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W. de Sitter, "The Expanding Universe," *Scientia*, Jan. 1931.

The Expanding Universe. By Abbé G. Lemaître.

(*Communicated by Sir A. S. Eddington.*)

1. *Introduction.*

Eddington has suggested that the expansion of a universe in equilibrium may be started by the formation of condensations. A preliminary investigation by W. H. McCrea and G. C. McVittie seems to point out an effect of opposite sense according to the nature of the condensations.* I find that the formation of condensations and the degree of concentration of these condensations have no effect whatever on the equilibrium of the universe. Nevertheless, the expansion of the universe is due to an effect very closely related to the formation of condensations, which may be named the "stagnation" of the universe. When there is no condensation, the energy, or at least a notable part of it, may be able to wander freely through the universe. When condensations are formed this free kinetic energy has a chance to be captured by the condensations and then to remain bound to them. That is what I mean by a "stagnation" of the world—a diminution of the exchanges of energy between distant parts of it.

In order to investigate the effect of condensations in a universe homogeneous in the mean, I consider a definite condensation of supposed spherical symmetry, and I average the outside condensations so that they also may be thought of as having spherical symmetry. The condensation under investigation is limited by a spherical shell which is the neutral zone between it and neighbouring condensations; a point on this neutral zone is not more within the gravitational influence of the interior condensation than of the condensations outside. The expansion of the neutral zone gives a measure of the expansion of the

* Sir A. S. Eddington, *M.N.*, **90**, 668, 1930; W. H. McCrea and G. C. McVittie, *M.N.*, **91**, 128, 1930; G. C. McVittie, *M.N.*, **91**, 274, 1931.

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whole universe. I find by this method exactly the same equations as were deduced directly from the equations of a homogeneous universe of variable radius. The result does not depend on any variation in the degree of concentration of the matter, and it shows that the pressure p of the homogeneous universe must be thought of in the case of condensations as the radial kinetic energy, the density of exchanges of energy between the condensation and the outside regions.

In order to study the effect of a variation of p it is convenient to choose as an auxiliary variable the gravitational potential M/R, where M is the total proper mass. Then equations may be reduced to quadratures, and any law of variation can be easily discussed. Variations of p in a universe in equilibrium are found to induce variations of R of the opposite sense.

In order to obtain quantitative formulæ for the expansion of a universe in equilibrium due to the stagnation process, I worked out the special case where the stagnation arose in an instant. The value of the actual radius of the universe depends on the observed receding velocity and mean density by formulæ practically independent of the degree of the initial stagnation. The epoch of the rupture of equilibrium is found for the capture of a millionth of the mass of the universe having velocity of 30 km./sec. to be of the order of 10^{11} years.

2. *Non-static Field of Spherical Symmetry: Birkhoff's Theorem.*

The problem of the condensations in a universe of variable radius can be considered as a problem of a non-static field with spherical symmetry. We shall therefore obtain the general equations of a spherical non-static field, and extend to the case where there is a cosmological constant a very important theorem due to Birkhoff; viz.—Schwarzschild's exterior solution is the general solution in empty space even if the field is not static.* Of course, this theorem makes abstraction of immaterial changes of co-ordinates; it supposes only that the spherical symmetry is conserved and that the exterior field remains empty. This is the relativistic transposition of Newton's theorem that the exterior field of a spherical body does not depend on its condensation or expansion; or, otherwise stated, that spherical pulsations do not induce gravitational waves.

Spherical symmetry is characterised by an interval

$$ds^2 = g_{11}dx_1^2 + g_{22}(d\theta^2 + \sin^2\theta d\phi^2) + 2g_{14}dx_1dx_4 + g_{44}dx_4^2$$

where g_{11}, g_{22}, g_{14}, and g_{44} are functions of x_1 and x_4 only. This expression is invariant for the group of transformations which keeps $d\theta^2 + \sin^2\theta d\phi^2$ invariant and which represents the rotations of a sphere on itself.

