cartesian background
$ds^2 = - dt^2 + dx^2 + dy^2 + dz^2 + \frac{2mr^3}{r^4 + a^2 z^2} \left( dt + \frac{r(x dx + y dy)}{a^2 + r^2} + \frac{a(y dx - x dy)}{a^2 + r^2} + \frac{z}{r} dz \right)^2$		
$x^2 + y^2 + z^2 = r^2 + a^2 \left( 1 - \frac{z^2}{r^2} \right), \quad r = r(x, y, z)$		
$\begin{aligned} x &= (r \cos \phi + a \sin \phi) \sin \theta \\ y &= (r \sin \phi - a \cos \phi) \sin \theta \\ z &= r \cos \theta \end{aligned}$		
<b>Boyer-Lindquist</b>	$(t, r, \theta, \phi)$	Schwarzschild like
$ds^2 = - \left( 1 - \frac{2mr}{\rho^2} \right) dt^2 - \frac{4mra \sin^2 \theta}{\rho^2} dt d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left( r^2 + a^2 + \frac{2mra^2 \sin^2 \theta}{\rho^2} \right) \sin^2 \theta d\phi^2$		
$\rho^2 := r^2 + a^2 \cos^2 \theta \quad \Delta := r^2 - 2mr + a^2 = (r - r_+)(r - r_-)$		
$\begin{aligned} dv &= dt + \frac{r^2 + a^2}{\Delta} dr \\ d\varphi &= d\phi + \frac{a}{\Delta} dr \end{aligned}$		
<b>Kerr original</b>	$(v, r, \theta, \varphi)$	Eddington-Finkelstein like
$ds^2 = - \left( 1 - \frac{2mr}{\rho^2} \right) (dv - a \sin^2 \theta d\varphi)^2 + 2 (dv - a \sin^2 \theta d\varphi) (dr - a \sin^2 \theta d\varphi) + \rho^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$		
$r_{E\pm} := m \pm \sqrt{m^2 - a^2 \cos^2 \theta} \quad r_{\pm} := m \pm \sqrt{m^2 - a^2}$		

As input for checking the *Kerr* solution, we use the *orthonormal coframe*[\[181\]](#)

$$\vartheta^{\hat{0}} := \frac{\sqrt{\epsilon\Delta}}{\rho} (dt - a \sin^2\theta d\phi) , \quad (128)$$

$$\vartheta^{\hat{1}} := \frac{\rho}{\sqrt{\epsilon\Delta}} dr , \quad (129)$$

$$\vartheta^{\hat{2}} := \rho d\theta , \quad (130)$$

$$\vartheta^{\hat{3}} := \frac{\sin\theta}{\rho} [(r^2 + a^2) d\phi - a dt] . \quad (131)$$

We introduced the sign function, which is convenient for discussing the different regions in the Penrose-Carter diagram:

$$\epsilon = \begin{cases} +1 & \text{for } r > r_+ \text{ or } r < r_- , , \\ -1 & \text{for } r_- < r < r_+ . \end{cases} \quad (132)$$

The metric can then be written in terms of the coframe as

$$ds^2 \equiv g = \epsilon(-\vartheta^{\hat{0}} \otimes \vartheta^{\hat{0}} + \vartheta^{\hat{1}} \otimes \vartheta^{\hat{1}}) + \vartheta^{\hat{2}} \otimes \vartheta^{\hat{2}} + \vartheta^{\hat{3}} \otimes \vartheta^{\hat{3}} . \quad (133)$$

From Table 4 it is not complicated to read off the Schwarzschild and the Lense-Thirring metric as special cases. In comparison to the Schwarzschild metric, the Kerr solution includes a new parameter  $a$  which will be related to the angular momentum. However, it should be noted that, by setting  $a = 0$ , the Kerr metric reduces to the Schwarzschild metric, as it should be ( $\rho^2 \rightarrow r^2$  and  $\Delta \rightarrow r^2 - 2mr$ ):

$$ds^2 = - \left(1 - \frac{2mr}{r^2}\right) dt^2 + \frac{r^2}{r^2 - 2mr} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 . \quad (134)$$

By canceling  $r^2$  in the  $dr^2$ -term, we immediately recognize the Schwarzschild metric.

Considering the parameter  $a$  we should note the following fact. For small values of the parameter  $a$ , where we may neglect terms of the order of  $a^2$ , we arrive at ( $\rho^2 \rightarrow r^2$  and  $\Delta \rightarrow r^2 - 2mr$ ) and

$$\begin{aligned} ds^2 = & - \left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \\ & - \frac{4ma \sin^2\theta}{r} dt d\phi . \end{aligned} \quad (135)$$

Since, in spherical coordinates we have  $y dx - x dy = r^2 \sin^2\theta d\phi$ , the crossterm may be rewritten as  $\frac{4ma \sin^2\theta}{r} dt d\phi = \frac{4ma}{r^3} (x dz - y dx)$ . Thus, in the limiting case  $a^2 \ll 1$ , the Kerr metric yields the Lense-Thirring metric, provided we identify  $ma = J_z$ .

### 3.4 The concept of a Kerr black hole

We come back to our Fig. 7 with “Schwarzschild” versus “Kerr”. The Kerr spacetime may be visualized by a vortex, where the water of the lake spirals towards the center. Much of the above said for the Schwarzschild case is still valid. However, one important difference occurs. The stationary limit and the event horizon separate, which will be illustrated by corresponding graphical representations.

In case of a vortex, the flow velocity of the in-spiraling water has two components. The radial component which drags the boat towards the center whereas the additional angular component forces the boat to circle around the center. Again, the *stationary limit* is defined by the distance at which the boat ultimately can withstand the radial and circular drag of the water flow. Beyond the stationary limit the situation is not as hopeless as in the Schwarzschild case. Using all its power, the boat may brave the inward flow. But then it has not enough power to overcome the angular drag and is forced to orbit the center. By means of a clever spiral course the boat may even escape beyond the stationary limit. The stationary limit is not necessary an event horizon. At some distance, nearer to the center than the stationary limit, also the pure radial flow of water will exceed the power of the boat. There, inside the stationary limit, is the event horizon.

In order to investigate the structure of the Kerr spacetime, we first look at “strange behavior” of the metric components in Boyer-Lindquist coordinates. The following cases can be distinguished:

$$\begin{aligned}\Delta = 0 & \quad g_{rr} \text{ becomes singular,} \\ \rho^2 = 2mr & \quad g_{tt} \text{ vanishes,} \\ \rho^2 = 0 & \quad g_{rr} \text{ and } g_{\theta\theta} \text{ vanish, the other components are singular.}\end{aligned}$$

As we have extensively discussed in the previous section, singularities of components of the metric *may* signify physical effects but, on the other hand, may only be due to “defective” coordinates. Thus, we will proceed along similar lines to investigate the nature of these singularities.

We will not address the geodesics of the Kerr metric in detail. For an elementary discussion the reader is referred to Frolov and Novikov[64] and to the more advanced discussion in Hackmann et al.[72].

## Depicting Kerr geometry

We draw a picture of the spatial appearances and relations of the various horizons and the singularity of the Kerr metric. From outside to inside these are, explicitly,

$$\begin{array}{llll}
 \text{outer} & \text{ergosurface} & r_{E+} & := m + \sqrt{m^2 - a^2 \cos^2 \theta} \\
 & & & \downarrow \text{joined at polar axis} \\
 \text{event} & \text{horizon} & r_+ & := m + \sqrt{m^2 - a^2} \\
 & & & \downarrow \text{merge for } a \rightarrow m \\
 \text{Cauchy} & \text{horizon} & r_- & := m - \sqrt{m^2 - a^2} \\
 & & & \downarrow \text{joined at polar axis} \\
 \text{inner} & \text{ergosurface} & r_{E-} & := m - \sqrt{m^2 - a^2 \cos^2 \theta} \\
 & & & \uparrow \text{lies on the rim for } \theta = \pi/2 \\
 & \text{singularity} & r & = 0
 \end{array}$$

For  $a = 0$ , inner ergosurface and Cauchy horizon vanish, whereas outer ergosurface and event horizon merge to the Schwarzschild horizon. To visualize the various surfaces we use Kerr-Schild quasi-Cartesian coordinates. The radial coordinate  $r$  of the Boyer-Lindquist coordinates is related to the coordinates  $x, y, z$  of the Kerr-Schild coordinates via, see Table 4,

$$x^2 + y^2 + \frac{r^2 + a^2}{r^2} z^2 = r^2 + a^2, \quad z = r \cos \theta. \quad (137)$$

Substituting  $r = 0$ ,  $r = r_{\pm}$ ,  $r = r_{E\pm}$ , and a little bit of algebra yields:

- **Singularity**  $r = 0$

Since  $r = 0$  leads to  $z = 0$ , we get the equation of a circle of radius  $a$  in the equatorial plane,

$$x^2 + y^2 = a^2. \quad (138)$$

For  $a = 0$ , the ring collapses to the Schwarzschild singularity.

A closer inspection shows that the structure of the singularity is more complex[66, 110].

- **Horizons**  $r = r_{\pm}$

In this case we arrive at the equation for an oblate (for  $a < m$ ) ellipsoid,

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{z^2}{a_3^2} = 1, \quad (139)$$

where  $a_1^2 = a_2^2 = r_{\pm}^2 + a^2 > a_3^2 = \frac{1}{r_{\pm}^2}$ .

