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The Search for the Topology of the Universe Has Just Begun

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Microwave background anomalies motivate further searches for cosmic topology. For manifolds other than the simple 3-torus, existing searches allow the shortest distance around the Universe to be much less than the distance to the horizon. Yet, to explain anomalies, the shortest distance through us is likely to just exceed the horizon size. While galaxy and 21cm surveys are potentially more powerful, a more thorough microwave background search is merited as previous ones considered only a subset of topologies and of their parameter spaces. Current limits on topology are much weaker than generally understood.

Introduction. — Standard cosmology combines general relativity and quantum mechanics to produce a simple model accounting for the distribution of matter over the history and volume of the observable Universe. The average spatial curvature of this model is constrained by observations to be flat, or very nearly so [1]. However, general relativity concerns only the local geometry of the spacetime manifold, not its topology. Quantum processes in the very early Universe may induce "non-trivial" (multiply connected) topology of spacetime [2, 3] that remains present today on very large physical scales, even when inflation occurs [4]. Indeed, the temperature variations in the cosmic microwave background (CMB) suggest the presence of statistically anisotropic correlations, much as would result from non-trivial topology. These include the anomalous statistical properties of the low-multipole (low-ℓ) harmonic coefficients [5–8], the lack of large-scale correlations [8–16], and the asymmetry of power on the sky [8, 15–33]. If topology is the explanation for CMB anomalies, there is detectable topological information in the CMB. While unambiguous indicators of topology have yet to be detected, we present evidence that prior searches for topology [5, 34–43] have far from exhausted the potentially significant possibilities. Much more can be done to discover, or constrain, the topology of space and investigate its potential effects.

If the Universe is a manifold with non-trivial spatial topology, then through any spatial point there are closed spacelike curves that are not continuously deformable to a point. An observer will perceive each object as having multiple copies, with relative locations determined by the details of the topology. This can be interpreted as space having finite extent in one, two, or three dimensions. The parameter space for such universes is much larger than what has been systematically tested. Even limited to spatially flat manifolds (i.e., curvature parameter $\Omega_K = 0$), there are 17 inequivalent topologies (in addition to the trivial case), each of which has multiple real parameters. Most attention so far has been confined to just one of these, the simple 3-torus (though see for example [44]), and even then has focused on a rectangular-prism fundamental domain, though a parallelepiped is permitted.

The observed CMB fluctuations encode information about topology, even when the topology scale exceeds the diameter of the visible Universe. CMB observations probe the last scattering surface (LSS) of cosmological photons, at a comoving distance of nearly the Hubble length H_0^{-1} (where H_0 is the present value of the Hubble expansion rate). Non-trivial topology, by breaking statistical isotropy, induces anisotropic correlations in the

CMB temperature and polarization fluctuations. When the scale of the topology is small compared to the LSS diameter, pairs of circles with matched temperature (and polarization) become visible in different parts of the sky [34]. These have not been observed [5, 35–43], but correlations can persist even when the scale of the topology is large enough to preclude matched circles [45, 46].

In this Letter, we demonstrate that (1) cosmic topology remains eminently detectable despite past negative searches of Refs. [5, 34–43]; (2) previous analyses have not considered a large number of topological degrees of freedom, even for spatially flat manifolds; (3) constraints due to the non-observation of matched circle pairs are less restrictive than widely believed [47]; (4) anisotropic correlations induced by topology can be detectable in the CMB even in the absence of matched circle pairs [46]; and (5) though significantly larger manifolds may be detectable in future observations of large-scale structure [48], if topology is the explanation for CMB anomalies the topology scale cannot greatly exceed the LSS diameter.

Cosmic topology.— Cosmic topology refers to the properties of spatial sections of the $3 + 1$-dimensional manifold describing the Universe on the largest scales. We assume the *geometry* of a Friedmann-Lemaître-Robertson-Walker (FLRW) metric, with the usual cases of negative (hyperbolic), zero (flat or Euclidean), and positive (spherical) spatial curvature. Topologies of these geometries have been widely explored mathematically. There are eighteen Euclidean topologies, with up to six real parameters each, and a countable infinity of both spherical and hyperbolic topologies, with just the curvature scale as a parameter of each [49]. Manifolds of the three curvatures do not transform smoothly into one another: a Euclidean topology is fundamentally different from an allowed topology for a curved manifold, even if the curvature is nearly zero. We concentrate on the flat case in this Letter, as it suffices to make the general case for a renewed search for topology.

