Detecting Planck-Scale Dark Matter with Quantum Interference

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(Received 15 January 2024; revised 10 May 2024; accepted 8 August 2024; published 9 September 2024)

In spite of the large astronomical evidence for its effects, the nature of dark matter remains enigmatic. Particles that interact only, or almost only, gravitationally, in particular with masses around the Planck mass —the fundamental scale of quantum gravity—are intriguing candidates. Here, we show that there is a theoretical possibility to directly detect such particles using highly sensitive gravity-mediated quantum phase shifts. In particular, we illustrate a protocol utilizing Josephson junctions.

DOI: 10.1103/PhysRevLett.133.111001

Introduction—There is clear astrophysical evidence that the energy density of the Universe includes a large component—denoted "dark matter" (DM)—that is not described by the standard model (SM) of particle physics and whose nature remains mysterious [1,2]. Weakly interacting massive particles, suggested by supersymmetric extensions of the SM, have been a favorite candidate, but negative results of several searches and the failure to detect supersymmetry in particle accelerators [3] have squeezed them to less appealing theoretic corners. Searches for other candidates, such as axions, predicted by extensions of the strong sector of the SM are currently underway [4].

An intriguing DM candidate is provided by particles that interact only or almost only gravitationally and have a mass of the order of the Planck mass ($m_P \sim 20$ μg): these could immediately account for the astrophysical evidence, where DM is precisely revealed by its gravitational interaction. The problem with this candidate is that—precisely because of the weakness of the gravitational interaction—direct detection is expected to be hard [5]. So, in a sense, a most natural DM candidate is the hardest to detect. In this Letter, we point out that detection could actually be within reach employing quantum interference techniques.

Why Planck-mass particles—The Planck scale is the fundamental scale in quantum gravity [6]. It is plausible to expect stable or quasistable objects at this scale as part of the spectrum. There are arguments indicating that quantum gravity could predict such particles [7] and stabilize Planck-mass black hole remnants at the end of the evaporation [8]. Hawking radiation theory predicts small black holes to radiate intensely, but the Planck scale is outside the domain of validity of Hawking's theory, which does not take quantum gravity phenomena into account [9]. Stable or semistable Planck-mass objects could therefore be a consequence of quantum gravity. This is a DM candidate that does not require modifications of the SM or modifications of general relativity above the Planck length scale. Furthermore, the strength of the interaction of such particles, combined with the assumption of a sufficiently hot big bang, leads to a density of these objects at decoupling whose order of magnitude is compatible with the present dark matter density [10–12].

Quantum phases—Inspired by recent developments in the area of tabletop experiments involving gravity and quantum phenomena, and the surrounding theoretical debate (see [13,14] and references therein), here we point out that direct detection may not be out of reach. Specifically, we make use of the fact that quantum phases can encode information about tiny momentum transfer, even if the corresponding displacement is too small to be detected.

We first consider an idealized detector where the center of detector mass is set in a superposition of locations. We then discuss a more concrete protocol that employs the collective quantum state of electrons in a suitable arrangement of Josephson junctions as the sensitive probe.

Idealized quantum protocol—Consider a quantum particle of mass m (the "detector," or D, particle) split into a superposition of two positions and then recombined. For concreteness, imagine it is a particle with spin $1/2$, prepared in the $|+\rangle_z$ eigenstate of the spin in the z direction, and split according to the eigenstates $|\pm\rangle_y$ of the spin in the y direction. Upon recombination, the particle will still be in the $|+\rangle_z$ state. But say a (classical) particle with mass M (the "dark matter" or DM particle) flies rapidly next to one of the two positions during the time the state was split. The DM particle transfers different amounts of momentum to the two branches of the D particle, altering the relative phase. Upon recombination, the phase shift can give rise to

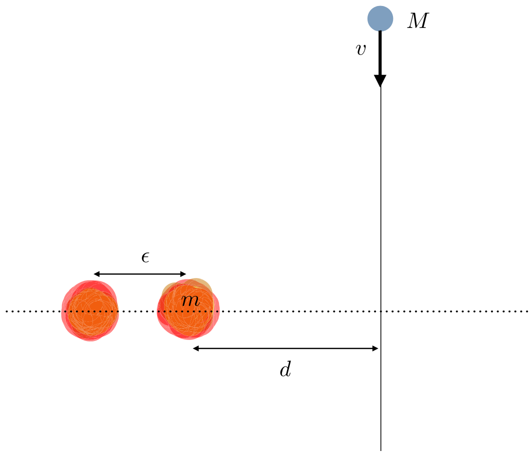

FIG. 1. A particle of mass m in a superposition state with separation ϵ. The DM particle passes by with velocity v.

a nonvanishing probability of measuring $|-\rangle_z$. Figure 1 illustrates the setting.