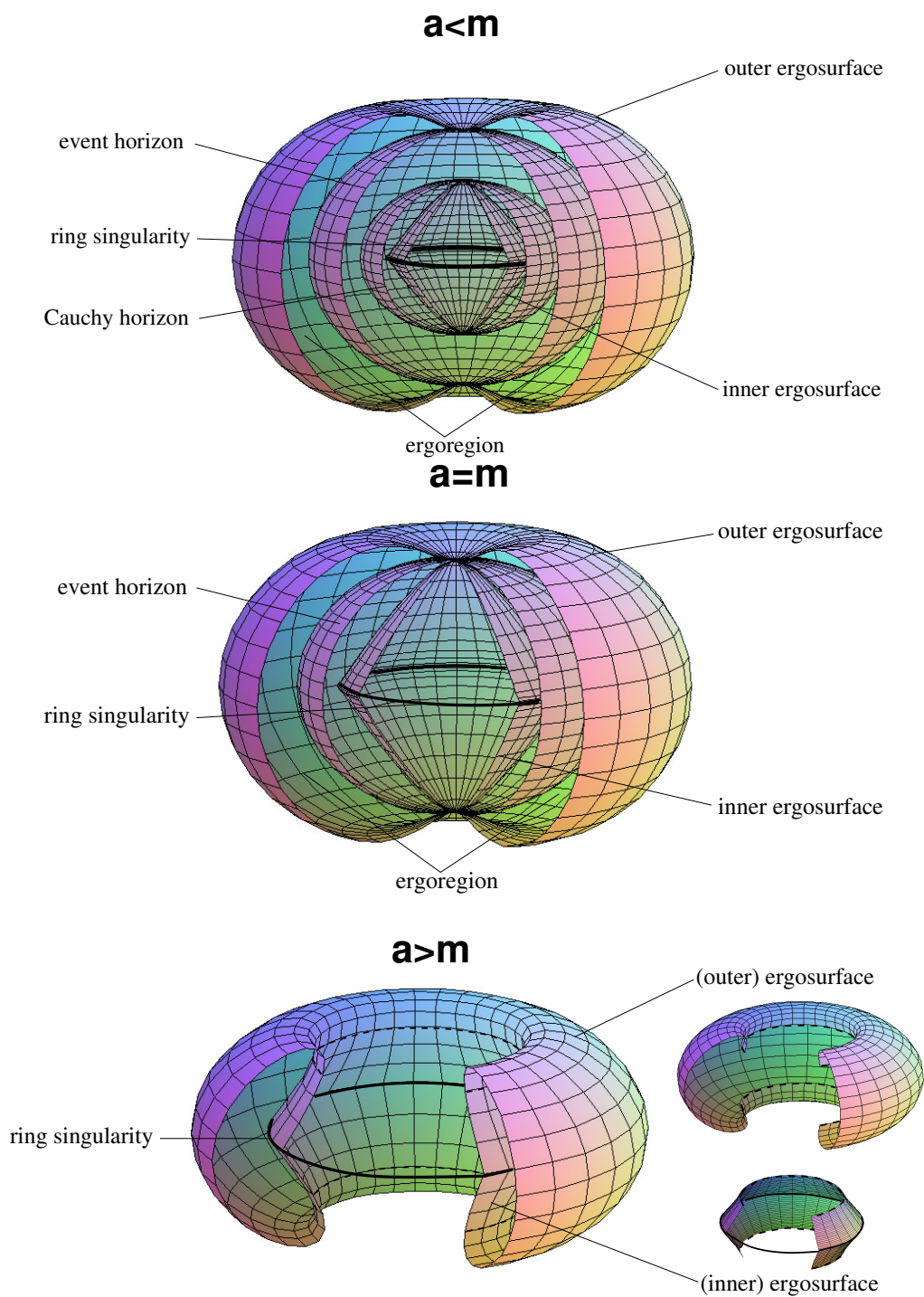


Fig. 12: Ergosurfaces, horizons, and singularity for slow, extremal ('critical'), and fast Kerr black holes.

- **Ergosurfaces**  $r = r_{E_{\pm}}(\theta)$

Things are a little bit more involved in this case because  $r$  is not constant. We can also derive a “ellipsoid-like” equation (for  $a \leq m$ ),

$$\frac{x^2}{a_1^2(\theta)} + \frac{y^2}{a_2^2(\theta)} + \frac{z^2}{a_3^2(\theta)} = 1, \quad (140)$$

now with

$$a_1^2(\theta) = a_2^2(\theta) = r_{E_{\pm}}^2(\theta) + a^2, \quad a_3^2(\theta) = \frac{1}{r_{E_{\pm}}^2(\theta)}. \quad (141)$$

The  $\theta$ -dependence will deform the ellipsoid. On the equatorial plane with  $\theta = \pi$  we have  $r_{E_{\pm}}^2 = 0$ . Hence,  $a_1 = a_2 = a$  and  $a_3$  diverges. This results in a non-regular rim on which the ring singularity is located.

For  $a > m$ , the  $r_{E_{\pm}}$  is partly not defined, since the term under the square-root changes sign, and becomes negative if

$$\cos(\theta) = \frac{m}{a}. \quad (142)$$

This defines two rings with  $\theta_1 = \arccos(m/a)$ ,  $\theta_2 = \pi - \theta_1$  and  $r = m$ . As a consequence, the outer ergosurface only extends to these rings from the outside, and the inner ergosurface up to the rings from the inside. The outcome is a kind of torus. The center-facing side is constituted by a part of the inner ergosurface (along with the ring singularity), whereas the outside facing parts are given by a part of the outer ergosurface.

An extensive discussion of the embedding of the ergosurfaces into Euclidean space, together with corresponding Mathematica-programs, can be found in Ref.[114].

The surfaces are visualized in Fig. 12. We did not use a faithful embedding but rescaled axes in order to achieve better visibility.

The presence of the term  $\sqrt{m^2 - a^2}$  requires the distinction of three different cases dependent on the values of the mass parameter  $m$  and the angular momentum parameter  $a$ :

$m > a \rightsquigarrow$  slow rotation     $m = a \rightsquigarrow$  critical rotation     $m < a \rightsquigarrow$  fast rotation

The slow rotating case shows the richest structure. Both ergosurfaces and both horizons are present and distinct from each other. As  $a$  approaches  $m$ , the event and the Cauchy horizon draw nearer and nearer. At critical rotation,  $a = m$ , both horizons merge into one single event horizon with  $r = m$ . Eventually, for fast rotation  $a > m$ , the event horizon disappears and reveals the naked ring singularity which now is located at the inner edge of the now toroidal shaped (outer) ergosurface.

### 3.5 The ergoregion

We explore the region between the outer ergosurface and the event horizon. There it is not possible to stand still, anything has to rotate, even the event horizon. The compulsory rotation in the ergoregion allows one to extract energy from the black hole. This so-called Penrose process leads to black hole thermodynamics.

#### Constrained rotation

The outer ergosurface,  $r = R_{E+}$ , is defined by the equation  $g_{00}(r_{E+}) = 0$ . Thus, it is a surface of infinite redshift and a Killing horizon. For a third characterization of the ergo surfaces we have to deal not only with radial but also with rotational motion. Consider the Kerr metric in Boyer-Lindquist coordinates with  $dr = d\theta = 0$ ,

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dt d\phi + g_{\phi\phi}d\phi^2,$$

or, after dividing by  $dt^2$ ,

$$\left(\frac{ds}{dt}\right)^2 = g_{tt} + 2g_{t\phi}\frac{d\phi}{dt} + g_{\phi\phi}\left(\frac{d\phi}{dt}\right)^2.$$

The explicit form of the metric components is not needed here. Note that  $\Omega = \frac{d\phi}{dt}$  is the angular velocity with respect to a distant observer,

$$\left(\frac{ds}{dt}\right)^2 = g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2. \quad (143)$$

The worldline of a particle has to be timelike,  $ds^2 < 0$ . Since the last equation is quadratic in  $\Omega$ , this is only possible between the roots  $ds^2 = 0$ ,

$$\Omega_{\min/\max} := -\frac{g_{t\phi}}{g_{\phi\phi}} \mp \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}.$$

What does

$$\Omega_{\min} < \Omega < \Omega_{\max}$$

mean?! In flat Minkowski spacetime (with Cartesian coordinates),  $\Omega_{\min/\max} = \pm 1$  implies that a particle, e.g., may freely circle around a point, restricted only by the condition  $|v| = |r \cdot \Omega| < c$ . In the Kerr spacetime, at  $r = r_{E\pm}$ , the smallest possible value of  $\Omega$  becomes 0. The particle may just stay at rest, but can rotate only in one direction, namely in direction of the angular



momentum of the black hole. Beyond  $r_{E_+}$ ,  $\Omega$  is forced to be larger than zero: The particle must co-rotate with the black hole.

The preceding statement is only correct for radially infalling particles. In general, the influence of the rotating black hole on the motion of particles is more complex[165].

### Rotation of the event horizon

The behavior of  $\Omega_{\pm}$  on the event horizon is quite remarkable. By using the identities

$$2mr_{\pm} = r_{\pm}^2 + a^2, \quad \rho^2 - r^2 = a^2 \cos^2 \theta = a^2 - a^2 \sin^2 \theta, \quad (144)$$

one finds for the event horizon ( $r = r_+$ ),

$$\Omega_+ = \Omega_- = \Omega_H := \frac{a}{2mr_+}. \quad (145)$$

To interpret this result we use (143) and write

$$g_{\mu\nu} l^{\mu} l^{\nu}|_{r=r_+} = 0, \quad \text{with } l^{\mu} = (1, 0, 0, \Omega_H). \quad (146)$$

The integral lines of  $l^{\mu} = \dot{x}^{\mu}$ ,

$$x^{\mu} = (t, r_+, \theta_0, \Omega_H t), \quad (147)$$

define a lightlike hypersurface rotating with a uniform angular velocity: The event horizon of a Kerr black hole rotates “rigidly” with  $\Omega_H$ , see in this context Frolov & Frolov[63]. A consequence of this finding is discussed in the next paragraph.

