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Coordinate confusion in conformal cosmology★

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ABSTRACT

A straightforward interpretation of standard Friedmann–Lemaître–Robertson–Walker (FLRW) cosmologies is that objects move apart because of the expansion of space, and that sufficiently distant galaxies must be receding at velocities exceeding the speed of light. Recently, however, it has been suggested that a simple transformation into conformal coordinates can remove superluminal recession velocities, and hence the concept of the expansion of space should be abandoned. This work demonstrates that such conformal transformations do not eliminate superluminal recession velocities for open or flat matter-only FLRW cosmologies, and all possess superluminal expansion. Hence the attack on the concept of the expansion of space based on this is poorly founded. This work concludes by emphasizing that the expansion of space is perfectly valid in the general relativistic framework; however, asking the question of whether space *really* expands is a futile exercise.

Key words: cosmology: theory.

1 INTRODUCTION

While it is almost a century since Hubble (1929) identified the expansion of the Universe, debate is still ongoing as to what this expansion physically means. The mathematics of cosmology are set within the framework of general relativity, and textbooks typically describe the expansion of the Universe as an expansion of space itself. However, while the concept of expanding space has recently been under fire (Whiting 2004; Peacock 2006), it is clear that what has been attacked is a particular picture of space expanding like a fluid and carrying galaxies along with it; Barnes et al. (2006) and Francis et al. (2007) have demonstrated that it is correct to talk about the expansion of space, as long as one clearly understands what the mathematics of general relativity is telling us.

However, some recent attacks on the picture of expanding space have been more forceful (e.g. Chodorowski 2005, 2006), with a typical line of criticism invoking a comparison between an explosion of massless particles in static, flat space–time (Milne model) and empty Friedmann–Lemaître–Robertson–Walker (FLRW) universes. In a recent paper, Chodorowski (2007) examines the nature of FLRW cosmologies in conformal coordinates, concluding that superluminal separation of objects can be removed through a simple change of coordinates, and hence that superluminal expansion is illusionary; this is in contrast to Davis & Lineweaver (2004), who point out that such superluminal expansion is a generic feature of general relativistic cosmologies. The goal of this short contribution is to clear up some of the confusion surrounding the concept of expanding space and conformal transformations, showing that superluminal expansion does not necessarily vanish in conformal coordinates. Furthermore, the concept of expanding space is reasserted as a valid description of the Universe, although discussion on whether space *really* expands is seen to be futile.

2 GENERAL RELATIVISTIC COSMOLOGIES

2.1 FLRW universes

The starting point for a standard, general relativistic model of the cosmos begins with the assumption of homogeneity and isotropy. With this, the space–time of the universe can be described by a FLRW metric with invariant interval of the form

$$ds^2 = dt^2 - a^2(t)\left[dx^2 + R_0^2 S_k^2(x/R_0)(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \quad (1)$$

where $S_k(x) = \sin x$, x and $\sinh x$ for spatial curvatures of $k = +1$ (closed), $k = 0$ (flat) and $k = -1$ (open) respectively, with the curvature given by R_0^{-2}; note that, throughout, $c = 1$. Also, $a(t)$ is the scale factor, the evolution of which depends upon the relative mix of energy density in the universe. It is clear from this form of the metric given by equation (1) that for a fixed coordinate time t, the physical separation of objects depends on the size of the scale factor $a(t)$, and the increase of $a(t)$ with t results in the increasing separation of objects; this is typically taken to be the expansion of the universe.

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2.2 Velocities in expanding universes

In order to understand superluminal recession, we must first be very clear about how we are defining recession velocity in an expanding universe. A fundamental definition of distance in general relativity is the proper distance, defined as the spatial separation between two points along a hypersurface of constant time. Given the form of the FLRW metric (equation 1), the radial distance from the origin to a coordinate x along a hypersurface of constant t is

$$D_{\rm p}(t) = a(t)\, x. \tag{2}$$

Taking the derivative with respect to coordinate time [which is synchronous for all comoving observers (fixed x) and is equivalent to their proper time τ], we obtain what we will refer to as the proper velocity

$$v_{\rm p} \equiv \frac{{\rm d}D_{\rm p}}{{\rm d}\tau} = \frac{{\rm d}D_{\rm p}}{{\rm d}t} = \frac{{\rm d}a}{{\rm d}t}x + a\frac{{\rm d}x}{{\rm d}t}. \tag{3}$$

For comoving observers with ${\rm d}x/{\rm d}t = 0$ this becomes the well-known distance–velocity law. However, universes that are open or flat are spatially infinite, and the above metric predicts that sufficiently distant objects will separate at velocities exceeding the speed of light; this issue has introduced a lot of confusion and discussion into the nature of the expansion (Davis & Lineweaver 2004).