The magnitude of the effect can be estimated as follows. Take the D particle as the source of an external potential for the DM particle. As shown in the Appendix [Eq. (A14)], the displacement of the D particle during the passage of the DM particle is of the order of

$$\Delta d \approx \frac{c^2}{v^2}\frac{M}{m_p}\ell_p, \tag{1}$$

where d and v are defined in Fig. 1 and ℓ_p, m_p, and c are the Planck length, the Planck mass, and the speed of light, respectively. For $M \approx m_p$ and $v \approx 10^{-3}$ c (the mean velocity of DM particles in the galactic halo [15]) Δd is of the order of $10^6 \ell_p \sim 10^{-27}$ cm. Thus, detection using the classical response would be extremely hard (see [5] for a detailed analysis).

On the other hand, the relative "quantum phase" between the two superimposed configurations can be estimated by evaluating the action difference between the two branches,

$$\Delta S = \int dt \left(\frac{GmM}{\sqrt{d^2 + (vt)^2}} - \frac{GmM}{\sqrt{(d+\epsilon)^2 + (vt)^2}} \right), \tag{2}$$

which only involves the difference of the integrated Newtonian potential in the two configurations of the D particle separated by the distance ϵ. G is the Newton constant. The integration of each term is logarithmically divergent, but the integration of the difference is finite. A direct evaluation gives

$$\Delta S = 2\frac{GmM}{v}\log(1+\epsilon/d) \approx 2\frac{GmM}{v}\frac{\epsilon}{d}. \tag{3}$$

An improved calculation that takes into account the modification of the trajectory of the DM particle is given in the Appendix. It changes the factor 2 in (3) into a 3.

The difference in the action gives a phase difference in the evolution of the two branches [16] of the quantum state

$$\Delta\phi = \frac{\Delta S}{\hbar} = 3\frac{mM}{m_p^2}\frac{c}{v}\frac{\epsilon}{d}. \tag{4}$$

This result can also be understood as follows. The difference of the action between the two branches is equal to the change of the Hamilton function for the motion of the DM particle in the field of the D particle. To first order, this is precisely encoded in the change in momentum by the general relation $\partial S/\partial x = -p$. Hence, the above calculation can be seen as an evaluation of the difference in momentum transfer between the two branches [17]. This can be detectable even if the displacement in position is imperceptible.

A non-negligible phase shift, in fact, gives rise to a nonvanishing probability P of measuring the recombined D particle in the state $|-\rangle_z$ as

$$P = \frac{1-\cos\Delta\phi}{2}. \tag{5}$$

If the dark matter particles have Planckian mass [8,10,11], $M \sim m_p$, then

$$\Delta\phi \sim \frac{\epsilon}{d}\frac{c}{v}\frac{m}{m_p}. \tag{6}$$

It is currently possible to put a mass of the order $m \approx 10^{-17}$ Kg $= 2\times 10^{-8} m_p$ into quantum superposition [18]. The speed of cold DM particles in the galactic halo leads to an expected mean velocity on earth of $v \approx 10^{-3}$ c [15]. This gives $\Delta\phi \approx 10^{-5}\epsilon/d$. Because of the amplifying nature of the factor c/v in Eq. (6), pushing technology to masses $m \sim 10^{-3} m_p$ is required, in order for the prefactor of ϵ/d to become order unity. This is far beyond current possibilities. Fortunately, a more realistic protocol can be designed using macroscopic collective quantum systems, as in, for instance, the one representing electrons in a superconductor.

Josephson protocol—The effect described above can be amplified when the phase shift (4) is induced in the wave function of a large number of particles in a coherent state. A device that allows to exploit this possibility is a superconducting Josephson junction (JJ) [19]. This realization of the detector has the advantage that the collective state of the electrons translates the probabilistic response of (5) into a directly measurable signal, circumventing the need of a statistical reconstruction of the phase. The standard theory of superconductors [22,23] yields two key equations,

$$I = I_c \sin(\Delta\phi_e), \qquad \frac{\partial \Delta\phi_e}{\partial t} = \frac{\Delta\Phi}{\hbar}, \tag{7}$$

where I is the electric current across the junction and $\Delta\Phi$ is the Cooper pairs potential energy difference between the two sides of the junction, ϵ is the insulator width of the JJ,