### Penrose process and black hole thermodynamics

The (outer) ergosurface is a Killing horizon, not an event horizon. It is possible for particles to pass from the inside to the outside. This allows for a peculiar scenario: Since inside the Killing horizon the particle is forced to spin around, it picks up an additional rotational energy. This energy can be partly extracted by means of the Penrose process. An infalling particle traverses the Killing horizon, picks up rotational energy and subsequently decays into two parts. If one part plunges into the event horizon, the other part, carrying away some of the rotational energy, can return to the outside of the Killing horizon. Thus, the region between Killing and event horizon is justly labeled as “ergoregion” (from Greek ergon = work).

The observation that energy can also be extracted from the black hole gave rise to black hole thermodynamics. The next question is then how the parameters change if the black hole is infinitesimally disturbed. It was Bekenstein[13] who established a relation between the variations of the mass, the angular momentum, and the area of the event horizon. Using the coframe (185), for  $\lambda = 0$ , we find for the area of the event horizon

$$A = \int_{\substack{r = r_+ \\ t = \text{const.}}} \vartheta^2 \wedge \vartheta^3 = \int_{\substack{r = r_+ \\ t = \text{const.}}} \sin \theta (r^2 + a^2) d\theta \wedge d\phi = 4\pi (r_+^2 + a^2). \quad (148)$$

We can rewrite (148), using (144) and  $J = ma$ ,

$$A = 8\pi m r_+ = 8\pi (m^2 - \sqrt{m^4 - J^2}). \quad (149)$$

The differential of this equation is

$$dA = \frac{\partial A}{\partial m} dm + \frac{\partial A}{\partial J} dJ = \frac{8\pi}{\kappa} dm - \frac{8\pi}{\kappa} \Omega_H dJ, \quad (150)$$

with

$$\kappa = \frac{1}{2m} \frac{\sqrt{m^4 - J^2}}{m^2 + \sqrt{m^4 - J^2}}, \quad \Omega_H = \kappa \frac{J}{\sqrt{m^4 - J^2}}. \quad (151)$$

The parameter  $\Omega_H$  is the angular velocity of the horizon (145). The parameter  $\kappa$  is the surface gravity. Eq.(150) can be rewritten as

$$dm = \frac{\kappa}{8\pi} dA + \Omega_H dJ. \quad (152)$$

The infinitesimal change of the mass,  $dm$ , is proportional to the the infinitesimal change of the energy,  $dE$ . The term  $\Omega_H dJ$  describes the infinitesimal change of the rotational energy. This suggests the identification of (152) with the first law of thermodynamics. The analogy is still more compelling by observing that, for a given black hole of initial (or irreducible) mass  $m$ , the area of the horizon is always increasing. Even by exercising a Penrose process, which extracts rotational energy from the black hole, a fragment of the incoming particle will fall into the black hole thereby increasing its mass and, in turn, the area of the horizon. Accordingly, the area  $A$  of the horizon behaves formally as if it is proportional to an entropy  $S$  and the surface gravity  $\kappa$  as if it is proportional to a temperature  $T$ . In fact, the *Hawking temperature* and the *Bekenstein-Hawking entropy* turn out to be

$$T = \frac{\hbar}{2\pi k_B} \kappa, \quad S = \frac{1}{4G\hbar} A, \quad (153)$$

with  $k_B$  as the Boltzmann constant. Eq.(152) together with its thermodynamical interpretation (153) can be considerably generalized thereby establishing the new discipline of “black hole thermodynamics”, see Heusler[85] and Carlip.[28]

### 3.6 Beyond the horizons

*In the Schwarzschild spacetime, event horizon and Killing horizon coincide. In the Kerr spacetime, for  $m > a$ , there is an outer Killing horizon, an event horizon, an inner Killing horizon and an inner horizon. So far, all the coordinate systems we used for the Kerr metric show singularities at the outer and inner horizons  $r = r_{\pm}$ . The construction of a regular coordinate system is possible along the same lines as for the Schwarzschild metric. Of course, the corresponding calculations are much more involved for the Kerr case. Therefore, we will give more a kind of heuristic approach to motivate Kruskal-like coordinates for the Kerr metric.*

#### Using light rays as coordinate lines

Our first task is to construct Eddington-Finkelstein like coordinates for the Kerr metric by considering radial light rays. We restrict ourselves to the case  $\theta = 0 = \phi$ . The Kerr metric in Boyer-Lindquist coordinates reduces to ( $\theta = 0 \rightarrow \rho^2 = r^2 + a^2 = \Delta + 2mr$ ):

$$ds^2 = -\frac{\Delta}{\rho^2} dt^2 + \frac{\rho^2}{\Delta} dr^2.$$

Hence, for in-/out-going light rays,  $ds^2 = 0$ , we find

$$dt = \pm \frac{\rho^2}{\Delta} dr = \pm \frac{r^2 + a^2}{(r - r_+)(r - r_-)} dr$$

or, explicitly,

$$\pm t = \int dr \frac{r^2 + a^2}{(r - r_-)(r - r_+)} = r + \frac{r_+^2 + a^2}{r_+ - r_-} \ln |r - r_+| - \frac{r_-^2 + a^2}{r_+ - r_-} \ln |r - r_-| + const. \quad (154)$$

Unlike in the Schwarzschild spacetime, there form **two** event horizons, at  $r = r_-$  and  $r = r_+$ , respectively. However, as  $a \rightarrow 0$ ,  $r_-$  goes to 0, whereas  $r_+$  approaches  $2m$  and the Schwarzschild situation is reproduced.

We next focus on the (Boyer-Lindquist) coordinates  $(t, r)$  and how the horizons etc. will appear in terms of the new coordinates. The other coordinates and the regularity of the metric is not addressed. However, all

the details can be found in the literature, see Refs.[20, 30, 78]. Using (154) analogously to (42), we introduce Eddington-Finkelstein like coordinates for Kerr,

$$v := t + r + \sigma_+ \ln|r - r_+| - \sigma_- \ln|r - r_-|, \quad (155)$$

$$u := t - r - \sigma_+ \ln|r - r_+| + \sigma_- \ln|r - r_-|, \quad (156)$$

where (according to the notation in Ref.[20])

$$\sigma_{\pm} := \frac{r_{\pm}^2 + a^2}{r_+ - r_-} = \frac{mr_{\pm}}{\sqrt{m^2 - a^2}}. \quad (157)$$

Again, we can get rid of the coordinate singularity by rescaling  $u$  and  $v$  analogously to (48). Since we have two horizons,  $r = r_+$  and  $r = r_-$ , we have to decide with respect to which singularity we rescale. We firstly choose  $r_+$  and define, see (48),

$$\tilde{v} := \exp\left(\frac{v}{2\sigma_+}\right) = \frac{|r - r_+|^{\frac{1}{2}}}{|r - r_-|^{\frac{\nu}{2}}} e^{\frac{r+t}{2\sigma_+}}, \quad (158)$$

$$\tilde{u} := -\exp\left(-\frac{u}{2\sigma_+}\right) = -\frac{|r - r_+|^{\frac{1}{2}}}{|r - r_-|^{\frac{\nu}{2}}} e^{\frac{r-t}{2\sigma_+}}, \quad (159)$$

with

$$\nu := \frac{\sigma_-}{\sigma_+} = \frac{r_-}{r_+} > 1. \quad (160)$$

Again, we go back to time- and space-like coordinates, exactly like in (50),

$$\tilde{t} := \frac{1}{2}(\tilde{v} + \tilde{u}), \quad \tilde{r} := \frac{1}{2}(\tilde{v} - \tilde{u}). \quad (161)$$

Then we work out the four coordinate patches exactly like (55) to (60). We arrive at a Kruskal like coordinate system. However, there arises an important difference: The coordinate system still is singular for  $r = r_-$ . This can be most easily seen from the analog to (59), the inverse transformation to  $r$ , which now reads

$$\tilde{r}^2 - \tilde{t}^2 = -\tilde{v}\tilde{u} = \frac{r - r_+}{(r - r_-)^{\nu}} e^{\frac{r}{\sigma_+}}. \quad (162)$$

The horizon  $r = r_+$  is regular in this coordinate system and is described by  $\tilde{r} = \pm\tilde{t}$ . The transformation(s) are valid in the domain  $r_- < r < +\infty$

$r = r_+$	:	$\tilde{r} = \pm\tilde{t}$	as for Schwarzschild
$r \rightarrow +\infty$	:	$\tilde{r}^2 - \tilde{t}^2 \rightarrow +\infty$	particularly $\tilde{r} \rightarrow \pm\infty$ for $\tilde{t} = 0$
$r \rightarrow r_-$	:	$\tilde{r}^2 - \tilde{t}^2 \rightarrow -\infty$	particularly $\tilde{t} \rightarrow \pm\infty$ for $\tilde{r} = 0$
$r = r_{E+}$	:	$\tilde{r}^2 - \tilde{t}^2 = \text{const.} > 0$	hyperbolas in I, II patches

In contrast to the Schwarzschild case, the full upper and lower halfplanes of the  $(\tilde{r}, \tilde{t})$  plane is covered. It is not limited by the hyperbolas of the Schwarzschild singularity  $r = 0$ !

We can regularize with respect to  $r_-$  by introducing

$$\tilde{v} := -\exp\left(-\frac{v}{2\sigma_-}\right), \quad \tilde{u} := \exp\left(\frac{u}{2\sigma_-}\right). \quad (163)$$

Now we find

$$\tilde{r}^2 - \tilde{t}^2 = \frac{r - r_-}{(r - r_+)^{\frac{1}{\nu}}} e^{-\frac{r}{\sigma_-}}. \quad (164)$$

This coordinate system covers the domain  $-\infty < r < r_+$ . Like the first coordinate system, it contains also the region between the horizons,  $r_- < r < r_+$ . This time,  $r \geq r_+$  is excluded.

