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Inflation, large branes, and the shape of space

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Abstract

Linde has recently argued that compact flat or negatively curved spatial sections should, in many circumstances, be considered typical in Inflationary cosmologies. We suggest that the "large brane instability" of Seiberg and Witten eliminates the negative candidates in the context of string theory. That leaves the flat, compact, three-dimensional manifolds—Conway's *platycosms*. We show that deep theorems of Schoen, Yau, Gromov and Lawson imply that, even in this case, Seiberg–Witten instability can be avoided only with difficulty. Using a specific cosmological model of the Maldacena–Maoz type, we explain how to do this, and we also show how the list of platycosmic candidates can be reduced to three. This leads to an extension of the basic idea: the conformal compactification of the entire Euclidean spacetime also has the topology of a flat, compact, four-dimensional space.

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1. *Nearly* flat or *really* flat?

Linde has recently argued [1] that, at least in some circumstances, we should regard cosmological models with flat or negatively curved *compact* spatial sections as the norm from an Inflationary point of view. Here we wish to argue that cosmic holography, in the novel form proposed by Maldacena and Maoz [2], gives a deep new interpretation of this

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idea, and also sharpens it very considerably to exclude the negative case. This focuses our attention on cosmological models with *flat, compact* spatial sections.

Current observations [3] show that the spatial sections of our Universe (as defined by observers for whom local isotropy obtains) are fairly close to being flat: the total density parameter Ω satisfies $\Omega = 1.02 \pm 0.02$ at 95% confidence level, if we allow the imposition of a reasonable prior [4] on the Hubble parameter. (See, however, [5] for a cautionary note.) The present era of "precision cosmology" [6] is based on the assumption that the true value of Ω is even closer to unity than the observations demand—see, for example, [7]. Applications of precision cosmology depend on this "almost exactly flat" assumption in a crucial way: for example, Wang and Tegmark [8] stress that without this assumption essentially nothing can be said about the evolution of the dark energy density. Turning to the theoretical situation, we find that the leading theory, Inflation [9,10], also demands values of Ω which are extremely close, though not exactly equal, to unity. Most versions require unity *plus or minus* some small number (typically [9] about 10^{-4}).

Of course, Inflation itself explains why the Universe currently *appears* to be flat: any local evidence of curvature is "inflated away". But here we wish to propose that this process merely restores the local spatial geometry to its initial and most natural global state, namely that of a *perfectly flat, compact three-dimensional manifold*. That is, we suggest that the fundamental value of Ω is *exactly*, not nearly, unity; this is proposed as an exact initial condition for stringy cosmology.

The reader is entitled to ask whether the distinction between approximate and exact initial flatness really has any content. For it is clear that ordinary, flat $\mathbb{R}^3$ can be given a constant negative curvature of any magnitude, however small, since hyperbolic space H^3 has this same $\mathbb{R}^3$ topology. Similarly, $\mathbb{R}^3$ can be consistently deformed so that it has positive Ricci curvature everywhere.[1] Thus flat $\mathbb{R}^3$ can be deformed in a way which leads to either positive or negative Ricci curvature, of any magnitude, at every point, and so it is hard to see how there can be a difference between extremely small curvature and exactly zero curvature.

This, however, is where the assumption of *compactness* is crucial. For the topology of an exactly flat compact manifold is radically different from that of either a positively or a negatively curved space, whether compact or not. A consequence of this is that it is impossible to deform a compact flat manifold in such a way that its sectional curvature is everywhere negative; on the other hand, it is also impossible to deform it so that even the *scalar* curvature becomes positive everywhere. (See [12] and p. 306 of [13].) Of course, such a space can be locally deformed (by the presence of a galaxy, say) but not in a way leading consistently to curvature of a definite sign. If the Universe had spatial sections of this kind, and if the matter content were smoothed out, then the geometry would *have* to be exactly flat, as a result of these extremely deep geometric theorems. Thus, the hypothesis of exact underlying flatness does make sense if the spatial sections are compact.

[1] Examples of this can be constructed, but of course in this case the Ricci curvature cannot be *constant*, that is, the metric cannot be Einstein, if the metric is complete. In fact [11], $\mathbb{R}^3$ is the only non-compact three-dimensional manifold which can accept a complete metric of positive Ricci curvature.

The suggestion that the spatial topology of our Universe is not trivial is of course an old one [14]. Some current interest in this idea focuses on the relation to the AdS/CFT correspondence [15–19]. Inflation explains why we probably cannot see [20] direct evidence of such non-triviality: the fundamental domain is inflated to a size larger than the current cosmological horizon. Nevertheless, the idea that the spatial sections may be compact continues to attract attention, from many different points of view. In the specific case of *flat*, compact sections, discussions include simple models of components of dark energy [21,22] and string/brane gas cosmology [23–28]; in particular it is interesting that, whether or not string/brane gas cosmology succeeds in explaining the dimensionality of observed space, the Brandenberger–Vafa scenario, with its toral model of *all* spatial directions, is still widely regarded as a natural initial condition for string cosmology.

Most relevantly for our work here, it has long been known [29,30] that flat or negatively curved *compact* spatial sections arise very naturally in quantum cosmology. More recently, Linde [31] has emphasised that such constructions are also natural from the Inflationary point of view; and, more recently still, as we mentioned earlier, he has strengthened this to the claim that compact but *not positively curved* spatial sections should be considered to be typical in Inflationary quantum gravity rather than exotic [1]. Linde stresses that there is no conflict, in Inflationary theory, between the assumption of compactness and the Inflationary prediction that the effects of compactness should not be *directly* observable. In fact, the compactness of the spatial sections may play a vital role in ensuring sufficient initial homogeneity for Inflation to begin. In this connection, it has recently been argued [32] that Inflation *requires* us to take a global viewpoint and not to ignore the structure beyond the horizon.

It is the objective of this work to argue that the hypothesis of exact spatial flatness, but *not* negative curvature, is natural from the *holographic* point of view.

The form of cosmic holography in which we are interested here, due to Maldacena and Maoz [2] is one which adapts the basic ideas of the AdS/CFT correspondence to the cosmological case. As in AdS/CFT, the starting point is anti-de Sitter spacetime, but now transformed into a cosmological spacetime by the introduction of some kind of matter [33–36]. The resulting cosmology has both a Bang and a Crunch, but its *Euclidean* version is entirely non-singular and has a well-behaved conformal infinity, on which the dual field theory is to be defined. Each connected component of this conformal infinity has *precisely the same topology* as the spatial sections of the Lorentzian version of the spacetime.[2]

If there is a holographic AdS/CFT-style duality here, it follows that the cosmological model is controlled by a field theory which does not "care" how large the spatial sections may be at any particular time, such as the present. Whatever their size, the field theory is still sensitive to their structure, including their topological structure. (For concrete examples of the profound ways in which non-trivial topology can affect the behaviour of field theories, see [38,39].) In short, cosmic holography allows us to probe the global form of the

[2] This picture is actually consistent with the hypothesis that the spatial sections are *compact*, for in the generalized Euclidean AdS/CFT correspondence it is usually desirable for the CFT to be defined on a compact space; see Section 2.3 of [37].

spatial sections, whether or not[3] the fundamental domain is far larger than the current horizon: it is capable of this because an AdS/CFT type of duality is a correspondence between the *entire* bulk and its infinity.

As an application of these ideas, we shall try to constrain the structure of the spatial sections. We do this with the aid of the "large brane instability" discussed by Seiberg and Witten [43]. We shall see that holography rules out negative curvature for *compact* cosmological spatial sections, *no matter how small* the curvature may be in magnitude; in fact, it is possible to make this argument even if the well-known "BKL" behaviour (describing the growth of anisotropies during the approach to cosmological singularities) is taken into account. We shall also see that holography does allow flat, compact spatial sections, but only if specific conditions are satisfied close to the singularities.

If the spatial sections of our world are flat and compact, then it is potentially important to determine which of the ten possible topologies [44] has been selected—and why. We shall not completely succeed in fixing the topology, but the list of candidates will be greatly reduced, from ten to three. One of the three survivors is the Hantzsche–Wendt space [45] or "didicosm" (in the naming system due to Conway [46], which we adopt here), the most complex of the ten.

We begin with a very brief introduction to a class of cosmological models [33,35] which generalize those proposed by Maldacena and Maoz by allowing for a period of acceleration, in accord with current observations [47]. Throughout this discussion, we shall for simplicity ignore all forms of matter other than the quintessence field; this includes the inflaton, though we stress that ultimately (as explained in [1]) we rely on Inflation to ensure detailed agreement with current observations. We then explain how these models are compatible with cosmic holography, laying particular stress on the stringent conditions imposed by the Seiberg–Witten instability [43]. Next, we argue that a holographic one-to-one bulk/infinity correspondence can be maintained only by extending our basic hypothesis to the entire spacetime: that is, we propose that the compactification of the (Euclidean) version is globally conformal to a four-dimensional flat compact manifold. We will see that this imposes conditions which only a few candidate topologies (for the three-dimensional sections) are able to meet. Because we are concerned with the topology (and not with the precise geometry) of these spaces, it is reasonable to hope that our conclusions are valid even though our concrete cosmological model is too simple to be realistic.