The coordinate velocity can also be defined as

$$v_{\rm c} = \frac{{\rm d}x}{{\rm d}t}. \tag{4}$$

For the FLRW metric, comoving observers have coordinate velocities of zero, and peculiar velocities $a\,{\rm d}x/{\rm d}t$ must be less than unity, to be consistent with special relativity (see Francis et al. 2007). It follows that all radial coordinate velocities in the FLRW metric will be subluminal. This reflects a feature of the coordinate system; what is important, however, is not how arbitrarily defined coordinates change with respect to one another, but how the proper distance between any two points changes with respect to the proper time of observers.

2.3 Conformal transformations

Conformal transformations are important in understanding the causal structure of space–time (Hawking & Ellis 1973). A conformal transformation maps from one set of coordinates to another while preserving angles and infinitesimal shape, and two space–times represented by metrics g' and g are conformally equivalent just if

$$g'(\boldsymbol{x}) = \Omega(\boldsymbol{x})^2\, g(\boldsymbol{x}), \tag{5}$$

where $\Omega(\boldsymbol{x})$ is a scalar function.[1] This function can be interpreted as a scalar field that influences perfect rulers and clocks to distort one space–time into the other. A metric that is conformally equivalent to the Minkowski metric is labelled ‘conformally flat’.

An examination of the FLRW metric (equation 1) reveals that it is conformally flat[2] and hence can be written in the form

$${\rm d}s^2 = \Omega^2(\boldsymbol{x})\, {\rm d}s^2_{\rm flat}, \tag{6}$$

[1] More precisely, two metrics are conformally equivalent if they possess the same Weyl tensor.

[2] For flat space–time, the Weyl tensor vanishes identically. This can be simply shown to be true for FLRW space–time using a symbolic mathematics package such as GRTENSOR (http://grtensor.phy.queensu.ca).

where ${\rm d}\, s_{\rm flat}$ represents the space–time of special relativity. The precise form of $\Omega(\boldsymbol{x})$ changes depending on whether flat, closed or open cosmologies are considered. This space–time mapping from the FLRW metric to the Minkowski metric also subsumes null geodesics (the motion of photons, which satisfy ${\rm d}s = 0$), i.e. the distorted light-cones seen in cosmological coordinates can be drawn on to the classical light-cones of special relativity (see fig. 1 in Davis & Lineweaver 2004).

Typically, conformal representations of FLRW universes consider only the radial motion of photons and neglect the angular components of the metric. With such a transformation, fundamental, or comoving, observers (with fixed x, θ and ϕ in equation 1) move on straight, vertical lines on an R–T representation of flat space–time, while photons move at 45° (the coordinate transformation from open FLRW coordinates to conformal coordinates for an open universe is discussed in detail in Section 2.4). Such an approach has proved to be very powerful in understanding cosmic causality and the nature of fundamental horizons in the universe (Rindler 1956; Ellis & Stoeger 1988). However, it is important to note that the consideration of purely radial paths results in a representation which is not fully conformal; the mathematical transformation of the full FLRW metric into conformally flat coordinates was tackled by Infield & Schild (1945). An important result from their study is that in fully conformal coordinates, fundamental observers (comoving observers in FLRW metrics) no longer travel along straight, vertical paths; this is examined in more detail in the next section.

2.4 An open universe

Chodorowski (2007) considers the question of the conformal representation of the FLRW metric, focusing, as a specific example, on an open universe. Starting with the FLRW metric (equation 1), he shows that the adoption of a change in coordinates

$$R = A{\rm e}^{\eta} \sinh\chi, \tag{7}$$

$$T = A{\rm e}^{\eta} \cosh\chi,$$

where η is the conformal time, defined such that ${\rm d}t = R_0\, a\, {\rm d}\eta$, and $\chi = x/R_0$, allows the FLRW metric to be written as

$${\rm d}s^2 = \frac{R_0^2\, a^2(\eta)}{T^2 - R^2}\left[{\rm d}T^2 - {\rm d}R^2 - R^2\left({\rm d}\theta^2 + \sin^2\theta\, {\rm d}\phi^2\right)\right], \tag{8}$$

which is just

$${\rm d}s^2 = \frac{R_0^2\, a^2(\eta)}{T^2 - R^2}\, {\rm d}s^2_{\rm flat}. \tag{9}$$

