EQUATION OF STATE AND EQUATION SYMMETRIES IN COSMOLOGY

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Symmetry groups of Friedman equations both without and with dissipation, and of Bianchi type I equations with dissipation are discussed. It is shown that the symmetry groups of these equations correctly single out corresponding equations of state.

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0. *Introduction*

In General Relativity there is a striking dualism of space-time and matter. The structure of space-time is governed by the field equations, physical properties of matter, on the other hand, do not result from the model but enter the picture, via energy-momentum tensor, from other departments of physics.

Mathematically, one proceeds in an analogous way to that adopted by Newton who defined a dual geometrico-physical concept of a mass-point. In General Relativity one defines particles as curves in space-time and associates with them physical quantities such as density and pressure. To be precise, a *particle* of iest-mass m is defined as a future directed curve $\gamma: I \to M$ in a space-time M, such that $g(\gamma_*, \gamma_*) = -m^2$. The tangent γ_* is the energy-momentum of the particle. Then, a *particle flow* of rest-mass m is defined to be a pair $(\vec{P}, \eta)$ where η is a function $\eta: M \to [0, \infty)$ called *world density*, and $\vec{P}$ is the energy-momentum vector field $\vec{P}: M \to T(M)$ such that each integral curve of $\vec{P}$ is a particle of

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rest-mass m. The *energy-momentum tensor* of a particle flow $(\vec{P}, \eta)$ is $T = \eta\vec{P} \otimes \vec{P}$; and pressure p, understood as a function on M, enters it through $\vec{P}$ (for a detailed formalism consult [1]). Mc Crea [2] has noticed that since this matter characterizing function is taken from outside of General Relativity, there are no a priori reasons — within the relativistic context — why p should be non-negative. The only way how p should be interpreted is through effects it produces in a model. For example, $p < 0$ may be interpreted as a creation of matter.

From the mathematical point of view, equation of state also plays an important role in General Relativity. As it is well known, Einstein's field equations, together with Bianchi identities, form an underdetermined system of equations. In order to render it fully integrable one more equation must be added. Usually it is done by assuming an equation of state. It turns out that Einstein's equations completed by an equation of state satisfy certain symmetries which form a group, called symmetry group of these equations (see [3] and [4]).

It was Collins [5] who noticed that the above procedure can be reversed. One could a priori assume a symmetry group for Einstein's equations and out of it deduce the condition of integrability which has the form of the equation of state. In fact, Collins applied this method to the case of classical and relativistic stars to obtain physically realistic equations of state [5, 6]. This illustrates — to use Collins' expression — a subliminal role of mathematics in physical considerations which consists in suggesting correct physics on grounds of logical beauty (such as, for example, symmetry principles). Collins writes: "In contrast to the incontestable logic of sublime mathematics, which in principle leads relentlessly to a definite answer to a well-posed physical problem, mathematics can also act by subtle suggestion, and at times this may even occur without our being aware of it. This is the subliminal role of mathematics in physics" [5].

In the following we will pursue Collins' way of thinking applying it to the field of cosmology. In Section 1 we sketch the theory of symmetry groups of differential equations, then we show that cosmological equations, together with a suitable equation of state admit a certain Lie group of symmetries or, vice versa, the invariance of equations with respect to a given symmetry group singles out a corresponding equation of state. In Sections 2–4 we show this for ordinary Friedman cosmological models, for Friedman models with dissipation and for Bianchi type I models with dissipation, respectively. In Section 5 we briefly comment upon the obtained results.

1. Symmetry group of a system of differential equations

Let us consider, in a Euclidean space $E^N(x, u)$, $x = (x^1, \ldots, x^n)$, $u = (u^1, \ldots, u^m)$, $n+m = N$, a system of differential equations

$$p^\alpha = f^\alpha(x, u), \quad \alpha = (1, \ldots, m), \tag{1.1}$$

$$p^\alpha = \frac{\partial u^\alpha}{\partial x}$$

and point-point transformations

$$T: \quad \begin{aligned} \bar{x} &= \bar{x}(x, u), \\ \bar{u}^\alpha &= \bar{u}^\alpha(x, u), \end{aligned} \tag{1.2}$$

which map each solution of system (1.1) into a solution of the same system. T is a Lie group. (All these considerations may be easily generalized to the case when differential equations are defined on a N-dimensional differential manifold; see, [4].)