Throughout this work we follow Maldacena and Maoz [2] in assuming that the background geometry, prior to the introduction of some kind of matter, is that of *anti*-de Sitter spacetime, AdS_4. (See [48–52] for relevant work on AdS-based cosmology.) Of course, many efforts have been made to develop cosmic holography on a de Sitter-like background; see [53] for a very clear analysis of the current state of such attempts. Constraints on cosmic topology can also be developed in that context: see [15–17].

[3] All current data are *compatible* with Inflationary expectations regarding spatial curvature and topology, but this is not to say that alternatives (see, for example, [40–42]) have been completely ruled out. For the sake of clarity, however, we shall assume here that they have been.

2. Flatness, acceleration, and Breitenlohner–Freedman

For our subsequent discussions it will be very helpful to have a concrete model of the various physical mechanisms to be considered. In this section, we introduce an extremely simple cosmological model which can play this role. No claim is made that this model itself is realistic, though possibly it could be made so by superimposing matter, radiation, inflaton and other fields on the simple spacetime to be defined below.

The basic cosmological model we shall consider is one with a Bang and a Crunch. There are in fact very general arguments [54] which suggest that the ultimate state of our Universe will be a Crunch of the kind that arises naturally when potentials are allowed to be negative [55]. If our Universe is *now* anti-de Sitter-like—something that is not excluded by observations, since such spacetimes can accelerate, though only temporarily [33]—then this is a straightforward consequence of having a negative cosmological constant in the background. But even if the present state of the Universe is de Sitter-like, this probably corresponds to a metastable state which eventually fluctuates or tunnels to an *anti*-de Sitter-like basin of attraction of some potential. (The alternative is decompactification, but this possibility only arises if one has some argument which rules out negative potentials.)

Maldacena and Maoz [2] analyse Bang/Crunch spacetimes with metrics of the form

$$g^{-}_{\mathrm{MM}} = -\left(\mathrm{d}t^{-}\right)^2 + a^{-}\left(t^{-}\right)^2 g^{+}(\Sigma), \tag{1}$$

where t^{-} is proper time, $a^{-}(t^{-})$ is the scale factor (which vanishes at both ends of some finite interval), and $g^{+}(\Sigma)$ is a metric on a *time-independent* three-dimensional Riemannian manifold Σ which acts as a model for the spatial sections. (Throughout this work, we use a '+' superscript to indicate a Euclidean coordinate or field, a negative sign for its Lorentzian counterpart.) Note that such metrics do not take into account the evolution of anisotropies, which we shall consider, in specific cases, in later sections of this work.

One way of obtaining spacetimes of this kind is to introduce matter into anti-de Sitter spacetime, allowing it to act on the geometry in accord with Einstein's equation. The result is typically a Bang/Crunch spacetime. Maldacena and Maoz observe that the Euclidean version will in general be non-singular and asymptotically hyperbolic (that is, asymptotically like Euclidean AdS). It will therefore have a well-defined conformal boundary. The hope is that, in some way that is not yet fully understood, the non-singular Euclidean boundary "replaces" the singularities of the Lorentzian version. A field theory on the boundary should give a holographic description of the bulk in the familiar way.[4] Notice that, by contrast, de Sitter spacetime does not have a holographic Euclidean version, since the usual Euclidean version of dS_4, the four-sphere, has no boundary. In this sense, Euclidean holography favours AdS_4 over dS_4 as the fundamental "background" for cosmology.

Maldacena and Maoz also observe that the Euclidean versions of their Bang/Crunch cosmologies are topologically non-trivial: they refer to such spaces as Euclidean "wormholes". For this reason they use particular matter configurations such as Yang–Mills merons and instantons to construct their cosmological models. Unfortunately, such matter cannot

[4] This is the sense in which we shall understand "holography". Note that other interpretations, involving entropy bounds, may not be consistent with Maldacena–Maoz cosmologies: see in particular [48].

lead to cosmic acceleration. On the other hand, ordinary scalar matter, in the guise of "quintessence", can easily lead to acceleration, but it cannot generate a topologically non-trivial Euclidean "spacetime" [56,57]. We are thus led, as in [58], to consider a *Euclidean axion* as the matter content of the Euclidean version of the spacetime; for axions appear to be unique in leading *both* to acceleration *and* to topological non-triviality. Other forms of matter and radiation, as well as the inflaton, have well-known effects on the expansion history, so for simplicity we shall not consider them here.

Motivated by the discussion of quintessence superpotentials in [59], in [35] we proposed to develop a Euclidean axion cosmology by postulating a superpotential. Since the potential should be periodic for an axion, the same applies to the superpotential; and since the axion field φ^+ is a pseudo-scalar, it is natural to restrict attention to superpotentials which are odd in φ^+. Thus we consider superpotentials of the form

$$W^+\bigl(\varphi^+\bigr) = \sum_{k=1}^{\infty} C_k \sin\left(k\sqrt{\frac{4\pi}{\varpi}}\,\varphi^+\right), \tag{2}$$

where ϖ is a positive constant. If we take only one term for simplicity, we can assume it to be the first; requiring the potential to yield the usual negative cosmological constant for pure Euclidean AdS_4, with all sectional curvatures equal to $-1/L^2$, when $(W^+)'$ vanishes, we can fix the constant C, and so we obtain

$$W^+\bigl(\varphi^+\bigr) = \frac{1}{16\pi L}\sin\left(\sqrt{\frac{4\pi}{\varpi}}\,\varphi^+\right). \tag{3}$$

Higher-order terms in the original expansion (2), which we shall consider later, are obtained by replacing ϖ by ϖ/k^2.

The potential corresponding to W^+ may be written as

$$V^+\bigl(\varphi^+\bigr) = -\frac{3}{8\pi L^2} + V^+_{\text{Axion}}, \tag{4}$$

where

$$V^+_{\text{Axion}} = \frac{3-\varpi^{-1}}{8\pi L^2}\cos^2\left(\sqrt{\frac{4\pi}{\varpi}}\,\varphi^+\right). \tag{5}$$

Thus we are effectively considering Euclidean AdS_4, with "energy" density $-3/(8\pi L^2)$, into which we have introduced a matter field with a potential V^+_{Axion}.

In accordance with our hypothesis that the spatial sections of our cosmological model are to be flat and compact, we recall [44] that every compact flat three-manifold can be expressed as T^3/F, where T^3 is the three-torus and F is a small finite group (which is in fact isomorphic to the holonomy group of the manifold). Locally, therefore, we can use the usual angular coordinates on a three-torus (taken to be cubic for convenience), and the Euclidean metric will have the general form

$$g^+ = \bigl(\mathrm{d}t^+\bigr)^2 + A^2a^+\bigl(t^+\bigr)^2\bigl[\mathrm{d}\theta_1^2 + \mathrm{d}\theta_2^2 + \mathrm{d}\theta_3^2\bigr], \tag{6}$$

where A measures the circumferences of the torus when $a^+(t^+)$, the Euclidean scale factor (which we can abbreviate to a^+), is equal to unity.

The solution for a^+ in the present case, obtained by solving [33] the Einstein equations with the potential (5) (and a canonical kinetic term) superimposed on Euclidean AdS_4, yields the metric

$$g^+(\varpi, A) = \left(dt^+\right)^2 + A^2 \cosh^{2\varpi}\left(\frac{t^+}{\varpi L}\right)\left[d\theta_1^2 + d\theta_2^2 + d\theta_3^2\right]. \tag{7}$$

This is entirely non-singular. If we embed this space as the interior of a manifold-with-boundary, then the boundary has two connected components at $t^+ = \pm\infty$. However, it was argued in [35] that holography dictates that these two components should be topologically identified, and that is what we propose to investigate below.

The Lorentzian version of all this is significantly different: we now have a Bang/Crunch spacetime with metric

$$g^-(\varpi, A) = -\left(dt^-\right)^2 + A^2 \cos^{2\varpi}\left(\frac{t^-}{\varpi L}\right)\left[d\theta_1^2 + d\theta_2^2 + d\theta_3^2\right]. \tag{8}$$

Contrary to what is often said, such cosmologies can be perfectly compatible with current observations, a point stressed recently by Wang et al. [60]. Notice that this metric allows for a period of accelerated expansion provided that ϖ is not too small; in fact there is such an interval if $\varpi > 1$. The Lorentzian version of φ^+, denoted φ^-, is a quintessence field [61–63] with an exponential-like potential given by

$$V^-_{\text{Quintessence}} = \frac{3 - \varpi^{-1}}{8\pi L^2} \cosh^2\left(\sqrt{\frac{4\pi}{\varpi}}\,\varphi^-\right); \tag{9}$$

this is superimposed on an AdS_4 geometry with cosmological constant $-3/L^2$. The energy density of φ^- can be computed in terms of the Lorentzian scale function $a^-(t^-)$ (which we abbreviate to a^-); the result [33] is

$$\rho\left(\varphi^-\right) = \frac{3}{8\pi L^2}\left(a^-\right)^{-2/\varpi}. \tag{10}$$

The total energy density is the sum of this and the energy density of the background AdS_4. If we had taken the k-th order term in (2) instead of the first, then the density of φ^- would vary as $(a^-)^{-2k^2/\varpi}$; so the $k = 1$ term dominates when the Universe is large, while the higher order terms in the Fourier expansion are important very near to the Bang and the Crunch.