Hence light-cones plotted in R–T coordinates will be the classical light curves of special relativity (see Fig. 1). Infield & Schild (1945) demonstrate that the motion of fundamental observers in the FLRW metric (χ = constant) is still mapped on to straight lines in R–T coordinates, and with this choice of coordinate transformations, Chodorowski (2007) demonstrates that such lines possess a slope of

$$\beta = \frac{{\rm d}R}{{\rm d}T} = \frac{R}{T} = \tanh\chi. \tag{10}$$

Hence the fundamental observers have a constant velocity across the R–T plane given by β, where $\beta \rightarrow 1$ as $\chi \rightarrow \infty$. This is taken to be evidence that the coordinate velocity is always less than the speed of light, so that the relative motion of the fundamental observers is always subluminal, no matter what their separation. In this manner, it appears that superluminal motion can be removed through a coordinate transformation.

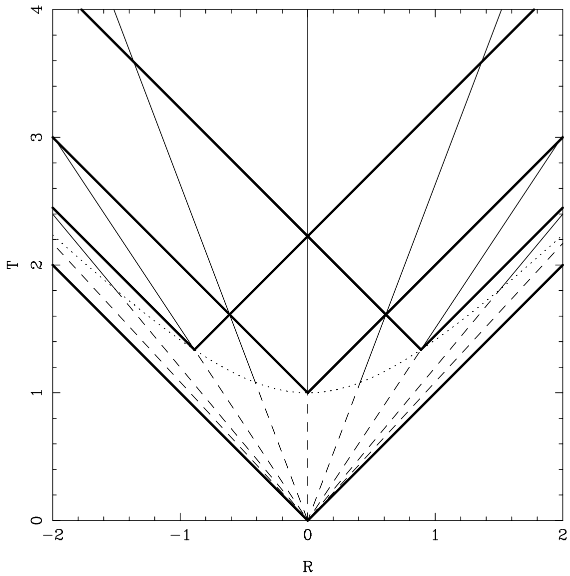

Figure 1. An open universe in conformal coordinates. The thick, solid lines denote the path of light rays in conformal coordinates, whereas the dashed and solid lines represent the paths of fundamental observers. The entire (infinite) open universe is contained within the outermost light cone. The dotted hyperbola represents the big bang.

3 INTERPRETATION

How are we to interpret this conclusion? Has superluminal expansion, and hence the expansion of space, been refuted? The argument against superluminal recession boils down to the finding, through conformal transformations, that the coordinate velocity is subluminal in conformal coordinates. However, as was shown in Section 2.2, all FLRW universes – even in the original coordinates – posses coordinate velocities that are subluminal. Of greater importance is the mapping of proper velocity to conformal coordinates. Since space–time has been sliced up differently, the surfaces of constant coordinate time – over which proper distance is measured – have been altered. The critical concern is therefore how this new proper distance changes relative to the new time coordinate. This was not addressed in Chodorowski (2007).

To answer this, it is useful to examine the picture of the example open universe in R–T coordinates (Fig. 1). As FLRW universes are conformally flat, light-cones in this picture are at 45°. As seen in the coordinate transformation given in equation (7), all fundamental observers (constant χ) sit on straight lines originating at the origin; note that the entire (infinite) universe is contained within the outer light-cone. It might be tempting to consider the point at $(R, T) = (0, 0)$ as the FLRW big bang, but in fact this ‘point’ ($\eta = 0$) is mapped to a hyperbola in the plane, from which the paths of fundamental observers extend; paths behind this curve have no physical equivalent in the FLRW universe.

What do we mean when we say that the universe is expanding? It does not mean that coordinates are changing in some particular fashion, as even in a standard FLRW universe, objects maintain spatial coordinate separation (i.e. the fundamental or comoving spatial coordinates are separate). In fact, universal expansion should be interpreted as an increase in the physical separation of objects with cosmic time, i.e. a galaxy at B is moving away from A at so many metres per second, with time being measured by A’s clock, and distance being the proper distance.