$r = r_-$	:	$\tilde{r} = \pm \tilde{t}$	as above
$r \rightarrow -\infty$	:	$\tilde{r}^2 - \tilde{t}^2 \rightarrow \infty$	particularly $\tilde{t} \rightarrow \pm\infty$ for $\tilde{r} = 0$
$r \rightarrow r_+$	:	$\tilde{r}^2 - \tilde{t}^2 \rightarrow -\infty$	particularly $\tilde{r} \rightarrow \pm\infty$ for $\tilde{t} = 0$
$r = r_{E-}$	:	$\tilde{r}^2 - \tilde{t}^2 = \text{const.} > 0$	hyperbola in I*, II* patches
$r = 0$	:	$\tilde{r}^2 - \tilde{t}^2 = -\frac{r_-}{r_+} < 0$	hyperbola in I*, II* patches

Again, the whole  $(\tilde{r}, \tilde{t})$  plane is covered. Note that the spacetime extends beyond the ring(!) singularity.

### 3.7 Penrose-Carter diagram and Cauchy horizon

*We compactify the Kruskal-like coordinate system for Kerr, yielding conformal Penrose-Carter diagrams. We discuss the analytical extension and the role of the inner horizon as Cauchy horizon.*

In order to draw a *Penrose-Carter diagram* for the Kerr spacetime, we compactify the coordinates via the tangent function like in Sec.2.6. The result looks at first quite similar to Schwarzschild in Fig. 11. However, the cutoff at  $r = 0$  vanishes. The diagrams Fig. 13 and Fig. 14 both show the entire compactified  $(\tilde{r}, \tilde{t})$ -space.

The two coordinate sets overlap in the region between the horizons. Thus, the corresponding coordinate patches have to be identified. And we can even draw beyond that ... Patch II is identified with patch IV\*, II\* with another patch IV\*\*. And so on: We find an infinite sequence of coordinate systems. Formally, this constitutes a *maximal analytic extension* of the Kerr spacetime. Alas, there are good reasons for not believing in such vast an extension.

The Kerr metric is a vacuum solution of Einstein's field equation—it describes a *totally empty* spacetime. To render it physically meaningful, we

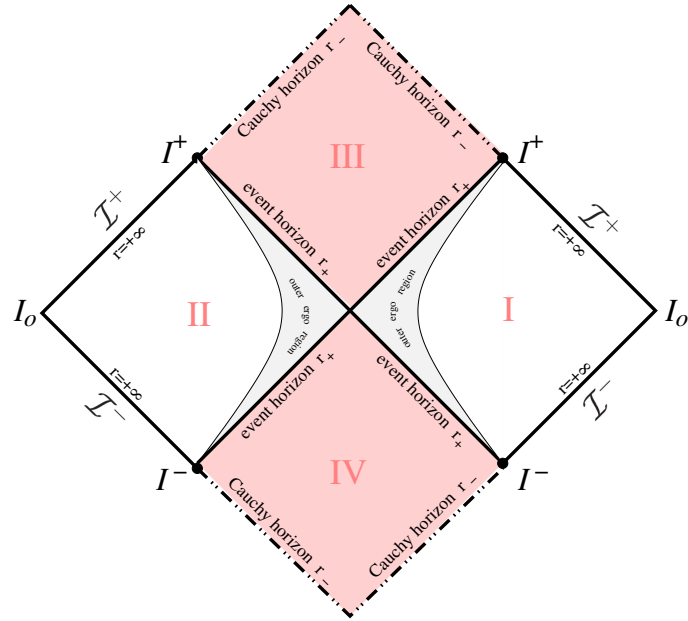


Figure 13: The Penrose diagram for the Kerr spacetime for  $r > r_-$ .

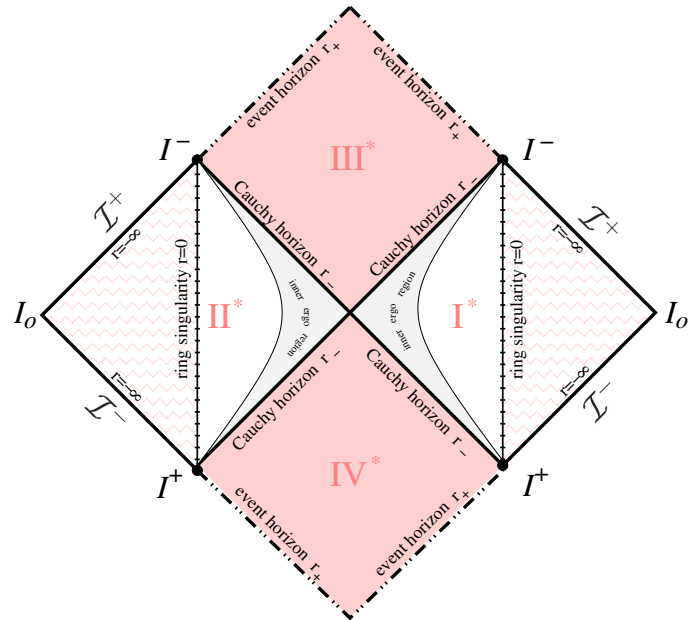


Figure 14: The Penrose diagram for the Kerr spacetime for  $r < r_-$ .

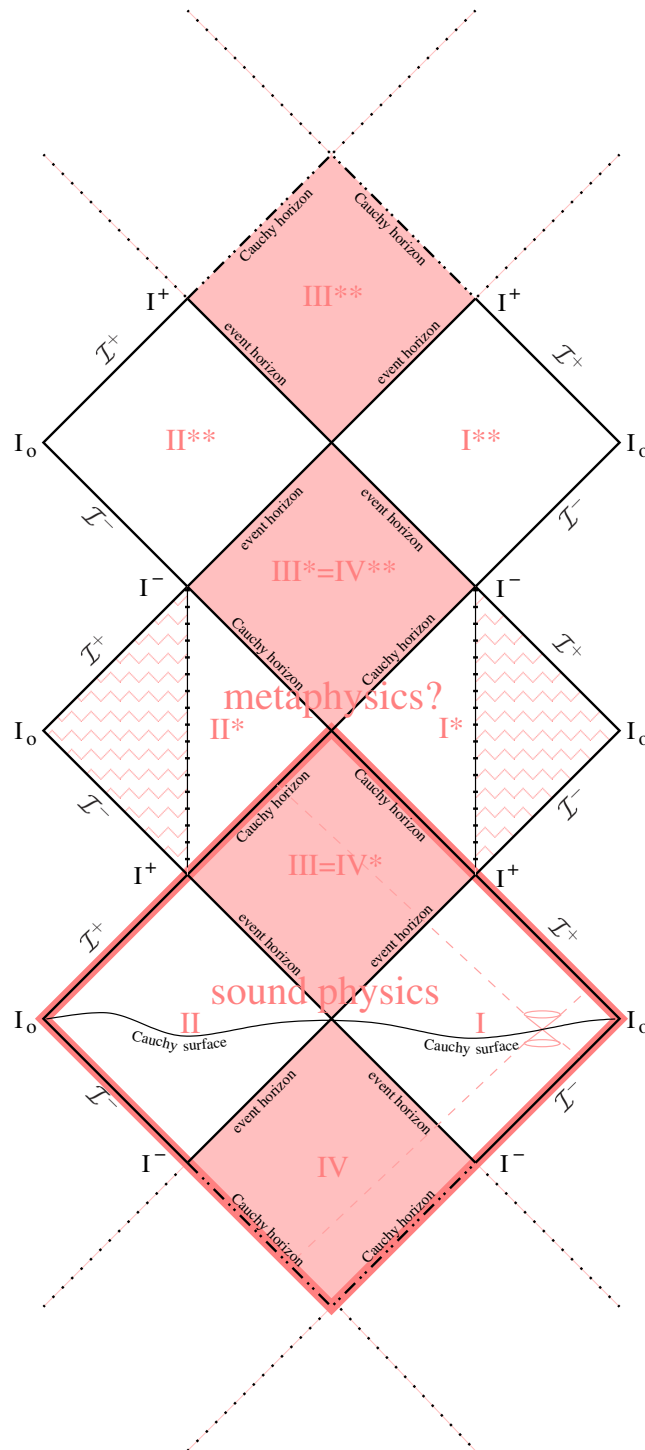


Figure 15: Maximal analytic extension of the Kerr spacetime

should regard it as the spacetime structure generated by a sensible physical source. One may ask then, why a single source should produce an infinite number of spacetimes. And it is even worse. The regions beyond the Cauchy horizon are exceptionally badly behaved. Consider the Cauchy surface in regions I+II of Fig. 15. All light rays and particle trajectories from the past intersect this surface only once. Then the field equations will tell us their future development, see Franzen[62], for example. In Fig. 15 this is roughly indicated by the little light cone. However, even *total* knowledge of the world in I+II does not determine what might be going on in regions I\*+II\*. That is why  $r = r_-$  is called a Cauchy horizon, see Fig. 16. Thus, I\* and II\* are not only beyond the Cauchy horizon but also beyond predictable, sound physics. Moreover, the zigzagged region beyond the singularity is physically doubtful.

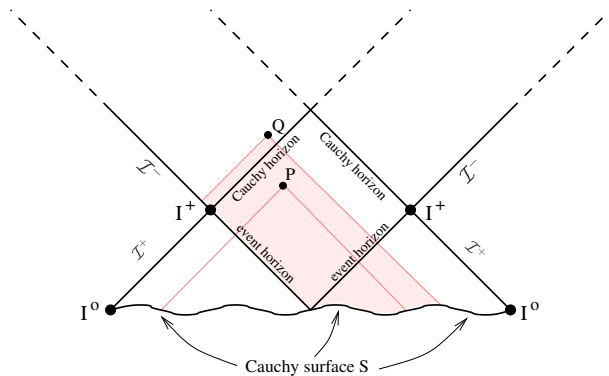


Figure 16: Cauchy horizon: The causal past of a point P *outside* the *Cauchy horizon* of S is entirely determined by the information given on the Cauchy surface S. A point Q *inside* the Cauchy horizon receives also information from  $\mathcal{I}^-$ . Evidently, initial data on S are not sufficient to uniquely determine events at point Q. The surface that separates the two regions “causally determined by S” and “not causally determined by S” is called Cauchy horizon.