Clearly the Lorentzian metric $g^-(\varpi, A)$ given by (8) is not asymptotically AdS. Nevertheless its Euclidean version, given by (7), *is* asymptotically hyperbolic, that is, asymptotically similar to Euclidean AdS. Since the Maldacena–Maoz formulation of cosmic holography is based on an interplay between the Euclidean and Lorentzian versions, any constraint on the parameters which we can derive from this fact must be accepted as physically relevant. A fundamental example of such a constraint is the Breitenlohner–Freedman bound [64], which, as explained in [37], is also valid in the Euclidean case. (Notice that the Euclidean space is compactified only in some directions: its volume is infinite towards either component of the boundary. The discussion in [37] applies here.) The BF bound imposes a very interesting condition on ϖ, as we now explain.

The field φ^+ does not decay to zero towards either $t^+ = \infty$ or $-\infty$, but rather to $\pm\frac{\pi}{2}\sqrt{\varpi/4\pi}$, respectively: this can be seen from the explicit solution for it,

$$\varphi^+ = \pm\sqrt{\frac{\varpi}{4\pi}}\cos^{-1}\left[\operatorname{sech}\left(\frac{t^+}{\varpi L}\right)\right]. \tag{11}$$

This behaviour is necessary in order to ensure that the total energy density should tend to the AdS_4 value $-3/8\pi L^2$ near infinity (see Eq. (5)). Concentrating on the $t \to +\infty$ end of the manifold, we therefore find it convenient to define a new field ψ^+ by

$$\psi^+ = \frac{\pi}{2}\sqrt{\varpi/4\pi} - \varphi^+. \tag{12}$$

Substituting this into Eq. (5) we see that the mass of ψ^+ is given by

$$m^2 = \frac{3 - \varpi^{-1}}{\varpi L^2}. \tag{13}$$

In general, one can expect an AdS/CFT-style correspondence to break down [65] if the Breitenlohner–Freedman bound fails; since we are of course ultimately interested in establishing a correspondence of this kind for cosmology, we must ensure that the BF bound is satisfied here. In four dimensions this bound is $m^2 \geqslant (3/4)\Lambda$, where Λ is the negative cosmological constant of an AdS background. That is, m^2 can be negative without causing any instability, as long as it is not too negative. Here this bound becomes

$$\frac{3 - \varpi^{-1}}{\varpi L^2} \geqslant \frac{-9}{4L^2}, \tag{14}$$

whence we have for positive ϖ

$$\varpi \geqslant \frac{2}{1+\sqrt{2}}\frac{1}{3}. \tag{15}$$

Thus the parameter ϖ is allowed to go below the value $1/3$, which means that V^+_{Axion} (Eq. (5)) is allowed to be negative. However, the lowest value of ϖ allowed by (15) is not very far below $1/3$; it is in fact equal to about 82.8% of $1/3$.

In fact, cosmological data [35] require the basic value of ϖ to be quite large; in particular, there is a period of cosmic acceleration, as observed, if and only if ϖ is greater than unity. However, our discussion here is based on the assumption that we take only the *first* term in the expansion (2). If we drop this assumption, then (15) can be interpreted as requiring us to truncate the expansion in such a way that if k labels the final term, then not just ϖ but also ϖ/k^2 satisfies the inequality. In view of the discussion around Eq. (10), this last term will be the dominant one near to the Bang and the Crunch in the Lorentzian version of the spacetime.

Combining all these results, we conclude that our Euclidean axion is governed by a superpotential given by a finite sum in Eq. (2). We can ignore all terms in the sum apart from the first (which dominates when the Universe is large) and the last (which dominates near to the Bang and the Crunch). The quintessence density will grow very rapidly near to the Bang/Crunch: it can in fact grow (as a^- tends to zero) more rapidly than $(a^-)^{-6}$. However, evaluating the right side of (15), we find that the maximum rate at which the

density can grow is as (approximately) $(a^-)^{-7.2426}$. This "window" between the exponent six and a value just over 7.2 will be considered in detail below.

Having introduced a concrete example of a holographic cosmology, we can turn to the question of how holography influences the structure of the spatial sections of spacetimes of this general kind.

3. Flatness, holography and Seiberg–Witten instability

Linde [1] argues that compact spatial sections are favoured by Inflationary theory. There are in fact several strong advantages in compact sections: for example, because compact sections are (under some circumstances) circumnavigable, it is easy and natural in such cosmologies to arrange for sufficient homogenization for Inflation to begin. On the other hand, *positive* curvature is generically disfavoured in quantum-gravitational studies of initial conditions for Inflation [30]. Thus Inflationary quantum gravity firmly directs our attention towards either flat or negatively curved compact spatial sections. There is, of course, an enormous number of such manifolds, but we shall now see that this number is drastically reduced when we study Bang/Crunch cosmologies from the AdS/CFT point of view.

In [43], Seiberg and Witten have studied the extension of the AdS/CFT correspondence to general geometries of the AdS type: that is, they considered the consequences of doing string theory on non-compact Euclidean spaces, with negative Ricci curvature, admitting a conformal compactification in the sense of Penrose. One of their more remarkable findings was that BPS branes "near" to the conformal boundary ("large branes") will give rise to an instability if the conformal structure at infinity is represented by a metric of negative *scalar* curvature. (When discussing compact conformal manifolds, we can, without loss of generality, assume that the scalar curvature is a constant of arbitrary magnitude but of a fixed sign [66].) The unexpected role of the scalar curvature is a strong hint that this instability is "holographic", for one knows that the scalar curvature is an essential component of the conformally invariant Laplace operator,

$$\Delta_{\text{CONFORMAL}} = \Delta + \frac{n-2}{4(n-1)}R, \tag{16}$$

defined by the conformal structure at (n-dimensional) infinity. (It is important to note that everything we say here is based on the assumption that n is *greater than 2*. The case of two-dimensional boundaries is special and will not be considered here.) Indeed, Seiberg and Witten were able to show that negative scalar curvature does induce the instability in the field theory at infinity that holography demands given the large brane instability in the bulk. Seiberg–Witten instability has been subjected to a deep study recently in [67] and [68]; it represents a fundamental constraint on possible boundary geometries and topologies in any generalized version of the AdS/CFT correspondence. For it is clear that it would not be consistent to ignore the effects of such unstable processes on the underlying geometry, and these effects could be drastic.

This comment applies with particular force in the context of Maldacena–Maoz holography [2]. For here the idea is that, however, singular the Lorentzian cosmology may be, its

Euclidean version should be sufficiently well-behaved that there are asymptotically AdS regions which are not, for example, cut off by some kind of disturbance resulting from the unrestrained growth of large branes in those regions. Thus Seiberg–Witten instability *must* be avoided in cosmic holography.

The relevance of all this arises from the following simple observation: the spatial sections of the particular spacetimes considered in the previous section, and by Maldacena and Maoz, have the *same* conformal geometry as the space on which the dual theory is defined; for example, it is clear that if the manifold with metric given by Eq. (7) is embedded as the interior of a manifold-with-boundary, then each component of the boundary has the structure of the flat space T^3/F, with its "flat" conformal structure. An analogous statement would hold if we considered a similar spacetime but with negatively curved spatial sections. If the Maldacena–Maoz cosmologies are a correct implementation of string theory in cosmology, it therefore follows that string theory *predicts that the spatial sections of our Universe cannot be negatively curved*; indeed, they cannot even have negative scalar curvature. However, this argument ignores perturbations. We will deal with these after introducing some mathematical machinery.

The first result we need is the *Kazdan–Warner classification* [69]—see [70] for a recent discussion—of all compact manifolds of dimension at least three. This is concerned with the following question: given such a manifold and any smooth function S on it, does there exist a metric on that manifold having S as its scalar curvature? This is ultimately a question about the "*deformability*" of the manifold.[5] For example, can a sphere (of dimension greater than two) be deformed to such an extent that its scalar curvature becomes negative *everywhere*? Such questions are answered by the Kazdan–Warner classification theorem:

Theorem (Kazdan–Warner). *All compact manifolds of dimension at least three fall into precisely one of the following three classes*:

[P] *On these manifolds, every smooth function is the scalar curvature of some Riemannian metric.*

[Z] *On these manifolds, a smooth function can be a scalar curvature of some Riemannian metric if and only if it either takes a negative value somewhere, or is identically zero.*

[N] *On these manifolds, a smooth function can be a scalar curvature of some Riemannian metric if and only if it takes a negative value somewhere.*

For example, spheres are evidently not in [Z] or [N], so they must be in [P]. (It follows that a sphere of dimension at least three *can* be deformed in such a way that its scalar curvature is negative everywhere—see [68] for an explicit construction.) It can be shown (using some deep theorems to be discussed below) that compact manifolds of negative sectional curvature are in [N]. This means that *every* conformal structure on such a manifold is represented by a metric of constant negative scalar curvature: *no matter how we deform it, its scalar curvature can never vanish or become positive everywhere*. Thus, the Seiberg–

[5] It is interesting that *Lorentzian* compact manifolds are probably [70] arbitrarily "deformable" in this sense.

Witten instability in this case is particularly radical, since it is independent of the choice of metric and must arise from the topology of the space—the Kazdan–Warner classification depends only on the (differential)[6] topology of the manifold. One says that the instability is *induced topologically* [71].