Chodorowski (2007) notes that for a spatially flat FLRW universe, the conformal representation is

$$\mathrm{d}s^2 = a^2(\eta)[\mathrm{d}\eta^2 - \mathrm{d}\chi^2 - \chi^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)], \tag{11}$$

so that the distance to a galaxy at comoving coordinate $\chi = X$ from a fundamental observer at $\chi = 0$ is taken along a hypersurface of constant cosmological time ($\mathrm{d}\eta = 0$) and is

$$D_\mathrm{p}(\eta) = \int \sqrt{-\mathrm{d}s^2} = a(\eta)\int_0^X \mathrm{d}\chi = a(\eta)X, \tag{12}$$

whereas the proper time τ as measured by the fundamental observer at the origin is related to the coordinate time t and conformal time η via

$$\mathrm{d}\tau = \mathrm{d}t = a(\eta)\,\mathrm{d}\eta. \tag{13}$$

The rate of change of the proper distance to a comoving observer at $\chi = X$ in terms of the proper time as measured at the origin is

$$\frac{\mathrm{d}D_\mathrm{p}}{\mathrm{d}\tau} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}(aX). \tag{14}$$

For a flat universe, the radial coordinate X is unbound and hence, even in this conformal representation, superluminal expansion remains a feature.

What about the conformal representation of the open universe considered by Chodorowski (2007)? As this is a coordinate transformation from the FLRW universes, the distance is a line integral

$$D_\mathrm{p}(\eta) = \int \sqrt{-\mathrm{d}s^2} = R_0 a(\eta)\int \frac{\sqrt{\mathrm{d}R^2 - \mathrm{d}T^2}}{\sqrt{T^2 - R^2}}, \tag{15}$$

with the condition that the path be restricted to a hyperbola in the R–T plane (η = constant), so that $T^2 - R^2 = A^2\mathrm{e}^{2\eta} \equiv k^2$. From this one obtains the relation $\mathrm{d}T = (R/T)\,\mathrm{d}R$; the integration proceeds from the origin along to a point $R(\chi) = R_\chi$:

$$\frac{D_\mathrm{p}(\eta)}{a(\eta)R_0} = \int_0^{R_\chi}\frac{\mathrm{d}R}{T} = \int_0^{R_\chi}\frac{\mathrm{d}R}{\sqrt{k^2 + R^2}} = \mathrm{asinh}\left(\frac{R_\chi}{k}\right) = \chi. \tag{16}$$

This physical separation – even in this conformal representation – is that expected from the standard FLRW analysis.

But of course, one of the joys of relativity is the ability to slice and dice space–time differently for differing observers, and we can instead calculate the distance along the spatial hypersurfaces defined by constant T in the conformal representation; this is the approach adopted by Chodorowski (2007). Does this remove superluminal expansion? Remembering that in an open, matter-only universe,

$$t(\eta) = \frac{\Omega_0}{2(1-\Omega_0)^{3/2}}\left(\sinh\eta - \eta\right), \tag{17}$$

$$a(\eta) = \frac{\Omega_0}{2(1-\Omega_0)}\left(\cosh\ \eta - 1\right), \tag{18}$$

where Ω_0 is the present-day normalized matter density (see Hobson, Efstathiou & Lasenby 2005). Hence the distance along the hypersurface is (taking $A = 1$ for convenience)

$$\begin{aligned} D_\mathrm{p}(T) &= \frac{R_0\Omega_0}{2(1-\Omega_0)}\int_0^R \frac{\cosh(\ln(\sqrt{T^2 - R'^2})) - 1}{\sqrt{T^2 - R'^2}}\,\mathrm{d}R' \\ &= \frac{R_0\Omega_0}{4(1-\Omega_0)}\left[R - 2\mathrm{atan}\left(\sinh\chi\right) + \frac{\chi}{T}\right]. \end{aligned} \tag{19}$$

Fig. 2 presents this proper distance as a function of the proper time experienced by an observer at $R = 0$ for three fiducial universes with $\Omega_0 = 0.01, 0.5$ and 0.99. In each, the solid lines represent the proper