All possible Euclidean topologies can be generated starting from 1-2 rectangular or hexagonal prisms [35, 50, 51]. In the simplest case, the 3-torus E_1, opposite faces of a right rectangular prism are identified, giving a finite volume and simple periodic boundary conditions. One can skew the prism to start instead with a parallelepiped—opposite faces are now parallelograms, and the translations carrying a face to its mate are not all normal to the face. We call E_1 built from a parallelepiped "tilted." Alternately, before identification, one can rotate a rhombic face by π (giving the E_2 space), a square face by $\pi/2$ (E_3), a hexagonal face by $2\pi/3$ (E_4) or $\pi/3$ (E_5). These rotations followed by translations are called "corkscrew" motions, which also names the axis of rotation. In some cases one flips opposite faces instead of rotating them, generating various versions of the Klein bottle space (E_7–E_{10}). One can also take one or two dimensions of a starting rectangular prism to have infinite length, so space is periodic in only one or two dimensions; in these cases one can still rotate or flip the remaining faces. These cases are E_{11}–E_{17}. The final possibilities are the Hantzsche-Wendt space (E_6), which starts from two adjacent rectangular prisms and matches faces between them (see, e.g., Ref. [44]), and the trivial topology E_{18}, also called the covering space—the 3-manifold of a given geometry in which any closed loop can be deformed to a point.

These choices determine various characteristics of the cosmic 3-manifold; it may be: orientable or non-orientable (roughly, whether or not a closed path around the space preserves or interchanges left and right handedness); finite in zero, one, two, or three dimensions; statistically homogeneous (all observers see the same pattern of their own images around them) or inhomogeneous [52].

This classification was first applied to cosmological manifolds in Ref. [53], but the discussion was limited to a subset of the parameters describing some topologies: the angular degrees of freedom (e.g., the ability to skew a right rectangular prism into a parallelepiped) have largely been ignored, as have the consequences of the lack of statistical homogeneity, while the parameterizations of possible translations between fundamental domains have been incomplete. The complete sets of parameters will be addressed in detail in Ref. [46].

One can fully describe the topology by determining the images (periodic repetitions) of a coordinate triad based at any point. These images need not (and in general will not) form a simple periodic lattice. We can equivalently think of topology as related to tilings of the full covering space of the associated geometry. The individual tiles are called the fundamental domain. Despite the name, although the *volume* of the fundamental domain is uniquely determined, its *shape* is not—different domains can tile the covering space yet embody precisely the same set of isometries and different choices may seem natural to different observers.

For many purposes, the most useful information about a specific topology is the eigenmodes of the Laplacian operator subject to the boundary conditions imposed by the topology—the analogues of Fourier modes on the Euclidean covering space. This is already clear in the case of a rectangular prism E_1. In this rectangular box with periodic boundary conditions, the modes are exactly the Fourier modes restricted to a countable set of wavevectors, as opposed to the continuum of allowed modes on the covering space. In other compact Euclidean topologies, the finite volume of the fundamental domain still restricts the Laplacian to a countable set of eigenmodes, with each being a linear combination of finitely many Fourier modes of different equal-magnitude wavevectors. In the non-compact topologies E_{11}–E_{17} the continuum is modified but not fully discretized. These changes to the eigenmodes modify the statistical properties of the matter fields, and are thereby potentially detectable in the CMB.