This topological aspect of Seiberg–Witten instability has a direct physical consequence, as follows. The classical Belinsky–Khalatnikov–Lifschitz analysis of the approach to cosmological singularities (see [72] for a recent discussion) would lead one to expect that, as a Bang or a Crunch is approached, the geometry of the spatial sections would become more and more anisotropic, and this distortion might well become so extreme that the precise nature of the conformal structure induced at Euclidean infinity would no longer be clear. Now, however, we see that such anisotropies are irrelevant: no matter how complicated they may be, the scalar curvature induced at Euclidean infinity can never be positive or zero—no amount of distortion can avert Seiberg–Witten instability in this case. For whatever happens to the conformal geometry during the evolution, the topology of the spatial sections does not change, and the topology of conformal infinity remains that of a space on which *every* metric defines a conformal structure with negative scalar curvature. We conclude that *holography totally forbids spatial sections of negative curvature*, even if perturbations are taken into account.

Notice that the theory forbids negative curvature of any magnitude, no matter how small, because in any case it does not make sense to speak of "small" curvatures on the boundary (which only has a conformal structure, not a Riemannian metric). Thus there is indeed a real distinction between *extremely small* negative curvature and *zero* curvature on the bulk spatial sections (which do of course have a Riemannian structure). This distinction is a direct reflection of the holographic nature of Maldacena–Maoz cosmology.

To summarize, we have here a very strong prediction from cosmic holography: the theory could not be saved if any value of Ω below unity were confirmed by observation. It is interesting to note that, until the discovery of cosmic acceleration, the cosmological data actually pointed strongly towards negative spatial curvature; so we have an example in which cosmic holography makes a statement which might easily have been falsified.

Now let us turn to the case of principal interest to us: cosmological models with flat, compact boundaries and spatial sections. Seiberg and Witten did not consider the case where the scalar curvature of the boundary is zero. Here the analysis depends on higher-order terms [68] in the expansion of the brane action, and unfortunately it is difficult to give a general statement of the precise conditions needed to avert instability. However, much can be learned regarding this case by studying ground states for AdS black holes with flat, compact event horizons; for these spacetimes have flat conformal structures on conformal infinity. The ground state for such black holes is not anti-de Sitter spacetime but

[6] By this we mean that, in some examples of high-dimensional topological spaces which can admit more than one differentiable structure, the KW class can change if the differentiable structure is changed, even if the underlying topological structure does not change. But this cannot happen in the cases considered in this work.

rather the "AdS instanton" with (Euclidean) metric [73] given in $n+1$ dimensions by

$$g^{+}(\mathrm{AdSI}) = \frac{L^2}{r^2}\left(1-\frac{r_0^n}{r^n}\right)^{-1}\mathrm{d}r^2 + \left(\frac{r^2}{L^2}\right)\left[\left(\mathrm{d}t^{+}\right)^2+\left(1-\frac{r_0^n}{r^n}\right)\mathrm{d}x^2+\sum_1^{n-2}\left(\mathrm{d}x^i\right)^2\right]. \tag{17}$$

Here x and x^i are coordinates on the circumferences of circles of various radii; that is, they are proportional to angles. In the Euclidean case, the "time" coordinate too is angular. The conformal structure at infinity ($r\to\infty$) is represented by the flat metric

$$g^{+}(\mathrm{AdSI},\infty) = \left(\mathrm{d}t^{+}\right)^2+\mathrm{d}x^2+\sum_1^{n-2}\left(\mathrm{d}x^i\right)^2, \tag{18}$$

and this is a metric on a *compact* manifold, since all of the coordinates are angular. Thus the structure at infinity for the AdS instanton is precisely a compact, flat, n-dimensional manifold. The very fact that the instanton *is* a well-behaved, unique ground state [74–76] for these black holes strongly suggests that vanishing scalar curvature on the boundary is compatible with a stable field theory there, dual to one of these physically well-defined bulk configurations. Thus, we do have a large class of examples in which zero scalar curvature at the boundary is not pathological. While there undoubtedly exist other examples in which it is, one expects that these examples must involve highly intricate geometric constructions, not the very simple structures we are considering here.

For concreteness, and in order to avoid giving an analysis which is too model-dependent, we shall assume that scalar-flat boundaries of Maldacena–Maoz cosmologies—which are after all geometrically much simpler than AdS black holes with flat event horizons—do not lead to large brane instabilities in the bulk. Under this assumption, the cosmological model we considered above is of course stable in the Seiberg–Witten sense, since it is clear that the conformal structure induced on both connected components of Euclidean infinity is represented by a flat, hence scalar-flat, metric. As in the negatively curved case, however, one has to consider whether perturbations can disturb this simple picture. For a flat manifold can be deformed: a generic distortion produces a new conformal structure not represented by a flat metric. To assess the consequences of this, we need some further results in global differential geometry.

First, we need the concept of an *enlargeable* manifold [13, p. 302]. These are n-dimensional manifolds M such that, given any positive ϵ, there exists an orientable Riemannian covering M^* and a map f (which is constant at infinity and of non-zero degree) from M^* to the Riemannian n-sphere of curvature unity, where f contracts all lengths by a factor of at least ϵ. In other words, M must have "arbitrarily large" covering spaces. Notice that enlargeability is a topological condition. Clearly all compact flat manifolds are enlargeable.

The work of Schoen, Yau [12], Gromov, and Lawson [13, p. 306] can be summarized as follows:

Theorem (Schoen–Yau–Gromov–Lawson). *There is no metric of positive scalar curvature on any compact enlargeable spin manifold.*

It follows that compact enlargeable spin manifolds can never be in Kazdan–Warner class [P]. Now tori are compact, enlargeable, and spin; hence, no matter how a torus is deformed, the scalar curvature can never become positive everywhere, and it follows that the same is true of any quotient of a torus. Since every flat compact manifold is a quotient of a torus, we see that this statement is true of any compact flat manifold. On the other hand, it is obvious that flat compact manifolds are not in Kazdan–Warner class [N]. It follows that they are in [Z]. But this means that the only way to avoid a negative scalar curvature metric on these spaces is to ensure that the scalar curvature is precisely zero everywhere. This appears to be a strong constraint. In fact, it is *far stronger* than it seems. For Gromov and Lawson, extending a theorem of Bourguignon, were able to prove [13, p. 308] the following result.

Theorem (Bourguignon–Gromov–Lawson). *If a metric on a compact enlargeable spin manifold has zero scalar curvature, then that metric must be exactly flat, that is, the curvature tensor must vanish everywhere.*

This is a remarkable result: the vanishing of a single scalar invariant, the scalar curvature, forces the *entire* curvature tensor to vanish exactly on these manifolds. Recall now Schoen's theorem [66] to the effect that any conformal structure on a compact manifold is represented by a metric with constant scalar curvature; recall also that a smooth function on a manifold in KW class [Z] has to be negative somewhere if it is the scalar curvature of some metric, unless it is exactly zero. Combining all these observations, we have the following statement:

Corollary. *Let g be a metric on a manifold with the topology of a compact flat manifold. Then unless g itself is conformal to a flat metric, it is conformal to a metric of constant negative scalar curvature.*

That is, if such a manifold is a component of the conformal boundary of a manifold of the kind considered by Seiberg and Witten, and if a flat metric on the boundary is distorted, however, slightly, so that it ceases to be conformally flat, then the system will become unstable to the production of large branes. The situation here regarding Seiberg–Witten instability is thus almost as severe as it is in the negatively curved case: the instability can be avoided only if the boundary is perfectly (conformally) flat.

These deep geometric results thus impose an extremely demanding self-consistency check on our proposal. For the conformal structure at Euclidean infinity is obtained by taking a suitable *limit* of the metric on the spatial sections, after removing the conformal factor. (See the following section for the details.) This means that, on the Lorentzian side, we have to ensure that the spatial sections tend to become increasingly flat (again after removing the conformal factor) as both the Bang and the Crunch are approached in cosmologies like the one discussed in the previous section, with the Lorentzian metric given by Eq. (8). That is, of course, trivial for this precise metric, but this simplicity is based on the assumption that no other form of matter is present. If we introduce small anisotropies

corresponding to local concentrations of matter or radiation, it is far from clear that the spatial sections will be so well-behaved near to the Bang and to the Crunch. Indeed, the Belinsky–Khalatnikov–Lifschitz analysis mentioned above indicates that under small perturbations a generic spacetime with ordinary matter sources can be expected to develop severe anisotropies as one approaches a Bang or a Crunch, and so one would *not* in general expect a more realistic version of (8) to induce flat metrics on the spatial sections at very early or very late times; therefore, it is far from clear that the conformal structure at infinity will be represented by a perfectly flat metric.

We shall now see how this problem is naturally avoided by the cosmological models introduced in the preceding section, for some values of the fundamental parameter ϖ but not for others.

