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Firewalls, Hawking (Tolman) Radiation, and a Tentative Resolution of the Firewall-Mass Problem

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Abstract: It has been theorized that black holes are surrounded by firewalls, although there is not universal agreement concerning this. We show that, if firewalls exist, they can originate via Hawking radiation—which had been anticipated, albeit for *non*-black holes, by Tolman—at the minimum possible ruler distance (the Planck length) beyond the Schwarzschild horizon, where it has not suffered any gravitational redshift, or, alternatively, suffered maximal gravitational blueshift. We also show that the firewall temperature is on the order of the Planck temperature, *in*dependently of the mass and hence also of the Schwarzschild radius of a Schwarzschild black hole. Then we explain the exponential nature of the gravitational frequency shift as a function of the gravitational potential. Next, we consider the firewall-mass problem, and provide an at least tentative resolution thereto based on: (i) the mass of a firewall being canceled by the negative gravitational mass =(negative gravitational energy)$/c^2$ accompanying its formation, (ii) the *un*changed observations of a distant observer upon formation of a firewall, and (iii) Birkhoff's Theorem. In concluding, we provide remarks concerning thermodynamics in gravitational fields, showing that equilibrium relativistic gravitational temperature gradients can*not* be exploited to violate the Second Law of Thermodynamics.

Keywords: Schwarzschild black holes; Ruler distance; Planck units; Hawking radiation; Tolman radiation; Gravitational frequency shift; Firewall mass; Negative gravitational mass-energy; Birkhoff's Theorem; Thermodynamic equilibrium

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1. Introduction

It has been theorized that black holes are surrounded by firewalls [1–7], although there is not universal agreement concerning this [1–7]. There is a vast literature exploring this topic, of which we have cited only a tiny sample. But the many works cited, and discussed, in our cited Refs. [1–7] could provide at least the beginning of a thorough literature search [1–7].

In Section 2, we review basic concepts pertaining to Schwarzschild black holes and Hawking radiation. In Section 3, we discuss Tolman's [8,9] anticipation of Hawking radiation, albeit for *non*-black holes, and Tolman's proof that *any* gravitator—black hole or *non*-black hole *must radiate*—and hence can*not* be in thermodynamic equilibrium with a surrounding vacuum at (or at least sufficiently close to) absolute zero (0 K), but *must eventually completely evaporate into that vacuum*. (See also Garrod [10].) This has been corroborated by recent research [11]. In Section 4, we show that, if firewalls exist, they can originate via Hawking (Tolman) radiation at the minimum possible ruler distance [12] (the Planck length [13–15]) beyond the Schwarzschild horizon, where it has not suffered any gravitational redshift [16,17], or, alternatively, suffered maximal gravitational blueshift. We also show that the firewall temperature is on the order of the Planck temperature [18], *in*dependently of the mass and hence also of the Schwarzschild radius of a Schwarzschild black hole. We emphasize and focus on *ruler* distance [12] because, unlike other distance measures in General Relativity [12], ruler distance—*uniquely!* [12]—is actual *physical* distance: the distance between two points as measured by rulers laid upon the shortest possible spatial path separating them and hence the *physical* distance separating them [12]. In Section 5, we explain the exponential nature of the gravitational frequency

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shift as a function of the gravitational potential. In Section 6, we consider the firewall-mass problem [3], and provide an at least tentative resolution thereto based on: (i) the mass of a firewall being canceled by the *negative* gravitational mass [16,17] $= (\textit{negative}$ gravitational energy$)/c^2$ [16,17] accompanying its formation, (ii) the *un*changed observations of a distant observer upon formation of a firewall, and (iii) Birkhoff's Theorem [19–22]—actually first discovered by Jørg Tofte Jebsen [22]. We show that the mass of a firewall is *exactly* counterbalanced by the (negative) gravitational mass-energy accompanying its formation. Perhaps this may complement other lines of reasoning [4] disputing massiveness [3] of firewalls. (There is a caveat [23–25][1] with respect to Birkhoff's Theorem [19–22], but it [23–25] is *not* relevant with respect to our considerations.[1]) In Section 7, we provide concluding remarks concerning thermodynamics in gravitational fields, showing that equilibrium relativistic gravitational temperature gradients can*not* be exploited to violate the Second Law of Thermodynamics.