We concentrate below on E_1, E_2 and E_3 because they

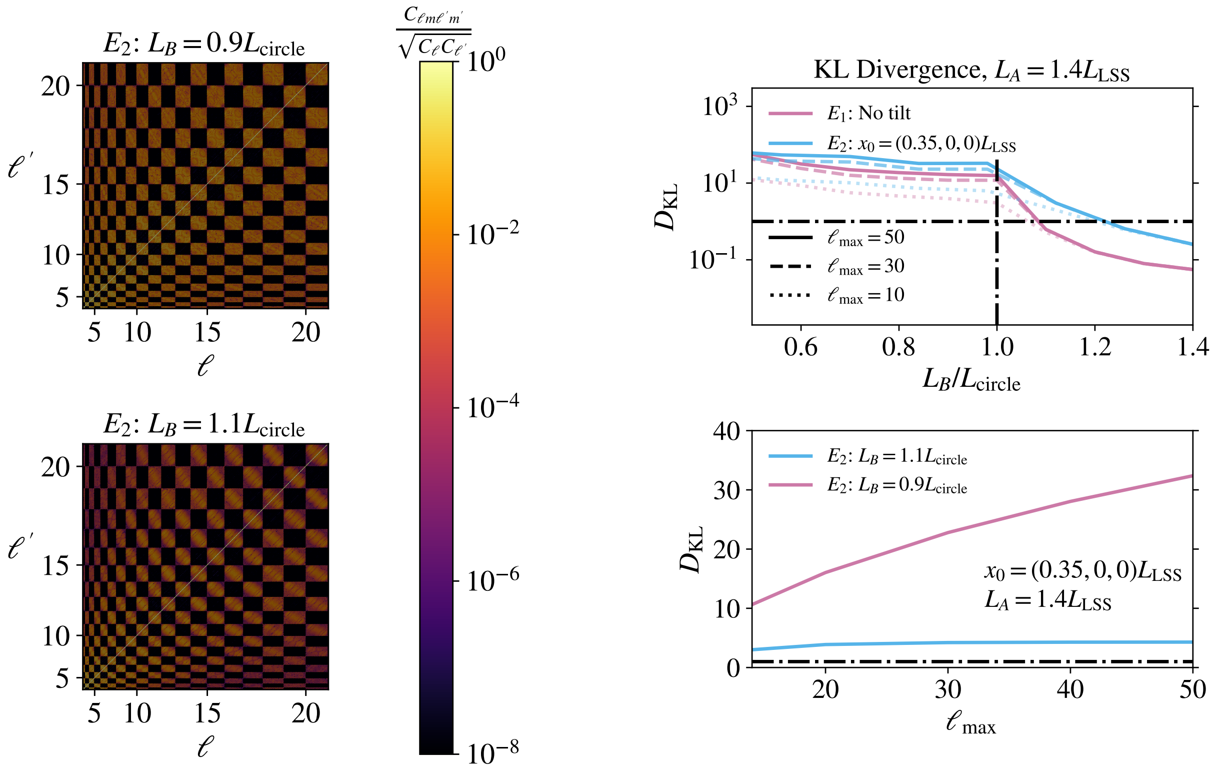

FIG. 1. **Left:** Portions of rescaled CMB temperature correlation matrices for a half-turn space (E_2) with $L_B/L_{\rm circle} = \{0.9, 1.1\}$ and an off-axis observer at $x_0 = (0.35, 0, 0)L_{\rm LSS}$. L_B is the length along the corkscrew axis and $L_{\rm circle}$ is the length scale below which matched circles would be detected. Here $L_A = 1.4L_{\rm LSS}$ is the other topological length scale. **Upper right:** The Kullback-Leibler (KL) divergence as a function of $L_B/L_{\rm circle}$ for E_2 and a simple untilted 3-torus (E_1). **Lower right:** The KL divergence as a function of $\ell_{\rm max}$ for E_2.

suffice to demonstrate our broader points. Each is characterized by two to six parameters describing the shapes of the faces, and the translations and rotations that carry them between one another. Another three to six parameters may be needed to characterize an observer's position and orientation in the space (see Ref. [46] for more details about these and other Euclidean spaces).

Correlations induced by topology.— For Gaussian fluctuations, two-point covariance matrices contain all the information available to determine the observational repercussions of the topology. The mode structure induced by topology breaks the isotropy of modes in the covering space: a continuous set of Fourier modes with uniform density becomes a grid in wavevector, inducing correlations between amplitudes of spherical harmonics. Even for the simple torus, skewing the fundamental domain to a parallelepiped skews the grid of allowed wavevectors. In cases with reflections and rotations, further correlations are induced between Fourier modes, and thus between spherical harmonic modes, enhancing the prospects for observational signatures in the CMB and in large-scale structure.

For the CMB in particular, the topological mode structure changes the distribution of spherical harmonic components from the diagonal $\langle a_{\ell m} a_{\ell' m'}\rangle = C_\ell \delta_{\ell\ell'}\delta_{mm'}$ to a full covariance matrix $C_{\ell m \ell' m'}$ with *off-diagonal* terms. Observations indicate that, even if modified by topology, initial perturbation amplitudes on scales small compared to H_0^{-1} today are well described by a Gaussian distribution following a slightly tilted Harrison-Zel'dovich spectrum depending only on the amplitude of the wavevector, e.g., as predicted by inflation [54]. Under linear evolution the CMB, as well as the cosmic large-scale structure, would then be described by an anisotropic Gaussian distribution.