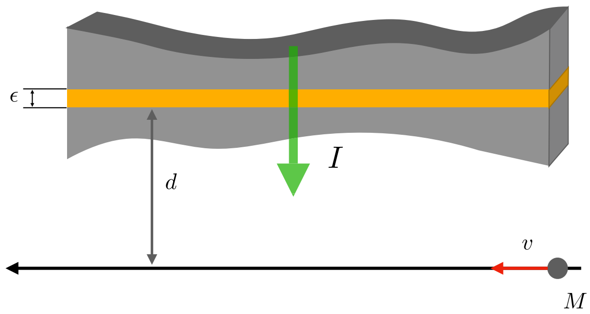

FIG. 2. A long JJ interacting with the DM particle. This optimal can be achieved by suitably orienting the detector.

and I_c the critical current. The phase $\Delta\phi_e$ induced by the passage of a DM particle is given by (2) with $m = 2m_e$. The spatial delocalization ϵ of the collective quantum state of the electrons is now given by the size of the insulating gap of the JJ; compare Figs. 1 and 2. Using (4) and the previous numbers, we get $\Delta\phi_e \approx 10^{-19}\epsilon/d$ as the electron mass is $m_e \approx 10^{-22} m_p$. For small ϵ one has that $I_c \approx e\hbar n_s a/(m_e \epsilon)$, where a is the area of the JJ and (at low temperatures) the density of superconducting electrons n_s approaches the Fermi density $n_s \approx n_f = (3\pi^2)^{-1}(2m_e\varepsilon_f/\hbar^2)^{3/2}$, with ε_f the Fermi energy of the material [24]. Present technology allows for the integration of transistors close to the nanometer scale [25] and it seems possible to produce 50 nm of width and (say) 1 m long JJs with $a \sim 50 \times 10^{-9}\ \mathrm{m}^2$ compactly packed in the geometry proposed (see detector geometry below). Using this, and the value $\varepsilon_f \sim 12$ eV (for aluminum), and in an ideal aligned configuration, the DM particle would induce a current $I_{\rm out} \approx 10^7 \epsilon/d s^{-1}$ (electrons per second) with a single DM event $[I_{\rm out} \approx 10^{-11}(\epsilon/d)\mathrm{A}]$ on such JJ.

Detector geometry—We illustrate for concreteness a possible macroscopic geometry for the detector. This will allow us to discuss the role of the parameters d and ϵ in the estimates above, and the most relevant sources of external perturbations. Call each single long JJ as described in the previous paragraph, and shown in Fig. 2, a "detecting cell." The alignment of the detecting cell with the DM particle trajectory can be fixed by the astrophysical knowledge of the velocity field of DM in our galaxy. The actual detector can consist of a three-dimensional array of detecting cells (see Fig. 3), arranged in a two-dimensional lattice, with separation ℓ. If appropriately oriented, a DM particle will maximally excite a few of the near neighboring cells for which the impact parameter is $d \lesssim \ell$. For the purposes of the present Letter, we assume the apparatus can be designed so that $\ell \sim \epsilon$. Therefore, for estimates we can take $\epsilon/d \sim 1$.

Noise—Here are some estimates of the most obvious sources of noise and some considerations on ways to deal with them.

A major source of noise is thermal. The thermal noise in the output of a cell is suppressed as $\exp(-\Delta/kT)$, where Δ the superconductor energy gap at $T = 0$ and $T \ll T_c$ [26]. For aluminium $\Delta/k \approx 3.9$ K with $T_c \approx 1.2$ K. The signal-to-noise ratio is then $\mathrm{snr} = I_{\rm out} = I/I_T > 1$ requires $T < 10^{-1}$ K. Much lower temperatures have been attained in the lab in small controlled environments [27], but achieving this at the space and timescales necessary for a realistic detector configuration is likely to be a key challenge.

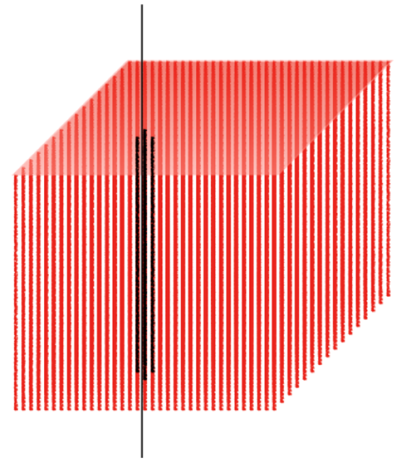

FIG. 3. Detecting cells 3D schematic configuration. A DM particle event excites local cells only (excited cells are represented as black lines).