We introduce standard co-ordinates r, T; first r in place of x_1 by

$$r^2 = -g_{22},$$

* G. D. Birkhoff, *Relativity and Modern Physics*, p. 253 (1923).

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and then with the new g—*i.e.* the g after this first transformation has been performed

$$\epsilon d\mathrm{T} = g_{14}dr + g_{44}dx_4,$$

where $1/\epsilon$ is an integrand factor of the expression on the right-hand side. Then we obtain expressions of the form

$$ds^2 = -e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^{\nu}d\mathrm{T}^2 \quad . \quad . \quad (1)$$

where λ and ν are functions of r and T.

For instance, if R is a function of t and

$$ds^2 = -\mathrm{R}^2(d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi\sin^2\theta d\phi^2) + dt^2,$$

we write

$$r = \mathrm{R}\sin\chi,$$

giving

$$ds^2 = -r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{1}{\mathrm{R}^2\cos^2\chi}(dr - \mathrm{R}'\sin\chi dt)^2 + dt^2,$$

and by

$$\mathrm{T} = \frac{1}{\cos\chi}e^{\int\frac{dt}{\mathrm{RR}'}}$$

we obtain

$$ds^2 = -r^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{-dr^2 + \mathrm{R}^2\mathrm{R}'^2\cos^2\chi}{\cos^2\chi - \mathrm{R}'^2\sin^2\chi}d\mathrm{T}^2$$

where R and χ must be replaced by their expressions in terms of r and T. Except in special cases as for the de Sitter universe, elimination of R and χ cannot be performed explicitly. Note that the transformation is only admissible when $\tan\chi < \mathrm{R}'$. For $\tan\chi = \mathrm{R}'$, we have apparent singularities as is well known in the de Sitter universe at the so-called "horizon" of the centre. Our process is thus to come back to such non-homogeneous co-ordinates as the original co-ordinates used by de Sitter.

Writing

$$p = -\mathrm{T}_1{}^1, \qquad \tau = -\mathrm{T}_2{}^2 = -\mathrm{T}_3{}^3, \qquad \rho = \mathrm{T}_4{}^4, \qquad q = \mathrm{T}_{14}$$

the computation of the energy tensor for interval (1) gives

$$-\lambda + \kappa p = e^{-\lambda}\nu_1/r + (e^{-\lambda} - 1)/r^2 \quad . \quad . \quad . \quad . \quad (2)$$

$$\lambda + \kappa\rho = e^{-\lambda}\lambda_1/r - (e^{-\lambda} - 1)/r^2 \quad . \quad . \quad . \quad . \quad (3)$$

$$\kappa q = \lambda_4/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$-\lambda + \kappa\tau = e^{-\lambda}[\tfrac{1}{2}\nu_{11} - \tfrac{1}{4}\lambda_1\nu_1 + \tfrac{1}{4}\nu_1\nu_1 - (\lambda_1 - \nu_1)/2r] - e^{-\nu}[\tfrac{1}{2}\lambda_{44} - \tfrac{1}{4}\lambda_4\nu_4 + \tfrac{1}{4}\lambda_4\lambda_4] \quad . \quad . \quad . \quad (5)$$

The λ on the left-hand side is the cosmological constant and not the function λ. Indices 1 and 4 mean partial derivatives with respect to r and T.

These equations are very convenient for studying the transition between static solutions. If we allow parameters in these solutions to vary with time, the energy tensor has the same value as that computed with parameters constant, with the addition of the impulse term

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q and the second line in the expression of τ. These two additional terms form a transversal elastic wave making the transfer of energy supposed by the variation of the parameters. When the function λ does not vary, this additional wave vanishes. For instance, in Schwarzschild's interior solution

$$ds^2 = -R^2(d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi \sin^2\theta d\phi^2) + (A - B\cos\chi)^2 dt^2$$

we can let A and B vary with time, the value of $p = \tau$ remains the same as for A and B constant. Note that Einstein and de Sitter universes are special cases of Schwarzschild's interior solution (B = 0, A = 0), but with different values of the cosmological constant.