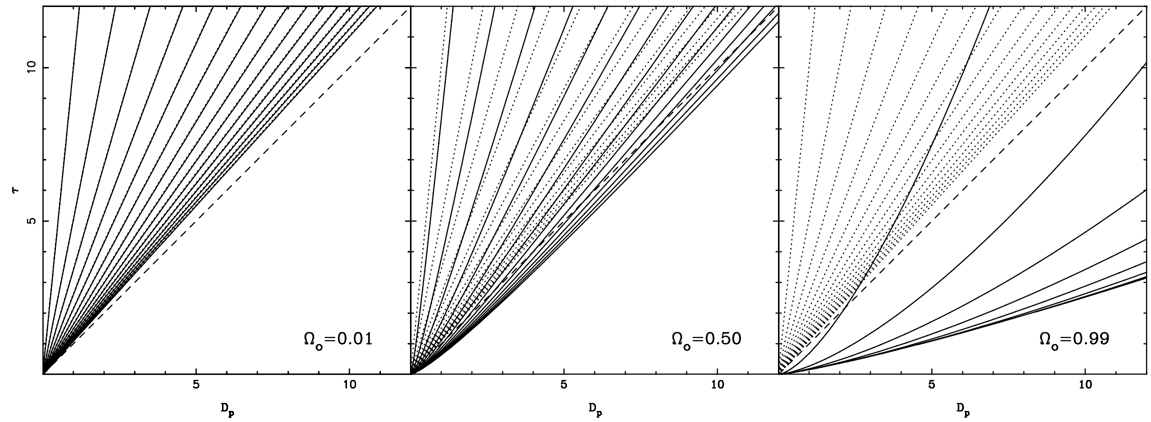

Figure 2. The proper distance to several comoving observers in several open matter-only universes (solid curves). The dashed line at 45° represents the speed of light. The dotted lines represent the recession paths of the fundamental observers integrated over the conformal coordinates without considering the conformal factor outside the metric.

distance, while the dashed lines at 45° represent the speed of light. The dotted lines represent the distance in terms of the conformal coordinates while neglecting the conformal factor outside the metric (i.e. over Minkowski space–time).

For the low-density case, the conformal factor tends to unity and the space–time becomes that of special relativity. Hence the proper distance increases as expected in this representation; the paths are subluminal and match those calculated in the *R*–*T* coordinates. However, as we increase the mass density of the universe, it is seen that the increase of the proper distance with proper time deviates from Minkowski space-time, in places being superluminal. This is very apparent in the case with $\Omega_0 = 0.99$, where the majority of paths are receding superluminally.

It is interesting to examine the properties of this proper velocity for constant *T* slices in a little more detail. Noting that the proper time τ for an observer at the originis related to the conformal coordinate time *T* via

$$\mathrm{d}\tau = \frac{R_0\, a(\eta)}{T}\,\mathrm{d}T, \tag{20}$$

it is straightforward to show that

$$\frac{\mathrm{d}D_\mathrm{p}}{\mathrm{d}\tau} = \frac{\mathrm{d}T}{\mathrm{d}\tau}\frac{\mathrm{d}D_\mathrm{p}}{\mathrm{d}T} = \left[\mathrm{e}^{\eta}\tanh\chi - \frac{\chi}{\mathrm{e}^{\eta}}\right]\frac{1}{(\mathrm{e}^{\eta}+\mathrm{e}^{-\eta}-2)}, \tag{21}$$

where η is the conformal time ticked off at the origin and is related to the proper time at the origin via $\mathrm{d}\tau = R_0\, a\, \mathrm{d}\eta$. Importantly, the form of the curve is independent of Ω_0 and hence is valid for all open ($0 < \Omega_0 < 1$) FLRW universes. The left-hand panel of Fig. 3 presents this function for several values of χ; as $\eta \to \infty$, $\mathrm{d}D_\mathrm{p}/\mathrm{d}\tau \to \tanh\chi$, the coordinate velocity, but it is clear from this figure that at early times the coordinate velocity is negative and superluminal, becoming subluminal before becoming positive and superluminal again; this is true for all values of χ.