2. Review of basic concepts pertaining to Schwarzschild black holes and Hawking radiation

The Schwarzschild metric of a Schwarzschild black hole of mass M and Schwarzschild radius $r_S = 2GM/c^2$ is [18]

$$\begin{aligned} ds^2 &= \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 - r^2\left(d\theta^2 - \sin^2\theta d\phi^2\right) \\ &= \left(1 - \frac{r_S}{r}\right) c^2 dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 - \sin^2\theta d\phi^2\right) \\ &= c^2 d\tau^2 - dl^2 - r^2\left(d\theta^2 - \sin^2\theta d\phi^2\right). \end{aligned} \tag{1}$$

Schwarzschild-coordinate radial distance r is *not* radial *ruler* distance but is radial area distance and also radial distance from apparent size [12]: in accordance with the *Euclidean* form of the angular term of the Schwarzschild metric [the last term in all three lines of Equation (1)], a spherical shell at r_{shell} has ruler-distance circumference $2\pi r_{\text{shell}}$ and ruler-distance surface area $4\pi r_{\text{shell}}^2$ [12]. At all $r \geq r_S$, l—*not* r—is radial *ruler* distance, t is Schwarzschild-coordinate time (proper time measured by a clock at rest at $r \to \infty$) [16,17], and τ is proper time measured by a clock at rest at *any* given $r \geq r_S$ [16,17]. [A clock at rest *at* $r = r_S$—if such a clock can exist—must be constructed entirely of photons (and/or other zero-rest-mass particles)!]

Although not necessary for our derivations, it may be helpful, as an aside, to briefly remark on the following three features of the Schwarzschild metric [Equation (1)]: (i) Setting $ds^2 = 0$ in Equation (1) shows that the *physical* radial velocity of light $V_{\text{phys,light}} = dl/d\tau = c$ at all $r \geq r_S$ [26]; but, by contrast, the Schwarzschild-*coordinate* radial velocity of light $V_{\text{coor,light}} = dr/dt = c\,[1 - (r_S/r)]$ decreases monotonically with decreasing r from c at $r \to \infty$ to zero at $r = r_S$ [26]. (ii) We focus on distance [12], especially on *ruler* distance [12], and most especially on *radial ruler* distance [12], *beyond* the Schwarzschild horizon r_S in Schwarzschild spacetime [26]. But it may be interesting to note that, in accordance with the *angular* term of the Schwarzschild metric [the last term in all three lines of Equation (1)] being of the *identical Euclidean* form at *all* $r \geq 0$ [26],[2] *even within* r_S,[2] where a spherical shell can*not* be at rest but *must* be collapsing, while falling through a given $r_{\text{shell}} < r_S$ it has ruler-distance circumference $2\pi r_{\text{shell}}$ and ruler-distance surface area $4\pi r_{\text{shell}}^2$ [26].[2] *Even within* r_S,[2] where r becomes time*like*, r does *not* become time itself:[2] *un*like time itself r still retains these *spatial* geometrical attributes [26].[2] Time itself has *no spatial* geometrical attributes. (iii) Because the gravitational field of a Schwarzschild black hole is purely radial, it seems intuitive that since this gravitational field stretches [27,28][3] space from the Euclidean [27,28],[3] the stretching occurs *only* in the *vertical*[3] radial r direction, *not* in the *horizontal* angular θ and ϕ directions: thus the *identical Euclidean* form of the angular term of the Schwarzschild metric [the last term in all three lines of Equation (1)] at *all* $r \geq 0$ [26]. Indeed, more generally, intuition suggests that stretching [27,28][3] of space from the Euclidean by *any* gravitational field occurs *only* in the *vertical* direction [27,28],[3] *not* in any *horizontal* direction [27,28].[3]

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This intuition augments the immediately preceding Item (ii), and is perhaps most clear in relation to Sakharov's elastic-strain theory of gravity [29–34].[4,5] (Thus, might space be the ether [35,36]?[4,5])

We will be concerned only with *radial* motions of photons in the gravitational fields of Schwarzschild black holes, because only *radial* motions can result in gravitational frequency shifts. In this regard we will be concerned only with the *temporal-radial* part of the Schwarzschild metric [Equation (1)], hence ignoring the angular term thereof (the last term in all three lines thereof).