In Fig. 1, we show (left) the rescaled correlation matrices of CMB temperature fluctuations, $\Xi_{\ell m \ell' m'} \equiv C_{\ell m \ell' m'}/\sqrt{C_\ell C_{\ell'}}$, induced by (unskewed) E_2 topology. (C_ℓ are elements of the diagonal covariance matrix for the Euclidean covering space computed with the same values of standard cosmological parameters.) We consider a right rectangular prism fundamental domain with length $L_A = 1.4L_{\rm LSS}$ in the two pure translation directions ($\hat{x}$,$\hat{y}$), and $L_B = \{0.9, 1.1\}L_{\rm circle}$ in the corkscrew direction ($\hat{z}$), where $L_{\rm circle} = 0.714L_{\rm LSS}$ is the minimum value of L_B for that manifold for which $\geq 95\%$ of observers would detect a matched circle pair. $L_{\rm LSS}$ is the diameter of the last-scattering surface of CMB photons. The observer is placed at $0.35L_{\rm LSS}\hat{x}$ from the corkscrew axis, and does (not) see circles in the top (bottom) panel. There are significant correlations between disparate ℓ. We discuss the right panels of Fig. 1 below.

Little work has been done on CMB polarization and topology, but Refs. [55, 56] suggest it provides an additional avenue for exploring topology, and may even have enhanced correlations compared to temperature because polarization originates more predominantly from the LSS.

Constraining the topology.— How can we search for the anisotropic correlations induced by topology? For sufficiently small topology scales, we would see clones of individual objects, such as galaxies or quasars; these have not been seen [57–64]. Similarly, in CMB temperature fluctuations, matching pairs of circles on the sky would be signatures of the self-intersection of the LSS. The size of such circles, the locations of their centers, and the phase of their matching along the circles depend on the details of the topology, but the existence of circles depends only on isotropic geometry and a small-enough topology scale [34]. The non-observation of such matched patterns limits the shortest closed paths across our fundamental domain, which could in turn be interpreted as a limit on the parameters of any given topology. This search is computationally challenging because of the large number of candidate circle centers, radii, and relative phases, but has been completed on both Wilkinson Microwave Anisotropy Probe (WMAP) and *Planck* data [5, 35–43].

Once the length of such a returning path exceeds the diameter of the LSS, there are no longer matching circles, and we must search for excess anisotropic correlation. This can be done by directly evaluating the likelihood as a function of the parameters of the topologies [42, 43, 65, 66]. However, this is computationally challenging because we cannot take advantage of the usual simplification of isotropy: the signal covariance matrix is diagonal in harmonic space or, equivalently, it is only a function of the angular distance between pixels. Even calculating and storing the $\mathcal{O}(\ell_{\max}^4)$ entries of the full $C_{\ell m \ell' m'}$ matrix is infeasible at high $\ell_{\max}$.

Current constraints.— To date, none of the tests outlined above have detected evidence of non-trivial topology.

The matched-circles search has the advantage of being generic: for a sufficiently small fundamental domain, every non-trivial topology of an FLRW cosmology predicts a pattern of repeated circles. However, translating the non-detection of matched circles into limits on topology depends on the details of each individual topology. In particular, for topologies in which any of the faces of the fundamental domain is rotated, the induced pattern of circles depends on the location of the observer with respect to the corkscrew axis—as the observer moves away from the axis, limits on the size of the domain weaken. In Fig. 2, we display illustrative limits on the parameter space of E_1–E_5 [47]: for the excluded parameter values, observers at $\geq 95\%$ of locations in the manifold would detect at least one matched circle pair of any size. These curves are indicative of the limits that can be derived