For a vacuum, we have $\rho = 0$, thus from (3)

$$e^{-\lambda} = 1 - \frac{2m}{r} - \frac{\lambda r^2}{3},$$

where m may be a function of T. But q vanishes and therefore, from (4), m is simply a constant.

Again, by (2) and (3)

$$\kappa(p + \rho) = \frac{e^{-\lambda}}{r}(\lambda_1 + \nu_1);$$

thus

$$\frac{\partial\lambda}{\partial r} + \frac{\partial\nu}{\partial r} = 0$$

and

$$\nu = -\lambda + f(T);$$

the function $f(T)$ may be absorbed in dT by using $\int e^{\frac{1}{2}f(T)}dT$ as a new time. Therefore

$$ds^2 = \frac{-dr^2}{1 - \frac{2m}{r} - \frac{\lambda r^2}{3}} - r^2(d\theta^2 + \sin^2\theta d\phi^2) + \left(1 - \frac{2m}{r} - \frac{\lambda r^2}{3}\right)dT^2 \quad . \quad (6)$$

is the solution in vacuum even when the field is not supposed static. This is Birkhoff's theorem.

The second derivative $\lambda_{11} \equiv \frac{\partial^2\lambda}{\partial r^2}$ does not appear in the equations. We can therefore accept discontinuities of λ_1 and we obtain

$$e^{-\lambda} = 1 - \frac{2m}{r} - \frac{\lambda r^2}{3} - \frac{\kappa}{r}\int \rho r^2 dr \quad . \quad . \quad . \quad (7)$$

where there is no inconvenience in supposing that ρ has points of discontinuity. Therefore the Schwarzschild interior and exterior field may be dealt with as a unique problem with a discontinuity of ρ at the boundary of the fluid sphere.

On the other hand, $\nu_{11} \equiv \frac{\partial^2\nu}{\partial r^2}$ appears in the equations so that ν_1 and p must be continuous. This is the mathematical justification of the physically evident condition $p = 0$ at the boundary of the material

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sphere. Otherwise, it would be necessary to suppose infinite values of τ at the boundary, which would be equivalent to supposing a rigid shell, a kind of vessel enclosing the fluid.

Therefore discontinuities of ρ in infinitely small regions do not modify the field, but discontinuities of p cannot be admitted. Also discontinuities of τ are of no consequence.

When $p + \rho = 0$ and $q = 0$, we have an immediate extension of Birkhoff's theorem, the field being defined by

$$e^{-\lambda} = e^{\nu} = 1 - \frac{2m}{r} - \frac{\lambda r^2}{3} + \frac{\kappa}{r}\int pr^2 dr \; ; \qquad (8)$$

but, when p does vary with time, m is a function of T such that λ_4 remains equal to zero.

Note that, for a fluid $p = \tau$, $p + \rho = 0$, $q = 0$ can be put in tensorial form as

$$T_{\mu}{}^{\nu} = - g_{\mu}{}^{\nu} p \qquad (9)$$

where p is an invariant.

3. *Expansion of the Universe deduced from the Motion of a Material Particle at the Neutral Zone between the Condensations.*

In order to study the motion of the neutral zone between a particular condensation and the rest of the universe we can consider a material particle placed at this neutral zone. It is clear that a material particle can stand by itself at the neutral zone and that therefore its motion is a geodesic. It is necessary to make definite assumptions as to the value of the material tensor in the neighbourhood of the particle. We have seen that discontinuities of ρ and τ can be admitted in an infinitely small volume without modification of the field; we can therefore choose in the neighbourhood of the particle an energy tensor according to relation (9).