Hawking radiation, the (at least essentially) blackbody radiation from a Schwarzschild black hole, is most typically construed to have the temperature [15,37–43]

$$T_{\mathrm{H},r\to\infty} = \frac{\hbar c^3}{8\pi GkM} = \frac{\hbar c^3}{8\pi Gk\frac{r_{\mathrm{S}}c^2}{2G}} = \frac{\hbar c}{4\pi k r_{\mathrm{S}}}, \tag{2}$$

where k is Boltzmann's constant, M is the mass of the black hole, and $r_{\mathrm{S}} = 2GM/c^2$ is its Schwarzschild radius [15,37–43]. Note that $T_{\mathrm{H},r\to\infty}$ given by Equation (2) is the temperature of Hawking radiation at a great distance from a Schwarzschild black hole, i.e., at $r \to \infty$, hence after Hawking radiation having suffered the maximum possible gravitational redshift [15–17,37–43]. For all non-primordial black holes, which are all of stellar mass or larger, $T_{\mathrm{H},r\to\infty}$ is extremely low compared to the current temperature $T_{\mathrm{CBR}} = 2.725\,\mathrm{K}$ [44] of the cosmic background radiation [44]. For sufficiently small primordial black holes [45–55] ($M < \frac{\hbar c^3}{8\pi GkT_{\mathrm{CBR}}} = 4.50\times 10^{22}\,\mathrm{kg} \iff r_{\mathrm{S}} < \frac{\hbar c}{4\pi kT_{\mathrm{CBR}}} = 6.69\times 10^{-5}\,\mathrm{m}$), $T_{\mathrm{H},r\to\infty} > T_{\mathrm{CBR}} = 2.725\,\mathrm{K}$ obtains. But as of this writing, to the best knowledge of the author, no such sufficiently small primordial black holes—indeed, no primordial black holes at all—have been discovered [45–55]. Moreover, while there are rationales according to which primordial black holes might contribute, perhaps significantly, to cold dark matter, there also are both theoretical and observational upper limits on their abundance [45–55] and therefore also on their actual contribution to cold dark matter [45–55]. Hence, while it is possible that they may contribute, perhaps significantly, to cold dark matter, as of this writing, to the best knowledge of the author, it is uncertain whether or not they actually exist [45–55].

Thus far we have considered $T_{\mathrm{H},r\to\infty}$. But $T_{\mathrm{H},r}$ at smaller values of r ($r_S \le r < \infty$) has been discussed as well [43]. Closer to r_{S} (at $r_S \le r < \infty$) [43] than at $r \to \infty$, Hawking radiation has suffered less [16,17,43] gravitational redshift [16,17] and hence has a higher [43] temperature [15,37–43]

$$T_{\mathrm{H},r} = T_{\mathrm{H},r\to\infty}\left(1-\frac{r_{\mathrm{S}}}{r}\right)^{-1/2} = \frac{\hbar c^3}{8\pi GkM}\left(1-\frac{r_{\mathrm{S}}}{r}\right)^{-1/2} = \frac{\hbar c}{4\pi k r_{\mathrm{S}}}\left(1-\frac{r_{\mathrm{S}}}{r}\right)^{-1/2}. \tag{3}$$

3. Tolman's anticipation of Hawking radiation: *all* gravitators—black holes *and non*-black holes—*must radiate*

It is *extremely important* to note that Equation (3) is a special case of the more general result derived by Tolman [8,9]. In accordance with the notation of Tolman's (identical) Equations (128.6) and (129.10)] of Ref. [9]:

$$T_{\mathrm{T}}\left(\vec{\mathbf{r}}\right) = T_{\mathrm{T},\infty}\left[\frac{|g_{tt,\infty}|}{\left|g_{tt}\left(\vec{\mathbf{r}}\right)\right|}\right]^{1/2} = T_{\mathrm{T},\infty}\left|g_{tt}\left(\vec{\mathbf{r}}\right)\right|^{-1/2}$$
$$\iff T_{\mathrm{T},\infty} = T_{\mathrm{T}}\left(\vec{\mathbf{r}}\right)\left|g_{tt}\left(\vec{\mathbf{r}}\right)\right|^{1/2}. \tag{4}$$

