Visually exploring the Einstein Field Equations

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Visualizations of spacetime curvature are developed in a flexible *Mathematica* notebook. The Einstein field equations are derived in part, and a program is used to show a number of strange stress energy momentum tensors. The experimental metrics include a truncated sinusoidal perturbation and a Gaussian distribution function perturbation to the flat spacetime metric. Visual images developed include simulating the geodesics of light rays, and graphing the stress energy momentum tensor components in matrices of density plots.

I. INTRODUCTION

The development of general relativity theory dates back to 1905 when Einstein first began writing and publishing work about the properties of light and the equivalence of gravitational and accelerated mass. Einstein was able to foresee that light would bend in a gravitational field and exhibit quantized energy and momentum. Such foresight was remarkable given the state of theoretical physics in the early 1900's. Einsten's thinking was characterized by his numerous thought experiments that guided and developed his intuition about gravity. He often would spend long hours in a boat contemplating the consequences of various physical interactions. He imagined what he might have seen riding on a light-wave, or what a person would feel if they were sealed inside an accelerating box. Often these thought experiments were simple intuitive extensions of Newtonian dynamics and classical electromagnetism. The insights Einstein gained from these thought experiments allowed him to devise perhaps the greatest of all modern physical theories.

The concept of general relativity is fairly simple to understand; it is a generalization of classical mechanics to account for the invariant speed of light and the equivalence of inertial and gravitational mass. However, the implementation of these ideas is astoundingly complicated. Ten coupled nonlinear partial differential equations compose the Einstein field equations that determine the curvature of spacetime. An exact solution to the Einstein field equations for a spherically symmetric mass was obtained by Schwarzschild in 1915, to the surprise of many including Einstein. A solution for a charged mass, the Reissner-Nordstrom solution, did not appear until 1918 [1]. In 1963 Kerr was able to solve the field equations for a rotating spherically symmetric mass, nearly 48 years after the equations were published [2]. In 1965, Newman synthesized the solutions for both the rotating and charged black hole, producing the Kerr-Newman solution [2]. These three solutions are some of the only known exact solutions to the Einstein field equations. The field equations are so complicated that a great deal is still unknown about their possible effects.

As a consequence of the complexity of the field equations, all theoretical gravity research involving arrangements of mass must be done computationally. The field equations can be written symbolically as relations between tensors $\mathbf{G} \sim \partial\Gamma + \Gamma\Gamma \sim \kappa\mathbf{T}$ where $\mathbf{G}$ is the Einstein tensor that governs curvature, $\mathbf{T}$ is the stress energy momentum tensor, the source of curvature, ∂ represents partial derivatives, and Γ is the connection that determines the world-lines of free particles in spacetime. Finding Γ with a specified $\mathbf{T}$ involves integrating the field equations a number of times. Finding the metric, the tensor that clearly denotes the warping of various aspects of spacetime, requires even more integration. Any attempt to integrate these equations for a number of different stress energy momentum tensors involves great computational power, time, and complexity.

To sidestep the difficult integration we have developed a program that efficiently produces the stress energy momentum tensor from a specified metric. This computational route involves only derivatives and is therefore much more simple and accurate to computationally perform. The program outputs easily understood and intuitive graphics that describe the stress energy momentum tensor, the metric, and the geodesics. We have used the program to explore a number of different metrics and draw conclusions about their behavior and physical realizability. The program has the adaptability to be generalized to a more systematic experimental approach guided by human input, and future research is suggested along those lines.

The importance of visualization and intuition in the evolution of the theory of general relativity cannot be understated. The research program presented here provides the user with a powerful visual tool to explore the implications of spacetime curvature.

II. GENERAL RELATIVITY PRIMER

The theory of general relativity is the final form of an elegant restructuring of classical mechanics. The discovery of the invariant speed of light c around the turn of the 20th century fundamentally changed the principal concepts that all theories of motion had been built upon. Time and space are unified by the invariant speed and are no longer immutable as they were in classical mechanics. The effects of this unification are described by the theory of special relativity, and the ultimate implication for inertial motion is described in the theory of general relativity.

