Friedmann cosmology with a generalized equation of state and bulk viscosity

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The universe media is considered as a non-perfect fluid with bulk viscosity and described by a more general equation of state. We assume the bulk viscosity is a linear combination of the two terms: one is constant, and the other is proportional to the scalar expansion $\theta = 3\dot{a}/a$. The equation of state is described as $p = (\gamma - 1)\rho + p_0$, where p_0 is a parameter. This model can be used to explain the dark energy dominated universe. Different choices of the parameters may lead to three kinds of fates of the cosmological evolution: no future singularity, big rip, or Type III singularity of Ref. [S. Nojiri, S.D. Odintsov, and S. Tsujikawa, Phys. Rev. D **71**, 063004 (2005)].

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I. INTRODUCTION

The cosmological observations indicate that the expansion of our universe accelerates [1]. Recently lots of work on extended gravity [2]such as modifying equation of state or by introducing the so called dark energy is to explain the cosmic acceleration expansion observed. To overcome the drawback of hydrodynamical instability, a linear equation of state of a more general form, $p = \alpha(\rho - p_0)$ is proposed [3], and this form incorporated into cosmological model can describe the hydrodynamically stable dark energy behaviors.

The observations also indicate that the universe media is not a perfect fluid [4] and the viscosity is concerned in the evolution of the universe [5, 6, 7]. In the standard cosmological model, if the equation of state parameter ω is less than -1, the universe shows the future finite singularity called Big Rip [8, 9]. Several ideas are proposed to prevent the big rip singularity, like by introducing quantum effects terms in the action.

In this paper, we show that the Friedmann equations can be solved with both a more general equation of state and bulk viscosity detailed as follows. The equation of state is

$$p = (\gamma - 1)\rho + p_0, \tag{1}$$

where p_0 and γ are two parameters. The bulk viscosity is expressed as

$$\zeta = \zeta_0 + \zeta_1 \frac{\dot{a}}{a}. \tag{2}$$

where ζ_0 and ζ_1 are two constants conventionally. The $\omega = p/\rho$ is constrained as $-1.38 < \omega < -0.82$ [10] by present observation data, so the inequality in our case should be

$$-1.38 < \gamma - 1 + \frac{p_0}{\rho} < -0.82. \tag{3}$$

The parameter p_0 can be positive (attractive force) or negative (repulsive force), and conventionally ζ_0 and ζ_1 are regarded as positive. To choose the parameters properly, it can prevent the Big Rip problem or some kind of singularity for the cosmology model, like in the phantom energy phase, as shown below.

This paper is organized as follows. In Sec. II we describe our model and give out the exact solution. In Sec. III we consider some special cases of the solution. In Sec. IV we discuss the acceleration phase and the future singularities in this model, and in the last section (Sec. V) we summarize our conclusions.

II. MODEL AND CALCULATIONS

We consider the Friedamnn-Roberson-Walker metric in the flat space geometry (k=0)

$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2 d\Omega^2), \tag{4}$$

and assume that the cosmic fluid possesses a bulk viscosity ζ. The energy-momentum tensor can be written as

$$T_{\mu\nu} = \rho U_\mu U_\nu + (p - \zeta\theta)H_{\mu\nu}, \tag{5}$$

where in comoving coordinates $U^\mu = (1, 0)$, $\theta = U^\mu_{;\mu} = 3\dot{a}/a$, and $H_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$ [11]. By defining the effective pressure as $\tilde{p} = p - \zeta\theta$ and from the Einstein equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$, we obtain the Friedmann equations

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho, \tag{6a}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3\tilde{p}). \tag{6b}$$

The conservation equation for energy, $T^{0\nu}_{;\nu}$, yields

$$\dot{\rho} + (\rho + \tilde{p})\theta = 0. \tag{7}$$

Using the equation of state to eliminate ρ and p, we obtain the equation which determines the scale factor $a(t)$

$$\frac{\ddot{a}}{a} = -\frac{3\tilde{\gamma} - 2}{2}\frac{\dot{a}^2}{a^2} + 12\pi G\zeta_0 \frac{\dot{a}}{a} - 4\pi G p_0, \tag{8}$$

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where the effective equation of state parameter is shifted from the original one as

$$\tilde{\gamma} = \gamma - 8\pi G\zeta_1. \tag{9}$$

So we can see that the equivalent effect of the second term in ζ is to change the parameter γ to $\tilde{\gamma}$ in the equation of state. As shown in Ref. [5], the barrier $\omega = -1$ between the quintessence region ($\omega > -1$) and the phantom region ($\omega < -1$) can be crossed, as a consequence of the bulk viscosity available.