4. Ensuring flatness at infinity

In order to discuss anisotropies, we need to recall some aspects of the metrics of "asymptotically AdS" Euclidean spaces. The formal definition of such metrics is discussed at length in [33], and we need not rehearse all the details here: the main point is simply as follows. Under conditions which will always be satisfied for the spaces discussed here, the metric of an asymptotically AdS Euclidean space M (with asymptotic sectional curvature $-1/L^2$) can be written, near to any connected component of the conformal boundary, as

$$g^+(M) = \frac{L^2}{\rho^2}\big[\mathrm{d}\rho^2 + g^+_\rho\big], \tag{19}$$

where ρ is a coordinate such that the given component of the conformal boundary is at $\rho = 0$. Here g^+_ρ is a metric on the spaces transverse to the boundary. The point we wish to stress is that g^+_ρ *does* in general have a non-trivial dependence on ρ; the conformal structure at this component of infinity is represented by a metric which is obtained by taking the *limit* of g^+_ρ as ρ tends to zero. In this sense, the metrics of the form (1) considered above were very special cases, since we did not need to take this limit. A good example of this limiting process is provided by Lorentzian AdS_4 itself: in global coordinates (t, r, θ, ϕ) the metric can be expressed as

$$\begin{aligned} g^-(\mathrm{AdS}_4) = \cosh^2(r/L)\big(&-\mathrm{d}t^2 + \operatorname{sech}^2(r/L)\,\mathrm{d}r^2 \\ &+ L^2\tanh^2(r/L)\big[\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\phi^2\big]\big), \end{aligned} \tag{20}$$

and we see that the metric still depends on r even after the divergent conformal factor $\cosh^2(r/L)$ is removed. (Here, of course, the boundary is obtained by letting r tend to infinity, so that $\tanh^2(r/L)$ tends to unity and we obtain the usual cylindrical conformal boundary of AdS_4.)

This kind of behaviour is actually quite well-adapted to the cosmological case, since it is well known (see, for example, [72]) that the approach to cosmological singularities is *ultralocal*: that is, ultimately, only the (proper) time dependence of the metric is important. Hence, in studying the very late or very early stages of a Bang/Crunch cosmology, we can indeed concentrate on metrics which resemble (19), in the sense that the metric at infinity

is obtained by stripping away a conformal factor and then taking the limit of a family of metrics parametrized by a single parameter. In the notation of [72], we can express the metric in the ultralocal phase as

$$g^{-}_{\text{Anisotropic}} = -\left(\mathrm{d}t^{-}\right)^2 + \left(a^{-}\right)^2 \sum_i \mathrm{e}^{2\beta_i} \left(\sigma^i\right)^2, \tag{21}$$

where $a^{-}(t^{-})$ is an overall scale factor, where the σ^i are orthogonal, time-independent one-forms on the spatial slices, where the β_i are three distinct functions of proper time satisfying

$$\beta_1 + \beta_2 + \beta_3 = 0, \tag{22}$$

and where all dependence on spatial position has been suppressed. For locally flat spatial sections one finds that

$$\frac{\mathrm{d}\beta_i}{\mathrm{d}t^{-}} = c_i \left(a^{-}\right)^{-3}, \tag{23}$$

and one can show [72] that the scale factor satisfies a FRW equation of the form

$$3H^2 = 8\pi \left[\rho + \frac{\sigma^2}{(a^{-})^6} \right], \tag{24}$$

where H is the Hubble parameter, where ρ is the total energy density and where

$$\sigma^2 = \frac{1}{2}\left[c_1^2 + c_2^2 + c_3^2\right]; \tag{25}$$

thus σ is a constant which is an overall measure of the extent of anisotropy in such a spacetime.

In our case, ρ is the sum of the energy density of the background AdS_4, namely, $-3/8\pi L^2$, with the energy density of the quintessence field. Now with regard to this latter, recall that we saw that the Breitenlohner–Freedman bound requires that the series in Eq. (2) should terminate, with the last value of k being the largest integer satisfying

$$\frac{2}{1+\sqrt{2}} \frac{1}{3} \leqslant \frac{\varpi}{k^2}. \tag{26}$$

For example, in the case of $\varpi = 10$ (see [35]), the last value of k is 6, and this means that the corresponding quintessence component has a density proportional to $(a^{-})^{-7.2}$. (Recall that the magnitude of the exponent must not exceed 7.2426.) In general, if the last value of k satisfies (26), then it *may* also satisfy $\varpi/k^2 < 1/3$. If this is so, then we see from equation (10) that the quintessence energy density grows, as a^{-} tends to zero, more rapidly than $(a^{-})^{-6}$. For example, in the case where $\varpi = 10$, this means that, extremely near to the Bang or the Crunch—*not* at other times—Eq. (24) becomes

$$3H^2 = 8\pi \left[\frac{-3}{8\pi L^2} + \frac{3}{8\pi L^2 (a^{-})^{7.2}} + \frac{\sigma^2}{(a^{-})^6} \right]. \tag{27}$$

Clearly, the second term on the right is the dominant one near to the Bang and the Crunch—and this would remain true even if we included the contributions of ordinary

matter, radiation, and so on. In particular, whatever the initial anisotropy σ may have been, it will be completely insignificant compared to this term: one has a kind of "cosmic no-hair" theorem.

The situation here is exactly analogous to the way, as one moves *away* from the initial singularity, the inflaton potential dominates all other terms in the Friedmann equation, so that anisotropies are "inflated away" by the inflationary expansion: here the "last" quintessence component has the same effect as the singularities are *approached*, because its density grows more rapidly than that of any other contribution. Since there is no limit to the contraction, there is no limit to this effect—all local anisotropies will be completely wiped out in the very last stages of the approach to the singularities. A very similar phenomenon plays a crucial role in the cyclic cosmologies [77], and we see that it is equally important here, although we stress that there is *no* "bounce" in our case: we need rapid density growth rates not to prepare for a phase of expansion succeeding a crunch, but to ensure that the metric induced on (Euclidean) infinity is indeed flat. (Because of this difference, it turns out that much larger values of the effective equation-of-state parameter are required in the cyclic case than here; as we know, in our case the magnitude of the largest exponent of the scale factor is never much larger than six.)

The essential point here is that a three-dimensional Riemannian manifold which is locally isotropic around each point—that is, there is a local isometry mapping any unit vector at any point to any other unit vector at that point—has a sectional curvature which is independent of direction. For in *three* dimensions each unit vector at a point uniquely determines a two-dimensional subspace of the tangent space, namely, the subspace perpendicular to it. But if the sectional curvature of a Riemannian manifold of dimension at least three is independent of direction, then [78, p. 202] it is also independent of position; that is, the curvature is constant. Since compact manifolds of constant negative curvature are in Kazdan–Warner class [N], while those of constant positive curvature are in [P], it follows that the only way that a metric on a manifold with the topologies we are considering here can be locally isotropic is by being perfectly flat. We conclude that the conformal structure induced at Euclidean infinity is represented by a perfectly flat metric, provided that the matter content of our spacetime is such that all local anisotropies are eliminated by a "final" quintessence component with $\varpi/k^2 < 1/3$.

We require, then, that the final value of k should satisfy

$$\frac{2}{1+\sqrt{2}}\frac{1}{3} \leqslant \frac{\varpi}{k^2} < \frac{1}{3}. \tag{28}$$

These inequalities express the competing demands of the Breitenlohner–Freedman bound (which requires the lower bound) and of Seiberg–Witten instability (which, via the Schoen–Yau–Gromov–Lawson theorems, requires the upper bound). It is striking that the allowed interval is so short.

The effect of (28) is to exclude certain values of ϖ; the only allowed values are those lying in intervals $[a, b)$ where (28) is satisfied for some integer k. These intervals are given in Table 1. The intervals are closed to the left, open to the right (so that, for example, $\varpi = 3$ is *not* permitted).

Notice that there is an upper bound on the values of ϖ so excluded, because the allowed interval for $k = 11$ overlaps the allowed interval for $k = 12$, and all subsequent allowed

Table 1

k	a	b
1	0.276142	0.333333
2	1.104570	1.333333
3	2.485281	3.000000
4	4.418278	5.333333
5	6.903559	8.333333
6	9.941126	12.000000
7	13.530976	16.333333
8	17.673112	21.333333
9	22.367532	27.000000
10	27.614238	33.333333
11	33.413227	40.333333
12	39.764502	48.000000

intervals overlap their successors. (One sees this either by consulting the table or by means of a simple calculation based on requiring the lower end of one interval to be smaller than the upper end of its predecessor.) This upper bound is given by

$$\varpi_{\text{forbidden}} < \frac{242}{3}\left(\sqrt{2}-1\right) \approx 33.4132. \tag{29}$$

That is, all values of ϖ above this number are allowed. Below it, there is a haphazard set of intervals which are allowed, alternating with intervals which are not. For example, $\varpi = 10$, an example studied in detail in [35], is allowed; on the other hand, $\varpi = 9.90$ is not; nor is $\varpi = 2$, also studied in [35]. In short, values of ϖ below 33.4132 entail careful fine-tuning; larger values do not. If we can argue on independent grounds that ϖ is large, then there are no difficulties with fine tuning.

We conclude that if we take ϖ to be large, then "cosmic baldness" will automatically ensure that the conformal structure induced at Euclidean infinity is represented by an exactly flat metric, and this can be achieved without violating the Breitenlohner–Freedman bound. In fact, the observations [35] require ϖ to be at least this large. Furthermore, there are general theoretical reasons for expecting ϖ to be larger still. In many string theory compactifications [79,80] there is a general tendency to predict that the fundamental length scale of our observed spacetime should be very short. But one sees from Eqs. (7) and (8) that the natural length scales of the Euclidean and Lorentzian versions of our spacetime are different: in the former case, the space is asymptotic to a Euclidean AdS_4 with "radius" L, whereas in the latter case the Universe is finite in all directions, including time, with a total lifetime of $\pi\varpi L$. Thus L can indeed be small without contradicting the observations, provided that ϖ is very large.