Any conversation about mechanics must involve a location of an observer. A reference frame of an observer consists of the coordinates, position, motion, and all other measurements made at the particular point in space (and time) where the observer exists in the Universe. An inertial reference frame (IRF) is subject to no external forces. A global reference frame is able to make consistent measurements at all locations and times in space and time. It is standard practice in special and general relativity to set the constants $c = 1$ and $G = 1$. We shall adopt this convention for the rest of the work.

Consider the paradox of relative motion created by an invariant speed. Two observers move in relation to a light source. Observer A in reference frame O moves toward the light source at a speed v. Observer B in reference frame $\bar{O}$ moves away from the light source. Each measures the speed of the light. In both the O and $\bar{O}$ reference frame, the speed of light is exactly c. Classical theory predicts that observer A should measure a faster speed because her velocity would be added to the speed of light. Similarly, observer B would classically measure a slower speed. In order for both these observations to be true, something must be happening to the way observers A and B are measuring space and time. The reference frames O and $\bar{O}$ must have relative units of measurement.

In special relativity, time and space warp in order to preserve the invariant speed c in all reference frames. Special relativity predicts that lengths will contract and time will dilate (slow). I now principally follow the lecture notes of Lindner [1].

Time dilation and length contraction indicate that events exist at locations in both space and time unique to each reference frame. The location can be represented by a 4-dimensional spacetime vector defined as

$$X = \begin{pmatrix} t[\lambda] \\ x[\lambda] \\ y[\lambda] \\ z[\lambda] \end{pmatrix}.$$

The affine parameter λ is a parameter that controls the evolution of the position in the 4-dimensions of spacetime. The evolution of a 4-vector through spacetime is called the world-line of the particle, event, or observer. The affine parameter determines where the particle is on the world-line. The world line can be visualized if restricted to two dimensions of space plus one dimension of time (2+1 dimensional spacetime), as shown in Fig.1. The line is surrounded by a light cone with a radius that represents the maximum distance light could travel in the elapsed time measured from the base of the cone. No world-line can escape its own light cone, and all events outside the light-cone are causally disconnected from the world-line.

Consider a chicken hatching from an egg being observed from many different IRFs. Every reference must agree on one reality for the chicken: the reality of the chicken must be sequential and causal. (Here I neglect the possibility of time travel until such concepts can be understood in greater detail.) This implies that there is some quantifiable interval that all observers must agree upon regardless of their reference frame. This invariant interval is the proper "wristwatch" aging of the chicken and is calculated by each observer in each reference frame. The invariant interval is defined by the relativistic dot product of two 4-vectors. The dot product is determined by the metric,

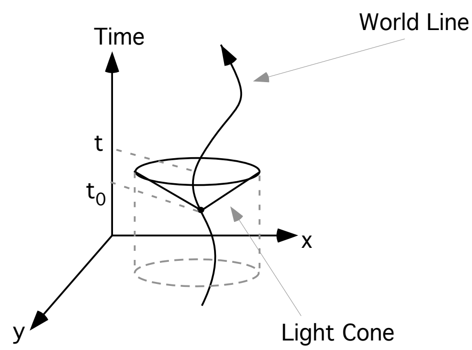

FIG. 1: Time evolves upwards. A particle moves along the world line in through 2+1 spacetime. This is seen as an movement in the x-y plane. The light cone shows the distance light can travel in the x-y plane from time t_0 to t. The top of the cone has been projected onto the x-y plane with dashed gray lines.