The subscript T refers to Tolman. Tolman also presents this result (equivalently, but in slightly different form) via the second equation in the Abstract and Equations (28), (29), (42), and (53) in the main text of Ref. [8], and in Equation (128.10) of Ref. [9]. (See also Garrod's [10] discussion of this point.) In

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Equation (4), $T_{\mathrm{T}}\left(\vec{\mathbf{r}}\right)$ is the thermodynamic-equilibrium temperature and $\left|g_{tt}\left(\vec{\mathbf{r}}\right)\right|$ is the magnitude of the time-time component of the metric at vector ruler distance [12] $\vec{\mathbf{r}}$ from the center of mass of a gravitator, and $T_{\mathrm{T},\infty}$ and $|g_{tt,\infty}| = 1$ are the same quantities in the limit $\vec{\mathbf{r}} \rightarrow \infty$: Owing to gravity vanishing and hence the metric approaching the Minkowskian in the limit $\vec{\mathbf{r}} \rightarrow \infty$ in any direction from the center of mass of a gravitator, $|g_{tt,\infty}| = 1$—this justifies the second step of Equation (4). The absolute value signs are employed in Equation (4) because some authors give the time-time component of the metric a negative sign and the spatial components a positive sign (other authors vice versa).

In Ref. [8] and in Sections 128 and 129 of Ref. [9], Tolman implies that our Equation (4) is valid in *any static* spacetime [56]. But given rotation at *constant* angular velocity, a *time-independent* centrifugal potential can be incorporated into the *time-independent* gravitational potential that obtains in *static* spacetime, i.e., into that which obtains neglecting the rotation [56]. This *time-independent* gravitational-centrifugal potential would then of course be a function of θ as well as of r, but at a *given* θ it can still be expressed as a function of r alone. Thus we can construe Equation (4) to be valid in *any static or stationary* spacetime [56]. Hence Tolman's (identical) Equations (128.6) and (129.10) of Ref. [9], rewritten as our Equation (4), imply that at thermodynamic equilibrium temperature increases downwards[6] in *any static or stationary* gravitational field (the centrifugal contribution construed as incorporated within the total field if there is rotation at *constant* angular velocity) [56]. But for simplicity and definiteness we focus on the *static* spacetimes at $r \geq r_{\mathrm{S}}$ of *non*-rotating spherically-symmetrical, i.e., Schwarzschild, gravitators.

Even more importantly, Tolman [8,9] *furthermore* implies *more than that*: as emphasized by the second line of our Equation (4), he *furthermore* implies that $T_{\mathrm{T},\infty}$ *must be finitely higher than absolute zero* (0 K)—because $T_{\mathrm{T},\infty} = 0\,\mathrm{K}$ would *in*correctly imply that $T_{\mathrm{T}}\left(\vec{\mathbf{r}}\right) = 0\,\mathrm{K}$ obtains *everywhere*—at least, everywhere that $\left|g_{tt}\left(\vec{\mathbf{r}}\right)\right| > 0$ or equivalently that $\left|g_{tt}\left(\vec{\mathbf{r}}\right)\right|^{-1/2} < \infty$: in the case of a Schwarzschild black hole, for which $g_{tt}\left(\vec{\mathbf{r}}\right) = |g_{tt}(r)| = 1 - \frac{r_{\mathrm{S}}}{r} \implies \left|g_{tt}\left(\vec{\mathbf{r}}\right)\right|^{-1/2} = \left(1 - \frac{r_{\mathrm{S}}}{r}\right)^{-1/2}$, in the region $r \geq r_{\mathrm{S}}$ *everywhere* beyond r_{S}, i.e., *everywhere* except at *exactly* $r = r_{\mathrm{S}}$. This in turn implies that *any* gravitator—black hole or *non*-black hole *must radiate*: a black hole surrounded by a vacuum colder than $T_{\mathrm{H},r\rightarrow\infty}$ and non-black hole surrounded by a vacuum colder than $T_{\mathrm{T},\infty}$ can*not* be in thermodynamic equilibrium with that vacuum, but *must eventually completely evaporate into that vacuum*! Thus at least the *qualitative* fact that Hawking (Tolman!) radiation emanates from *all* gravitators—not only from black holes but also from *non*-black holes—(even if not also the *quantitative* value of $T_{\mathrm{H},r\rightarrow\infty}$ [15,37–43]) was discovered at least as early as 1930 [8,9]! *Any* gravitator—black hole or *non*-black hole—surrounded by a 0 K vacuum can*not* be at thermodynamic equilibrium unless it is enclosed within an opaque thermally insulating shell [57,58] and thereby insulated from that vacuum: otherwise it will eventually *completely* Hawking- (Tolman!-) evaporate into that vacuum! This has been corroborated by recent research]11].

