BIANCHI TYPE I COSMOLOGICAL MODEL WITH BULK VISCOSITY

Wung-Hong HUANG
Department of Physics, National Cheng Kung University, Tainan, Taiwan 700, ROC

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We present some solutions of Bianchi type I cosmological models with bulk viscosity. Exact solutions are given when the universe is filled with stiff matter while the viscosity coefficients are constant or proportional to the energy density. Effects of the bulk viscosity are found to smooth out the anisotropy of the universe and create matter by the gravitational field in the course of evolution. At the final stage, cosmologies are driven to the infinite expansion state, the de Sitter spacetime, or the isotropic Friedmann universe.

1. Introduction

Cosmological evolution with a fluid containing viscosities has already attracted the attention of many investigators. Misner [1] suggested that neutrino viscosity may considerably reduce the anisotropy of the black-body radiation. Murphy [2] obtained an exactly soluble cosmological model of the zero curvature Friedmann model in the presence of bulk viscosity alone. The solutions exhibit the interesting feature that the big bang type singularity appears in the infinite past. Belinskii and Khalatnikov [3] analyzed Bianchi type I cosmological models under the influence of viscosity. They then found the remarkable property that near the initial singularity the gravitational field creates matter.

The viscosity mechanism in cosmology can account for the high entropy of the present universe [4,5]. Bulk viscosity associated with the grand-unified-theory phase transition [6] may lead to an inflationary scenario [7,8]. (The inflationary cosmology formulated by Guth in 1981 [9] is used to overcome several important problems in the standard big bang cosmology.)

Exact solutions of the isotropic homogeneous cosmology with general viscosity for the open, closed and flat universe had been found by Santos et al. [10]. Using certain simplifying assumptions, Banerjee and Santos [11,12] obtained some exact solutions for the homogeneous anisotropic model. Recently Banerjee et al. [13] obtained some Bianchi type I solutions for the case of stiff matter by using the assumption that shear viscosity coefficients are power functions of the energy density. However, the bulk viscosity coefficients adopted in their model are zero or constant.

In this paper, without introducing shear viscosity, we shall examine Bianchi type I cosmological models with bulk viscosity alone. The exact solutions are obtained for the stiff matter if the viscosity coefficient is constant or proportional to the energy density. We find that the isotropic de Sitter spacetime is an attractor state as $t\to\infty$ if the viscosity coefficient is constant. It is thus in accord with the "cosmic no hair" theorem [14–16] even though the strong energy condition [17] is violated [18]. However, contrary to the finding of Murphy [2], whose investigation of the isotropic model leads to the big bang model without singularity, our model exhibits the $R=0$ singularity with infinite energy density at finite past [3,10,13]. Effects of bulk viscosity proportional to the energy density are found to reduce the anisotropy of the universe and create matter by the gravitational field in the course of the evolution of cosmology. Finally, just as in the isotropic model, the cosmology evolves to an infinite expansion state, the de Sitter spacetime, or the Friedmann universe, in agreement with previous results [3,13].

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2. Einstein's field equations

The space time we considered is described by the metric

$$ds^2 = -dt^2 + X^2\,dx^2 + Y^2(dy^2 + dz^2)\,, \tag{1}$$

where X and Y are functions of time alone. The energy–momentum tensor of a bulk viscous fluid is written in the form

$$T_{\mu\nu} = (\epsilon + \bar{p})u_\mu u_\nu + \bar{p}g_{\mu\nu}\,, \tag{2}$$

$$\bar{p} = p - \eta u^\lambda{}_{;\lambda}\,, \tag{3}$$

$$u^\lambda u_\lambda = -1\,, \tag{4}$$

where ϵ is the energy density, p the pressure, η the bulk viscosity coefficient, and u_μ is the four-velocity. Choosing a comoving frame where $u_\mu = \delta^0{}_\mu$, Einstein's field equations become