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

an operator that determines how space is measured. The relativistic dot product is then

$$\tau^2 = \eta X \circ X.$$

For spacetime in special relativity this interval is simple and unchanging,

$$\tau^2 = t^2 - x^2 - y^2 - z^2. \tag{1}$$

It is useful to consider the implications of 4-vector velocity and momentum. The 4-velocity is found by dividing by the proper time τ

$$V^\alpha = \begin{pmatrix} dt[\lambda]/d\tau \\ dx[\lambda]/d\tau \\ dy[\lambda]/d\tau \\ dz[\lambda]/d\tau \end{pmatrix} = \begin{pmatrix} \gamma \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}.$$

The 4-momentum is 4-velocity multiplied by the mass m

$$P^\alpha = \begin{pmatrix} \gamma m \\ \gamma m v_x \\ \gamma m v_y \\ \gamma m v_z \end{pmatrix}.$$

The invariant quantity of the 4-momentum is mass, preserving the conservation of mass from Newtonian mechanics. The first component of the 4-momentum is the total energy. This is apparent in the Newtonian approximation to the first component:

$$\begin{aligned}\gamma m &= m\left(1 - \left(\frac{v}{c}\right)^2\right)^{-1/2} \\ &= m\left(1 + \frac{v^2}{c^2}\right) \\ &= mc^2 + \frac{1}{2}mv^2, \end{aligned} \tag{2}$$

where the SI units have been inserted for clarity. The second term of Eq.2 is the kinetic energy and the first term is the energy of the mass. Therefore, $E = mc^2$ for objects at rest.

Gravity invalidates the idea of a non-accelerating inertial reference frame. Objects in free fall around a large mass do not feel a force yet are subject to acceleration. Inertial motion in a gravitational field is an acceleration rather than an unchanging velocity. Special relativity assumes that every inertial reference frame has a constant velocity with respect to the global reference frame. Near large massive objects there will always exist some point in any reference frame that is subject to acceleration, unless we assume a spacetime that is locally free of all mass. This is to assume a special relativity, and hence the name of the prior theory.

Einstein proposed the architecture of curved spacetime to describe the curving world-lines of inertial reference frames. In technical terms, the world-lines of IRFs are always parallel in special relativity. In general relativity the world-lines of IRFs follow the geodesics of the curved spacetime. The geodesics are both the straightest curves between two points and the curves with stationary length. Acceleration causes geodesics to diverge and converge.

The description of curved spacetime is formulated mathematically with tensor equations. Tensors are collections of values and equations organized by indices. A tensor of rank n is a collection of D^n components where D is the dimension of the tensor. All general relativity has a 4-component dimension representative of the 3 + 1 dimensions of spacetime. For example, a 4-vector is a rank one tensor in 4 dimensions.

The addition of mass will change the invariant spacetime interval. Where space and time were given equal footing by the metric of special relativity, they must be generalized to account for inertial acceleration. Spacetime curvature specifies the metric, and the metric governs the warp of 4-vector components In general relativity, the invariant interval is defined by the equation

TABLE I: The Stress Energy Momentum Tensor with component parts. p stands for pressure and S for shear.

Energy Density	E_x Flux	E_y Flux	E_z Flux
P_x Density	p_x	S_x in xz plane	S_x in xy plane
P_y Density	S_y in yz plane	p_y	S_y in xy plane
P_z Density	S_z in yz plane	S_z in xz plane	p_z

$$\tau^2 = g_{\alpha\beta} X^\alpha X^\beta,$$

where g is the metric, a rank 2 tensor. With an arbitrary metric it is possible to calculate the invariant interval of any inertial reference frame. Alternately, one can think of the metric as specifying the curvature of spacetime and hence the mass distribution.

The connection Γ determines the geodesics of curved spacetime. The connection is defined by the derivatives of the metric, symbolically $\Gamma \sim \partial g$.

In order to quantify exactly the type of curvature we need for physical representation of spacetime, we must consider the source of the curvature. From special relativity $E = mc^2$ and therefore all forms of energy will gravitate. Energy in general relativity is embodied by the 4-momentum. We must define the curvature in terms of the 4-momentum density over the surface area of 4-dimensional spacetime. This is in close analogy to deriving acceleration from gravitational field flux due to mass in newtonian theory. Just as gravitational field flux determines acceleration in Newtonian theory, surface 4-momentum density determines curvature in general relativity. The surfaces in 4 dimensions are 3 volumes, and therefore we must consider the 4-momentum density for each 3D volume of spacetime. We define $d^3\mathbf{V}_i$ to be the 3D surface for the specified dimension i of the 4-vector X. For example, $d^3\mathbf{V}_t = dxdydz$ is the 3D volume in the time dimension. With this convention we can define the components of the stress energy momentum tensor