In this region, the asymptotics is reversed, see the permutation of  $I^+$  and  $I^-$ . As a consequence, the asymptotic mass in  $I^*$  picks up a minus sign as compared to I. So the *same source* possesses a positive mass  $+m$  in I and a negative mass  $-m$  in  $I^*$ , which seems strange. Moreover, it turns out that these regions are crowded with closed timelike curves. The whole extension is not globally hyperbolic. Thus one should restrict to the “diamond of sound physics”, I+II+III+IV. To do this consistently, one has to devise a physical mechanism preventing traveling beyond the Cauchy horizon, that is, the Cauchy horizon should become singular in some sense (cosmic censorship, see Penrose[150]).



### 3.8 Gravitoelectromagnetism, multipole moments

*The curvature tensor of the Kerr metric is calculated. By squaring it suitably, we find the two quadratic curvature invariants. Subsequently, we determine the gravitoelectric and the gravitomagnetic multipole moments of the Kerr metric, and we mention the Simon-Mars tensor the vanishing of which leads to the Kerr metric.*

The analogy between gravity and electrostatics became apparent when the Coulomb law was discovered in 1785. The gravitational and the electrostatic forces both obeyed an inverse-square law, with the difference that the mass can only be positive whereas the electric charge exists with both signs. Equal electric charges repel, opposite ones attract; in contrast, gravity is always attractive.

In 1820 electromagnetism was discovered by Oersted, and the emerging unified theory, called “electrodynamics” by Ampère, eventually found its expression in the Maxwell equations of 1864. Besides the electric field  $\mathbf{E}$  related to charge, we have the magnetic field  $\mathbf{B}$  related to *moving* charge. These fields, together with the electric and magnetic excitations  $\mathbf{D}$  and  $\mathbf{H}$ , respectively, obey the Maxwell equations.

Newton’s gravitational theory was only superseded in 1915/16 by Einstein’s gravitational theory, general relativity. However, already in the 1870s physicists began to speculate whether, besides Newton’s “gravitoelectric” field, related to mass at rest, there may also exist a new “gravitomagnetic” field, accompanying moving mass; for more details and references see Mashhoon[116]. As we saw above, these speculations became a solid basis in general relativity. In (90), the gravitomagnetic Lense-Thirring term surfaced, which found solid experimental verification in the meantime. Thus, we can speak with justification of gravitoelectromagnetism[116] (GEM), a notion which can guide our intuition, see in this context also Ni and Zimmermann[137].

#### Gravitoelectromagnetic field strength

Electrodynamics is a linear theory, GR a nonlinear one. Still, if we take a linearized version of GR, there are those strong analogies between electrodynamics and gravitodynamics, as worked out, for instance, nicely in Rindler’s[165] book. However, the analogies go even further, as pointed out particularly by Mashhoon[116]. Even in an arbitrary gravitational field, if referred to a Fermi propagated reference frame with coordinates  $(T, \mathbf{X})$ , GEM is a useful concept. If we apply the geodesic deviation equation (33) to such a frame, the gravitoelectromagnetic field strength, representing the tidal forces,

turns out to be[116]<sup>20</sup>

$${}^{\text{GEM}}F_{\alpha\beta} = -R_{\alpha\beta 0i}(T) X^i. \quad (165)$$

If we develop (33) up to the order linear in the velocity  $\mathbf{V} := d\mathbf{X}/dT$ , we find

$$\frac{d^2 X^i}{dT^2} = -R_{0i0j} X^j + 2R_{ik0j} X^j V^k = -{}^{\text{GEM}}F_{i0} - 2{}^{\text{GEM}}F_{ki} V^k. \quad (166)$$

Now we recall that in electrodynamics the electric and the magnetic fields  $\mathbf{E}$  and  $\mathbf{B}$ , respectively, are accommodated in the 4d electromagnetic field strength tensor according to

$$(F_{\alpha\beta}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ \diamond & 0 & B_3 & -B_2 \\ \diamond & \diamond & 0 & B_1 \\ \diamond & \diamond & \diamond & 0 \end{pmatrix} = -(F_{\beta\alpha}). \quad (167)$$

The diamond symbol  $\diamond$  denotes matrix elements already known because of the antisymmetry of the matrix involved. The corresponding 2-form reads  $F = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta$ . Keeping (167) in mind, Eq.(166) can be rewritten as a vector equation

$$\frac{d^2 \mathbf{X}}{dT^2} = -{}^{\text{gr}}\mathbf{E} - 2\mathbf{V} \times {}^{\text{gr}}\mathbf{B}. \quad (168)$$

In accordance with the equivalence principle, this equation of motion is *independent* of the mass. The analogy with electromagnetism requires that the *gravitoelectric* charge, in terms of the mass  $m$ , is  $-1$  and the *gravitomagnetic* charge  $-2$ . In electrodynamics, both quantities are  $+1$ . The difference comes from the vector nature of the electromagnetic potential  $A_\alpha$  as compared to the tensor nature of the gravitational potential  $g_{\alpha\beta}$ , that is, helicity 1 as compared to helicity 2. The relation between the gravitomagnetic to the gravitoelectric charge, that is, the *gyrogravitomagnetic ratio*, is two:  ${}^{\text{gr}}\gamma = 2$ . Note that in Gravity Probe-B the authors specify the gyrogravitomagnetic ratio as 1. However, their gyros carried only orbital angular momentum rather than spin angular momentum. Hence this is to be expected; for more detailed discussions on this difference, see Ref.[80, 138].

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<sup>20</sup>Alternatively, we could generalize the Newtonian tidal force matrix of (9) to the gravitoelectric and gravitomagnetic tidal force matrices,  $\mathcal{E}_{ij} = R_{i0j0}$  and  $\mathcal{B}_{ij} = \epsilon_{ikl} R_{klj0}$ , respectively, see Scheel & Thorne[169]. Both matrices are symmetric and trace-free. Note that  ${}^{\text{GEM}}F_{\alpha\beta}$  is an antisymmetric  $4 \times 4$  matrix and  $\mathcal{E}$  and  $\mathcal{B}$  are both symmetric trace-free  $3 \times 3$  matrices.

It has been pointed out by Ni[139] that the “measurement of the gyrogravitational ratio of [a] particle would be a further step [138] towards probing the microscopic origin of gravity. GP-B serves as a starting point for the measurement of the gyrogravitational factor of particles.”

## Quadratic invariants

In electrodynamics, we have two quadratic invariants [81]:

$$\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} = \star(\star F \wedge F) = B^2 - E^2, \quad \frac{1}{2}F_{\alpha\beta}F^{*\alpha\beta} = \star(F \wedge F) = 2\mathbf{E} \cdot \mathbf{B}, \quad (169)$$

where we used for the tensor dual the notation  $F^{*\alpha\beta} := \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$ . We also employed the very concise notation of exterior calculus with the Hodge star operator.<sup>21</sup>

The first invariant is proportional to the Maxwell vacuum Lagrangian and is an ordinary scalar, whereas the second one corresponds to a surface term and is a pseudoscalar (negative parity).

Turn now directly to the Kerr metric and list for this example the tidal gravitational forces, which are represented by the curvature tensor. With its 20 independent components, it can be represented by a trace-free symmetric  $6 \times 6$  matrix, see (32). The collective indices  $A, B, .. = 1, \dots, 6$  are defined as follows:  $\{\hat{t}\hat{r}, \hat{t}\hat{\theta}, \hat{t}\hat{\phi}; \hat{\theta}\hat{\phi}, \hat{\phi}\hat{r}, \hat{r}\hat{\theta}\} \longrightarrow \{1, 2, 3; 4, 5, 6\}$ . We throw the orthonormal Kerr coframe (128) to (133) into our computer and out pops the  $6 \times 6$  curvature matrix,

$$(R_{AB}) = \begin{pmatrix} -2\mathbb{E} & 0 & 0 & 2\mathbb{B} & 0 & 0 \\ \circ & \mathbb{E} & 0 & 0 & -\mathbb{B} & 0 \\ \circ & \circ & \mathbb{E} & 0 & 0 & -\mathbb{B} \\ \circ & \circ & \circ & 2\mathbb{E} & 0 & 0 \\ \circ & \circ & \circ & \circ & -\mathbb{E} & 0 \\ \circ & \circ & \circ & \circ & \circ & -\mathbb{E} \end{pmatrix} = (R_{BA}), \quad (170)$$

with

$$\mathbb{E} := mr \frac{r^2 - 3a^2 \cos^2 \theta}{(r^2 + a^2 \cos^2 \theta)^3}, \quad \mathbb{B} := ma \cos \theta \frac{3r^2 - a^2 \cos^2 \theta}{(r^2 + a^2 \cos^2 \theta)^3}. \quad (171)$$

It is straightforward to identify  $\mathbb{E}$  as the gravitoelectric and  $\mathbb{B}$  as the gravitomagnetic component of the curvature. This is in accordance with (165).