A second source of noise is given by the macroscopic gravitational perturbations produced by nearby mass displacements (these can range from seismic modifications of the local gravitational field to the motion of massive bodies in the vicinity of the experimental setup). Phase differences due to nearly constant perturbations grow linearly in time at a rate given by the gradient of the gravitational potential (gravitational force) times ϵ. This will produce a slow modulation of the current that can be distinguished from the fast DM particle signal with a timescale approximately given by their flying time $\Delta t \sim 10^{-5}$ s (in the setup used as illustration here). In addition, macroscopic gravitational perturbations can be discerned from DM signals due to their global effect on the measuring cells. The smoking gun of a DM event being a local excitation of a few detecting cells along the track (Fig. 3). Thus, it is reasonable to expect macroscopic perturbations of the gravitational field to act on a very different time and space scale. This fact permits the development of filtering techniques to extract the DM signal from this type of noise.

Flying-by charged particles constitute a standard noise source for regular particle detectors, such as those used in searches for weakly interacting massive particles. We expect standard techniques that deal with this issue to be applicable also to the present case. Furthermore, note that if any charged particle does enter the detector (see next paragraph), it would excite a very large collection of cells due to the overwhelming strength of the electromagnetic interaction over the gravitational one. Therefore, we are in the setting where the weakness of gravity compared to electromagnetism comes to our advantage as the local excitation of only a few cells should identify unambiguously a DM particle event.

Electrically neutral particles can penetrate the detector and produce charged particles inside via ionization. As in the previous case, experience with standard particle detectors suggest that this can also be under control in our setting. Neutrons can be prevented from entering the devise using standard shielding techniques [28]. Neutrinos cannot be shielded and can produce ionization. But these events are rare and do not seem to represent a problem. Using the data from [29] one can estimate fewer than 10^2 ionizations per year in a 1 m side detecting cube. Assuming that it takes 1 h to reset the system, the probability that a DM particle crosses the detector and is missed due to such noise source is less than one event in 10^6.

Conclusion—We have pointed out the possibility that Planck-mass particles making up dark matter that only interact gravitationally could be detected using quantum interference. We have illustrated an idealized setup where a mass is set in a superposition of locations, and a realistic setup using Josephson junctions. The central idea is to detect the relative quantum phase acquired by a macroscopic quantum system whose wave function is spatially delocalized. Concretely, we propose a large number of long JJs arranged in an orientable detector. Given the numbers we have used, this implies that each detector would require manufacturing about 10^{16} long JJs, which may pose a significant technological challenge. Such arrangement allows for an amplification of the signal sufficient for a one-shot detection scheme. This seems necessary given the low rate of such events, as the flux of such DM particles on Earth is expected to be of the order of 1 $\mathrm{particle/m^2/yr}$ [5]. Covering a total area of several square meters with such detectors can give a significant rate. We have crudely estimated the most obvious sources of noise. The measurement may be within technological reach, but a more detailed feasibility study is needed. The challenge is significant, yet it is remarkable that quantum mechanics can amplify effects, which classically amount to hardly detectable Planck scale displacements (1), to macroscopic observable levels. Rapidly evolving quantum technologies combined with the growing interest in experiments testing the interface of gravity and quantum mechanics can be used to address crucial questions in astrophysics, and possibly provide direct validation of certain implications of quantum gravity.

Acknowledgments—We thank J. P. Ernenwein, G. Higgins, T. Jonckheere, L. Raymond, J. Sofo, D. Sudarsky, A. Verga, and M. Zemlicka for fruitful discussions. We acknowledge support from the No. 61466 and No. 62312 grants from the John Templeton Foundation, as part of the "Quantum Information Structure of Spacetime (QISS)" project (qiss.fr).

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End Matter

Appendix: Exact classical nonrelativistic calculation—The general relativistic expression for the action of a test particle (here the D particle) is proportional to its proper time,

$$S = -mc^2 \int d\tau, \tag{A1}$$

where $c d\tau = \sqrt{-ds^2}$, and we have set the speed of light to unit. In the weak field approximation the line element ds^2 defining the gravitational field generated by the DM particle is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) c^2 dt^2 + \left(1 + \frac{2GM}{r}\right) d\vec{x}^2, \tag{A2}$$

where G is the Newton constant. Using t as integration variable along the trajectory of the DM particle gives

$$S = -mc^2 \int dt \sqrt{\left(1 - \frac{2GM}{r}\right) - \left(1 + \frac{2GM}{r}\right)\frac{\dot{\vec{x}}^2}{c^2}}, \tag{A3}$$

whose leading order expansion in $\dot{\vec{x}}$ and M/r gives the expected nonrelativistic action up to a constant, namely

$$S = \int \left(\frac{1}{2} m \dot{\vec{x}}^2 - V(r) - mc^2\right) dt, \tag{A4}$$

with $V(r) = -GM/r$ the Newton's potential.