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Black holes evaporate ever more rapidly as they lose mass, and thus *completely* evaporate into a vacuum at absolute zero (0 K) in a *finite* time $\Delta t_{\mathrm{BH,evap}}$. For evaporation of black holes into a 0 K vacuum [37–43]:

$$\begin{aligned}
\frac{dM}{dt} &= -\frac{1}{c^2}\frac{dE}{dt} = -\frac{1}{c^2}A\sigma T^4_{\mathrm{H},r\to\infty} \\
&= -\frac{1}{c^2}4\pi r_{\mathrm{S}}^2\sigma\left(\frac{\hbar c^3}{8\pi GkM}\right)^4 \\
&= -\frac{1}{c^2}4\pi\left(\frac{2GM}{c^2}\right)^2\sigma\left(\frac{\hbar c^3}{8\pi GkM}\right)^4 \\
&= -\frac{\sigma\hbar^4c^6}{256\pi^3G^2k^4M^2} \equiv -\frac{C_1}{M^2} \\
&\implies dt = -\frac{M^2}{C_1}dM \\
&\implies \Delta t_{\mathrm{BH,evap}} = \int dt = \frac{-\int_{M_{\mathrm{initial}}}^{0} M^2 dM}{C_1} = \frac{\int_0^{M_{\mathrm{initial}}} M^2 dM}{C_1} = \frac{M^3_{\mathrm{initial}}}{3C_1} \\
&\doteq 2.0972\times10^{67}\left(\frac{M_{\mathrm{initial}}}{M_\odot}\right)^3 \mathrm{y},
\end{aligned} \tag{5}$$

where the minus signs account for M *de*creasing during evaporation, the dot-equal sign ($\doteq$) means very nearly equal to, $\sigma = 5.670374419\times10^{-8}\,\mathrm{W}\,/\,\mathrm{m}^2\,\mathrm{K}^4$ is the Stefan-Boltzmann constant [59], $C_1 \equiv \sigma\hbar^4c^6/256\pi^3G^2k^4 \doteq M^3_\odot/\left(6.2916\times10^{67}\,\mathrm{y}\right)$, and $M_\odot$ is the mass of the Sun.

By contrast, *non*-black holes evaporate ever more slowly as they lose mass. But the time rate of this slowdown is itself sufficiently slow that they, too, *completely* evaporate into a vacuum at absolute zero (0 K) in a *finite* time. For a weak-field ($M_{\mathrm{initial}} \ll r_{\mathrm{S}}c^2/2G \iff r_{\mathrm{initial}} \gg r_{\mathrm{S}} = 2GM/c^2$) non-rotating spherical, i.e., Schwarzschild, *non*-black hole, $|g_{tt}(r)|$ and therefore also $|g_{tt}(r)|^{-1/2}$ is essentially equal to unity, and hence also by Equation (4) the Tolman [8,9] temperature $T_{\mathrm{T}}\left(\vec{\mathbf{r}}\right)$ is essentially constant at $T_{\mathrm{T},\infty}$, as M decreases from M_{initial} to $M_{\mathrm{final}} = 0$ and r decreases from r_{initial} to $r_{\mathrm{final}} = 0$ during the *entire* evaporation process. Moreover, $A \propto r^2$, and, assuming uniform density for simplicity (justified in the weak-field limit because gravity is too weak to significantly compress material with depth), also $A \propto M^{2/3}$. Hence in the weak-field limit for evaporation of non-rotating spherical (Schwarzschild) uniform-density non-black holes into a 0 K vacuum:

$$\begin{aligned}
\frac{dM}{dt} &\propto -AT^4_{\mathrm{T},\infty} \propto -r^2T^4_{\mathrm{T},\infty} \propto -M^{2/3}T^4_{\mathrm{T},\infty} = -C_2M^{2/3} \\
&\implies dt = -\frac{M^{-2/3}}{C_2}dM \\
&\implies \Delta t_{\mathrm{NBH,evap}} = \int dt = \frac{-\int_{M_{\mathrm{initial}}}^{0} M^{-2/3}dM}{C_2} = \frac{\int_0^{M_{\mathrm{initial}}} M^{-2/3}dM}{C_2} = \frac{3M^{1/3}_{\mathrm{initial}}}{C_2},
\end{aligned} \tag{6}$$

where the minus signs account for M *de*creasing during evaporation, and C_2 is a constant. $\Delta t_{\mathrm{NBH,evap}}$ is *finite* because although dM/dt decreases with decreasing M, it does so only proportionately to $M^{2/3}$. (In order to render $\Delta t_{\mathrm{NBH,evap}}$ infinite, dM/dt would have to decrease with decreasing M at least proportionately to M itself.) That $\Delta t_{\mathrm{NBH,evap}}$ is *finite* is corroborated by recent research [11].

Comparing Equations (5) and (6) with the *same* M_{initial} for both a Schwarzschild black hole and a non-rotating spherical (Schwarzschild) weak-field uniform-density non-black hole:

$$\frac{\Delta t_{\mathrm{NBH,evap}}}{\Delta t_{\mathrm{BH,evap}}} = \frac{\frac{3M^{1/3}_{\mathrm{initial}}}{C_2}}{\frac{M^3_{\mathrm{initial}}}{3C_1}} = \frac{9}{C_1C_2M^{8/3}_{\mathrm{initial}}} \equiv \frac{C_3}{M^{8/3}_{\mathrm{initial}}}. \tag{7}$$

Equation (5) provides a numerical value for C_1, namely $C_1 \equiv \sigma\hbar^4 c^6/256\pi^3 G^2 k^4 \doteq M_\odot^3/\left(6.2196\times 10^{67}\,\mathrm{y}\right)$. But without a numerical value for $T_{\mathrm{T},\infty}$—except that, as we showed in the third paragraph of this Section 3, $T_{\mathrm{T},\infty}$ *must be finitely higher than absolute zero* (0 K)—and consequently without a numerical value for C_2 and hence also for $C_3 \equiv 9/C_1C_2$, the best that we can do with respect to Equation (7) is a qualitative evaluation: If $M_{\mathrm{initial}} < C_3^{3/8}$, $\Delta t_{\mathrm{NBH,evap}} > \Delta t_{\mathrm{BH,evap}}$; if $M_{\mathrm{initial}} = C_3^{3/8}$, $\Delta t_{\mathrm{NBH,evap}} = \Delta t_{\mathrm{BH,evap}}$; if $M_{\mathrm{initial}} > C_3^{3/8}$, $\Delta t_{\mathrm{NBH,evap}} < \Delta t_{\mathrm{BH,evap}}$.

Thus *all* gravitators—*black holes and non-black holes*—are enveloped by atmospheres of equilibrium blackbody radiation. Because both $T_{\mathrm{H},r\to\infty} > 0\,\mathrm{K}$ [15,37–43] and $T_{\mathrm{T},\infty} > 0\,\mathrm{K}$ [8,9], neither a black hole nor a non-black hole can be in thermodynamic equilibrium with a surrounding vacuum at absolute zero (0 K), but *must* radiate into that vacuum and eventually *completely* evaporate into that vacuum, unless shielded from that vacuum by being enclosed within an opaque thermally-insulating shell [57,58].

The Tolman-Hawking evaporation of a black hole into a vacuum colder than $T_{\mathrm{H},r\to\infty}$ and of a non-black hole into a vacuum colder than $T_{\mathrm{T},\infty}$ is in accordance with the Second Law of Thermodynamics. The entropy of a black hole is large [37–43], but the entropy of the radiation dispersed into a vacuum colder than $T_{\mathrm{H},r\to\infty}$ by its Hawking-evaporation is even larger. The entropy of a non-black hole is not as large as that of a black hole of the same mass, providing even more scope for an entropy increase as it Tolman-evaporates into a vacuum colder than $T_{\mathrm{T},\infty}$.

