USEFUL FORMULAE IN DIFFERENTIAL GEOMETRY

Differential forms:

$$\alpha = \frac{1}{p!}\alpha_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}; \qquad \alpha \in \wedge^p. \tag{1}$$

$$\alpha \wedge \beta = (-)^{pq} \beta \wedge \alpha; \qquad \alpha \in \wedge^p, \quad \beta \in \wedge^q. \tag{2}$$

Exterior derivative, d:

$$d\alpha \equiv \frac{1}{p!}\,\partial_{[\nu}\alpha_{\mu_1\ldots\mu_p]}\,dx^v \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \tag{3}$$

d maps p-forms to $(p+1)$-forms:

$$d : \wedge^p \to \wedge^{p+1}; \qquad d^2 = 0. \tag{4}$$

Defining the components of $d\alpha$, $(d\alpha)_{\mu_1\cdots\mu_{p+1}}$, by

$$d\alpha \equiv \frac{1}{(p+1)!}\,(d\alpha)_{\mu_1\ldots\mu_{p+1}}\,dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}}, \tag{5}$$

we have

$$(d\alpha)_{\mu_1\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1}\,\alpha_{\mu_2\ldots\mu_{p+1}]}, \tag{6}$$

where

$$T_{[\mu_1\ldots\mu_q]} \equiv \frac{1}{q!}\Big(T_{\mu_1\ldots\mu_q} + \text{even perms} - \text{odd perms}\Big). \tag{7}$$

Leibnitz rule:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-)^p \alpha \wedge d\beta, \qquad \alpha \in \wedge^p, \quad \beta \in \wedge^q. \tag{8}$$

Stokes' Theorem:

$$\int_M d\omega = \int_{\partial M} \omega, \tag{9}$$

where M is an n-manifold and $\omega \in \wedge^{n-1}$.

Epsilon tensors and densities:

$$\varepsilon_{\mu_1\ldots\mu_n} \equiv (+1, -1, 0) \tag{10}$$

if $\mu_1 \ldots \mu_n$ is an (even, odd, no) permutation of a lexical ordering of indices $(1 \ldots n)$. It is a tensor density of weight $+1$. We may also define the quantity $\varepsilon^{\mu_1\cdots\mu_n}$, with components given numerically by

$$\varepsilon^{\mu_1\cdots\mu_n} \equiv (-1)^t\,\varepsilon_{\mu_1\cdots\mu_n},$$

where t is the number of timelike coordinates. NOTE: This is the *only* quantity where we do not raise and lower indices using the metric tensor. $\varepsilon^{\mu_1\cdots\mu_n}$ is a tensor density of weight -1. We define epsilon *tensors*:

$$\epsilon_{\mu_1\ldots\mu_n}=\sqrt{|g|}\,\varepsilon_{\mu_1\ldots\mu_n}\,,\qquad \epsilon^{\mu_1\cdots\mu_n}=\frac{1}{\sqrt{|g|}}\,\varepsilon^{\mu_1\cdots\mu_n}\,,\tag{11}$$

where $g\equiv\det(g_{\mu\nu})$ is the determinant of the metric tensor $g_{\mu\nu}$. Note that the tensor $\epsilon^{\mu_1\cdots\mu_n}$ *is* obtained from $\epsilon_{\mu_1\ldots\mu_n}$ by raising the indices using inverse metrics. (Note that in some conventions, the upstairs epsilon is taken to be always positive for a lexical ordering of indices, so that the lexically-ordered downstairs epsilon will be negative in a spacetime with an odd number of timelike coordinates. This seems to be illogical, since for antisymmetric tensors (differential forms), indices are naturally downstairs. Also, as we will see below, positivity for the lexically-ordered downstairs epsilon is needed in order to have the Hodge dual of unity be the positive volume element.)