<sup>21</sup>The Hodge star  $\star\omega$  of a p-form  $\omega = (1/p!)\omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$  is an  $(n-p)$ -form  $\star\omega$ , with the components  $(\star\omega)_{\mu_1 \dots \mu_{n-p}} = (1/p!)\varepsilon^{\nu_1 \dots \nu_p}_{\mu_1 \dots \mu_{n-p}} \omega_{\nu_1 \dots \nu_p}$ , where  $\varepsilon$  is the totally antisymmetric unit tensor and  $n$  the dimension of the space, see Eq.(C.2.90) in Ref.[81]

It is obvious how we should continue. Our gravitoelectromagnetic invariants will be<sup>22</sup>

$$K := \frac{1}{2} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = -\star(\star R_{\alpha\beta} \wedge R^{\alpha\beta}), \quad (172)$$

$$\mathcal{P} := \frac{1}{4} \varepsilon^{\gamma\delta\mu\nu} R_{\alpha\beta\gamma\delta} R^{\alpha\beta}_{\mu\nu} = \star(R_{\alpha\beta} \wedge R^{\alpha\beta}). \quad (173)$$

Again, our program determines the *Kretschmann*<sup>23</sup> scalar  $K$  and the *Chern-Pontryagin* pseudoscalar  $\mathcal{P}$  to be[19]

$$K = -24(\mathbb{B}^2 - \mathbb{E}^2), \quad \mathcal{P} = -48 \mathbb{E} \mathbb{B}. \quad (174)$$

The similarity to (169) is impressive. The GEM analogy quite apparently applies to the full nonlinear theory. The results in (174), partly in more involved representations, can be found in the literature, see, for instance, the books of de Felice & Clarke[42] and of Ciufolini & Wheeler[36], but compare also de Felice & Bradley[41], Henry[84], and Cherubini et al.[32].

Thus, the quadratic invariants  $K$  and  $\mathcal{P}$  confirm that the Kerr metric is the exterior field of a rotating mass distribution. In order to get more information about this distribution, we proceed, like in electrodynamics, and look into the gravitoelectromagnetic multipole moments of this rotating mass.

### Gravitomagnetic clock effect of Mashhoon, Cohen, et al.

According to the results of Lense-Thirring, the rotation of the Sun changes the spacetime around it by inducing gravitomagnetic effects. As we saw above, in a similar way the temporal structure around a Kerr metric is affected by the angular momentum of the Kerr source. Thus, a gravitomagnetic clock effect should emerge,<sup>24</sup> the measurability of which requires very accurate clocks. The effect can be demonstrated by two clocks that move on equatorial orbits, one in prograde and the other in retrograde orbit around the Kerr metric. It turns out[117] that the *prograde* equatorial clock is *slower* than the retrograde one. This is not necessarily what our intuition would

<sup>22</sup>In exterior calculus, we have the Euler 4-form  $E := R_{\alpha\beta} \wedge \star R^{\alpha\beta}$ , with  $K = \star E$ . Analogously, we have the Chern-Pontryagin 4-form  $P := -R^{\alpha}_{\beta} \wedge R^{\beta}_{\alpha}$ , which is an exact form, with  $\mathcal{P} := \star P$ , cf. Obukhov et al.[141].

<sup>23</sup>Usually in the literature[36, 42], the Kretschmann scalar is defined as  $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$ , even though the electrodynamics analogy would suggest to include the factor 1/2.

<sup>24</sup>This was first predicted by Cohen and Mashhoon[37] and worked out in greater detail by Mashhoon et al.[118, 117], see also Bonnor & Steadman[18] and the review papers in the workshop of Lämmerzahl et al.[103]. In a similar way, there emerges also a gravitomagnetic *time delay*, see Ciufolini et al.[33].

tell us. It is connected with the fact that the dragging of frames in a Kerr metric can sometimes turn out to be an “antidragging”, thus making this notion less intuitive,[165] as we already recognized in Sec.3.5.

Generalizations of this clock effect were studied, for example, by Hackmann & Lämmerzahl[71]. The recent discussion of the *Clocks around Sgr A\**, by Angéilil & Saha[7] is, in effect, just one more manifestation of the gravitomagnetic clock effect.

### Multipole moments: gravitoelectric and gravitomagnetic ones

In Newton’s theory, one gets a good idea about a mass distribution and its gravitational field by determining the multipole moments of the mass distribution  $M$ . In GR, because of the existence gravitomagnetism, we have to expect a new type of multipole moments, namely the moments  $J$  of the angular momentum distribution.

If a stationary axially symmetric line element of the form (93) is asymptotically flat, then it is possible[180] to define two sets of multipole moments, the gravitoelectric moments  $M_s$  (“mass multipole moments”) and the gravitomagnetic moments  $J_s$  (“angular momentum multipole moments”), for  $s = 0, 1, 2, \dots$ . These moments were found by Geroch[67] for the static and by Hansen[73] for the stationary case. They were reviewed by Quevedo[160] and used for constructing new exact solutions by Quevedo & Mashhoon[159, 161]. Hansen computed the multipole moments for the Kerr solution and found

$$s = 0 \quad M_0 = -m \quad J_1 = ma \quad (175)$$

$$s = 1 \quad M_2 = ma^2 \quad J_3 = -ma^3 \quad (176)$$

$$s = 2 \quad M_4 = -ma^4 \quad J_5 = ma^5 \quad (177)$$

$$s = 3 \dots \quad M_6 = ma^6 \dots \quad J_7 = -ma^7 \dots \quad (178)$$

More compactly, we have

$$M_{2s} = (-1)^{s+1} m a^{2s}, \quad M_{2s+1} = 0; \quad (179)$$

$$J_{2s} = 0, \quad J_{2s+1} = (-1)^s m a^{2s+1}. \quad (180)$$

It is possible to introduce normalized multipole moments, see Meinel et al.[123], such that for Kerr we have  $\widetilde{M}_s + i\widetilde{J}_s = m(ia)^s$ . Then the mass monopole  $\widetilde{M}_0 = m$  is positive. Apparently, the Kerr metric has a simple multipolar structure or, formulated differently, only very specific matter distributions can represent the interior of the Kerr metric.

Quevedo[160] compiled a number of theorems which illustrate the use of the multipole moments:

- 1) A stationary spacetime is *static* if and only if all its *gravitomagnetic* multipole moments vanish (Xanthopoulos 1979).
- 2) A static metric is flat if and only if all its *gravitoelectric* multipole moments vanish (Xanthopoulos 1979).
- 3) A stationary metric is axisymmetric if and only if all its multipole moments are axisymmetric (Gürsel 1983).
- 4) Two metrics with the same multipole moments have the same geometry at large distances from the source (Beig & Simon 1981; Kundu 1981; Van den Bergh & Wils 1985).
- 5) Any stationary, axisymmetric, asymptotically flat solution of Einstein's vacuum equation approaches the Kerr solution asymptotically (Beig & Simon 1980).
- 6) Any static, axisymmetric, asymptotically flat vacuum solution approaches the Schwarzschild solution asymptotically (Beig 1980).

In the formulation of Stephani, Kramer, et al.[180]:

- 7) A given asymptotically flat stationary vacuum spacetime is uniquely characterized by its multiple moments.

We recognize that the knowledge of the multipole moments provides a lot of insight into the physical properties of an exact solution.

From the point of view of the Kerr solution, theorem 5), see Beig & Simon[12], is perhaps the most interesting one. It underlines the central importance of the Kerr solution. The considerations in the context of theorem 5) were further developed by Simon[174, 175]. On the 3-dimensional spatial slices of a stationary axially symmetric metric, he defined the 3d “*Simon tensor*”, [15] a kind of complexified generalized Cotton-Bach tensor[65]. The vanishing of the Simon tensor then leads to the multipole moments of the Kerr solution. Later, Mars[112], see also Mars[111] and Mars & Senovilla[113], generalized this approach and was led to the 4d “*Simon-Mars tensor*”. In Ionescu & Klainerman[88], one can find a more extended discussion of the Simon-Mars tensor, see also Wong[192]. More recently, Bäckdahl & Valiente Kroon[9] have proposed replacing the Simon-Mars tensor by another measure of “non-Kerrness”, namely a scalar parameter.

### 3.9 Adding electric charge and the cosmological constant: Kerr-Newman metric

Enriching the Kerr metric by an electric charge is straightforwardly possible. We start from the metric (133) with coframe (128) to (132). This coframe can accommodate the Kerr, the Schwarzschild, and the Reissner-Nordström solutions. The different forms of the function  $\Delta$  suggest how a charged Kerr solution should look like ...

Schwarzschild	$(m)$	$\rho = r^2$	$\Delta = r^2 - 2mr$	
Reissner-Nord.	$(m, q)$	$\rho = r^2$	$\Delta = r^2 - 2mr$	$+q^2$
Kerr	$(m, a)$	$\rho = r^2 + a^2 \cos^2 \theta$	$\Delta = r^2 - 2mr$	$+a^2$
Kerr-Newman	$(m, a, q)$	$\rho = r^2 + a^2 \cos^2 \theta$	$\Delta = r^2 - 2mr$	$+q^2 + a^2$

Charging the Schwarzschild solution is achieved by adding  $q^2$  to the function  $\Delta$ . Since the charged Kerr solution should encompass the Reissner-Nordström solution, we tentatively keep the term  $q^2$  for the case  $a \neq 0$ . Now, we can indeed find a potential,

$$A = -\frac{qr}{\rho^2} (dt - a \sin^2 \theta d\phi), \quad (181)$$

such that the Einstein-Maxwell equations are fulfilled. The potential describes a line-like charge distribution at  $\rho = 0$ , that is, on the ring singularity of the Kerr spacetime, which is quite satisfying[135]. This charged Kerr solution was first worked out by Newman, Couch, Chinnapared, Exton, Prakash, and R. Torrence[134] (1965), using “methods which transcend logic”, as Ernst[54] puts it. He, in turn, proceeded from (120). Replacing<sup>25</sup>  $\xi$  by  $\sqrt{1 - qq^*} \xi$  generates a solution of the Einstein-Maxwell equations with potential  $A_t + iA_\phi = q/(\xi + 1)$ .

The Kerr and the Kerr-Newman solution behave quite similarly. We can adopt most of the discussion of the Kerr metric by substituting  $a^2 + q^2$  for  $a^2$ .

We can further generalize the Kerr-Newman metric to include also a cosmological constant, see Sec.4.1., and even more parameters, see Fig. 17.

<sup>25</sup>Here,  $q$  is *not* the charge but a complex parameter in the solution of the Ernst equation.