$$\mathbf{T}^{\mu\nu} = \frac{dP^\mu}{d^3\mathbf{V}_\nu}, \tag{3}$$

$$\mathbf{T}^{\mu\nu} = \begin{pmatrix} \frac{dE}{dxdydz} & \frac{dE}{dydzdt} & \frac{dE}{dxdzdt} & \frac{dE}{dxdydt} \\ \frac{dP^x}{dxdydz} & \frac{dP^x}{dydzdt} & \frac{dP^x}{dxdzdt} & \frac{dP^x}{dxdydt} \\ \frac{dP^y}{dxdydz} & \frac{dP^y}{dydzdt} & \frac{dP^y}{dxdzdt} & \frac{dP^y}{dxdydt} \\ \frac{dP^z}{dxdydz} & \frac{dP^z}{dydzdt} & \frac{dP^z}{dxdzdt} & \frac{dP^z}{dxdydt} \end{pmatrix}. \tag{4}$$

The stress energy momentum tensor (SEM) has the general form given in Table I.

The SEM tensor, $\mathbf{T}^{\nu\mu}$, is a rank 2 symmetric tensor that determines the Einstein tensor. The Einstein tensor describes the curvature of spacetime and satisfies the condition

$$\mathbf{G}^{\mu\nu} = \kappa \mathbf{T}^{\mu\nu}. \tag{5}$$

Equation 5 is known as the Einstein field equation.

Symbolically, the field equations can be written as the dependencies of the Einstein tensor set equal to the dependencies of the SEM tensor: $\mathbf{G} \sim \partial\partial\partial P \sim \mathbf{T}$. It can be shown that the Einstein tensor is symmetric. Therefore, Eq.5 represents 10 independent coupled nonlinear differential equations that must be solved for the 4 unknown components of the 4-momentum. Einstein spoke of the SEM tensor with great confidence in its physical groundwork. However, he was unsure about his final conception of the curvature. Since the 1920's other formulations of the curvature have been attempted with limited success.

III. PROGRAM DESIGN

In order to visualize spacetime, I chose to specify the metric and solve for the stress energy momentum tensor and the geodesics. This involved creating the entire architecture of general relativity on a program platform with symbolic algebra. *Mathematica* was a natural choice because of its extensive graphing and integrating capabilities and symbolic algebra manipulation. The design philosophy of the program was to create a diverse group of images that would elucidate the dynamics of the curved spacetime specified by a user defined metric. This entailed allowing the user to visualize any plane in 3 dimensional space and view both the curve of the geodesics in conjunction with a given stress energy momentum component density plot.

Once the metric has been entered, the notebook has the ability to create density plots of both the metric and the stress energy momentum tensor components. A *Mathematica* density plot creates a plot of a given function colored according to its magnitude at any given point. The plots are created in a grid representing the individual components of the metric. This allows the user to witness the stress energy tensor magnitude of the shear, pressure, energy flux, momentum flux, and total energy components over the specified plane in three dimensional space. The user can also view the magnitude of the warping of spacetime for each component of the metric. These pictures represent a powerful tool for understanding the dynamics of the Einstein field equations.

The geodesics provide a way of visualizing how space is curving. The *NDSolve* algorithm is used to integrate the four coupled nonlinear differential equations that determined the geodesics of the light rays. Setting the appropriate starting position and velocity 4-vectors equal to the zero affine parameter value develops initial conditions for the light rays. These initial conditions are the unified with the geodesic equations of motion and fed into NDSolve. Interpolating functions are returned as solutions.

The final output of the program is an ultimate synthesis between the representational styles, a combination of both the stress energy momentum tensor and the geodesics. This combination graph allows the user to see the direct result of the desired stress energy momentum component on the curving of spacetime. With this output, the functionality of the visualization has been pushed to the limits of the notebook. The images take a very long time to render.