Let $M\colon u = u^\alpha(x)$ be a solution of (1.1). M defines a submanifold in E^N; if $\overline{M} \equiv T(M)$ we have

$$\overline{M}: \quad \begin{aligned} \bar{x} &= \bar{x}(x, u(x)), \\ \bar{u}^\alpha &= \bar{u}^\alpha(x, u(x)). \end{aligned} \tag{1.3}$$

The derivatives $p^\alpha = \dfrac{\partial u^\alpha}{dx}$ and $\bar{p}^\alpha = \dfrac{\partial \bar{u}^\alpha}{d\bar{x}}$ satisfy the condition

$$\bar{p}^\alpha D\bar{x} = D\bar{u}^\alpha, \tag{1.4}$$

where $D = \dfrac{\partial}{\partial x} + p^\alpha \dfrac{\partial}{\partial u^\alpha}$. By solving (1.4) we obtain $\bar{p}^\alpha = \bar{p}^\alpha(x, u, p)$. By joining these solutions to transformations T, one gets a new set of transformations $\tilde{T}$ which is called "extension of T to the first derivatives".

If the infinitesimal operator of the group T has the form

$$X = \xi(x, u) \frac{\partial}{\partial x} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \tag{1.5}$$

then that of the group $\tilde{T}$ is

$$\tilde{X} = X + (D\eta^\beta - p^\beta D\xi) \frac{\partial}{\partial p^\beta}. \tag{1.6}$$

Functions $\mathscr{F}$ defined on $\tilde{E}^N(x, u, p)$, which are preserved under transformations $\tilde{T}$, are called invariants of the group T; they satisfy differential equations

$$\tilde{X}(\mathscr{F}) = 0 \tag{1.7}$$

and, vice versa, functions satisfying (1.7) are invariants of the group generated by X.

Now, we may apply equations (1.7) to our original equations (1.1) to obtain

$$\tilde{X}(p^\alpha - f^\alpha(x, u)) = 0. \tag{1.8}$$

Putting (1.6) into (1.8) and taking into account $p^\alpha = f^\alpha(x, u)$, we get

$$\begin{aligned} &\frac{\partial \eta^\alpha}{\partial x} + \frac{\partial \eta^\alpha}{\partial u^\beta} f^\beta - \frac{\partial \xi}{\partial x} f^\alpha - \frac{\partial \xi}{\partial u^\beta} f^\beta f^\alpha \\ &= \xi \frac{\partial f^\alpha}{\partial x} + \eta^\beta \frac{\partial f^\alpha}{\partial u^\beta}, \qquad \alpha, \beta = (1, \ldots, m). \end{aligned} \tag{1.9}$$

Equations (1.9) give us the conditions of the existence of the operator X for system (1.1). Algebra of these operators characterizes the symmetry group T of system (1.1).

In the following we shall consider autonomous dynamical systems for which $\partial f^{\alpha}/\partial x = 0$, and infinitesimal transformations are generated by $X \equiv \xi(x)\partial/\partial x + \eta^{\alpha}(u)\partial/\partial u^{\alpha}$. In this case equations (1.9) simplify to the form

$$\frac{\partial \eta^{\alpha}}{\partial u^{\beta}} f^{\beta} - \frac{\partial \xi}{\partial x} f^{\alpha} = \eta^{\beta} \frac{\partial f^{\alpha}_{\alpha}}{\partial u^{\beta}} . \tag{1.10}$$