With this assumption, the calculation is extremely simple. Readers who would find this assumption too artificial, as it involves negative values of ρ, will find in the next section a solution of the problem, directly admissible from the physical point of view.

As λ and ν do not depend on T in the neighbourhood of the particle, the equations of a geodesic are the same as for a statical field, and we have

$$\frac{dT}{ds} = ce^{\lambda},$$

where c is a constant of integration. e^{λ} is given by (8), and using this value in (1), we have for a radial geodesic

$$\left(\frac{dr}{ds}\right)^2 = - e^{-\lambda} + c^2,$$

or

$$\left(\frac{dr}{ds}\right)^2 = - 1 + c^2 + \frac{\lambda}{3}r^2 + \frac{2m}{r} - \frac{\kappa}{r}\int pr^2 dr \qquad (10)$$

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For $p = 0$, m does not vary with time, and writing

$$r = \sqrt{1 - c^2}R \qquad . \quad . \quad . \quad . \qquad (11)$$

$$2m = a\left(\frac{r}{R}\right)^3,$$

we have

$$\left(\frac{dR}{ds}\right)^2 = -1 + \frac{\lambda}{3}R^2 + \frac{a}{R} \qquad . \quad . \quad . \qquad (12)$$

which is, for $ds = dt$, the equation of a constant mass universe of vanishing pressure.

When p depends on T, m is a function of T, but λ depends only on r. We have therefore by derivation and suppression of the factor dr/ds

$$2\frac{d^2r}{ds^2} = \frac{2\lambda r}{3} - \frac{2m}{r^2} + \frac{\kappa}{r^2}\int pr^2dr - \frac{\kappa}{r}pr^2,$$

and by elimination of m and the integral

$$2r\frac{d^2r}{ds^2} + \left(\frac{dr}{ds}\right)^2 = -1 + c^2 + \lambda r^2 - \kappa pr^2 \qquad . \quad . \qquad (13)$$

or from (11),

$$\frac{2}{R}\frac{d^2R}{ds^2} + \frac{1}{R^2}\left(\frac{dR}{ds}\right)^2 + \frac{1}{R^2} = \lambda - \kappa p \qquad . \quad . \qquad (14)$$

which is the correct expression of p in a homogeneous universe of variable radius.

This result is completely independent of the degree or the variation of the condensations; it depends only on the radial pressure p at the neutral zone.

4. *Expansion of the Universe deduced from the Motion of a Material Shell at the Neutral Zone.*

The foregoing investigation, although mathematically correct, may be thought rather artificial. I shall therefore investigate the motion of the neutral zone by supposing that it is materialised by a thin spherical shell of negligible mass and no interior stress. This shell separates altogether the matter inside the condensation from the matter outside. Matter crossing the shell from outside or from inside is forced to rebound on its boundary.

In order to put the problem in mathematical form, we have to find tensorial expressions of the properties of the shell. We define the motion of the shell, by a four-vector u^μ of unit length

$$g_{\mu\nu}u^\mu u^\nu = 1 \qquad . \quad . \quad . \quad . \qquad (15)$$

and the normal to the shell (in four dimensions) by a co-variant a^μ such that

$$a_\mu u^\mu = 0, \qquad g^{\mu\nu}a_\mu a_\nu = 1 \qquad . \quad . \qquad (16)$$

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Using a local Galilean system of co-ordinates, we have for a particular point of the shell $\alpha_1 = 1$, $u^4 = 1$, the other components vanishing. Then we easily find expressions of the properties of the shell at this point and for this particular system of co-ordinates, and we transpose them in tensorial form by introducing α_1 and u^μ.