Now we explicitly compute the action evaluated on classical solutions. Success is granted by the fact that the Newtonian two-body problem is exactly solvable using conservation laws. The relativistic case can be equally solved, but we do not need it since relativistic corrections are negligible for cold dark matter particles. We fix the center of mass frame and assume, for simplicity, that $M \gg m$, implying that the center of mass coincides with the position of of the DM particle M. The action difference is invariant under Galilean transformations.

In spherical coordinates—and ignoring the constant term in the Lagrangian—the action (A4) becomes

$$S = \int \left(\frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 - V(r)\right) dt, \tag{A5}$$

where we have used the fact that in spherical coordinates $\dot{\vec{x}} = \dot{r}\hat{e}_r + r\dot{\phi}\hat{e}_\phi$ when one assumes (without loss of generality) the motion to happen on the $\theta = \pi/2$ plane. Because of spherical symmetry, angular momentum is conserved,

$$L = mr^2\dot{\phi} = \text{constant}. \tag{A6}$$

Using this and r as integration parameter (A5) becomes

$$S = \int \left(\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - V(r)\right)\frac{dr}{\dot{r}}. \tag{A7}$$

Finally we get $\dot{r}$ from energy conservation, namely

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = E, \tag{A8}$$

from which we get that

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - \frac{L^2}{2mr^2} - V(r)\right)}. \tag{A9}$$

Writing $V(r) = -GmM/r = -c^2 mM\ell_p/(m_p r)$ (where we wrote G in terms of Planck mass m_p, Planck length ℓ_p, and c), and substituting the previous two equations in (A7) we get

$$S = 2\int_{\infty}^{r_0} \frac{E + 2c^2\ell_p\frac{mM}{m_p r}}{\sqrt{\frac{2}{m}(E - \frac{L^2}{2mr^2} + c^2\ell_p\frac{mM}{m_p r})}}dr, \tag{A10}$$

where we have split the integral into the two symmetric branches around the point of closest approach r_0 corresponding to the situation where $\dot{r} = 0$ or equivalently

$$E - \frac{L^2}{2mr_0^2} + c^2\ell_p\frac{mM}{m_p r_0} = 0. \tag{A11}$$

Introducing the impact parameter d via the relation $L = dmv$ with $v = \dot{r}|_\infty$ or, using the conserved quantities,

$$L^2 = m^2 d^2 v^2 = 2Emd^2, \tag{A12}$$

we can write the condition (A11) as

$$\frac{1}{2}v^2 - \frac{d^2}{2r_0^2}v^2 + \frac{M\ell_p}{m_p r_0}c^2 = 0, \tag{A13}$$

where we replaced $E = mv^2/2$. The value of r_0 is

$$r_0 = \frac{-c^2\ell_p M + \sqrt{c^4\ell_p^2 M^2 + d^2 m_p^2 v^4}}{m_p v^2}. \tag{A14}$$

For $d \gg \ell_p$ one has

$$r_0 = d - \frac{c^2}{v^2}\frac{M}{m_p}\ell_p + O\left[\left(\frac{c^2}{v^2}\frac{M}{m_p}\right)^2\frac{\ell_p}{d}\right]\ell_p, \tag{A15}$$

and the impact parameter coincides with the parameter d in Fig. 1 to leading order. And the action (A10) reads

$$S = \int_R^{r_0} \frac{2(\frac{1}{2}mv^2 + 2mc^2\frac{M\ell_p}{m_p r})}{\sqrt{(v^2 - \frac{d^2}{r^2}v^2 + 2\frac{M\ell_p}{m_p r}c^2)}}dr, \tag{A16}$$

which is divergent when the cutoff $R \to \infty$ limit is taken. This is normal as the action of an unbounded trajectory is infinite [the action is proportional to proper time, as in (A1)]. We are interested in the change in the action with respect with d. To first order in ϵ [recall notation from Fig. (1)] we get

$$\begin{aligned}\frac{\Delta S}{\hbar} &= \lim_{R\to\infty} -\frac{\epsilon}{\hbar}\frac{\partial S}{\partial d}\\ &= \frac{3dcmMm_p v^3\epsilon}{2(d^2 m_p^2 v^4 + c^4\ell_p^2 M^2)}.\end{aligned} \tag{A17}$$

Assuming that $d^2 m_p^2 v^4 \ll c^4\ell_p^2 M^2$ one obtains

$$\frac{\Delta S}{\hbar} \approx 3\frac{c}{v}\frac{mM}{m_p^2}\frac{\epsilon}{d}, \tag{A18}$$

which is the relation in the Letter.