2. *Symmetry group of Friedman equations*

As it is well known, the Friedman equations may be written in the form of an autonomous dynamical system on the plane (H, ε), where H is the Hubble parameter and ε is the energy density [7, 8]

$$\dot{H} = -H^2 - \tfrac{1}{6}(\varepsilon + 3p) + \frac{\Lambda}{3} = f_1(\varepsilon, H),$$

$$\dot{\varepsilon} = -3H(\varepsilon + p) = f_2(\varepsilon, H). \tag{2.1}$$

All symbols have their usual meaning. This set of equations will be called Friedman's dynamical system. If we assume that the operator of the infinitesimal transformation for system (2.1) is of the form

$$X = \xi(t)\frac{\partial}{\partial t} + \eta^1(H)\frac{\partial}{\partial H} + \eta^2(\varepsilon)\frac{\partial}{\partial \varepsilon}, \tag{2.2}$$

then equations (1.10) may be reduced to

$$\frac{\partial \eta^1}{\partial H}(H) - \frac{\partial \xi}{\partial t}(t) = X(\ln f_1),$$

$$\frac{\partial \eta^2}{\partial \varepsilon}(\varepsilon) - \frac{\partial \xi}{\partial t}(t) = X(\ln f_2). \tag{2.3}$$

By assuming, in (2.1), the equation of state $p = \gamma\varepsilon$, $\gamma = \text{const}$, solving equations (2.3), and applying the theory presented in the previous section, one obtains the following

Theorem 1: Friedman's dynamical system (2.1) with $p = \gamma\varepsilon$, $\gamma = \text{const}$, admits a Lie group with the infinitesimal operator $X = -At\partial/\partial t + AH\partial/\partial H + 2A(\varepsilon + \Lambda)\partial/\partial\varepsilon$; and, vice versa, from the invariance of system (2.1) with respect to the operator X it follows the equations of state $p = \gamma\varepsilon + p_0$, $p_0 \sim \Lambda$. The equation of the group invariant is that of the flat model trajectory $\varepsilon - 3H^2 + \Lambda = 0$. Finite transformations of the group are: $\bar{H} = He^{\tau}$, $\varepsilon + \Lambda = (\varepsilon + \Lambda)e^{2\tau}$, $\bar{t} = te^{-\tau}$, $(H, \varepsilon) \neq (0, 0)$.

In cosmological applications, one usually assumes $0 \leqslant \gamma \leqslant 1$ with an emphasis on the special cases: $\gamma = 0$ is the equation of state for a dust filled universe, $\gamma = 1/3$ for a radiation filled universe.

Introducing to (2.1) the new variable $\tilde{H} = \varepsilon - 3H^2$ we easily obtain the solution

$$\varepsilon - 3H^2 + \Lambda = 3k \left(\frac{\varepsilon}{\varepsilon_0}\right)^{\frac{2}{3} \cdot \frac{1}{1+\gamma}}, \tag{2.4}$$

where $k = \pm 1, 0$ is the space curvature constant, and $\varepsilon_0 = \text{const}$. This gives us the phase trajectory on the phase plane. Phase portraits on the compactified (H, ε) plane are shown in Figs. 1–3. From these pictures we can see that if we perturb the dust model ($p = 0$) by adding to it a small amount of matter with $p = \gamma\varepsilon$, $\gamma > 0$, the dynamics of the model does not change qualitatively (i.e. there is a time orientation preserving homeomorphism between perturbed and unperturbed model trajectory). This means that Friedman's dynamical system is structurally stable with respect to such changes of its matter content. On the other hand, by comparing Figs. 1–3, we easily infer that introducing of small constant terms (cosmological constant or constant pressure) into the equation of state $p = \gamma\varepsilon$ does change the structure of the phase plane, i.e. Friedman's dynamical system with $p_0 = 0$ is structurally unstable against such changes in the equation of state.

