

# Four Dimensionality in Non-compact Kaluza-Klein Model

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## Abstract

Five dimensional model with extended dimensions investigated. It is shown that four dimensionality of our world is the result of stability requirement. Extra component of Einstein equations giving trapping solution for matter fields coincides with the one of conditions of stability.

## 1 Introduction

An investigation of possibility that dimension of our world is more than four is not new. Nearly all papers on this direction are done with the framework of standard Kaluza-Klein models where extra dimensions are curled up to an unobservable size (see for example review [1]). Besides of obvious achievements this approach encounters some problems such as: Why four dimensions are extended and others are curled; How to choose the signature of multidimensional space; Physical meaning of extra components of Einstein's equations is unclear; There exists the problem of stability.

An alternative proposal that the extra dimensions are extended and the matter is trapped in 4-dimensional submanifold was advanced in papers [2, 3, 4, 5]. This approach has properties similar to four dimensions - all dimensions are extended and equal at the beginning and the signature has the form (+,-,-, ... ,-).

Models of this kind also do not contradict to present time experiments [1]. Multidimensionality in these models was used to solve several problems, such as, cosmological constant, dark matter, non-locality or hierarchy problems [2, 3, 4, 6, 7].

For the simplicity, we investigate here only the case of five dimensions. The general procedure immediately generalizes to arbitrary dimensionality.

Using extended dimensions approach we want to consider our Universe as a 3-shell expanding in 5-dimensional space-time [7]. This model supported by at least two observed facts. First is the isotropic runaway of galaxies, which only for obviousness usually explained as an expansion of a 3-sphere in five dimensions. Second is the existence of a preferred frame in the Universe where the relict radiation is isotropic. In the framework of the closed-Universe model without boundaries this can also be explained if Universe is a bubble and the mean velocity of the background radiation is zero with respect to its center in the fifth dimension.

## 2 Stability Condition

In models of large extra dimensions we need the mechanism of confining a matter inside of the 4-dimensional submanifold which must be sufficiently narrow along the extra dimensions and flat along four others. It is natural to think that such a splitting of 5-dimensional space with

trapping of a matter into four dimensions is the result of existence of the special solution of multidimensional Einstein equations

$${}^5R_{AB} - \frac{1}{2}g_{AB} {}^5R = 6\pi^2 GT_{AB} . \quad (2.1)$$

Here  $G$  is 5-dimensional gravitational constant and  $A, B, \dots = 0, 1, 2, 3, 5$ .

We need stabile macroscopic solution, so it is natural to consider only classical gravitational, electromagnetic and scalar fields which can form extended solutions. For the beginning let us consider only gravitational and electromagnetic fields with the Lagrangian

$$L = -\sqrt{g} \left[ \frac{1}{12\pi^2 G} {}^5R + \frac{1}{4} F_{AB} F^{AB} \right] . \quad (2.2)$$

Generalisation for any Yang-Mills field is obvious.

To obtain stabile splitting of multidimensional space momentum toward the extra - fifth dimensions must be zero

$$P_5 = \int T_5^5 dt dV + \int T_5^\alpha dS_\alpha = 0 \quad (2.3)$$

(Greek indices  $\alpha, \beta, \dots = 0, 1, 2, 3$  numerate coordinates in four dimensions). Other dimensions can expand and our world can be expanding bubble.

For our choice of gravitational Lagrangian the energy-momentum tensor of gravitation and electromagnetic fields in five dimensions  $T_{AB}$  has the form of so named Lorentz energy-momentum complex

$$T_A^B = t_A^B + \tau_A^B = \frac{1}{12\pi^2 G \sqrt{g}} \partial_C X_A^{BC} , \quad (2.4)$$

where

$$X_A^{BC} = -X_A^{CB} = \sqrt{g} [g^{BD} g^{CE} (\partial_D g_{AE} - \partial_E g_{AD})] . \quad (2.5)$$

In (2.4)

$$\begin{aligned} t_A^B &= \frac{1}{12\pi^2 G} (g^{BD} \partial_A \Gamma_{DE}^E - g^{ED} \partial_A \Gamma_{DE}^B + \delta_A^B {}^5R) , \\ \tau_A^B &= -F^{BC} F_{AC} + \frac{1}{4} \delta_A^B F^{DE} F_{DE} \end{aligned} \quad (2.6)$$

are energy-momentum tensor of gravitational and electromagnetic fields respectively.