IV. EXPLORING METRIC PERTURBATIONS

It is helpful to begin exploration with simple metrics and work towards the visualization of more complicated perturbations. We will begin with a flat spacetime metric to familiarize the reader with the visualizations of the geodesics, $\mathbf{T}$, and $\mathbf{G}$. We will progress to a discrete perturbation that is forced to zero outside some boundary. Finally, we will consider the implications of a metric that has a Gaussian distribution.

We are viewing a cross section in the equatorial plane for all images that follow. However, it is possible to view a cross section through any plane the user would like to specify. The density plots depict the x-y plane with the x axis on the bottom of the plot, y axis on the left hand side, and the origin in the center. Flat spacetime has the metric

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

In the flat spacetime visualization shown in Fig.2 Box A, the metric is displayed as an array of component plots corresponding to the matrix representation above. The blue represents a positive value of 1 everywhere in space, the red represents a negative 1, and green represents 0. Likewise, in flat spacetime there is no source of curvature and therefore the stress energy momentum tensor $\mathbf{T} = 0$ everywhere, as shown in Fig.2 Box A. The stress energy momentum plot is shown in exactly the same fashion as the metric plot.

Each component density plot of the metrics are scaled according to a single color palette that spans the entire range of values of all plots combined. We can refer to this as a global color palette that allows direct and meaningful comparison of each of the components. In most situations with the density plots of the stress energy momentum tensor, it is beneficial to scale each plot individually. If there is a great difference in the magnitude of the components then one component plot could completely dominate the scaling and eliminate the details of the other component plots. The scaling of the color function remains a problem for future developers of the program. Problems with color scaling will be addressed as they arise for different visualizations.

The geodesic plots graph the path of photons in either a dashed or colored line. The photons begin their paths of to the left of the picture as indicated by the arrow in Fig.2 Box B. The plots are in the equatorial plane.

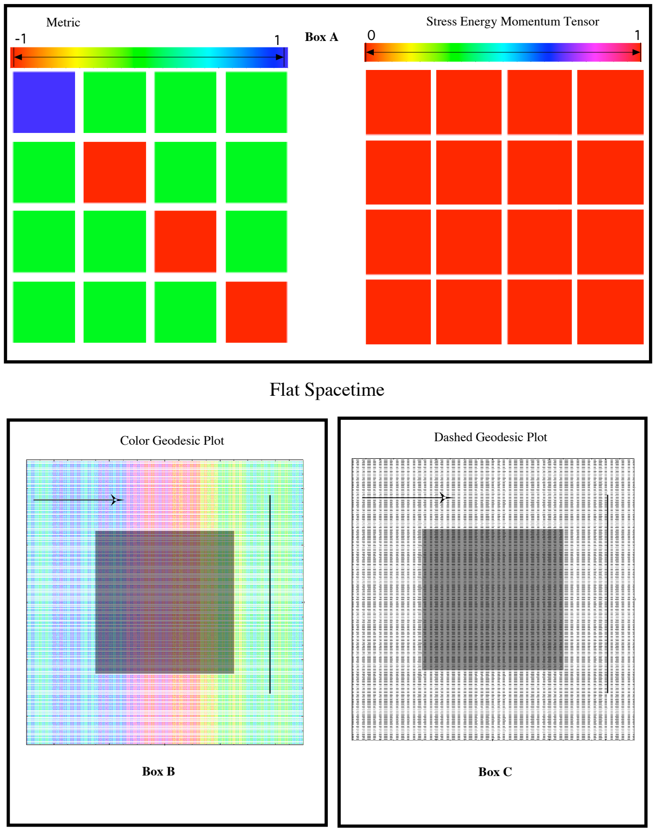

FIG. 2: In Box A the diagonal elements are negative when red and positive when blue. The stress energy tensor is zero everywhere. Coordinates have been added in the top right box of the stress energy momentum to indicate direction for each individual plot. Box B, and C show straight geodesics as expected. There is no time warp because the colors and dashes are all exactly in phase as we expect in flat spacetime.