Tolman was aware of the concept of black holes (even if not of the moniker "black hole"): see the last paragraph of Section 96 of Ref. [9]. Yet nowhere does this enter into Tolman's derivations [8,9] that at thermodynamic equilibrium temperature increases downwards[6] in *any* static, or even stationary, gravitational field [56]. Indeed, despite early contemplations of the concept of black holes [60–64], this concept [60–64] (and the moniker "black hole" [60–64]) was not mainstream until the 1960s [60–64]. Hence if Tolman's [8,9] discovery had borne fruit in 1930 (or shortly thereafter), it would have (i) initially been construed with respect to *non*-black holes and (ii) dubbed Tolman radiation instead of Hawking radiation: Hawking radiation would then initially have been construed as emanating from *non*-black holes—and dubbed Tolman radiation rather than Hawking radiation!

Note that with the substitution $T_{\mathrm{T},\infty} \to T_{\mathrm{H},r\to\infty} = \hbar c^3/8\pi GkM = \hbar c/4\pi k r_{\mathrm{S}}$ *with respect to black holes*, Tolman's (identical) Equations (128.6) and (129.10) of Ref. [9], rewritten as our Equation (4) [and his equivalent equations in slightly different form: the second equation in the Abstract and Equations (28), (29), (42), and (53) in the main text of Ref. [8], and in Equation (128.10) of Ref. [9]] reduce to our Equation (3) for the Schwarzschild metric, for which $g_{tt}\left(\vec{\mathbf{r}}\right) = 1 - \frac{r_{\mathrm{S}}}{r}$, and to our Equation (2) in the limit $r \to \infty$. These topics, and related ones, will be further discussed in Sections 5, 6, and 7. [Of course this substitution is *un*physical and hence can*not* be made with respect to *non*-black holes: as per the paragraph containing Equation (6), $T_{\mathrm{T}}\left(\vec{\mathbf{r}}\right)$ is essentially constant at $T_{\mathrm{T},\infty}$ as M decreases from M_{initial} to $M_{\mathrm{final}} = 0$ and r decreases from r_{initial} to $r_{\mathrm{final}} = 0$ during the *entire* evaporation process of a (weak-field-limit) Schwarzschild *non*-black hole.]

At this point, it is worthwhile to note the similarities[7]—owing to the equivalence principle [65,66][7]—between Hawking (Tolman) radiation and Unruh radiation; but also a caveat.[7]

4. All firewalls are at the Planck temperature

In Section 4, it may be helpful to envision a Schwarzschild black hole enclosed concentrically within an opaque thermally-insulating spherical shell at $r_{\mathrm{shell}} \to \infty$ [57,58].[8] Hawking radiation at temperature $T_{\mathrm{H},r\to\infty}$ as per Equation (2) is reradiated and/or reflected downwards[6] from the inner surface of this spherical shell, suffering increasing gravitational blueshift with decreasing r in accordance with Equations (3) and (4) [8–10,15–17,37–43,57,58].[7,8] Since thermodynamic equilibrium obtains *perfectly* within the shell [57,58],[8] the caveat "(at least essentially)" can be deleted from the sentence containing Equation (2): radiation within the shell is *exactly* blackbody [57,58].[8] Indeed,

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enclosure of *any* radiation—whether emanating from a source or freely existing in space[9]—within an opaque thermally-insulating shell ensures *perfect* thermodynamic equilibrium and hence an *exactly* Planckian blackbody spectrum [57,58].[8] For example, if the Sun was so enclosed, the currently *approximately* blackbody radiation [67–71] at its photosphere would become *exactly* blackbody [57,58].[8] Without enclosure within an opaque thermally-insulating shell, radiation *can* be *exactly* blackbody; with enclosure, it *must* be *exactly* blackbody [57,58].[8] (Of course, *exactly* blackbody radiation incorporates the cutoff of the Planckian blackbody spectrum for wavelengths exceeding the size of an enclosure or cavity [72,73]. But this is not a consideration for our spherical shell, because it is at $r_{\text{shell}} \to \infty$ [57,58].[8]) Moreover, it should be noted that the Planckian form of *any* exactly-blackbody spectrum, and thus its having an *exactly* well-defined temperature, survives gravitational frequency shifting [74]—and also motional Doppler frequency shifting [74], cosmological frequency shifting [74], and any combination of any two or all three types of frequency shifting [74].