To obey the stability condition (2.3) for the solutions we must have

$$T_5^\alpha = T_5^5 = 0 . \quad (2.7)$$

Using (2.5) from (2.7) we obtain

$$\partial_\beta g_{5A} = \partial_5 g_{5\alpha} = 0 . \quad (2.8)$$

Simple solution of (2.8) is

$$g_{5\alpha} = 0 , \quad g_{55} = const = -1 . \quad (2.9)$$

In general  $g_{55}$  can be any function of  $x^5$ , but this function in all formulae will appear as a coefficient and would not influence on the dynamic of the theory.

From the other hand from first condition of (2.7) using explicit form (2.6) we obtain

$$\partial_5 \Gamma_{\beta\gamma}^\alpha = 0 , \quad F_{5\alpha} = 0 \quad (2.10)$$

and equation of electromagnetic field has standard Maxwell 4-dimensional form

$$D_\nu F^{\mu\nu} = 0 \quad . \quad (2.11)$$

Here we want to notice that in model of Visser [3] electromagnetic field don't obey the condition (2.10) ( $F_{5\alpha}$  not equal to zero) and therefore his solution is unstable.

Finally from (2.9) and first of (2.10) we obtain the metric tensor corresponding to stable splitting of multidimensional space-time

$$g_{\alpha\beta} = \lambda^2(x^5)\eta_{\alpha\beta} \quad , \quad g_{55} = -1 \quad , \quad g_{5\beta} = 0 \quad , \quad (2.12)$$

where  $\eta_{\alpha\beta}$  is the 4-dimensional metric tensor and  $\lambda^2(x^5)$  in the meanwhile is the arbitrary function of fifth coordinate. This solution which we received from stability conditions exactly coincide with the anzats of Rubakov-Shaposhnikov [2].

Splitting (2.12) was made in the frame of the 4-dimensional wall. If we consider our Universe as an expanding bubble in the frame of the bubble center (2.12) needs Lorentz transformation. Bubble expansion  $\lambda(x^5) \rightarrow \lambda(x^5 - vt)$ , where  $v$  is velocity of the wall, means conformal transformation of 4-dimensional metric  $\eta_{\alpha\beta}$ . To keep stability, the theory must be invariant under conformal transformations in the submanifold. This condition fixes dimension of our world. Indeed using formulae (2.11) and (2.12) Lagrangian of electromagnetic field in any dimensions can be written in the form

$$L = \sqrt{\lambda^{2n}\eta} \frac{1}{4\lambda^4} \eta^{\alpha\gamma} \eta^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta} \quad , \quad (2.13)$$

where  $n$  is dimension of submanifold of trapping. Only in case of four dimensions ( $n = 4$ ) we have conformal invariance and stable splitting is possible. Thus only 3-dimensional expanding bubbles can be survived for a long time and our Universe can be one of them.

### 3 Trapping

In previous section it was shown that in Gaussian normal coordinates the 5-dimensional metric of our Universe can be written in the form

$$ds^2 = -(dx^5)^2 + \lambda^2(x^5)\eta_{\alpha\beta} dx^\alpha dx^\beta \quad . \quad (3.1)$$

In this coordinates components of Christoffel symbols with two or three indices 5 are equal to zero, while with the one index 5 forms the tensor of extrinsic curvature [8]

$$\begin{aligned} K_{\alpha\beta} &= \Gamma_{\alpha\beta}^5 = \frac{1}{2} \partial_5 g_{\alpha\beta} = \lambda \lambda' \eta_{\alpha\beta} \quad , \\ K^{\alpha\beta} &= -\frac{1}{2} \partial_5 g^{\alpha\beta} \quad , \end{aligned} \quad (3.2)$$

where prime denotes derivative with respect of the coordinate  $x^5$ . Also we want to represent some useful formulae

$$\begin{aligned} g^{\alpha\gamma} K_{\gamma\beta} &= \Gamma_{5\beta}^\alpha = \lambda \lambda' \delta_\beta^\alpha \quad , \\ K &= g^{\alpha\beta} K_{\alpha\beta} = g_{\alpha\beta} K^{\alpha\beta} = 4\lambda' / \lambda \quad , \\ \partial_5 K &= g^{\alpha\beta} \partial_5 K_{\alpha\beta} - 2K^{\alpha\beta} K_{\alpha\beta} \quad . \end{aligned} \quad (3.3)$$