Bending of the lines indicates warping of spacetime. As the affine parameter increases, the photons move with exactly the same time position in spacetime. When some of the photons pass through a given perturbation their time position will be altered. The altered photons will have gone through a slowing or speeding of time. The colors and dashes are plotted using the time parameter. Therefore, a change in the time positions of the photons will be seen as a misalignment of the dashes or colors relative to the straight line shown in both Fig.2 Boxes B and C, and Fig.4 Boxes A and B. If no perturbation exists then the colors and dashes should be completely in line vertically, as shown in Fig.2. Box B. The dark gray boxes represent the perturbation of the discrete function to be considered next, and are placed there for future reference.

We now add a perturbation into the metric of the form

$$g_{\alpha\beta} = \begin{pmatrix} 1-\delta g & 0 & 0 & 0 \\ 0 & -1+\delta g & 0 & 0 \\ 0 & 0 & -1+2\delta g & 0 \\ 0 & 0 & 0 & -1+3\delta g \end{pmatrix}, \quad (6)$$

where $\delta g[x,y,z] = \kappa \sin^2[x]\sin^2[y]\sin^2[z]$ and κ is a scaling constant. In Fig. 3 Box A the slight perturbation in the metric visible as a discoloration along the diagonal elements of the density plot matrix. Next, notice the increased complexity and off diagonal terms of the covariant $\mathbf{T}$. These terms are highly complex. I would expect to find something more uniform for such a simple perturbation, but what we observe is a complex petal formation of negative and positive energy. Apparently, the Einstein field equations demand elaborate constructions of energy to create a discrete gravitational field. Also, even though our metric is smooth to the first derivative, $\mathbf{T}$ appears to cut off at the edges of the perturbation. This highlights the highly dynamic and non-linear behavior of the field equations. I suspect that any discontinuity in any order derivative will produce some form of cutoff in the stress energy momentum.

The geodesic plots are shown in Fig.4. Look closely at the divergence of the light rays in comparison to the artificially drawn straight lines. Apparently they remain unaffected when passing through the front portion of the perturbation, but are immediately caused to diverge in the latter half. Comparing Fig.4 Box B and Fig.3 Box B indicates a negative energy and shear in the spatial stress causes the divergence of the geodesics. Furthermore, there appears to be a slight bow in the dashed plot, indicating that time has sped up for the photons passing through the perturbation. These two effects are both hallmarks of antigravity and exotic matter. Exotic matter is matter with a total energy density less than zero or energy less than the addition of all the pressures along the diagonal of the stress energy momentum tensor. Physically, no substance has these properties, though electromagnetic fields come close. A material of this kind would allow for time machines and faster than light speed travel. The behavior of the geodesics is of the type we would expect given the negative components in $\mathbf{T}$. Hence, this is an example of a non-physical metric.

The Gaussian distribution represents the most complicated case of this brief study. The distribution does not truncate quickly, but exponentially decays at a rate quick enough to allow for a very good approximation to flat spacetime at far distances. This allows for the use of the same program architecture for the geodesic light rays. Before we consider the images, note that because gravity is a $1/r^2$ field law, we should expect that the metric components of any physically realizable mass concentration should behave like some kind of binomial or exponential function. We can use this program to test whether a Gaussian distribution to the metric satisfies the function described by real matter.