To summarize: Seiberg–Witten instability allows us, in the context of cosmic holography, to draw several surprisingly strong conclusions regarding the spatial sections of the Universe. The first is that negatively curved compact spatial sections are completely ruled out in string theory. In this case, the instability is particularly persistent, because it is *topological*: the spatial sections can be arbitrarily deformed as we trace them back to the Bang or forward to the Crunch, yet the system is still subject to instabilities arising from the

nucleation of branes which lower their action as they are moved towards the (Euclidean) boundary.

Combining this with Linde's [1] analysis discussed earlier, we find that *the flat compact three-manifolds are unique* in their ability to satisfy all of the strictures imposed by the requirements of Inflation (which accommodates positively curved sections only with great difficulty) and large brane instability (which even more firmly rules out negatively curved sections). Even the flat manifolds only narrowly escape Seiberg–Witten instability. They escape it if we can make the boundary *exactly* conformally flat; we saw that our "toy model" of an accelerating holographic cosmos was able to perform this feat, provided that the parameter ϖ is sufficiently large, as is naturally the case.

The conclusions we have reached here, while developed in the context of a particular model, are in fact extremely robust. That is, they do not depend on the particular choice we made—a Euclidean axion—for the matter content of our cosmology: for example, the prohibition on negatively curved spatial sections is extremely general, since it depends only on the topology of these spaces and not on their geometry. Our ability to avoid Seiberg–Witten instability in the case of a boundary in KW class [Z] did depend on the ability of our matter model to flatten the sections as the singularities are approached, but this is attainable for many matter models—see [77].

However, we shall now see that the list of candidates for the spatial geometry of the world can be still further reduced if we do adopt the specific matter model introduced earlier. Thus the findings of the next section should not be considered to be as general as those of this section.

5. Finite—and (conformally) flat—in all directions

With the help of Seiberg–Witten instability [43], cosmic holography [2], and Inflationary arguments [1], we have reduced the number of candidates for the spatial sections of the Universe to a mere ten. That is nevertheless nine too many.

If we knew precisely which of these ten has been chosen by Nature, then we would have a valuable clue as to the true nature of the initial state. The ten candidates, dubbed the *platycosms* by Rossetti and Conway [46] (a term we adopt here as a useful abbreviation), are of varying degrees of complexity. Among those which are orientable, we have the torus T^3, the dicosm $T^3/\mathbb{Z}_2$, the tricosm $T^3/\mathbb{Z}_3$, the tetracosm $T^3/\mathbb{Z}_4$, the hexacosm $T^3/\mathbb{Z}_6$, and the didicosm or Hantzsche–Wendt space $T^3/[\mathbb{Z}_2 \times \mathbb{Z}_2]$.

For all that we know, the spatial sections of our Universe could have the structure of the didicosm. Unlike the torus, this space has non-trivial holonomies, of two different kinds: the holonomy group is $\mathbb{Z}_2 \times \mathbb{Z}_2$, a finite subgroup of SO(3). The fundamental domain here is a rhombic dodecahedron, and, if this domain were small enough to be observable, the resulting patterns in the microwave sky would be remarkable indeed [45]. Even if it is not directly observable, a theoretical deduction that the spatial sections have such a complicated structure would surely be a strong hint that the initial state has been selected with great precision, presumably by something very much more intricate than a simple classical singularity. But how can such a theoretical deduction be made? In this section, we shall show how our toy model, with Euclidean metric (7) (where the transverse sections are

not necessarily globally T^3) leads to a partial answer to this question. The hope, of course, is that a more realistic matter model would yield a more complete answer.

Eq. (7) indicates that *if* we interpret the underlying manifold as the interior of a manifold-with-boundary, then that boundary is disconnected: it has two connected components. It was emphasised by Maldacena and Maoz [2] themselves that the status of Euclidean manifolds with a disconnected boundary is very problematic from a holographic point of view. In general, this apparent failure of a one-to-one correspondence is a very deep question (see [81–83] for discussions), but in [35] we suggested that it may have a very simple resolution *in the particular case* with which we are concerned here. The argument is as follows.

The two-dimensional open cylinder $(0,1)\times S^1$ can be compactified in (at least) two different ways. The first is to regard it as the interior of the compact manifold-with-boundary $[0,1]\times S^1$ (the closed cylinder); the second is to regard it as an open submanifold of the torus $S^1\times S^1 = T^2$ (obtained from T^2 by deleting a circle). Neither option is "correct": one makes a choice depending on the circumstances. The difference, of course, is that in the first case we have to add *two* circles, whereas in the second case we only need to add one. This led us, in [35], to suggest that the second kind of interpretation is required by cosmic holography.

In the case at hand, we can reexpress the metric (7) in the following way. Define a constant c_ϖ by

$$c_\varpi = \frac{\varpi}{\pi}\int_0^\infty \operatorname{sech}^\varpi(\zeta)\,\mathrm{d}\zeta, \tag{30}$$

and a new coordinate θ by

$$c_\varpi L\,\mathrm{d}\theta = \pm\operatorname{sech}^\varpi\left(\frac{t^+}{\varpi L}\right)\mathrm{d}t^+, \tag{31}$$

where the sign is '+' when t^+ is positive, '−' when t^+ is negative. The range of θ is $-\pi$ to $+\pi$. Now solve for t^+ in terms of θ and use this to express $\operatorname{sech}^\varpi(\frac{t^+}{\varpi L})$ in terms of θ. Denote this function by $G_\varpi(\theta)$; then $G_\varpi(\theta)$ vanishes at $\pm\pi$, and $g^+(\varpi, A)$ is given in terms of the coordinate θ as

$$g^+(\varpi,A) = c_\varpi^2 L^2 G_\varpi^{-2}(\theta)\left[\mathrm{d}\theta^2 + \left(\frac{A}{c_\varpi L}\right)^2\left(\mathrm{d}\theta_1^2+\mathrm{d}\theta_2^2+\mathrm{d}\theta_3^2\right)\right]. \tag{32}$$

As it stands, the coordinate θ cannot be extended to the whole circle: we have to delete the (*single*) point $\theta=\pm\pi$, because $g^+(\varpi,A)$ is singular there. However, removing the prefactor on the right side of (32) by a conformal transformation, we obtain precisely the standard local metric for a non-cubic four-dimensional torus. (The number $A/c_\varpi L$ can in fact be constrained by observational data: see [35].)

If we had begun with (32) instead of (7), we would undoubtedly have declared that the natural compactification of our Euclidean space is a space with the local geometry of a torus. Infinity here is not a boundary; it is instead a "submanifold at infinity". The real point, however, is that infinity is *connected* in this interpretation. Clearly the "double boundary" problem simply does not arise if we adopt this viewpoint.

Thus, we propose an extremely simple extension of our hypothesis of flat, compact spatial sections: not only the spatial sections, but also (the compactified Euclidean version of) *the entire four-dimensional spacetime* should have the topology of a compact flat manifold. In short, the spatial sections are flat, compact three-manifolds, while the compactified spacetime is globally conformal (Eq. (32)) to a flat, compact four-manifold.

Now recall that the adoption of the local three-dimensional metric $A^2(\mathrm{d}\theta_1^2+\mathrm{d}\theta_2^2+\mathrm{d}\theta_3^2)$ in Eq. (6) did not commit us to the global geometry and topology of the three-dimensional torus: *many* distinct compact flat three-dimensional manifold have this local metric, since all such manifolds can be expressed topologically as T^3/F, for some finite group F. (Recall that we assumed for simplicity that the covering torus was cubic, but trivial modifications allow us to consider the most general case.) In the same way, the appearance of the metric of a four-dimensional torus in (32) does not mean that we have here a manifold with the topology of T^4 or even that of $S^1\times T^3/F$. In constructing a general compact flat manifold, the procedure is as in the familiar case of a torus, but one is free to apply various isometries before performing the identifications that produce a compact space. In three dimensions, there are ten ways of doing this; in four dimensions [44], there are no fewer than 75, though we hasten to add that most of these 75 cannot be constructed from manifolds of the form T^3/F in the above way.

Return temporarily to the interpretation of $g^+(\varpi,A)$ as a metric on the interior of a manifold-with-boundary. That boundary consists of two copies of T^3/F. What we are proposing here is that these two copies should be identified. But, before performing the identification, we are free to apply an isometry, as above, to one of the copies. If we do this, we shall obtain a space with the same local metric as $S^1\times T^3/F$ (which is what we get if the isometry is trivial); that is, we obtain a metric of the form (32). The spaces we obtain will not be fully general flat compact four-dimensional manifolds, partly because the metric in (32) has a special form (the torus is rectangular and has three dimensions of the same length, while the fourth in general has a different length) and partly because we have already fixed part of the topology by specifying the group F. Nevertheless some freedom remains, because T^3/F still has a non-trivial isometry group even after the factoring by F. Can we reduce this freedom in a physical way? The answer is that we can, by exploiting the fact that our matter field is of a very specific kind: it is a (Euclidean) *axion*.