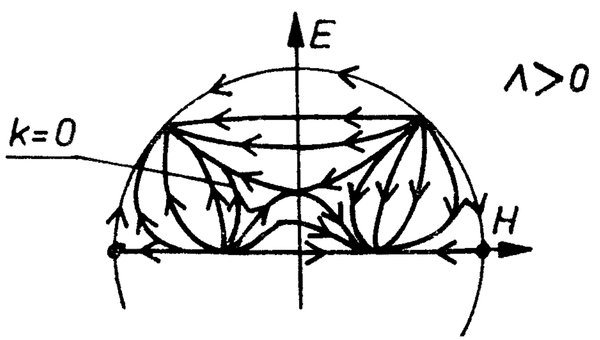

Fig. 1. Phase portrait of Friedman's dynamical system with $\Lambda > 0$

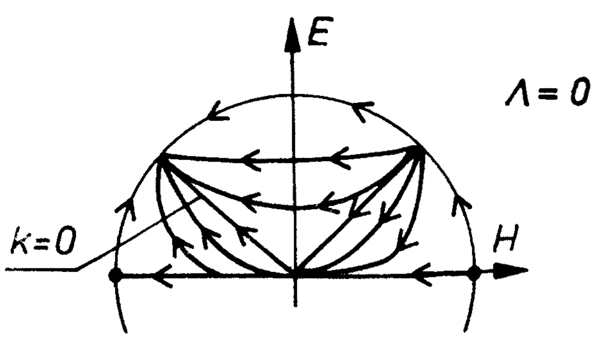

Fig. 2. Phase portrait of Friedman's dynamical system with $\Lambda = 0$

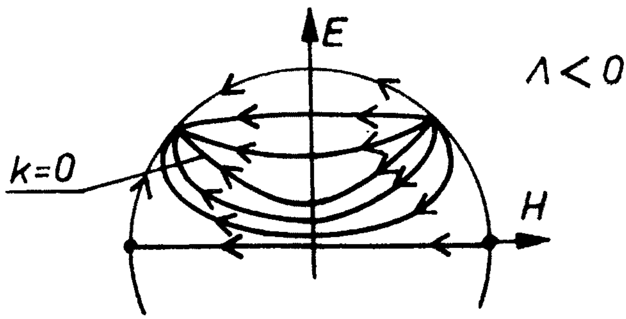

Fig. 3. Phase portrait of Friedman's dynamical system with $\Lambda < 0$

One should observe an interesting fact. Let us assume that, from observational data, we know that our universe is correctly described by a model with $k \approx 0$, $\Lambda \approx 0$ (which is not very far from what we know about our universe at present), then, basing on these data, we cannot predict whether our universe will recollapse to the final singularity or smoothly evolve to plus infinity. We are met here with a typical bifurcation situation. Happily enough, our retrodictions have more chances to succeed: the behaviour of models at early epochs is less sensitive to the changes in the equation of state. (For a more detailed discussion of structural stability properties of Friedman's cosmology consult [8].)

3. Symmetry group of Friedman equations with bulk viscosity

Inclusion of dissipative terms into left hand side of Einstein's equations is not only desirable for generality reasons [9] but also because of the fact that it may structurally stabilize the solutions [8]. Dissipative effects in cosmological models with Robertson--Walker types of symmetry mean the dependence $p = p(H)$ which gives the bulk viscosity coefficient $\zeta(\varepsilon) = -\frac{1}{3}\frac{\partial p}{\partial \varepsilon}$. Friedman's dynamical system with dissipation has form (2.1) with the substitution $p \leftrightarrow \tilde{p} = p - 3\zeta(\varepsilon)H$. We shall consider the case with $\Lambda = 0$.

Using a similar procedure as that employed in the preceding section one can easily establish the following

Theorem 2: Friedman's dynamical system with dissipation, with $p = \gamma\varepsilon$, $\gamma = \text{const}$, and $\zeta(\varepsilon) \sim \varepsilon^{1/2}$ admits a Lie group with the infinitesimal operator $X = -At\partial/\partial t + AH\partial/\partial H + 2A\varepsilon\partial/\partial\varepsilon$; and, vice versa, from the invariance of the system with respect to the operator X it follows the equation of state $\tilde{p} = \gamma\varepsilon - \mu\varepsilon^{1/2}H$, $\gamma, \mu = \text{const}$. The equation of the group invariant is that of the flat model trajectory $\varepsilon = 3H^2$ which is independent of the equation of state.