Any vector and tensor naturally is split-up into its components orthogonal and tangential to the shell. Using decomposition of the curvature tensor

$$\begin{aligned} {}^5R_{\alpha\beta} &= R_{\alpha\beta} + \partial_5 K_{\alpha\beta} - 2K_{\alpha}^{\gamma} K_{\gamma\beta} + K K_{\alpha\beta} \quad , \\ {}^5R_{55} &= -\partial_5 K - K^{\alpha\beta} K_{\alpha\beta} \quad , \\ {}^5R &= R + K^{\alpha\beta} K_{\alpha\beta} + K^2 + 2\partial_5 K \end{aligned} \quad (3.4)$$

one can find decomposition of Einstein's equations

$$\begin{aligned} R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}R + \partial_5 K_{\alpha\beta} - 2K_{\alpha}^{\gamma} K_{\gamma\beta} + K K_{\alpha\beta} - \frac{\lambda^2}{2}\eta_{\alpha\beta}(K^{\gamma\delta} K_{\gamma\delta} + K^2 + 2\partial_5 K) = \\ = \frac{6\pi^2 G}{\lambda^2}(-F_{\alpha}^{\delta} F_{\delta\beta} + \frac{1}{4}\eta_{\alpha\beta} F^{\gamma\delta} F_{\gamma\delta}) \quad , \\ \frac{1}{2\lambda^2}R + \frac{1}{2}(K^2 - K^{\gamma\delta} K_{\gamma\delta}) = -\frac{3\pi^2 G}{2\lambda^4} F^{\alpha\beta} F_{\alpha\beta} \quad . \end{aligned} \quad (3.5)$$

Using last formula of (3.3) we noticed that since

$$t_5^5 = \frac{1}{12\pi^2 G}(-\partial_5 K - g^{\alpha\beta} \partial_5 K_{\alpha\beta} + {}^5R) \quad , \quad (3.6)$$

fifteenth Einstein's equation (last equation of (3.5)) is nothing else than stability condition (2.7) -  $T_5^5 = 0$ . So extra component of Einstein's equations, whose physical meaning is unclear in standard Kaluza-Klein models, in our approach coincide with the condition of stability.

Using the explicit form of extrinsic curvature tensor (3.3) from the system of Einstein's equations (3.5) we find

$$-R = 12\lambda\lambda'' \quad . \quad (3.7)$$

Then from second equation of (3.5) for function  $\lambda(x^5)$  we have

$$\lambda\lambda'' - \lambda'^2 - \frac{\pi^2 G}{4} F^{\gamma\delta} F_{\gamma\delta} = 0 \quad . \quad (3.8)$$

This equation gives trapping solution

$$\lambda = \cosh(Ex^5) \quad , \quad (3.9)$$

where

$$E = \sqrt{\frac{\pi^2 G}{4} F^{\gamma\delta} F_{\gamma\delta}} \quad . \quad (3.10)$$

Width of our world  $\Delta \sim 1/E$  depended on gravitational constant and density of electromagnetic field.

To see how gravitational trapping works let us consider simple example of the real scalar field  $\phi$  in the background metric (2.12) with  $\lambda$  expressed by (3.9). If we put

$$\phi = \lambda u(x^\alpha) \quad (3.11)$$

to the 5-dimensional Klein-Gordon equation

$$\frac{1}{\sqrt{g}} \partial_A (\sqrt{g} g^{AB} \partial_B \phi) + m^2 \phi = 0 \quad , \quad (3.12)$$

for  $u$  we find

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta u + \frac{m^2 \lambda^4 - 1}{\lambda^2} u = 0 \quad . \quad (3.13)$$

According to (3.9) -  $\lambda = \cosh(Ex^5)$  and we see that "mass" of the scalar field in this equation has its minimum in the 4-dimensional submanifold and growth rapidly far from the wall. So field  $\phi$  is in the potential well.

## 4 Scalar fields

Now let us consider more complicate model adding to (2.2) the Lagrangian of complex scalar fields

$$L_\psi = \sqrt{g}[D_A\bar{\psi}D^A\psi - \xi|\psi|^2 {}^5R - U(|\psi|)] , \quad (4.1)$$

where  $U(|\psi|)$  in the meanwhile is any function of  $\bar{\psi}\psi$ . Variation of this Lagrangian by metric tensor gives energy-momentum tensor of scalar fields

$$T_{AB} = t_{AB} + 2\xi({}^5R_{AB} - \frac{1}{2}g_{AB} {}^5R + D_AD_B - g_{AB}D_CD^C)|\psi|^2 , \quad (4.2)$$

where

$$t_{AB} = D_A\bar{\psi}D_B\psi + D_B\bar{\psi}D_A\psi - g_{AB}[D_C\bar{\psi}D^C\psi - U(|\psi|)] . \quad (4.3)$$