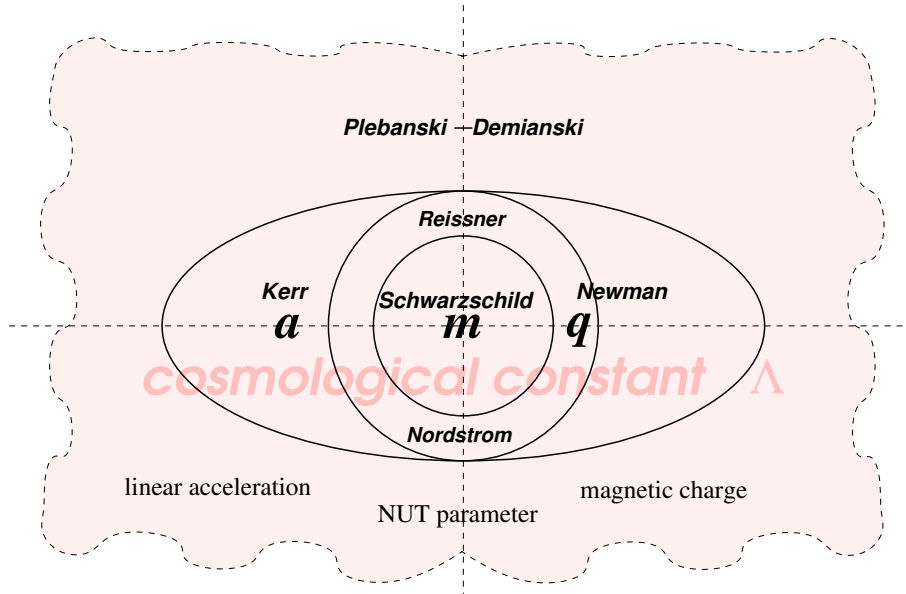


Figure 17: Schematics of Petrov D solutions: The spherically symmetric Schwarzschild solution with mass parameter  $m$  is located in the center. Adding an electric charge  $q$  brings us to the Reissner-Nordström solution. It is still spherically symmetric but adds a second horizon. The distance between the horizons increases with the charge  $q$ . Setting the black hole into rotation, the angular momentum parameter emerges,  $a \neq 0$ , and reduces the spherical symmetry to an axial one. An oblate ergosurface (two, actually) forms. Event horizon and ergosurface meet at the polar axis, the equatorial distance increases with  $a$ . All these solutions can be *deSittered*, that is, a cosmological constant  $\Lambda$  is added. All presented solutions are subcases of the Plebanski-Demianski solution, which adds three more parameters[69].

### 3.10 On the uniqueness of the Kerr black hole

*The Kerr black hole, up to some technical assumptions, is the unique solution for the stationary, axially symmetric case. We point to some of the literature where these results can be found.*

Because of the Birkhoff theorem, the Schwarzschild solution (mass parameter  $m$ ) represents the general spherically symmetric solution of the Einstein vacuum field equation. The analogous is true in the Einstein-Maxwell case for the 2 parameter Reissner-Nordström solution (mass and charge parameters  $m$  and  $q$ , respectively). Thus, for spherical symmetry, we have a fairly simple situation.



In contrast, in the axially symmetric case, there does *not* exist a generalized Birkhoff theorem. The 2-parameter Kerr solution (mass and rotation parameters  $m$  and  $a$ , respectively), is just a particular solution for the axially symmetric case. As we saw in Sec.3.8, the Kerr solution has very simple gravitoelectric and gravitomagnetic multipole moments (179,180). Numerous solutions are known that represent the exterior of matter distributions with different multipole moments. The analogous is valid for the 3 parameter Kerr-Newman solution (parameters  $m, a, q$ ), see Stephani et al.[180] and Griffiths & Podolský[69].

However, one can show under quite general conditions that the Kerr-Newman metric represents the most general asymptotically flat, stationary electro-vacuum *black hole solution* (“no-hair theorem”), see Meinel’s short review[122]. Important contributions to the subject of black hole uniqueness were originally made by Israel[90, 91], Carter[30, 31], Hawking & Ellis[78], Robinson[166, 167], and Mazur[120] (1967-1982), for details see the recent review of Chruściel et al.[38].

More recently Neugebauer and Meinel[130, 133] found a constructive method for proving the uniqueness theorem for the Kerr black hole metric. This was extended to the Kerr-Newman case by Meinel.[121] By inverse scattering techniques, they showed how one can construct the Ernst potential of the Kerr(-Newman) solution amongst the asymptotically flat, stationary, and axially symmetric (electro-)vacuum spacetimes surrounding a connected Killing horizon.

Let us then eventually pose the following questions[27]:

- (i) Are axially symmetric, stationary vacuum solutions outside some matter distribution “Kerr”? The answer is “certainly not”, and it makes sense to figure out ways to characterize the Kerr metric, see Sec.3.8.
- (ii) Is the Kerr solution the unique axially symmetric, stationary vacuum black hole? The answer is essentially “yes” (modulo some technical issues)—see, for example Mazur[120].

The general tendency in the recent development of the subject is to use additional scalar or other matter fields. They weaken the uniqueness theorems, which is probably not too surprising.

Let us conclude with a quotation that may make you curious to learn still more about the beauty of the Kerr metric: We have many different axially symmetric solutions. The Kerr solution is characterized by “*stationary, axially symmetric, asymptotically flat, Petrov type D vacuum solution of the vanishing of the Simon tensor, admitting a rank-2 Killing-Stäckel (KS) tensor of Segre type [(11)(11)] constructed from a (non-degenerate) rank-2 Killing-Yano (KY) tensor*”, see Hinoui et al.[87].

### 3.11 On interior solutions with material sources

*To match the Kerr (vacuum) metric to a material source consistently is one of the big unsolved problems. Only the rotating disc solution of Neugebauer & Meinel provides some hope.*

This section is added in order to draw your attention to an unsolved problem, to the solution of which you might want to contribute. Find a realistic material source for the Kerr metric in the sense of an exact solution. Many unsuccessful attempts have been made, see the early review of Krasiński[101] of 1978. More recently, in 2006, Krasiński[156] concludes “*that a bright new idea is needed, as opposed to routine standard tricks tested so far.*” This statement was not made lightheartedly, Krasiński knows what he is talking about.

Many axially symmetric vacuum solutions were constructed. Quevedo & Mashhoon[161], for example, deformed the multipole moments of the Kerr(-Newman) metric and constructed appropriate solutions of the Einstein(-Maxwell) equation that describe the exterior gravitational field of a (charged) rotating mass. It is always the hope that somebody may find a suitable matter distribution with the multipole moments of the Kerr solution—but this did not happen so far; for another approach see Marsh[115].

We are only aware of one exact solution that fits into this general context: It is the *infinitesimally thin and rigidly rotating dust* solution of Neugebauer & Meinel[131, 132] (1993). It is an exact analytical solution of the Einstein equation *with* matter. It depends on 2 independent parameters, the radius  $\rho_0$  of the disk and its angular velocity  $\Omega$ . Petroff & Meinel[151] developed, by means of an iterative procedure, a post-Newtonian approximation of the solution that helps to understand the Newtonian limit.

We recall that in electrostatics in flat space, for example, we prescribe an electric charge distribution and we are used to solve the corresponding boundary value problem within Maxwell’s theory. Similarly, Neugebauer & Meinel specified a very thin rotating disk of dust and solved the boundary value problem within GR. This is a well-defined procedure. The problem is, however, that within a non-linear theory, such as GR, it is extremely hard to implement. Remarkably, for certain parameter values, the gravitational field of the disk approach the extremal Kerr case. Accordingly, there exists a certain relation to the Kerr problem. The desideratum would be to find a rotating matter distribution the external field of which coincides with the *complete* Kerr field.

Driven by the fact that the electrically charged Kerr solution, the Kerr-Newman solution, has a g-factor of 2, exactly like the electron (see also Pfister

& King[153]), Burinskii[24, 25] speculated that a soliton like solution of the Dirac equation may be the source of the Kerr metric, see also Burinskii & Kerr[26]. Is that the “bright new idea” Krasinski was talking about? We do not know but a hard check of the Burinskii ansatz seems worthwhile.

## 4 Kerr beyond Einstein

*In generalizations of Einstein’s theory of gravity, the Riemannian geometry of spacetime is often extended to a more general geometrical framework. We describe two such examples in which the Kerr metric still plays a vital role.*

### 4.1 Kerr metric accompanied by a propagating linear connection

*We display the Kerr metric with cosmological constant that, together with an explicitly specified torsion, represents an exact vacuum solution of the two field equations of the Poincaré gauge theory of gravity with quadratic Lagrangian.*

In gauge theories of gravitation, see Blagojević et al.[17], the linear connection becomes a field that is at least partially independent from the metric. It can be either metric-compatible, then it is a connection with values in the Lie-algebra of the Lorentz group  $SO(1, 3)$  and the geometry is called a Riemann-Cartan geometry, or it can be totally independent, then it resides in a so-called metric-affine space and the connection is  $GL(4, R)$ -valued. For simplicity, we concentrate here on the former case, the Poincaré gauge theory of gravity, but the latter case is also treated in the literature[187, 11].

Let us shortly sketch the theory. Gauging the Poincaré group leads to a spacetime with torsion  $T^\alpha$  and curvature  $R^{\alpha\beta}$  (Riemann-Cartan geometry<sup>26</sup>):

$$T^\alpha := D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta^\alpha \wedge \vartheta^\beta = \frac{1}{2}T_{ij}^\alpha dx^i \wedge dx^j, \quad (182)$$

$$R^{\alpha\beta} := d\Gamma^{\alpha\beta} - \Gamma^{\alpha\gamma} \wedge \Gamma_\gamma^\beta = -R^{\beta\alpha} = \frac{1}{2}R_{ij}^{\alpha\beta} dx^i \wedge dx^j. \quad (183)$$

Besides the coframe 1-form  $\vartheta^\alpha$ , the Lorentz connection 1-form  $\Gamma^{\alpha\beta} = \Gamma_i^{\alpha\beta} dx^i = -\Gamma^{\beta\alpha}$  is a second field variable of the gauge theory. For a Riemannian space, torsion  $T^\alpha = 0$  and  $\Gamma^{\alpha\beta}$  becomes the Levi-Civita connection.