Introducing the new variable $\bar{H} = H\varepsilon^{-1/2}$, Friedman's dynamical system with dissipation can be easily integrated. The structure of the phase plane (H, ε) for this case is shown in Fig. 4. One can see that the solutions presenting (a) all closed models ($k = +1$), (b) all flat models ($k = 0$), and (c) those open expanding models ($k = -1$) which satisfy the

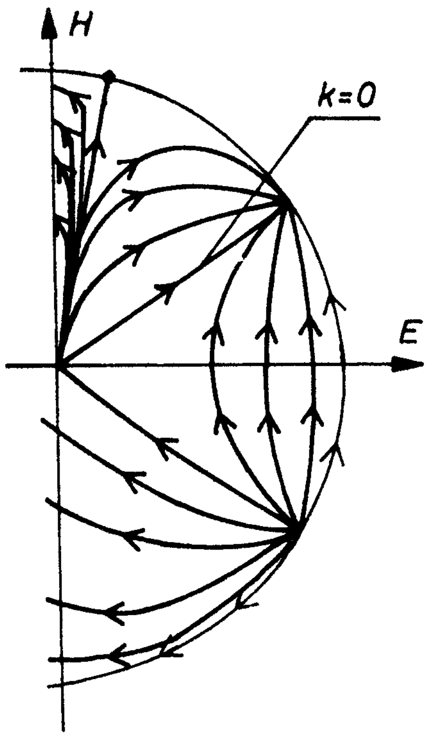

Fig. 4. Phase portrait of Friedman's dynamical system with dissipation

strong energy condition $\varepsilon + 3p \geqslant 0$, are structurally stable against small viscous perturbations. Moreover, world models with $\zeta(\varepsilon) \sim \varepsilon^{1/2}$ are interesting in that they have initial singularity but have no horizons, and consequently in these models Misner's paradox disappears [10].

4. *Symmetry group of the Bianchi type I cosmological models*

This class of cosmological models was first considered as a dynamical system with dissipation by Khalatnikov and Belinsky [11]. Bianchi type I metric is

$$ds^2 = dt^2 - R_1^2(t)dx_1^2 + R_2^2(t)dx_2^2 + R_3^2(t)dx_3^2 \tag{4.1}$$

and the components of the energy momentum tensor with dissipative terms may be written in the form

$$T_0^0 = -\varepsilon, \quad T_\alpha^0 = 0, \quad T_\alpha^\beta = p'\delta_\alpha^\beta - \eta\kappa_\alpha^\beta, \quad (\alpha, \beta = 1, 2, 3), \tag{4.1}$$

$$\kappa_\alpha^\beta = g^{\beta\gamma}\dot{g}_{\gamma\alpha}, \quad p' = p - (\zeta - \tfrac{2}{3}\eta)(\ln\sqrt{g})^{\cdot},$$

where η and ζ are coefficients of shear and bulk viscosity, respectively. In this case, Einstein's field equations may be reduced to the form of a dynamical system

$$\dot{H} = \varepsilon - 3H^2 - \tfrac{1}{2} w + \tfrac{3}{2} \zeta H,$$

$$\dot{\varepsilon} = 9\zeta H^2 + 4\eta(3H^2 - \varepsilon) - 3Hw, \tag{4.3}$$

where $w = \varepsilon + p$, and $H = (R_1 R_2 R_3)^{\cdot} = (\ln(\sqrt{g}))^{\cdot}$. The first of these equations is the Raychaudhuri equation, the second — the conservation principle $T_{i\ ;k}^{k} = 0$. Now, we have the following

Theorem 3: Dynamical system (4.3) with $p = \gamma\varepsilon$, $\gamma = \text{const}$, and $\zeta(\varepsilon) \sim \varepsilon^{1/2}$, $\eta(\varepsilon) \sim \varepsilon^{1/2}$ admits a Lie group with the infinitesimal operator $X = -At\partial/\partial t + AH\partial/\partial H + 2A\varepsilon\partial/\partial\varepsilon$, and, vice versa, from the invariance of system (4.3) with respect to the operator X it follows that $\zeta(\varepsilon) = \mu\varepsilon^{1/2}$, $\eta(\varepsilon) = \tilde{\mu}\varepsilon^{1/2}$ where μ, $\tilde{\mu} = \text{const}$. The equation of the group invariant is that of the flat Friedman model trajectory $\varepsilon = 3H^2$.