Since the dimension of the two terms $12\pi G\zeta_0$ and $-4\pi G p_0$ is [time]$^{-1}$ and [time]$^{-2}$, respectively, we define

$$12\pi G\zeta_0 = \frac{1}{T_1}, \tag{10}$$

$$-4\pi G p_0 = \frac{1}{T_2^2}, \tag{11}$$

then Eq. (8) becomes

$$\frac{\ddot{a}}{a} = -\frac{3\tilde{\gamma}-2}{2}\frac{\dot{a}^2}{a^2} + \frac{1}{T_1}\frac{\dot{a}}{a} + \frac{1}{T_2^2}. \tag{12}$$

Here T_1 and T_2 are criteria to determine whether we should concern the ζ and p_0. If $T_1 >> t, thecosmictimescale$, the effect of ζ can be neglected, and if $T_2 >> t$, the effect of p_0 can be neglected likewise.

Concerning the initial conditions of $a(t_0) = a_0$ and $\theta(t_0) = \theta_0$, if $\tilde{\gamma} \neq 0$, the solution can be obtained as

$$a(t) = a_0\left\{\frac{1}{2}\left(1+\tilde{\gamma}\theta_0 T - \frac{T}{T_1}\right)\exp\left[\frac{t-t_0}{2}\left(\frac{1}{T}+\frac{1}{T_1}\right)\right] + \frac{1}{2}\left(1-\tilde{\gamma}\theta_0 T + \frac{T}{T_1}\right)\exp\left[-\frac{t-t_0}{2}\left(\frac{1}{T}-\frac{1}{T_1}\right)\right]\right\}^{2/3\tilde{\gamma}}. \tag{13}$$

And we obtain directly

$$\frac{\dot{a}}{a} = \frac{1}{3\tilde{\gamma}}\frac{(1+\tilde{\gamma}\theta_0 T - \frac{T}{T_1})(\frac{1}{T}+\frac{1}{T_1})\exp(\frac{t-t_0}{T}) - (1-\tilde{\gamma}\theta_0 T + \frac{T}{T_1})(\frac{1}{T}-\frac{1}{T_1})}{(1+\tilde{\gamma}\theta_0 T - \frac{T}{T_1})\exp(\frac{t-t_0}{T}) + (1-\tilde{\gamma}\theta_0 T + \frac{T}{T_1})}. \tag{14}$$

Here we define

$$T = \frac{T_1}{\sqrt{1+6\tilde{\gamma}(T_1/T_2)^2}}. \tag{15}$$

We can see that when $T_2 \to \infty$, $T = T_1$; when $T_1 \to \infty$, $T = T_2/\sqrt{6\tilde{\gamma}}$.

III. $\tilde{\gamma} = 0$ AND SPECIAL CASES

For $\tilde{\gamma} = 0$, we should use the mathematical L'Hospital's rule to calculate the limit of Eq. (13) rigously and note

$$\lim_{\tilde{\gamma}\to 0}\frac{dT}{d\tilde{\gamma}} = -\frac{3T_1^3}{T_2^2}. \tag{16}$$

The limit of solution $a(t)$ when $\tilde{\gamma} \to 0$ is

$$a(t) = a_0 \exp[\left(\frac{1}{3}\theta_0 T_1 + \frac{T_1^2}{T_2^2}\right)\left(e^{(t-t_0)/T_1} - 1\right) - \frac{T_1(t-t_0)}{T_2^2}]. \tag{17}$$

Directly solving Eq. (12)

$$\frac{\ddot{a}}{a} = \frac{\dot{a}^2}{a^2} + \frac{1}{T_1}\frac{\dot{a}}{a} + \frac{1}{T_2} \tag{18}$$

gives the same result. So Eq. (13) is consistent for $\tilde{\gamma}$ crossing zero. The $T_2 \to \infty$ limit of Eq. (17) is

$$a(t) = a_0 \exp\left[\frac{1}{3}\theta_0 T_1\left(e^{(t-t_0)/T_1} - 1\right)\right]. \tag{19}$$

and the $T_1 \to \infty$ limit of Eq. (17) is

$$a(t) = a_0 \exp\left[\frac{1}{3}\theta_0(t-t_0) + \frac{(t-t_0)^2}{2T_2^2}\right]. \tag{20}$$

These two special limits are also consistent with directly solving Eq. (12), by checking.