The shell is opaque, thus in the local system $T_4{}^1 = 0$ and in tensorial form

$$T_\mu{}^\nu \alpha_\nu u^\mu = 0 \qquad (17)$$

It is at rest in the local system, $\dfrac{d^2x_1}{ds^2} = 0$ or $\dfrac{du^1}{ds} = 0$, therefore in tensorial form

$$\alpha_\sigma\left[\frac{du^\sigma}{ds} + \Gamma_{\mu\nu}{}^\sigma u^\mu u^\nu\right] = 0 \qquad (18)$$

The normal pressure P is the same on both faces in the local system, and has in tensorial form the value

$$P = T^{\mu\nu}\alpha_\mu\alpha_\nu \qquad (19)$$

The problem is first to eliminate α_μ, then to replace the Christoffel symbols and the material tensor by their expressions in terms of λ and ν, finally to eliminate λ and ν.

In spherical co-ordinates, we have

$$u^1 = \frac{dr}{ds} = u, \qquad u^4 = \frac{dT}{ds} = v \qquad (20)$$

the other component vanishing as the movement is clearly radial. Then (16) gives

$$\frac{\alpha_1}{v} = \frac{\alpha_4}{-u} = e^{\frac{\lambda+\nu}{2}}, \qquad \alpha_2 = \alpha_3 = 0 \qquad (21)$$

The equilibrium equation (18) becomes

$$v\left[2\frac{du}{ds} + \lambda_1 u^2 + e^{\nu-\lambda}\nu_1 v^2 + 2\lambda_4 uv\right] = u\left[2\frac{dv}{ds} + \nu_4 v^2 + e^{\lambda-\nu}\lambda_4 u^2 + 2\nu_1 uv\right] \quad (22)$$

and (15)

$$-e^\lambda u^2 + e^\nu v^2 = 1 \qquad (23)$$

Differentiating (23) and eliminating dv/ds with (22), we find

$$2\frac{du}{ds} + \lambda_1 u^2 + 2\lambda_4 uv + e^{\nu-\lambda}\nu_1 v^2 = 0 \qquad (24)$$

The opacity equation (17) is

$$(p+\rho)uv + T_{14}[e^{-\nu}u^2 + e^{-\lambda}v^2] = 0,$$

or from the values (2), (3), (4) of the material tensor

$$e^{-\lambda}(\lambda_1 + \nu_1)uv + \lambda_4 e^{-\lambda-\nu}[1 + 2e^\lambda u^2] = 0.$$

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Elimination of ν with (24) gives

$$2u\frac{du}{ds} + e^{-\lambda}(\lambda_1 u + \lambda_4 v) = 0.$$

Now

$$\lambda_1 u + \lambda_4 v = \frac{\partial\lambda}{\partial r}\frac{dr}{ds} + \frac{\partial\lambda}{\partial T}\frac{dT}{ds} = \frac{d\lambda}{ds};$$

therefore, c being a constant of integration,

$$e^{-\lambda} = c - u^2 \quad . \quad . \quad . \quad . \quad (25)$$

The expression of the proper pressure P is, from (19),

$$P = p + e^{\lambda}(p+\rho)u^2 + 2T_{14}uv,$$

or

$$kP - \lambda = \frac{e^{-\lambda} - 1}{r^2} + e^{-\lambda}\frac{\nu_1}{r} + \frac{\lambda_1 + \nu_1}{r}u^2 + \frac{2\lambda_4}{r}uv \quad . \quad (26)$$

Eliminating ν from (24) and λ from (25) we have

$$kP - \lambda = \frac{c - 1 - u^2}{r^2} - \frac{2}{r}\frac{du}{ds},$$

or writing

$$r = \sqrt{1 - c}R$$

we have finally

$$\frac{2}{R}\frac{d^2R}{ds^2} + \frac{1}{R^2}\left(\frac{dR}{ds}\right)^2 + \frac{1}{R^2} = \lambda - \kappa P \quad . \quad . \quad (27)$$

which is equation (14) written with P in place of p.

It is clear that the expansion of the neutral zone depends only on P and not on any process of condensation which is altogether eliminated from the final result.

This method provides a very intuitional way of considering the equations of the universe, as the radius R of the universe is proportional to the radius of the neutral zone surrounding a condensation.