$$\frac{dH}{dt} + HW = \tfrac{1}{2}(\epsilon - \bar{p})\,, \tag{5}$$

$$\frac{dh}{dt} + hW = \tfrac{1}{2}(\epsilon - \bar{p})\,, \tag{6}$$

$$W^2 - (2H^2 + h^2) = 2\epsilon\,, \tag{7}$$

where

$$H = \frac{dY/dt}{Y}\,, \tag{8a}$$

$$h = \frac{dX/dt}{X}\,, \tag{8b}$$

$$W = 2H + h\,. \tag{8c}$$

A linear combination of eq. (5) with eq. (6) gives

$$\frac{dW}{dt} + W^2 = \tfrac{3}{2}(\epsilon - \bar{p})\,, \tag{9}$$

$$\frac{d}{dt}(h - H) + (h - H)W = 0\,. \tag{10}$$

The Bianchi identities imply

$$\frac{d\epsilon}{dt} + (\epsilon + \bar{p})W = 0\,. \tag{11}$$

We also know that

$$\epsilon - \bar{p} = (2 - \gamma)\epsilon + \eta W\,, \tag{12}$$

$$\epsilon + \bar{p} = \gamma\epsilon - \eta W\,, \tag{13}$$

where γ is defined by the equation of state

$$p = (\gamma - 1)\epsilon, \quad 1 \leqslant \gamma \leqslant 2\,. \tag{14}$$

In the following sections we only discuss stiff matter, i.e. $\gamma = 2$, which is a possible relevance of the equation of state $p = \epsilon$ as regards the matter content of the early universe [19,20].

3. Solutions

Consider first the simplest case with constant viscosity, i.e. $\eta = \eta_0$. Then the field equations (5) and (6) become

$$\frac{dH}{dt} = (\tfrac{1}{2}\eta_0 - H)(2H + h)\,, \tag{15}$$

$$\frac{dh}{dt} = (\tfrac{1}{2}\eta_0 - h)(2H + h)\,. \tag{16}$$

The zeros of eqs. (15) and (16) give the fixed points in the phase plane $h \times H$. A stable one is the de Sitter state with $H = h = \frac{1}{2}\eta_0$. Dividing eq. (16) by eq. (17), we find a simple equation relating h and H,

$$\frac{dh}{dH} = \frac{\frac{1}{2}\eta_0 - h}{\frac{1}{2}\eta_0 - H}\,. \tag{17}$$

The above equation can be easily integrated to give

$$h = CH + \tfrac{1}{2}\eta_0(1 - C)\,, \tag{18}$$

where C is the integration constant. Substituting eq. (18) into eq. (10) we have

$$\frac{dH}{dt} + (1 + \tfrac{1}{2}C)(H - \tfrac{1}{2}\eta_0)\left(2H + \eta_0\frac{1-c}{2+c}\right) = 0\,. \tag{19}$$

After integration we finally obtain the exact solution

$$H = \frac{3\eta_0}{2(2+C)[\exp(\frac{3}{2}\eta_0 t) - 1]} + \tfrac{1}{2}\eta_0 \xrightarrow{t\to\infty} \tfrac{1}{2}\eta_0\,, \tag{20}$$

$$h = \frac{3C\eta_0}{2(2+C)[\exp(\frac{3}{2}\eta_0 t) - 1]} + \tfrac{1}{2}\eta_0 \xrightarrow{t\to\infty} \tfrac{1}{2}\eta_0\,. \tag{21}$$

The solutions tell us that bulk viscosity can be used to drive the anisotropic cosmology to the isotropic de Sitter universe. However, an initial singular phase of expansion occurs at finite past ($t=0$) where both Hubble parameters and energy density are infinitely large. Therefore, contrary to the claims of Murphy [2], bulk viscosity cannot remove the cosmological singularity in our model. These properties are also found in the other models [3,10,13] in which the shear viscosities have been included.

We next consider the case where the viscosity coefficient is proportional to the energy density, i.e. $\eta=\eta_0\epsilon$. From eqs. (5)–(8), (12) we obtain

$$\frac{dH}{dt}=\tfrac{1}{2}H(H+2h)(H+2h-2)\,, \tag{22}$$

$$\frac{dh}{dt}=\tfrac{1}{2}(H+2h)[H(H+2h)-2h]\,, \tag{23}$$

where H and h have been rescaled by a factor η_0. From the above two equations we find that there are three classes of fixed points in the phase plane $h\times H$: the de Sitter universe with $h=H=2/3$, the Minkowski spacetime with $h=H=0$ and states with $H+2h=0$. Using eq. (7) we know that the energy density vanishes on the lines $H=0$ and $H+2h=0$, so that the de Sitter state is the only fixed point with non-zero energy density.