Let us discuss two special cases in the following. When the constant term in the equation of state is not concerned, i.e. $T_2 \to \infty$,

$$a(t) = a_0\left[1 + \frac{1}{2}\tilde{\gamma}\theta_0 T\left(e^{(t-t_0)/T} + 1\right)\right]^{2/3\tilde{\gamma}}. \tag{21}$$

and when the constant term in the bulk viscosity is not concerned, i.e. $T_1 \to \infty$,

$$a(t) = a_0\left(\cosh\frac{t-t_0}{2T} + \tilde{\gamma}\theta_0 T \sinh\frac{t-t_0}{2T}\right)^{2/3\tilde{\gamma}}. \tag{22}$$

When $T \to \infty$, the two cases become

$$a(t) = a_0\left[1 + \frac{1}{2}\tilde{\gamma}\theta_0(t-t_0)\right]^{2/3\tilde{\gamma}}. \tag{23}$$

For $\tilde{\gamma} \to 0$, the limit case is

$$a(t) = a_0 e^{\theta_0 (t-t_0)/3}, \tag{24}$$

which corresponds to the de Sitter universe with accelerating cosmic expansion.

Additional notions: Eqs. (6a) and (6b) can be rewritten as

$$H^2 = \frac{8\pi G}{3}\rho, \tag{25a}$$
$$\dot{H} = -4\pi G(\tilde{p} + \rho). \tag{25b}$$

From these equations, the relation among viscosity, the scalar factor a and Hubble parameter H is

$$aH\frac{dH}{da} = -\frac{3\tilde{\gamma}}{2}H^2 + 12\pi G\zeta H - 6\pi G p_0. \tag{26}$$

which reflects the viscosity functions for dark energy and matter dominated universe evolution.

IV. ACCELERATION AND BIG RIP

If the universe accelerates, then mathematically

$$\frac{\ddot{a}}{a} > 0. \tag{27}$$

From Eq. (8), we can qualitatively see that the bulk viscosity and a negative density p_0 can cause the universe to accelerate. Since the expression of $\ddot{a}/a$ is too complicated in this situation, now we only discuss a special case, with $p_0 = 0$. Here $\ddot{a}/a > 0$ yields

$$\omega = \gamma - 1 > \frac{2}{3}e^{(t-t_0)/T} - 1 + \frac{2}{\theta_0 T} + 8\pi G\zeta_1. \tag{28}$$

As we know, if the bulk viscosity is zero as in the standard FriedammnRobertson-Walker cosmology model, an accelerating expansion universe corresponds to $\omega < -1/3$. Inequality (28) tells us that if the bulk viscosity is large enough, the universe expansion can accelerate even if $\omega > -1/3$.

According to [9], the future singularities can be classified in the following way:

- Type I ("Big Rip"): For $t \to t_s$, $a \to \infty$, $\rho \to \infty$ and $|p| \to \infty$
- Type II ("sudden"): For $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$ and $|p| \to \infty$
- Type III: For $t \to t_s$, $a \to a_s$, $\rho \to \infty$ and $|p| \to \infty$
- Type IV: For $t \to t_s$, $a \to a_s$, $\rho \to 0$, $|p| \to 0$ and higher derivatives of H diverge.

In this paper, $\rho \to \infty$ means $p \to \infty$ (we assume $\gamma \neq 1$ generally). In the following we show that different choices of the parameters may lead to three fates of the universe evolution: no future singularity, big rip, or the Type III singularity.

A. $\tilde{\gamma} < 0$

From Eq. (6a), we see

$$\sqrt{\rho} \propto \frac{\dot{a}}{a}. \tag{29}$$

If the denominator of Eq. (14) is zero,

$$\left(1 + \tilde{\gamma}\theta_0 T - \frac{T}{T_1}\right)\exp\left(\frac{t-t_0}{T}\right) + 1 - \tilde{\gamma}\theta_0 T + \frac{T}{T_1} = 0. \tag{30}$$

then $a \to \infty$, $\rho \to \infty$, so the big rip occurs. The solution for t is

$$t_{\rm s} = T\ln\left(-\frac{1 - \tilde{\gamma}\theta_0 T + \frac{T}{T_1}}{1 + \tilde{\gamma}\theta_0 T - \frac{T}{T_1}}\right) + t_0 \tag{31}$$

If we want to prevent the big rip, there should be no real solution for $t > t_0$, so

$$-\frac{1 - \tilde{\gamma}\theta_0 T + \frac{T}{T_1}}{1 + \tilde{\gamma}\theta_0 T - \frac{T}{T_1}} < 1. \tag{32}$$

The inequality is equivalent to

$$1 + \tilde{\gamma}\theta_0 T - \frac{T}{T_1} > 0. \tag{33}$$

The above inequality can be satisfied in some conditions for the phantom energy, so the big rip will not occur. Furthermore, we can see that even if the dark energy is in the quintessence region, there also can be future singularity. For example, if $\tilde{\gamma} > 0$ and $p_0 < 0$, it is possible that inequality (33) is not satisfied, so the future singularity may occur, which will be discussed in the next subsection below.