5. *Homogeneous Universe with Variable Mass.*

In the foregoing sections we have seen that a universe with condensations depends on equations essentially the same as those of a homogeneous universe, the only modification being that the density of energy ρ and the density of matter δ have no more direct interpretation, although p refers to a normal direction at the neutral zone between condensations. With this interpretation, we can deal with the universe as if it was strictly homogeneous.

For comparison with observation, it is convenient, following de Sitter, to measure the cosmical receding velocity by the de Sitter radius R_S according to

$$\frac{R'}{R} = \frac{v}{rc} = \frac{1}{R_S} \quad . \quad . \quad . \quad . \quad (28)$$

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and the proper density δ by the Einstein radius R_E, by

$$\kappa\delta = \frac{2}{R_E^2} \quad . \quad . \quad . \quad . \quad (29)$$

Using as a unit of distance $A = 10^{24}$ cm. $\simeq 10^6$ light years we have $R_S = 2000$ A and $R_E = 2000$ to $80,000$ A.

We shall also represent the cosmological constant by a radius R_C, the cosmological radius, defined by

$$\lambda = \frac{3}{R_C^2} \quad . \quad . \quad . \quad . \quad (30)$$

The equation of the field in a homogeneous universe of variable radius may be put in a form very convenient for discussion by writing

$$x = \frac{R^2}{R_C^2}, \quad \tau = \frac{t}{R_C}, \quad y = \frac{\kappa\delta R^2}{3} = \frac{\kappa}{3\pi^2}\frac{M}{R} = \frac{2}{3}\frac{R^2}{R_E^2} \quad (31)$$

We find easily
$$\frac{d^2x}{d\tau^2} - 4x = 4y - 2 \quad . \quad . \quad . \quad . \quad (32)$$

and p may be computed by

$$4\kappa p R^2 = \left(\frac{dx}{d\tau}\right)^2 - (2x - 1)^2 + 1 - 4y \quad . \quad . \quad (33)$$

Let us consider first the particular case where $y = 0$—*i.e.* a universe empty of matter, but which may be full of radiation. We find

$$x = \tfrac{1}{2} + c_1 e^{2\tau} + c_2 e^{-2\tau},$$

which gives, for $c_1 = c_2 = \frac{1}{4}$ (de Sitter universe),

$$x = \cosh^2 \tau.^*$$

In general we have

$$x = \tfrac{1}{2} + e^{2\tau}\left(c_1 + \int_0^\tau e^{-2\tau} y d\tau\right) + e^{-2\tau}\left(c_2 - \int_0^\tau e^{2\tau} y d\tau\right) \quad . \quad (34)$$

The equilibrium solution is found for y to equal a constant y_0.

$$x = \tfrac{1}{2} - y_0.$$

If y constant for $\tau < 0$ begins to vary from $\tau = 0$ we have

$$x = \tfrac{1}{2} - y_0 + e^{2\tau}\int_0^\tau e^{-2\tau}(y - y_0)d\tau - e^{-2\tau}\int_0^\tau e^{2\tau}(y - y_0)d\tau \quad . \quad (35)$$

or for τ small

$$x = \tfrac{1}{2} - y_0 + \tfrac{1}{2}y_0'(\sinh 2\tau - 2\tau) + \tfrac{1}{4}y_0''\left(\cosh 2\tau - 1 - \frac{(2\tau)^2}{2\,!}\right) + \ldots$$
$$+ \frac{1}{2^{2k-1}} y_0^{(2k-1)}\left(\sinh 2\tau - 2\tau - \ldots - \frac{(2\tau)^{2k-1}}{(2k-1)\,!}\right)$$
$$+ \frac{1}{2^{2k}} y_0^{2k}\left(\cosh 2\tau - 1 - \ldots - \frac{(2\tau)^{2k}}{(2k)\,!}\right) + \ldots \quad . \quad (36)$$

* Prof. de Sitter found also the case $c_2 = 0$, *P.N.A.S.*, **16**, 485, 1930.