Epsilon-tensor identities:

$$\epsilon_{\mu_1\ldots\mu_n}\epsilon^{\nu_1\ldots\nu_n}=(-1)^t\,n!\,\delta^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_n}\ .\tag{12a}$$

From this, contractions fo indices lead to the special cases

$$\epsilon_{\mu_1\ldots\mu_r\mu_{r+1}\cdots\mu_n}\epsilon^{\mu_1\cdots\mu_r\nu_{r+1}\ldots\nu_n}=(-1)^t\,r!(n-r)!\,\delta^{\nu_{r+1}\ldots\nu_n}_{\mu_{r+1}\cdots\mu_n}\ ,\tag{12b}$$

where again t denotes the number of timelike coordinates. The multi-index delta-functions have unit strength, and are defined by

$$\delta^{\nu_1\cdots\nu_p}_{\mu_1\cdots\mu_p}\equiv\delta^{[\nu_1}_{[\mu_1}\cdots\delta^{\nu_p]}_{\mu_p]}\ .\tag{13}$$

(Note that only one set of square brackets is actually needed here; but with our "unit-strength" normalisation convention (7), the second antisymmetrisation is harmless.) It is worth pointing out that a common occurrence of the multi-ndex delta-function is in an expression like $B_{\nu_1}\,A_{\nu_2\cdots\nu_p}\,\delta^{\nu_1\cdots\nu_p}_{\mu_1\cdots\mu_p}$, where $A_{\nu_2\cdots\nu_p}$ is totally antisymmetric in its $(p-1)$ indices. It is easy to see that this can be written out as the p terms

$$B_{\nu_1}\,A_{\nu_2\cdots\nu_p}\,\delta^{\nu_1\cdots\nu_p}_{\mu_1\cdots\mu_p}=\frac{1}{p}\Big(B_{\mu_1}\,A_{\mu_2\cdots\mu_p}+B_{\mu_2}\,A_{\mu_3\cdots\mu_p\mu_1}+B_{\mu_3}\,A_{\mu_4\cdots\mu_p\mu_1\mu_2}+\cdots+B_{\mu_p}\,A_{\mu_1\cdots\mu_{p-1}}\Big)$$

if p is odd. If instead p is even, the signs alternate and

$$B_{\nu_1}\,A_{\nu_2\cdots\nu_p}\,\delta^{\nu_1\cdots\nu_p}_{\mu_1\cdots\mu_p}=\frac{1}{p}\Big(B_{\mu_1}\,A_{\mu_2\cdots\mu_p}-B_{\mu_2}\,A_{\mu_3\cdots\mu_p\mu_1}+B_{\mu_3}\,A_{\mu_4\cdots\mu_p\mu_1\mu_2}-\cdots-B_{\mu_p}\,A_{\mu_1\cdots\mu_{p-1}}\Big)\ .$$

Hodge $*$ operator:

$$*(dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p})\equiv\frac{1}{(n-p)!}\,\epsilon_{\nu_1\ldots\nu_{n-p}}{}^{\mu_1\ldots\mu_p}\,dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{n-p}}.\tag{14}$$

The Hodge $*$, or dual, is thus a map from p-forms to $(n-p)$-forms:

$$*: \qquad \wedge^p \to \wedge^{n-p}. \tag{15}$$

Note in particular that taking $p=0$ in (14) gives

$$*1 = \epsilon = \frac{1}{n!}\epsilon_{\mu_1\cdots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n} = \sqrt{|g|}\, dx^1\wedge\cdots\wedge dx^n. \tag{16}$$

This is the general-coordinate-invariant volume element $\sqrt{|g|}\, d^n x$ of Riemannian geometry. It should be emphasised that conversely, we have

$$dx^{\mu_1}\wedge dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_n} = (-1)^t\, \varepsilon^{\mu_1\mu_2\cdots\mu_n}\, d^n x = (-1)^t\, \epsilon^{\mu_1\mu_2\cdots\mu_n}\, \sqrt{|g|}\, d^n x\ .$$

This extra $(-1)^t$ factor is tiresome, but unavoidable if we want our definitions to be such that $*1$ is always the *positive* volume element.

From these definitions it follows that

$$*\alpha\wedge\beta = \frac{1}{p!}\,|\alpha\cdot\beta|\,\epsilon, \tag{17}$$

where α and β are p-forms and

$$|\alpha\cdot\beta| \equiv \alpha_{\mu_1\cdots\mu_p}\beta^{\mu_1\cdots\mu_p}. \tag{18}$$

Applying $*$ twice, we get

$$**\,\omega = (-)^{p(n-p)+t}\,\omega, \qquad \omega\in\wedge^p. \tag{19}$$

In even dimensions, $n=2m$, m-forms can be eigenstates of $*$, and hence can be self-dual or anti-self-dual, in cases where $**=+1$. From (19), we see that this occurs when m is even if t is even, and when m is odd if t is odd. In particular, we can have self-duality and anti-self-duality in $n=4k$ Euclidean-signature dimensions, and in $n=4k+2$ Lorentzian-signature dimensions.