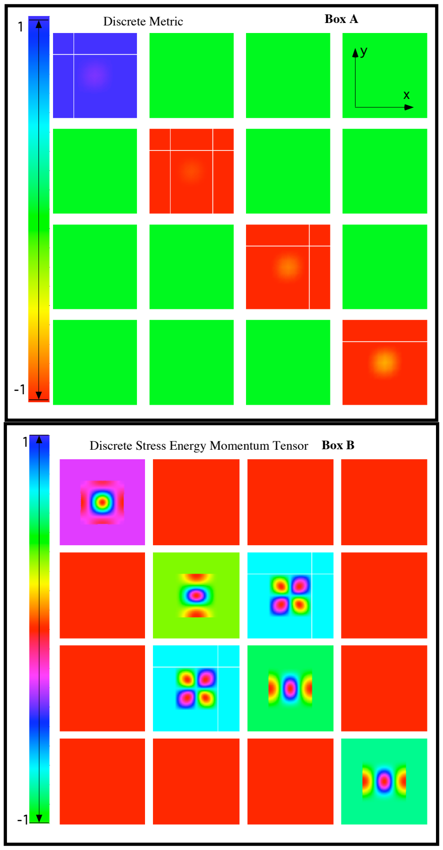

FIG. 3: In Box B the stress energy tensor is complicated with an alternating petal pattern of negative and positive energy. The scale is not fixed from component to component so direct comparison is invalid. The color palette wraps, so multiple colors indicate increasing or decreasing energy density, not oscillating energy density. The zero value for each plot can be found by examining the color outside the perturbation.

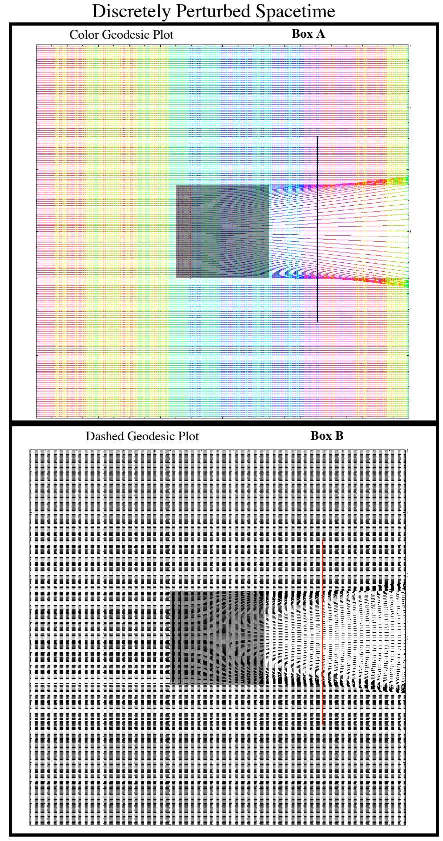

FIG. 4: The perturbance is represented by the gray box. The geodesics only diverge in the latter half of the perturbance. A black and red line has been added to the color and dashed images, respectively, to show the time shift.

The metric perturbation has the form of Eq. 6, with

$$\delta g[x,y,z] = \kappa \exp\left[\frac{-((x^2+y^2+z^2)-\mu)^2}{2\sigma^2}\right].$$

The perturbation can be seen as a faint discoloration at the origin from the flat spacetime metric. The complex petal formation in the discrete **T** reappears here with a smooth continuity. However, the total energy density changes dramatically between the discrete and gaussian cases. This is an interesting effect. If we compare the discrete and Gaussian perturbations, what we find is that the field equations produce roughly the same type of stress energy momentum fields. This may indicate that what matters is the shape of the perturbation rather than the way in which it drops to zero. This may imply that physically realizable solutions to the Einstein field equations define a characteristic curvature shape for all matter. Rather than any range of curvature, there is only one specific style of curvature that mass can produce.

The geodesic plots of Fig.6 show a much smoother divergence of the light rays. The circle gives a rough idea of where the perturbation is. The perturbation exists everywhere so it cannot be strictly encompassed by a border. Again, we observe the effects of antigravity as seen in the divergence of the geodesics.

V. FUTURE WORK

Future work could be taken in any number of directions. The visualization of the curved spacetime allows human intuition to guide the exploration of the Einstein field equations. Most extensions of the project would use the notebook to validate some external computations or guide the research to a meaningful computational regime. *Mathematica* is too slow to extensively work with the field equations. Notebooks use interpreted code, a much slower compiler design that limits massive amounts of computation needed for gravity research. However, *Mathematica* has the ability to explore a diverse range of metrics with trivial recoding. Other code platforms would require extensive recoding for each dramatically different metric.