Stability condition for the shell (2.7) -  $T_{5\alpha} = 0$ , now except of conditions  ${}^5R_{5\alpha} = F_{5\alpha} = 0$  gives the new condition for scalar fields

$$D_5\psi = (\partial_5 - iA_5)\psi = 0 . \quad (4.4)$$

Maxwell equations now take the standard 4-dimensional form with the source

$$D_\mu F^{\mu\nu} = j^\nu = \bar{\psi}\partial^\nu\psi - \psi\partial^\nu\bar{\psi} , \quad (4.5)$$

and the equation for the scalar field is

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu + \xi {}^5R)\psi + \lambda^2\frac{\partial U(|\psi|)}{\partial\bar{\psi}} = 0 . \quad (4.6)$$

Using formulae (3.3) for the extrinsic curvature tensor  $K_{\mu\nu}$ , splitting of Einstein's equations (3.5) now has the form

$$\begin{aligned} & \left(\frac{1}{6\pi^2G} + 2\xi|\psi|^2\right)(R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}R) + \frac{1}{2\pi^2G}\eta_{\alpha\beta}\lambda'' = \\ & = \frac{1}{\lambda^2}(-F_\alpha^\delta F_{\delta\beta} + \frac{1}{4}\eta_{\alpha\beta}F^{\gamma\delta}F_{\gamma\delta}) + D_\alpha\bar{\psi}D_\beta\psi + D_\beta\bar{\psi}D_\alpha\psi - \\ & - \eta_{\alpha\beta}[\eta^{\mu\nu}D_\mu\bar{\psi}D_\nu\psi - \lambda^2U(|\psi|)] + 2\xi(D_\alpha D_\beta - \eta_{\alpha\beta}D_\mu D^\mu)|\psi|^2 , \end{aligned} \quad (4.7)$$

$$\begin{aligned} & \left(\frac{1}{6\pi^2G} + 2\xi|\psi|^2\right)\frac{1}{2\lambda^2}R + \frac{1}{\pi^2G}\frac{(\lambda')^2}{\lambda^2} = \\ & = -\frac{1}{4\lambda^4}F^{\alpha\beta}F_{\alpha\beta} + \frac{1}{\lambda^2}\eta^{\mu\nu}D_\mu\bar{\psi}D_\nu\psi - U(|\psi|) - \frac{2\xi}{\lambda^2}D_\mu D^\mu|\psi|^2 . \end{aligned} \quad (4.8)$$

Substituting of (4.6) and (4.8) in trace of equation (4.7) one can find that solution of the system (4.6) - (4.8) is

$$\begin{aligned} 6\xi &= 1 , \\ R &= -12\lambda\lambda'' , \\ \lambda &= \cosh(Ex^5) . \end{aligned} \quad (4.9)$$

We see that constant of coupling gravitational and scalar fields  $\xi$  fixed on the value corresponding to conformal invariance of scalar field equation in four dimensions. So again we received 4-dimensionality from the condition of stability towards the extra dimensions. Only conformal invariant form of function  $U(|\psi|)$  in four dimensions, when  $\psi = \lambda(x^5)/u(x^\nu)$ , is

$$U(|\psi|) = \mu|\psi|^4/2 , \quad (4.10)$$

where  $\mu$  is coupling constant. Finally equation of massless scalar field in five dimensions (4.6) has the form

$$(\eta^{\mu\nu} D_\mu D_\nu + 2E^2 + \mu|\psi|^2)\psi = 0 \quad . \quad (4.11)$$

We see that because of coupling with gravitational field in four dimensions scalar field has "mass"  $E^2$  expressed with gravitational constant and density of electromagnetic field by (3.10).

## References

- [1] J. M. Overduin and P. S. Wesson, Phys. Rept. **283**, 303 (1997).
- [2] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **B125**, 139 (1983).
- [3] M. Visser, Phys. Lett. **B159**, 22 (1985).
- [4] E. J. Squires, Phys. Lett. **B167**, 286 (1986).
- [5] A. Barnaveli and O. Kancheli, Sov. J. Nucl. Phys. **51**, 901 (1990);  
Sov. J. Nucl. Phys. **52**, 920 (1990).
- [6] G. Kalbermann and H. Halevi, gr-qc/9810083.
- [7] M. Gogberashvili, hep-ph/9812296; hep-ph/9812365.
- [8] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (W.H.Freeman and Co., San Francisco, 1973).