The more explicit form of inequality (33) is

$$1 + \frac{\tilde{\gamma}\theta_0 T_1}{\sqrt{1 + 6\tilde{\gamma}(T_1/T_2)^2}} - \frac{1}{\sqrt{1 + 6\tilde{\gamma}(T_1/T_2)^2}} > 0. \tag{34}$$

From this inequality, we obtain that

(i) $p_0 < 0$, i.e. $T_2^2 > 0$: inequality (33) is always unsatisfied, so there will be a big rip at time $t_{\rm s}$.

(ii) $p_0 > 0$, i.e. $T_2^2 < 0$: inequality (33) is not always unsatisfied. If it is unsatisfied, there will be a big rip at time $t_{\rm s}$; if it is satisfied, there is no future singularity.

B. $\tilde{\gamma} > 0$

If the denominator of inequality (14) is zero, then $a \to 0$, $\rho \to \infty$, so the Type III singularity occurs. Following the same steps as before, we obtain that

(i) $p_0 < 0$, i.e. $T_2^2 > 0$: inequality (33) is always satisfied, so there is no future singularity.

(ii) $p_0 > 0$, i.e. $T_2^2 < 0$: inequality (33) is not always satisfied. If it is satisfied, there is no future singularity; if it is unsatisfied, there will be Type III singularity at time $t_{\rm s}$.

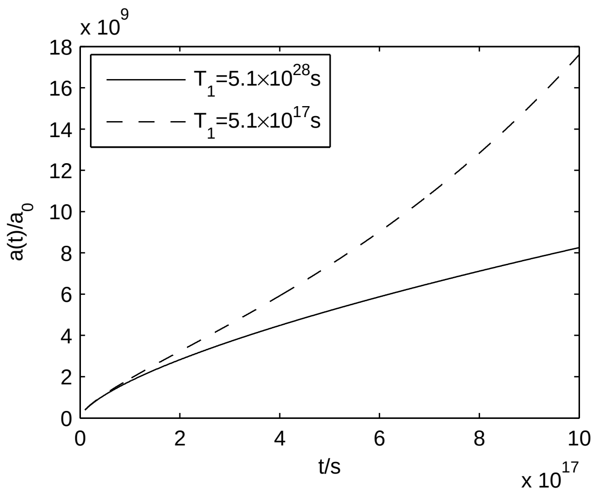

FIG. 1: $p_0 = 0$, $\gamma = 1$, the dash line corresponds to $T_1 = 5.1\times10^{28}$s, and another curve corresponds to $T_1 = 5.1\times10^{17}$s.

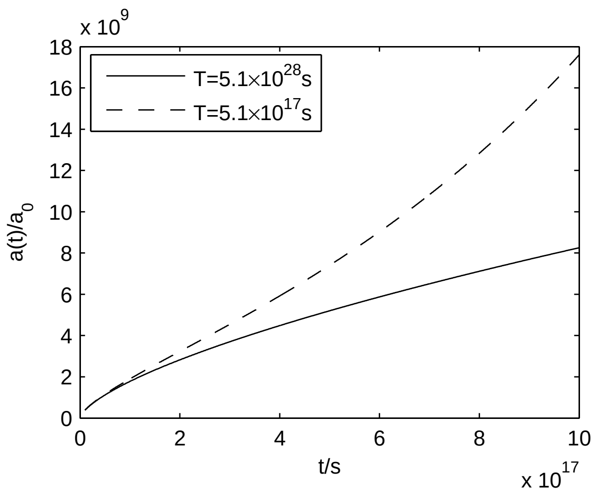

FIG. 2: $\zeta = 0$, $p_0 = 0$, the dash line corresponds to $\gamma = 0.181$, another curve corresponds to $\gamma = 0.18$ case.