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Therefore, if the Taylor development of y begins by

$$y = y_0 + \frac{1}{k!} y_0^{(k)} \tau^k + \ldots,$$

we have for $x \sim R^2$

$$x = x_0 + \frac{4y_0^{(k)}}{(k+2)!} \tau^{k+2} + \ldots,$$

and for $y\sqrt{x} \sim M$

$$y\sqrt{x} = y_0\sqrt{x_0} + \frac{1}{k!}\sqrt{x_0}\, y_0^{(k)} \tau^k + \ldots$$

If, in a universe in equilibrium, the proper mass begins to vary, the radius of the universe varies in the same sense.

From (33) we can also say: If, in a universe in equilibrium, the pressure begins to vary, the radius of the universe varies in the opposite sense.

Therefore stagnation processes induce expansion.

The above formulæ provide a method for studying in detail the expansion of the universe for any given law of y.

In the next section we shall work out the special case where the diminution of pressure responsible for the expansion arises suddenly.

6. *Expansion induced by a Sudden Stagnation.*

For the equilibrium before the stagnation arose, we have by (31), (32), (33),

$$\frac{6p_0}{\delta_0} = \frac{3x - 1}{1 - 2x} \quad . \quad . \quad . \quad . \quad (37)$$

with

$$x = \frac{R_1^2}{R_C^2} \quad . \quad . \quad . \quad . \quad . \quad (38)$$

R_1 is the equilibrium radius, R_C the cosmological radius. For $x = \frac{1}{3}$ the pressure vanishes; for $x = \frac{1}{2}$ the density vanishes.

Let us suppose that for $t = 0$, the pressure p_0 suddenly drops to zero. From $t = 0$ the solution is a zero pressure, and thus a constant mass solution, from (12) and (30)

$$t = R_C \int_{R_1}^{R} dR \sqrt{\frac{R}{R^3 - R_C^2 R + \frac{\alpha}{3} R_C^2}} \quad . \quad . \quad (39)$$

where the constant α is related to the actual radius R of the universe by

$$\alpha = \kappa \delta R^3 = \frac{2R^2}{R_E^2};$$

for $t = 0$, $R = R_1$, $R' = 0$, and therefore R_1 is a root of the cubic

$$R_1^3 - R_C^2 R_1 + \frac{2}{3}\frac{R_C^2}{R_E^2} R^3 = 0,$$

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and we have
$$\frac{R^3}{R_C R_E^2} = \frac{3}{2}(1 - x)\sqrt{x} = \gamma^3 \quad . \quad . \quad . \quad (40)$$
for $x = \frac{1}{3}$, γ^3 has a maximum $\frac{1}{\sqrt{3}} = 0 \cdot 58$ and drops to $\frac{3}{4\sqrt{2}} = 0 \cdot 53$ for the extreme case $x = \frac{1}{2}$.

The value of the cosmological radius R_C may be computed by
$$u = \left(\frac{R_E}{R_C}\right)^{\frac{2}{3}} \quad . \quad . \quad . \quad . \quad (41)$$
and
$$u^3 - \frac{1}{\gamma^2}u + \frac{2}{3} - \left(\frac{R_E}{R_S}\right)^2 = 0 \quad . \quad . \quad . \quad (42)$$

Some values are given in the following table (for $x = \frac{1}{3}$) :—

u.	1.	2.	3.	4.	∞.
R_E/R_S	0·47	2·41	4·83	7·67	∞
R_c/R_S	0·47	0·85	0·93	0·96	1

It remains to compute t by (39). This can be written
$$t = R_C\int_{R_1}^{R}\frac{dR}{\sqrt{(R - R_1)(R - R_2)}}\sqrt{\frac{R}{R + R_1 + R_2}} \quad . \quad (43)$$
or writing
$$R = \frac{R_1 - R_2}{2}\cosh\phi + \frac{R_1 + R_2}{2}$$
we find
$$t = R_C\int_0^{\phi}d\phi\sqrt{\frac{R}{R + R_1 + R_2}};$$
thus
$$\phi = \frac{\theta t}{R_C}$$
where θ is a function of ϕ, the value of which lies between $\sqrt{3}$ and 1 (because $R > R_1 > R_2$).