There exist numerous possibilities to study the nonlinear dynamics of the field equations, perhaps leading to

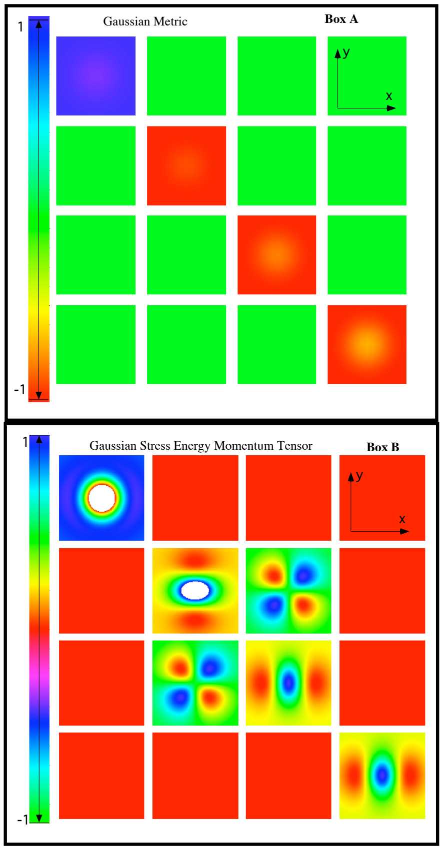

FIG. 5: In Box B the stress energy tensor is complicated with an alternating petal pattern of negative and positive energy like that seen in the discrete case. The color scale is not global as in the discrete case. White color represents a magnitude that is too large to be displayed by the color palette.

insightful dynamics of large gravitating objects. *Mathematica* computes the SEM tensor fairly easily. It may be possible to examine the dependencies of the SEM tensor by creating movies of its dependence on a changing metric.

The most advanced program for future work would involve simulating the interaction between a large gravitating mass and a small metric perturbation. Since there is no global time to parameterize the movement of the objects it is hard to say what sort of results would be valid. Checking the evolution of the perturbation will require a number of advanced and arcane mathematical tests. This course of research seems extremely well suited to the type of algebraic manipulation championed by *Mathematica*.

VI. CONCLUSION

The notebook is a novel way to explore the intricacies of the general relativity. Given more time, it would be possible to take the research in a number of different directions. We have demonstrated that common intuition is not a very good guide to developing solutions for the field equations. Historically, thought experiments have played an important role in the development of field theories. Only recently have field theories reached the point that human are no longer capable of understanding their complex dynamics. This simulation moves in a direction to restore the ability of the experimenter to actively think about general relativity.

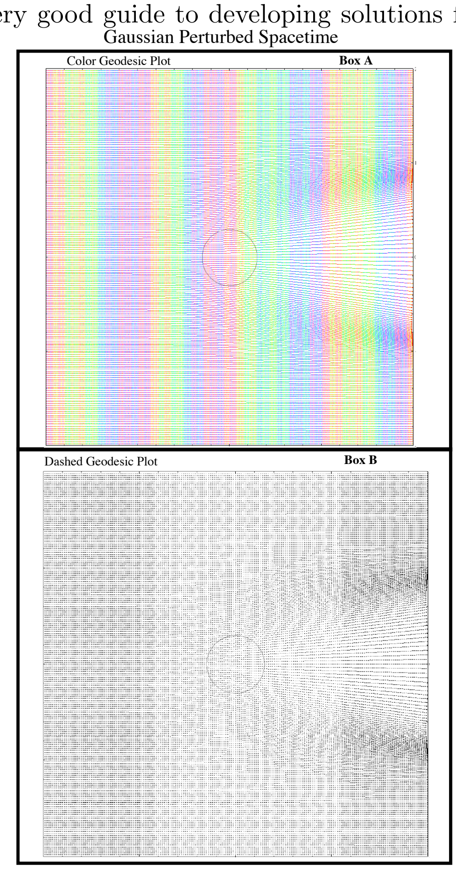

FIG. 6: Notice that the geodesics begin to diverge before they have passed through the largest part of the perturbance.

[1] J. Lindner, *Relativistic Gravity & Relativistic Cosmology*, Fall 2002 Edition, Department of Physics, The College of Wooster.

[2] Kip S. Thorne, *Black Holes and Time Warps*, W. W. Norton & Co., New York, Copyright 1994.

[3] Wikipedia, The Free Encyclopedia, special relativity.