The remaining issue is the relation between the FLRW conformal time η and the cosmological time *t*; this is given by equation (18) and is presented in the right-hand panel of Fig. 3. As expected from Fig. 2, in the $\Omega_0 = 0.01$ universe, the conformal time approaches 5 in a fraction of a Hubble time (i.e. $t < 1$) and hence the superluminal motion occurred in the very early universe and is not apparent given

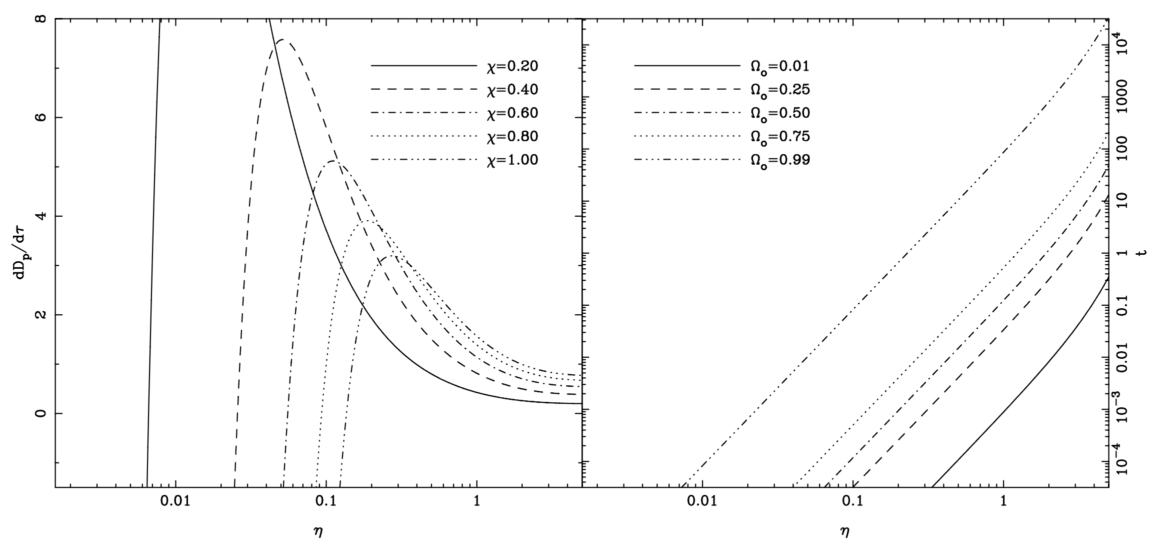

Figure 3. The left-hand panel presents the proper velocity in open matter-only FLRW universes, for a range of conformal times η; clearly, at early times, the universe possess superluminal contraction and then expansion. The right-hand panel presents the relationship between the conformal time and the cosmic time of the standard FLRW universe for a range of present-day values of the matter density.

the resolution of Fig. 2. For the $\Omega_0 = 0.99$ universe, on the other hand, this conformal time of $\eta \sim 5$ is not approached until after several hundred Hubble times and the superluminal expansion is apparent over cosmic history. However, in the distant universe, this superluminal motion will be lost as the proper velocity tends to the coordinate velocity. Note that as $\Omega_0 \to 0$, the excessive superluminal motion is pushed back to earlier epochs of cosmic time t until $\Omega_0 = 0$; the expansion is that of an empty, special relativistic universe, with the same proper and coordinate velocity.

4 CONCLUSIONS

In short, a recent interpretation of the nature of the expansion of the universe in conformal coordinates concludes that superluminal expansion, a staple of FLRW universes, is nothing but a coordinate effect of general relativity and it can be removed through a simple coordinate transformation. The current Letter has examined this claim and has found the conclusion to be erroneous, and that objects in the universe can still physically separate at superluminal velocities, even in conformal coordinates. It should be noted that the incorrect interpretation of conformal coordinates is not new: Querella (1998) attacked a series of papers which claimed that cosmology in conformal coordinates can even remove the need for a big bang (Endean 1994, 1995, 1997). As ever, in relativity, one should be careful about the interpretation of coordinates and the definition of distances.

In a companion paper, Francis et al. (2007) has discussed a number of issues relating to the recent discussions on the meaning and use of expanding space as a concept in cosmology, and we reiterate the most important of these now. The FLRW metric of the cosmos contains a term, the scale factor, which grows with time in an expanding universe. It is perfectly acceptable to talk of this metric expansion as the expansion of space, but one's intuition must be led by the mathematical framework of general relativity. If, however, one wishes to adopt the conformal metric with the flat space–time of special relativity (although a changing conformal factor in front of it), that is equally acceptable. The choice of coordinates is down to personal preference, as both must give the same predictions. From all of this, it should be clear that it is futile to ask the question 'is space *really* expanding?'; the standard FLRW metric and its conformal representation are the same space–time. No experiment can be formulated to differentiate one personal choice of coordinates from another.

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