The axion field φ^+ is able to distinguish $t^+=-\infty$ from $t^+=+\infty$, because from Eq. (11) we have

$$\lim_{t^+\to\infty}\varphi^+=\sqrt{\frac{\varpi}{4\pi}}\frac{\pi}{2}=-\lim_{t^+\to-\infty}\varphi^+, \tag{33}$$

and this sign difference is physical because from Eq. (3) we have

$$W^+\left(-\sqrt{\frac{\varpi}{4\pi}}\frac{\pi}{2}\right)=-W^+\left(+\sqrt{\frac{\varpi}{4\pi}}\frac{\pi}{2}\right). \tag{34}$$

This is important, because it apparently puts a stop to our plan of identifying the two boundary components: how can we do so when the field and its superpotential take different values on each component? But recall that an axion naturally reverses sign under a reversal of orientation. Thus the problem is solved in a natural, geometric way if—and

only if—we arrange for the orientation of one boundary component to be reversed before identifying it with the other. This is precisely the way, in two dimensions, a Klein bottle [44] is defined, and we can proceed in much the same way here, allowing, however, for the greater complexity of T^3/F. Let us see to what extent this requirement reduces our freedom in constructing our four-dimensional Euclidean "spacetime".

The point is simply that not every compact flat three-manifold admits an orientation-reversing isometry. In essence, factoring a Riemannian manifold by a discrete group usually reduces the "size" of the isometry group, since not all isometries of the original space are compatible with the factoring. In the present case, the reduction can obstruct the procedure we outlined above. The isometry groups of all of the platycosms are listed in [46], and the result we need can simply be stated using that list; however, we can gain more insight by means of the following elementary argument.

Suppose that one has a manifold M admitting a group $G(M)$ of diffeomorphisms (such as isometries, conformal symmetries, and so on). Let Γ be a subgroup of $G(M)$ and let $N(\Gamma)$ be the *normalizer* of Γ in $G(M)$. That is,

$$N(\Gamma) = \{g \in G(M) \mid g\gamma g^{-1} \in \Gamma \ \forall \gamma \in \Gamma\}. \tag{35}$$

Clearly $N(\Gamma)$ contains all those elements of $G(M)$ which descend to well-defined diffeomorphisms of M/Γ. But notice that every element of Γ itself has no effect on each element of M/Γ. Thus the symmetry group of M/Γ, which we denote by $G(M/\Gamma)$, is not $N(\Gamma)$ but rather the quotient $N(\Gamma)/\Gamma$:

$$G(M/\Gamma) = N(\Gamma)/\Gamma. \tag{36}$$

See [16] for more details and for other applications of this formula. The main point to bear in mind here is that an isometry of M can descend to an isometry of M/Γ only if it normalizes Γ.

Let us see how this works in some concrete examples. First, the torus T^3 is defined as follows [44, p. 117]. First recall that any isometry of $\mathbb{R}^3$ can be expressed as (B, a), where B is an orthogonal matrix and a is a vector, and where (B, a) means that we let B act first, followed by a translation through a. Let a_i, $i = 1, 2, 3$, be a fixed basis for $\mathbb{R}^3$. If Γ_3^* is generated over the integers by the isometries $\tau_i = (I_3, a_i)$, where I_3 is the identity matrix, then $T^3 = \mathbb{R}^3/\Gamma_3^*$. Now consider the isometry $\Omega = (-I_3, 0)$. It is easy to see that conjugation by Ω just maps each element of Γ_3^* to its inverse. Thus Ω does normalize Γ_3^* and so it projects to an isometry of T^3. Of course, Ω reverses the orientation of $\mathbb{R}^3$, so we see that T^3 does admit an orientation-reversing isometry, which is what we need.

Next, the platycosm of the form $T^3/\mathbb{Z}_3$ (the tricosm) is obtained as follows. First we constrain the vectors a_i: we require a_1 to be orthogonal to the other two, and we require a_2 and a_3 to be of the same length and to be inclined at an angle of $2\pi/3$. Next, we set $\alpha[\text{tri}] = (A_3[2\pi/3], a_1/3)$, where $A_3[2\pi/3]$ is a 3×3 matrix corresponding to a rotation through $2\pi/3$ in the a_2–a_3 plane: that is, $A_3[2\pi/3]$ maps a_1 to itself, a_2 to a_3, and a_3 to $-a_2 - a_3$. Then the tricosm is $\mathbb{R}^3/\Gamma[\text{tri}]$, where $\Gamma[\text{tri}]$ is obtained by adjoining $\alpha[\text{tri}]$ to the generators of Γ_3^*. Now conjugation by Ω still maps each element of Γ_3^* to its inverse, but it does not have this effect on $\alpha[\text{tri}]$; instead we have

$$\Omega\alpha[\text{tri}]\Omega^{-1} = \big(A_3[2\pi/3], -a_1/3\big). \tag{37}$$

The isometry on the right is not the inverse of α[tri] and is *not* an element of Γ[tri]. Thus Ω does not descend to an isometry of the tricosm. In fact, in order to do this, an isometry of $\mathbb{R}^3$ would have to reverse orientation in the plane defined by a_2 and a_3, while also reversing a_1; but such an isometry could not be orientation-reversing.

A similar argument works also for the tetracosm $T^3/\mathbb{Z}_4$ and the hexacosm $T^3/\mathbb{Z}_6$. It does not work for the dicosm $T^3/\mathbb{Z}_2$. To see why, note that this space is defined much as the tricosm, except that apart from being orthogonal to a_1, the other conditions on a_2 and a_3 are dropped, and α[tri] is replaced by $\alpha[\mathrm{di}] = (A_3[\pi], a_1/2)$, where $A_3[\pi]$ is a 3×3 matrix rotating the a_2–a_3 plane through π; then the dicosm is $\mathbb{R}^3/\Gamma$[di], where Γ[di] is obtained by adding α[di] to the generators of Γ_3^*. Again, conjugation by Ω maps Γ_3^* to itself, but now we have

$$\Omega\alpha[\mathrm{di}]\Omega^{-1} = \big(A_3[\pi], -a_1/2\big), \tag{38}$$

and this *is* in Γ[di] since it is the inverse of α[di] (because $A_3[\pi]$ is of order two). Thus Ω does descend to an orientation-reversing isometry of the dicosm. The didicosm $T^3/[\mathbb{Z}_2 \times \mathbb{Z}_2]$ can be constructed in much the same way as the dicosm, but with three additional generators instead of one, each involving a rotation by π in some plane. One can show that Ω descends to an orientation-reversing isometry in this case also. In the case of the non-orientable platycosms, a different argument applies, but for those platycosms it is in any case obvious that there can be no orientation-reversing isometries. *Thus the torus, the dicosm, and the didicosm are the only survivors*, that is, they are the only platycosms which can be used to construct a "generalized Klein bottle" of the kind we need in order to compensate for the fact that the axion field has opposite signs on the two connected components of infinity in our model.

Let us now show how to construct the final compact flat four-manifolds which, as explained above, are the possible underlying spaces of the conformal compactification corresponding to the metric (32). We shall concentrate on the dicosm: the construction for the torus is just a simpler version, while that for the didicosm is somewhat more complicated but introduces no essentially new difficulties.

Let a_μ, where $\mu = 0$ through 3, be an orthogonal basis for $\mathbb{R}^4$, where we take a_0 to be of length $2\pi c_\varpi L$, while a_1, a_2, and a_3 are of length $2\pi A$. Define 4×4 matrices A_4 and B_4 by $A_4 = \mathrm{diag}(1, 1, -1, -1)$ and $B_4 = \mathrm{diag}(1, -1, -1, -1)$, and then define a pair of $\mathbb{R}^4$ isometries as follows.

$$\alpha = (A_4, a_1/2), \qquad \beta = (B_4, a_0/2). \tag{39}$$

Let $\Delta_4(\Gamma[\mathrm{di}])$ be the group generated by these isometries, together with the translations $\tau_\mu = (I_4, a_\mu)$. Then $\Delta_4(\Gamma[\mathrm{di}])$ can be presented as follows:

$$\begin{aligned}
&\alpha^2 = \tau_1, && \beta^2 = \tau_0, && \beta\alpha\beta^{-1} = \alpha^{-1},\\
&\alpha\tau_0\alpha^{-1} = \tau_0, && \alpha\tau_2\alpha^{-1} = \tau_2^{-1}, && \alpha\tau_3\alpha^{-1} = \tau_3^{-1},\\
&\beta\tau_1\beta^{-1} = \tau_1^{-1}, && \beta\tau_2\beta^{-1} = \tau_2^{-1}, && \beta\tau_3\beta^{-1} = \tau_3^{-1}.
\end{aligned}$$

$\Delta_4(\Gamma[\mathrm{di}])$ is so named because it contains a subgroup Γ[di], generated by α, τ_1, τ_2, and τ_3, which corresponds to the fundamental group of the dicosm. One can see that $\Delta_4(\Gamma[\mathrm{di}])$ is a non-abelian infinite group with no element of finite order (other than the identity) and

with a maximal free abelian subgroup Γ_4^* (generated by the τ_μ) consisting of four copies of $\mathbb{Z}$. From the relations given, it is clear that Γ_4^* is normal in $\Delta_4(\Gamma[\mathrm{di}])$; the quotient $\Delta_4(\Gamma[\mathrm{di}])/\Gamma_4^*$ is of finite order (it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$); one says that Γ_4^* is of index 4 in $\Delta_4(\Gamma[\mathrm{di}])$ and of rank 4. By the relevant version of the Bieberbach theorems [44, p. 105] it follows that $\mathbb{R}^4/\Delta_4(\Gamma[\mathrm{di}])$ is a four-dimensional manifold, covered by a four-torus $\mathbb{R}^4/\Gamma_4^*$ with the flat metric

$$g^+_{\mathrm{flat}}(c_\varpi L, A) = c_\varpi^2 L^2\,\mathrm{d}\theta^2 + A^2\big(\mathrm{d}\theta_1^2 + \mathrm{d}\theta_2^2 + \mathrm{d}\theta_3^2\big); \tag{40}$$

here θ is an angular coordinate corresponding to a_0, while the θ_i correspond to the a_i.