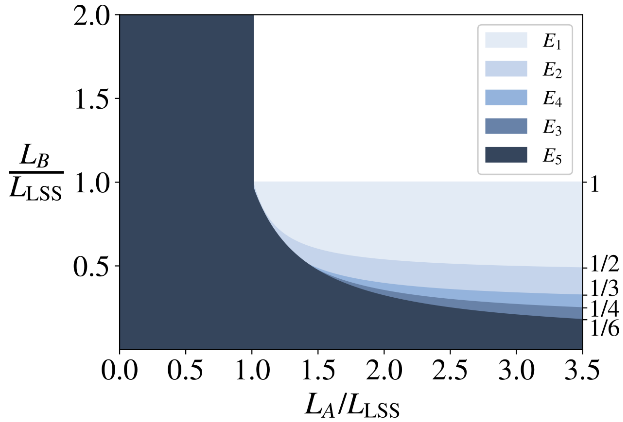

FIG. 2. Regions of topology parameter space where observers would or would not see matched circle pairs. In the shaded region for a topology (and topologies above it on the legend) $\geq$ 95% of observers have a clone closer than $L_{\rm LSS}$. In the exterior (white) region, no topology will produce matched circles. For the E_2 manifold we choose the right rectangular prism. E_2-E_5 are arranged in increasing order of their associated rotation.

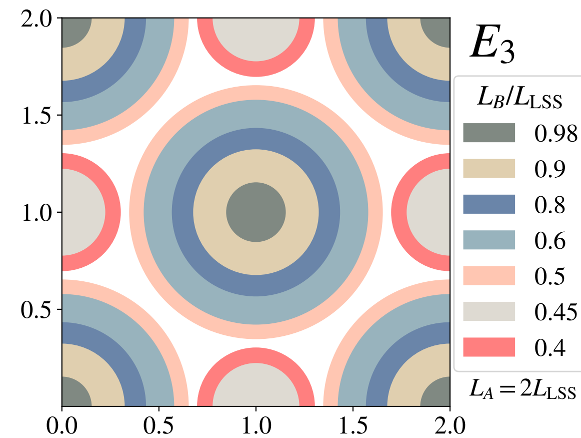

FIG. 3. For E_3 manifold with $L_A = 2L_{\rm LSS}$, locations $(x/L_{\rm LSS}, y/L_{\rm LSS})$ of observers in planes of constant z (the corkscrew axis) in which matched circle pairs would be detected, as a function of L_B. Even for $L_B = 0.4L_{\rm LSS}$ 20% of observers would still not see matched circles (white regions).

from a detailed analysis of matched circles in the CMB temperature from WMAP and *Planck* [5, 34–43, 67]. For the simple right-angled E_1 this straightforwardly limits the length of the shortest side to the diameter of the LSS sphere, but the rotations in E_2 through E_5 weaken the limits on the length L_B along the corkscrew axis—as the size of the square or hexagonal face perpendicular to that rotation, L_A, increases, observers in more and more of the manifold would not observe matched circles.

In Fig. 3 we display the square cross-section through the E_3 manifold with dimensions $L_A = 2L_{\rm LSS}$ along both axes, in a plane perpendicular to the axis of rotation at $(x, y) = (L_{\rm LSS}, L_{\rm LSS})$, where x, y are coordinates in this plane. We shade the regions in which an observer would detect one or more matched circle pairs as a function of L_B, the length of the translation associated with the $\pi/2$ rotation. In regions of a given color observers would see matched circle pairs for all values of L_B less than or equal to the value listed on the legend. As L_B is reduced, more and more observers see matched circles, but even for $L_B = 0.4L_{\rm LSS}$ a substantial fraction of the volume is "circle-free," and hence allowed by current observations.

Conversely, the likelihood search is not generic—the covariance matrix must be calculated anew for each topology and each set of topological parameters. Hence, only a very small fraction of testable parameter values [68] of a subset of Euclidean manifolds have been tested [69].

Future constraints. — We have already seen in the left panel of Fig. 1 that anisotropic correlations persist even when the size of the fundamental domain is larger than the diameter of the LSS and there are no matched circles. Can we detect this richer correlation structure? The presence of matched circles is a geometric effect, independent of the statistical properties of the fluctuations themselves. Once the scale of the topology is outside of the LSS, we depend on details of statistical properties to detect the induced correlations. We expect this to be more tied to the cosmological parameters than circles. In the following we specialize to models in which the fluctuations are described by the *Planck* 2018 cosmology [1].