C. $\tilde{\gamma} = 0$

For the case $\tilde{\gamma} = 0$,

$$\frac{\dot{a}}{a} = \frac{1}{3}\theta_0 e^{(t-t_0)/T_1} + \frac{T_1}{T_2^2}\left(e^{(t-t_0)/T_1} - 1\right). \tag{35}$$

So there is no future singularity in this case.

To illustrate the parameters in the general solution Eq (13) more clearly, we draw some graphics in Fig. 1-4. The initial condition is [11] $t_0 = 1000$s, $\theta_0 = 1.5\times10^{-3}\text{s}^{-1}$. At this time, the bulk viscosity $\zeta = 7.0\times10^{-3}\text{g/cm}\cdot\text{s}$, the corresponding $T_1 = 5.1\times10^{28}$s. We assume $\zeta_1 = 0$ for simplicity.

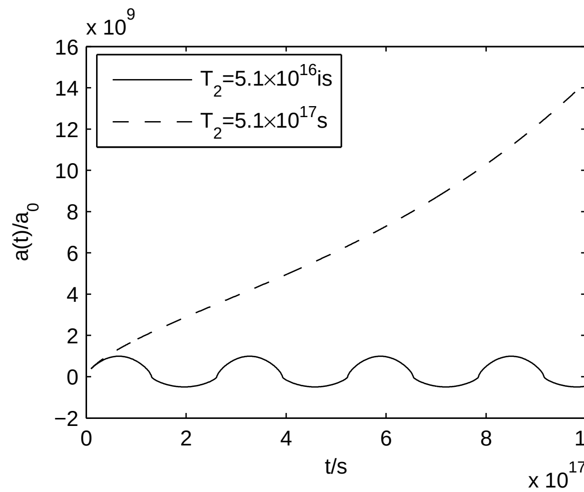

FIG. 3: $\zeta = 0$, $\gamma = 1$, the dash line corresponds to $T_2 = 5.1\times10^{16}i$s, while another curve corresponds to $T_2 = 5.1\times10^{17}$s case.

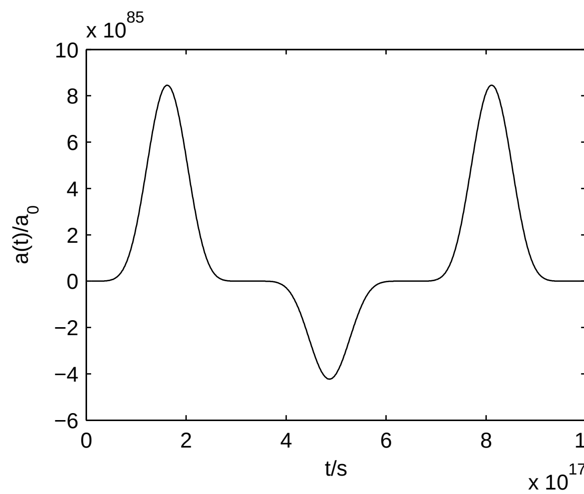

FIG. 4: $\zeta = 0$, $T_2 = 5.1\times10^{16}i$s, $\gamma = 0.1$.

V. CONCLUSION

In conclusion, we have solved the Friedmann equations with both a more general equation of state and bulk viscosity, and discussed the acceleration expansion of the universe evolution and the future singularities for this model. Compared with the standard model of cosmology, this model has had three additional parameters, ζ_0, ζ_1 and p_0: choices of ζ_0 and negative p_0 can cause the universe accelerate; ζ_1 can drive the cosmic fluid from the quintessence region to the phantom one [5], and positive p_0 may both prevent the big rip for phantom phase and lead to the Type III singularity of Ref. [9] for the quintessence phase. The relation between the choices of parameters and the future singularities of the cosmological evolution in this extended model is summarized as in the following table and we expect more detail investigations on viscosity effects to be carried out.

Parameters			Future singularity (at $t \to t_s$)
$\tilde{\gamma} < 0$	$p_0 < 0$		$a \to \infty, \rho \to \infty$
	$p_0 > 0$	$1 + \tilde{\gamma}\theta_0 T - T/T_1 > 0$	No
		$1 + \tilde{\gamma}\theta_0 T - T/T_1 < 0$	$a \to \infty, \rho \to \infty$
$\tilde{\gamma} = 0$			No
$\tilde{\gamma} > 0$	$p_0 < 0$		No
	$p_0 > 0$	$1 + \tilde{\gamma}\theta_0 T - T/T_1 > 0$	No
		$1 + \tilde{\gamma}\theta_0 T - T/T_1 < 0$	$a \to 0, \rho \to \infty$

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