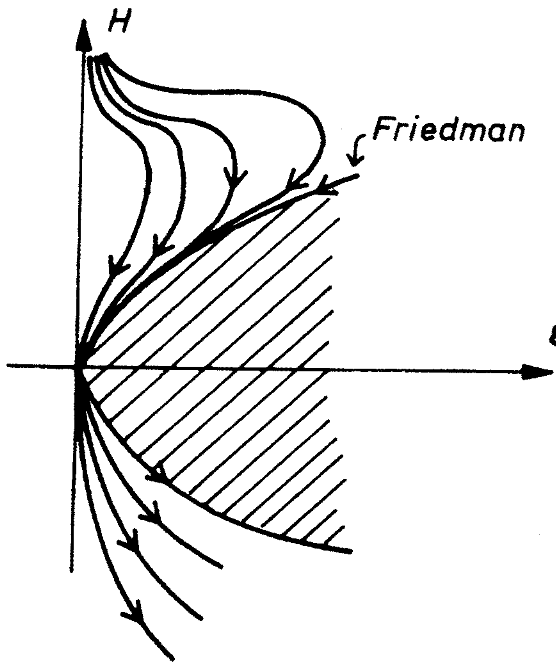

Fig. 5. Phase portrait of Bianchi I dynamical system

The phase portrait of the system is shown in Fig. 5. These models drastically differ from the flat Friedman model at infinity of the phase plane; however, at later stages, they approximate it in the structurally stable way.

5. A short comment

As we can see, symmetry groups of Friedman's equations — both viscous and non-viscous — correctly select physically meaningful equations of state. One could look at this fact in the following way. Usually equations of state are taken from local (non-cosmological) physics without gravitation (from outside of General Relativity). One should ask about compatibility conditions of such local physics without gravitation with the gravitation theory without local physics. It has turned out that there are symmetry groups of cosmological equations which provide correct compatibility conditions. We think that it is a non-trivial fact, the significance of which might have important implications for cosmology.

The symmetry group of Bianchi type I dynamical system selects equation of state which has no such clear physical meaning. However, one should remember that Bianchi I cosmological models do not describe as directly the observed universe as Friedman's solutions do, and usually they are considered because of their mathematical simplicity within the class of anisotropic homogeneous models.

As we have seen in the preceding sections, the considered problem is evidently connected with the question of structural stability of world models against perturbations of the equation of state. The logical connection is the following. The symmetry groups of the considered equations are groups of similarity, admissible rescallings being homeomorphic expansions and compressions of the phase plane ($\bar{H} = He^{\tau}$, $\bar{\varepsilon} = \varepsilon e^{2\tau}$, $\bar{t} = te^{-\tau}$) such that the trajectory of the flat model ($k = 0$) is left unchanged, every closed model (inside the parabola $k = 0$) goes into a closed model, every open model (outside the parabola $k = 0$) goes into an open model, and there are no transformations from a closed model into an open one, and vice versa. On the other hand, a system is structurally stable if its small perturbations give topologically equivalent (homeomorphic) dynamics. From this follows that the equations of state, being obtained from the symmetry groups, cannot spoil the structural stability of the system. (One should remember that, from purely mathematical point of view, transformations are admissible which send phase trajectories into the half-plane $\varepsilon < 0$.)

The structural stability problem of cosmological models has been considered by us elsewhere [8]; we are inclined to think that it is another crucial problem for methodology of physics and cosmology.

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