As a Riemannian manifold, $\mathbb{R}^4/\Delta_4(\Gamma[\mathrm{di}])$ can be expressed as $T^4/[\mathbb{Z}_2 \times \mathbb{Z}_2]$, where T^4 is the rectangular torus with aspect ratio given by $A/(c_\varpi L)$, and where $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the linear holonomy group of this space. One of the two independent non-trivial holonomies reverses orientation, while the other does not. All this can be repeated beginning with the three-torus instead of the dicosm, resulting in a flat four-manifold with the structure $T^4/\mathbb{Z}_2$, or with the didicosm (Hantzsche–Wendt space), resulting in a flat four-manifold with the structure $T^4/[\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2]$.

The overall picture, then, is this. The underlying structure of the compactified Euclidean four-dimensional space is that of $T^4/\mathbb{Z}_2$, $T^4/[\mathbb{Z}_2 \times \mathbb{Z}_2]$, or $T^4/[\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2]$. Let us take this last space as a concrete example. The local metric is given in (40); it is indistinguishable from that of a four-torus, except that the coordinates are not global. However, if we move around the "time" direction (the θ direction), we find that orientation is reversed once per cycle. If we arbitrarily select $\theta = \pm\pi$ to be the (single) point on the θ-circle where orientation is reversed, then a pseudoscalar such as our Euclidean axion will automatically reverse sign there. Now suppose that we single out $\theta = \pm\pi$ by performing the conformal transformation that maps the metric in (40) to the one given in Eq. (32). Now $\theta = \pm\pi$ has to be excised, and the space has the topology of a Bang/Crunch spacetime with spatial sections having the structure of a didicosm. Transforming to the Lorentzian version, we have the Bang/Crunch cosmology with metric (8). In the reverse direction, we transform the metric in (8) to its Euclidean version, which is apparently a space with a conformal infinity consisting of two components, each having the structure of the didicosm; however, we can naturally identify these after performing an orientation-reversing twist, so as to obtain the compact four-manifold $T^4/[\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2]$ as the compactification. The dual field theory resides on the *single*, orientable, didicosm section at $\theta = \pm\pi$, that is, at infinity.

Linde's considerations of quantum gravity [1], with which we began, allowed the spatial sections of our universe to have any one of the infinite variety of structures possible for compact three-manifolds of negative or zero curvature. We have narrowed this vast array down to just three candidates: the torus T^3, the dicosm $T^3/\mathbb{Z}_2$, and the didicosm $T^3/[\mathbb{Z}_2 \times \mathbb{Z}_2]$. We do not know how to reduce this list to a single candidate. It is noteworthy, however, that although they seem rather similar, the *homology* groups of the three surviving candidates are very different: in particular, their first homology groups (with integer coefficients) are given on p. 122 of [44] as

$$H_1\big[T^3\big] = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z},$$
$$H_1\big[T^3/\mathbb{Z}_2\big] = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

$$H_1\big[T^3/[\mathbb{Z}_2\times\mathbb{Z}_2]\big]=\mathbb{Z}_4\times\mathbb{Z}_4. \tag{41}$$

Notice that this last group is *finite*; thus the cycles around which branes may be wrapped have a very different structure in the didicosm from those in the torus or the dicosm. It also follows that the first (and therefore second) Betti numbers are quite different:

$$b_1\big[T^3\big]=3,\qquad b_1\big[T^3/\mathbb{Z}_2\big]=1,\qquad b_1\big[T^3/[\mathbb{Z}_2\times\mathbb{Z}_2]\big]=0. \tag{42}$$

The vanishing of $b_1[T^3/[\mathbb{Z}_2\times\mathbb{Z}_2]]$ means that, in sharp contrast to the torus and the dicosm, the didicosm has no harmonic one-forms or two-forms. A further study of all of these properties may lead to a physical way of distinguishing the didicosm from all other candidates.

6. Conclusion

What is the shape of space? This question, even in its modern form, has exercised leading minds for over a century ([84]; see [15] for a discussion of the views of de Sitter and Schwarzschild). For a time there was hope that it would be settled by direct observation, but, as this hope has begun to fade, we may have to turn to theory for guidance. There is a very large literature on observational aspects of topologically non-trivial cosmological models, but very little is known about the basic physical principles which might prefer one topology over another. Indeed, one of the main motivations of this work is to persuade the reader that it is possible to find such principles.

It is striking and exciting that Inflation, which tells us that we should probably not hope to see direct evidence of non-trivial spatial topology, nevertheless also tells us [1] that this topology probably *is* non-trivial. It then becomes a pressing question to determine which topological structure has been chosen—and how.

In this work, we have argued that our best theories of fundamental physics do allow us to narrow the field of candidates. The lessons we have learned vary in their degree of generality.

The most general lessons are based on the assumption that *some kind of bulk–boundary duality is valid in cosmology*. This very general assumption already has strong consequences. Most importantly, it tells us that we cannot ignore the most distant regions of our world, those beyond cosmological horizons: for those regions are just as surely part of the bulk as the regions near to us in space and time, and their role in the boundary dual theory cannot be excluded or neglected.

Slightly less generally, if we assume that the spacetime conformal boundary lies to the future and the past (as in de Sitter spacetime, or in any of the cosmologies of the general form considered by Maldacena and Maoz [2]), then typically each connected component of the "dual" space has the same topology as the spatial sections. But if we further assume that string theory controls the bulk-boundary relationship, then the very general arguments of Seiberg and Witten [43] apply. We are still at a very high level of generality at this point, but already we can, with the help of the theorem of Kazdan–Warner, make an extremely strong deduction about the nature of the spatial sections: *they cannot be negatively curved*. The startling feature of this argument is precisely the fact that it is *topological*: once it

has been established that a manifold is in KW class [N], its scalar curvature must remain negative (ensuring Seiberg–Witten instability) no matter how it may be deformed by the subsequent evolution of spacetime.

If Seiberg–Witten instability rules out negative curvature, and some of the most interesting versions of Inflation disfavour positive curvature [1], then of course we are directed towards the flat, compact three-manifolds: the *platycosms* [45,46]. Already this is a great reduction, since there are infinitely many compact negatively curved spaces, but only ten platycosms.

All of these conclusions are very general, since they do not depend on using a specific cosmological model. If we are willing to be more specific, then we can reduce the list still further. The particular cosmological model considered here, combined with the (holographically motivated) requirement that disconnected boundaries must be avoided, leads to a demand that the spatial sections should have a specific geometric property; and we found that only *three* candidates satisfy this condition. We do not ask the reader to take this particular conclusion very seriously, since it is based on specific properties of a specific matter model. The more important conclusion is that *we have shown explicitly that it is indeed possible to use physical principles to effect a vast reduction in the range of candidates*. It does not seem too far-fetched to imagine that a more sophisticated and realistic matter model might well succeed in reducing the list to a *single* candidate. It may be that this is how the shape of space will be discovered: perhaps only one topology is consistent with our best theories.

Recent work on the (surprisingly deep) geometry of the platycosms sheds interesting light on this observational/theoretical interplay for cosmic topology. It has been found [85] that it is possible for two platycosms with different topologies to be *isospectral*, that is, the spectra of their Laplace operators can be placed into a one-to-one correspondence. This is remarkable, because two three-dimensional tori can be isospectral only if they have the same shape and size. Since the analysis of CMB data involves precisely these spectra, it could be very difficult to distinguish these two spaces by means of CMB observations, even if the fundamental domain were small enough for direct observations to be possible. Yet they are certainly distinguished by our theoretical analysis above. For the isospectral pairs are obtained by taking a certain specific torus T_0^3 of a fixed shape (it is a "two-storey" rectangular torus), and then taking quotients. The quotient of the form $T_0^3/[\mathbb{Z}_2 \times \mathbb{Z}_2]$ is a particular example of a didicosm, named "Didi" in [85]; the quotient of the form $T_0^3/\mathbb{Z}_4$ is an example of a tetracosm, named "Tetra".[7] Didi and Tetra are isospectral, but we saw above that Didi is acceptable in our specific cosmology while Tetra must be excluded. Thus we have a situation where theoretical arguments are able to distinguish candidates which may be difficult to separate observationally.

Of course, the main task now is to determine or at least constrain the boundary field theory. Because of the special asymptotic properties of the Maldacena–Maoz cosmologies in the accelerating case [33], there is reason to believe that the bulk/boundary correspondence will be unusual here. This is currently under investigation.

[7] The name Dexter has also been suggested.

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