We can use the Kullback-Leibler (KL) divergence to compare the probability distribution functions for the $\{a_{\ell m}\}$ in the trivial topology, $p(\{a_{\ell m}\})$, and in a non-trivial topology, $q(\{a_{\ell m}\})$,

$$D_{\rm KL}(p||q) = \int {\rm d}\{a_{\ell m}\}\; p(\{a_{\ell m}\}) \ln\left[\frac{p(\{a_{\ell m}\})}{q(\{a_{\ell m}\})}\right] . \quad (1)$$

With p and q isotropic and anisotropic Gaussian distributions, $D_{\rm KL} = \frac{1}{2}\sum_i \left(\ln|\lambda_i| + \lambda_i^{-1} - 1\right)$ [45], where λ_i are the eigenvalues of $\Xi_{\ell m \ell' m'}$, for $\ell, \ell' \geq 2$. Given data $a_{\ell m}$ from an experiment, $D_{\rm KL}$ favors non-trivial topology if $D_{\rm KL} > 1$. Hence $D_{\rm KL} = 1$ serves as a threshold for the detectability of non-trivial topology in a perfect experiment with no noise, foreground emission, and mask.

The right panels of Fig. 1 show the KL divergence as a function of $L/L_{\rm circle}$ and $\ell_{\rm max}$. They convey a somewhat more optimistic message than Ref. [45] which considered a simple cubic torus E_1. We see here (top right) that E_2 domains larger than E_1 domains have detectable KL divergence. However, we also see that, in E_2, once an observer is unable to detect matched circle pairs ($L_B > L_{\rm circle}$) only $\ell \lesssim 30$ add significantly to the KL divergence. Though $D_{\rm KL}$ is challenging to calculate for large ℓ, a signal-to-noise calculation of off-diagonal correlations as signal and statistically isotropic Gaussian random fields as noise gives a similar level of potential detectability [46].

The dependance of $D_{\rm KL}$ on $\ell_{\rm max}$ and on topology parameters depends on the specific manifold, and on the observer location within that manifold. This is a matter for ongoing investigation, as is the maximum size of each manifold that can be observationally identified. However, it is important to realize that if topology is the physical cause of CMB large-angle anomalies then *ipso facto* the CMB must contain detectable topological information. Thus the KL divergence would inform us what values of various topological parameters we need to explore for each manifold, avoiding those values for which there is no topological information. A topological explanation for any observed CMB anomaly that exhibits parity-violating $\Delta\ell$-odd correlations would only arise from a manifold that encodes parity violation, precluding E_1, E_{11}, E_{16}, and, of course, E_{18}. The behaviour of $D_{\rm KL}$ with $\ell_{\rm max}$ implies that correlations between different $a_{\ell m}$ induced by topology with $L_B > L_{\rm circle}$ are limited to low ℓ, a potential explanation for large-scale CMB anomalies, and holds out the promise for testable predictions with future observations.

Preliminary calculations with two-dimensional tori suggest that information from the interior of the last scattering surface can significantly increase $D_{\rm KL}$, including for $L_B > L_{\rm circle}$, in particular by making available the information from shorter wavelength modes. This is presumably because the projection from the full 3-space to the 2-dimensional surface of the LSS mixes many uncorrelated Fourier modes, especially at higher wavenumber. Future galaxy and 21cm surveys thus hold the promise of increasing the range of the topology parameter space that can be explored observationally.

Expanding the reach of past topology searches will be discussed in upcoming work. We anticipate that the dimensionality of the parameter space, the size of the covariance matrix, and its lack of sparsity will make an exhaustive likelihood calculation untenable, even for just Euclidean manifolds. Instead we anticipate developing machine learning techniques to accelerate the likelihood calculation and to be used in the framework of likelihood-free inference [70] to more generally address the challenge of mining CMB data for evidence of non-trivial topology.

Conclusions. — We have shown that current observations of the CMB have not been comprehensively translated to limits on the allowed large-scale topology of the Universe. We report that for generic Euclidean manifolds, whose isometry groups include rotations or reflections, the lower limit on the topology scale is smaller than the diameter of the LSS by factors of $2-6$, and potentially much more. All of the Euclidean manifolds, other than the covering space, violate statistical isotropy; most of them also violate statistical homogeneity. The ability to detect topology even in the absence of explicit matched circles depends on the induced statistical anisotropy. The KL divergence suggests that there is information in CMB temperature correlations even when an observer does not see circles, although the topology scale cannot be much larger than $L_{\rm LSS}$ if topology is to explain CMB anomalies. How much this depends on the manifold, its size and other parameters, and the observer's position, is unknown. Large-scale structure information from future surveys will provide still more information which appears likely to offer a qualitative improvement on CMB temperature correlations. These possibilities will be explored in a series of forthcoming papers.

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