Adjoint operator, δ:

First define the inner product

$$(\alpha,\beta) \equiv \int_M *\alpha\wedge\beta = \frac{1}{p!}\int_M |\alpha\cdot\beta|\epsilon = (\beta,\alpha), \tag{20}$$

where α and β are p-forms. Then δ, the adjoint of the exterior derivative d, is defined by

$$(\alpha, d\beta) \equiv (\delta\alpha, \beta), \tag{21}$$

where α is an arbitrary p-form and β is an arbitrary $(p-1)$-form. Hence

$$\delta\alpha = (-)^{np+t}{*d*}\alpha, \qquad \alpha\in\wedge^p. \tag{22}$$

(We assume that the boundary term arising from the integration by parts gives zero, either because M has no boundary, or because appropriate fall-off conditions are imposed on the fields.)

δ is a map from p-forms to $(p-1)$-forms:

$$\delta : \qquad \wedge^p \to \wedge^{p-1}; \qquad \delta^2 = 0. \tag{23}$$

Note that in Euclidean signature spaces, δ on p-forms is given by

$$\begin{aligned} \delta\alpha &= *d{*}\alpha \qquad \text{if at least one of } n \text{ and } p \text{ even,} \\ \delta\alpha &= -{*}d{*}\alpha, \qquad \text{if } n \text{ and } p \text{ both odd.} \end{aligned} \tag{24}$$

The signs are reversed in Lorentzian spacetimes.

In terms of components, the above definitions imply that for all spacetime signatures, we have

$$\delta\alpha = -\frac{1}{(p-1)!}(\nabla_\nu\, \alpha^\nu{}_{\mu_1\cdots\mu_{p-1}})\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p-1}}, \tag{25}$$

where

$$\nabla_\nu\, \alpha^{\nu\mu_1\cdots\mu_{p-1}} \equiv \frac{1}{\sqrt{g}}\partial_\nu\Big(\sqrt{g}\alpha^{\nu\mu_1\cdots\mu_{p-1}}\Big) \tag{26}$$

is the covariant divergence of α. Defining the components of $\delta\alpha$, $(\delta\alpha)_{\mu_1\cdots\mu_{p-1}}$, by

$$\delta\alpha \equiv \frac{1}{(p-1)!}(\delta\alpha)_{\mu_1\cdots\mu_{p-1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p-1}}, \tag{27}$$

we have

$$(\delta\alpha)_{\mu_1\cdots\mu_{p-1}} = -\nabla_\nu \alpha^\nu{}_{\mu_1\cdots\mu_{p-1}}. \tag{28}$$

Hodge-de Rham operator:

$$\Delta \equiv d\delta + \delta d = (d+\delta)^2. \tag{29}$$

Δ maps p-forms to p-forms:

$$\Delta : \qquad \wedge^p \to \wedge^p. \tag{30}$$

On 0-, 1-, and 2-forms, we have

$$\begin{aligned} &\text{0-forms:} \qquad \Delta\phi = -\nabla_\lambda\nabla^\lambda\phi, \\ &\text{1-forms:} \qquad \Delta\omega_\mu = -\nabla_\lambda\nabla^\lambda\omega_\mu + R_\mu{}^\nu\omega_\nu, \\ &\text{2-forms:} \qquad \Delta\omega_{\mu\nu} = -\nabla_\lambda\nabla^\lambda\omega_{\mu\nu} - 2R_{\mu\rho\nu\sigma}\omega^{\rho\sigma} + R_\mu{}^\sigma\omega_{\sigma\nu} - R_\nu{}^\sigma\omega_{\sigma\mu}, \end{aligned} \tag{31}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor and

$$R_{\mu\nu} \equiv R^\rho{}_{\mu\rho\nu} \tag{32}$$

is the Ricci tensor.

Hodge's theorem:

We can uniquely decompose an arbitrary p form ω as

$$\omega = d\alpha + \delta\beta + \omega_H, \tag{33}$$

where $\alpha \in \wedge^{p-1}$, $\beta \in \wedge^{p+1}$ and ω_H is harmonic, $\Delta\omega_H = 0$.