Dividing eq. (23) by eq. (22) we obtain

$$\frac{dh}{dH}=\frac{H(H+2h)-2h}{H(H+2h)-2H}\,. \tag{24}$$

This equation can be written as

$$\frac{dL}{dH}=\frac{L(3H-2)}{H(L-2)}\,, \tag{25}$$

where

$$L=H+2h\,. \tag{26}$$

The general solutions of eq. (25) are

$$h-H-\ln(1+2h/H)+C=0\,, \tag{27}$$

where C is the integration constant. We can furthermore express h and H in terms of the variables r and θ,

$$h=r\sin\theta\,, \tag{28}$$

$$H=r\cos\theta\,, \tag{29}$$

and eq. (27) reduces to the elegant expression

$$r=\frac{C-\ln(1+2\tan\theta)}{\cos\theta-\sin\theta}\,. \tag{30}$$

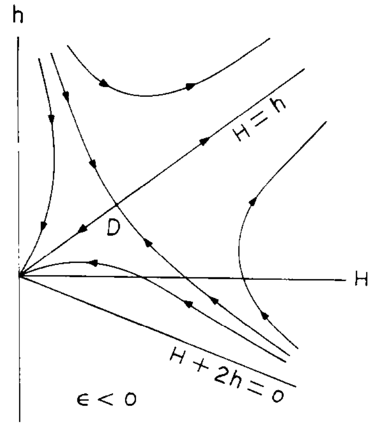

Fig. 1. The phase plane trajectories determined by eq. (30). Arrows denote the directions of increasing cosmic time. D is the saddle point which corresponds to the de Sitter spacetime.

This is an exact solution of the anisotropic extended Murphy model.

Using the above equation we can determine the flows in the phase space $h\times H$ and thus find the dynamical evolutions of cosmology. Some solutions are shown in fig. 1 for a variety of possible values of C. The point D in fig. 1 is the de Sitter state which becomes a saddle point now. The arrows in the trajectories can be easily determined by eqs. (22) and (23). The evolution of a cosmology are known to start from a fixed point or at infinity and end in another fixed point or infinity in the phase plane. (We only consider the physical plane where the trajectories evolute to positive Hubble parameters in the latter stage. The regions where $\epsilon<0$ violate the dominate energy condition [17] are also neglected.)

We then find that, except in the isotropic model ($h=H$) which had been investigated by Murphy [2], the cosmologies always begin with zero energy density at the initial phase of singularity. During the evolution, the energy density is increasing subsequently and the anisotropies of the universe are smoothed out. At the final stage as $t\to\infty$, depending on the integration constant C in eq. (30), there are three classes of states which may be approached asymptotically:

(1) Both the energy density and Hubble parameters go to infinity if $H>h$, $C>\ln 3$ or $h>H$, $\ln 3>C>0$.

(2) The energy density is finite and spacetime is

attracted to the de Sitter universe if $C=\ln 3$.

(3) Both the energy density and Hubble parameters decrease to approach zero and the model is driven to the isotropic Friedmann universe if $H>h$, $\ln 3>C>0$ or $h>H$, $C>\ln 3$.

It is noted that these final states are also found in isotropic models.

4. Conclusion

Assuming that the bulk viscous coefficient is constant or proportional to the energy density we have presented the exact solutions of the Bianchi type I cosmologies with a stiff fluid. Many characteristic traits, such as isotropization of cosmology, evoluting towards the de Sitter space and the creation of matter during the evolution, appear in our model. Although only two cosmic scale functions are used in our metrics, we expect that these traits will still exist in the Bianchi type I models with three cosmic scale functions. However, as the characteristic features of the cosmological models depend on the functional form of the viscosity coefficients [3,13], searching for the exact solution with general viscosity is certainly worth while. It remains to be found.

Finally, we want to mention that the solutions studied in this paper have been discussed qualitatively (but not given exactly) by Barrow [21–23]. There it is pointed out that the $\eta\propto\epsilon$ case results in a violation of the dominate energy condition $\epsilon+p>0$ as $t\to 0$. Barrow [24] also gave a further discussion of bulk viscous models in theories possessing quadratic curvature.

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