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<sup>26</sup>Experimental limits of a possible torsion of spacetime were recently specified in a remarkable paper by Obukhov et al.,[142] see also the literature given there.

We choose a model Lagrangian quadratic in torsion and curvature, in actual fact (for  $\hbar = 1, c = 1$ ),

$$V = -\frac{1}{2\kappa}(T^\alpha \wedge \vartheta^\beta) \wedge \star(T_\beta \wedge \vartheta_\alpha) - \frac{1}{2\rho}R^{\alpha\beta} \wedge \star R_{\alpha\beta}, \quad (184)$$

with Einstein's gravitational constant  $\kappa$  (dimension length-squared) and a dimensionless *strong* gravity coupling constant  $\rho$ . One can calculate the two vacuum field equations by varying with respect to  $\vartheta^\alpha$  and  $\Gamma^{\alpha\beta}$ . In 1988, for these two field equations, a Kerr metric with torsion[10] was found as an exact solution.

We display here the orthonormal coframe and the torsion: The coframe  $\vartheta^\alpha$ , in terms of Boyer-Lindquist coordinates  $(t, r, \theta, \phi)$ , reads (in the conventions used in Ref.[10])

$$\vartheta^{\hat{0}} := \frac{\sqrt{\Delta}}{\rho} (dt + a \sin^2\theta d\phi), \quad (185)$$

$$\vartheta^{\hat{1}} := \frac{\rho}{\sqrt{\Delta}} dr, \quad (186)$$

$$\vartheta^{\hat{2}} := \frac{\rho}{\sqrt{F}} d\theta, \quad (187)$$

$$\vartheta^{\hat{3}} := \frac{\sqrt{F} \sin\theta}{\rho} [adt + (r^2 + a^2) d\phi]. \quad (188)$$

As before, we have  $\rho^2 := r^2 + a^2 \cos^2\theta$ . However, the other structure functions pick up a cosmological constant  $\lambda$ :

$$F := 1 + \frac{1}{3}\lambda a^2 \cos^2\theta, \quad \Delta := r^2 + a^2 - 2Mr - \frac{1}{3}\lambda r^2(r^2 + a^2). \quad (189)$$

The corresponding metric is called a Kerr-deSitter metric. The coframe is orthonormal. Then the metric reads

$$g = -\vartheta^{\hat{0}} \otimes \vartheta^{\hat{0}} + \vartheta^{\hat{1}} \otimes \vartheta^{\hat{1}} + \vartheta^{\hat{2}} \otimes \vartheta^{\hat{2}} + \vartheta^{\hat{3}} \otimes \vartheta^{\hat{3}}. \quad (190)$$

It is a characteristic feature of these exact solutions that even though the Lagrangian (184) does *not* carry a cosmological constant, in the coframe and the metric there emerges such a constant, namely  $\lambda := -3\rho/(4\kappa)$ . This could be of potential importance for cosmology.

The torsion  $T^\alpha$  of this stationary axially symmetric solution of the Poincaré gauge theory reads ( $\vartheta^{\alpha\beta} := \vartheta^\alpha \wedge \vartheta^\beta$ ),

$$\begin{aligned} T^{\hat{0}} &= \frac{\rho}{\sqrt{\Delta}} \left[ -v_1 \vartheta^{\hat{0}\hat{1}} + \frac{\rho}{\sqrt{\Delta}} \left[ v_2 (\vartheta^{\hat{0}\hat{2}} - \vartheta^{\hat{1}\hat{2}}) + v_3 (\vartheta^{\hat{0}\hat{3}} - \vartheta^{\hat{1}\hat{3}}) \right] - 2v_4 \vartheta^{\hat{2}\hat{3}} \right], \\ T^{\hat{1}} &= T^{\hat{0}}, \\ T^{\hat{2}} &= \frac{\rho}{\sqrt{\Delta}} \left[ v_5 (\vartheta^{\hat{0}\hat{2}} - \vartheta^{\hat{1}\hat{2}}) + v_4 (\vartheta^{\hat{0}\hat{3}} - \vartheta^{\hat{1}\hat{3}}) \right], \\ T^{\hat{3}} &= \frac{\rho}{\sqrt{\Delta}} \left[ -v_4 (\vartheta^{\hat{0}\hat{2}} - \vartheta^{\hat{1}\hat{2}}) + v_5 (\vartheta^{\hat{0}\hat{3}} - \vartheta^{\hat{1}\hat{3}}) \right], \end{aligned} \quad (191)$$

with the following gravitoelectric and gravitomagnetic functions:

$$v_1 = \frac{M}{\rho^4} (r^2 - a^2 \cos^2 \theta), \quad v_5 = \frac{Mr^2}{\rho^4}; \quad (192)$$

$$v_2 = -\frac{Ma^2 r \sin \theta \cos \theta}{\rho^5} \sqrt{F}, \quad v_3 = \frac{Mar^2 \sin \theta}{\rho^5} \sqrt{F}, \quad v_4 = \frac{Mar \cos \theta}{\rho^4}. \quad (193)$$

Metric and torsion of this exact solution are closely interwoven. Note, in particular, that the leading gravitoelectric part in the torsion, for small  $a$ , is  $\sim M/r^2$ , a definitive Coulombic behavior proportional to the mass. For  $a = 0$ , we find a Schwarzschild-deSitter solution with torsion.

One may legitimately ask, why is it that the Lagrangian (184) yields an exact solution with a Kerr-deSitter metric? The answer is simple: The Lagrangian was devised such that the torsion square-piece, in lowest order in  $\kappa$ , encompasses a Newtonian approximation. This is already sufficient in order to enable the existence of a Kerr-deSitter metric. One could even add another torsion-square piece to  $V$  for getting an Einsteinian approximation, but this is not even necessary. Thus, only a Newtonian limit of some kind seems necessary for the emergence of the Kerr structure.

## 4.2 Kerr metric in higher dimensions and in string theory

*There exist also Schwarzschild and Kerr metrics in higher dimensional spacetimes. These investigations are mainly motivated by supergravity and string theory.*

Tangherlini[183] (1963) started to investigate higher dimensional Schwarzschild solutions, with  $n - 1$  spatial dimensions. He studied the (“planetary”) orbits in an  $n$ -dimensional Schwarzschild field (“Sun”) and found that only for  $n = 4$  we have stable orbits, see also Ortin[146]. According to Tangherlini, this is then the only case that is interesting for physics. Nowadays, however,

many physicists hypothesize that higher dimensions do exist because string theory suggests it.

Somewhat later, Myers and Perry[128] (1986) generalized these considerations to higher-dimensional Kerr metrics. In the meantime a plethora of such higher-dimensional objects have been found, see Allahverdizadeh et al.[5] and Frolov & Zelnikov[64]. Recently Keeler et al.[97] investigated, in the context of string theory, the separability of Klein-Gordon or Dirac fields on top of a higher-dimensional Kerr type solutions. Lately Brihaye et al.[21], for example, discussed the exact solution of a 5d Myers-Perry black holes as coupled to a massive scalar field. The physical interpretations of these results remain to be seen.

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## Appendix

### A Exterior calculus and computer algebra

We want to use as input the Papapetrou metric (93). We take the equivalent representation in the form of the orthonormal coframe of the Eqs.(94) to (98). How such a Reduce-Excalc program can be set up, is demonstrated in Stauffer et al.[178] and in Socorro et al.[176], for the Einstein 3-form, see Heinicke[82]:

```
%*****
% Coframe of Andress-Lewis-Papapetrou-Buchdahl metric
%*****
% file Buchdahl03.exi, 29 July 2014, fwh & chh
% in "Buchdahl03.exi";

load_package excalc;
off exp$
pform f=0, omega=0, gamma=0 $
fdomain f=f(rho,z), omega=omega(rho,z), gamma=gamma(rho,z) ;

coframe o(0) = sqrt(f) * (d t - omega * d phi),
        o(1) = sqrt(f)**(-1) * exp(gamma) * d rho,
        o(2) = sqrt(f)**(-1) * exp(gamma) * d z,
        o(3) = sqrt(f)**(-1) * rho * d phi
        with signature (1,-1,-1,-1);

displayframe;
frame e$

%*****
% Connection, curvature, and Einstein forms
%*****
pform conn1(a,b)=1, curv2(a,b)=2$
antisymmetric conn1, curv2$
factor o(0), o(1), o(2), o(3)$

conn1(-a,-b) := (1/2)*( e(-a)_|d o(-b) - e(-b)_|d o(-a)
                    - (e(-a)_|(e(-b)_|d o(-c))) * o(c))$
curv2(-a,b) := d conn1(-a,b) - conn1(-a,c) ^ conn1(-c,b)$
```

```

% Einstein tensor = Einstein 0-form
pform einstein3(a)=3, einstein0(a,b)=0$
symmetric einstein0$

einstein3(-a) := -(1/2) * curv2(b,-c) ^ # (o(-a) ^ o(-b) ^ o(c))$
einstein0(a,-b):= #( o(a) ^ einstein3(-b))$

on exp, gcd$
factor ^$
on nero;

einstein0(a,-b):= #( o(a) ^ einstein3(-b));

off nero;

% by inspection, we find
einstein0(1,-1) + einstein0(2,-2); % equals 0
einstein0(0,-0) - einstein0(3,-3); % eliminates gamma

out "Buchdahl03.exo";

load_package tri;
on tex;
on TeXBreak;
einstein0(a,-b):=einstein0(a,-b);
off tex;
einstein0(a,-b):=einstein0(a,-b);
omega:=0;
einstein0(a,-b):=einstein0(a,-b);

shut "Buchdahl03.exo";
;end;

```



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