The value of R_2 is easily computed in term of R_C and x, we find
$$\frac{4R}{R_C} = [3\sqrt{x} - \sqrt{4 - 3x}]\cosh\frac{\theta t}{R_C} + \sqrt{x} + \sqrt{4 - 3x} \quad . \quad (44)$$
or for $x \sim \frac{1}{3}$ using (37) ; the approximative formula for p_0 small
$$\frac{4R}{R_C} = \frac{2}{3}\frac{p_0}{\delta_0}\cosh\frac{\theta t}{R_C} + 1 + \frac{1}{\sqrt{3}} \quad . \quad . \quad . \quad (45)$$
or for t large
$$t = \frac{R_C}{\theta} \times 2 \cdot 303 \log_{10}\left[3\frac{\delta_0}{p_0}\left(4\frac{R}{R_C} - 1 - \frac{1}{\sqrt{3}}\right)\right] \quad . \quad . \quad (46)$$
The expression under the logarithmic sign is always positive as the minimum of $\frac{4R}{R_C}$ is $\frac{4}{\sqrt{3}}$.

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The values of R_C and R are practically independent of the importance of the initial stagnation. It is very curious to see how its influence on t is slow. Computing t for $\theta = 1$ and $R/R_C = 20$, we find

for
$$p_0/\delta_0 = \infty, \quad t = 5R_C \cong 10^{10} \text{ years},$$
$$p_0/\delta_0 = 10^{-8}, \quad t = 24R_C \cong 5 \times 10^{10} \text{ years},$$
$$p_0/\delta_0 = 10^{-14}, \quad t = 38R_C \cong 7\cdot5 \times 10^{10} \text{ years}.$$

The first case means that all the energy was in the form of electromagnetic radiation and suddenly condensed in matter; the second case supposes a radial free energy of 30 km./sec. suddenly bounded in condensations; the third case that one millionth of the mass was moving freely with this velocity and was captured by the rest of the matter.

Nevertheless, if p_0 tends to zero, t tends to infinity, the limiting case being the solution emphasised in our 1927 paper. As was pointed out by Eddington, such logarithmic infinities have no real physical significance.

Louvain, le 7 mars 1931.

Some Further Remarks on the Rotation of the Stars.

By E. A. Kreiken.

1. In a recent note * evidence was given that there exists a close correlation between the diameter of a star and its period of axial rotation. From the stars in Elvey's list † the following relation was obtained: ‡

$$\log P = 1\cdot49, \log D - 1\cdot25 \quad . \quad . \quad . \quad (1)$$

where P represents the period of axial rotation and D the hypothetical diameter of a star at the distance $\pi = 0''\cdot100$. The values of log P as derived from (1) still require a constant correction, which is due to the fact that only a fraction of the rotational velocity is observed. In the following we shall see how the numerical value of this constant may be determined. In the paper just mentioned we tentatively derived from (1) the values of log P for some spectroscopic binaries, some Cepheid variables, and some long-period M-type variables. The periods thus obtained were compared with the periods of revolution and the periods of light variation. The result was that with the spectroscopic binaries a strong correlation between the periods of axial rotation and of revolution was found. With the Cepheid and

* E. A. Kreiken, *M.N.*, **91**, 251 (1930 December).
† Elvey, *Astr. Journ.*, **71**, 221, 1930.
‡ A. Pannekoek drew my attention to the fact that the coefficient of log D is almost exactly what would be expected from Kepler's third law. This question will be considered in a future paper.

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