Prima facie, by Equations (3) and (4), it might seem that arbitrarily close to the Schwarzschild radius r_S (but still at $r > r_S$) $T_{\text{H},r\to r_S} \to \infty$. But this is *not* so. Thus far, we have *not* taken into account that, if at $r > r_S$, it is *not* possible, even in principle (let alone in practice) to be arbitrarily close to r_S, because owing to quantum fluctuations spacetime breaks down as ruler distance [12] on the order of the Planck length [13–15]

$$l_{\text{Planck}} = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.616255\,(18) \times 10^{-35}\,\text{m} \tag{8}$$

is approached. (The standard uncertainty is 0.000018×10^{-35} m [15].) Thus, even in principle (let alone in practice), it is *not* possible, if at $r > r_S$, to be any closer to r_S than at minimum radial ruler distance [13–15]

$$(\delta l)_{\min} = l_{\text{Planck}} = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.616255\,(18) \times 10^{-35}\,\text{m} \tag{9}$$

beyond r_S.

We now derive $T_{\text{H},r}$ as a function of radial ruler distance [12] δl beyond r_S, which we denote as $T_{\text{H},r_S+\delta l}$, focusing on regions just barely beyond r_S, i.e., where $\delta l \ll r_S$. Letting $\delta r = r - r_S$ be Schwarzschild-coordinate radial distance (which is also radial area distance and radial distance from apparent size [12]) beyond r_S, we have

$$1 - \frac{r_S}{r} = 1 - \frac{r_S}{r_S + \delta r} = \frac{r_S + \delta r - r_S}{r_S + \delta r} = \frac{\delta r}{r_S + \delta r} \overset{\delta r \ll r_S}{=} \frac{\delta r}{r_S}. \tag{10}$$

Applying Equations (1) and (10) [12],

$$\begin{aligned} dl &= \left(1 - \frac{r_S}{r}\right)^{-1/2} dr \\ \Longrightarrow\; d\,(\delta l) &= \left(1 - \frac{r_S}{r}\right)^{-1/2} d\,(\delta r) \overset{\delta r \ll r_S}{=} \left(\frac{\delta r}{r_S}\right)^{-1/2} d\,(\delta r) = \left(\frac{r_S}{\delta r}\right)^{1/2} d\,(\delta r)\,. \end{aligned} \tag{11}$$

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The last step of Equation (10) and the second-to-last step of Equation (11) are justified because we focus on regions just barely beyond r_S, where $\delta r \ll r_S$. Applying Equation (11), if $\delta r \ll r_S$:

$$\begin{aligned}
&d\,(\delta l) \overset{\delta r \ll r_S}{=} \left(\frac{r_S}{\delta r}\right)^{1/2} d\,(\delta r) \\
&\implies \delta l \overset{\delta r \ll r_S}{=} \int_0^{\delta r} \left(\frac{r_S}{\delta r'}\right)^{1/2} d\,(\delta r') = 2\,(r_S \delta r)^{1/2} \implies \delta r \ll \delta l \ll r_S \\
&\implies \delta r \overset{\delta r \ll \delta l \ll r_S}{=} \frac{(\delta l)^2}{4 r_S} \\
&\implies \left(\frac{r_S}{\delta r}\right)^{1/2} \overset{\delta r \ll \delta l \ll r_S}{=} \frac{\delta l}{2\delta r} = \frac{\delta l}{2\frac{(\delta l)^2}{4 r_S}} = \frac{2 r_S}{\delta l}.
\end{aligned} \tag{12}$$

Hence, applying Equations (2), (3), (4), (10), (11), and (12), if $\delta r \ll \delta l \ll r_S$:

$$\begin{aligned}
&T_{\mathrm{H},r} = T_{\mathrm{H},r_S+\delta r} \overset{\delta r \ll \delta l \ll r_S}{=} T_{\mathrm{H},r\to\infty} \left(\frac{r_S}{\delta r}\right)^{1/2} \\
&\implies T_{\mathrm{H},r_S+\delta l} \overset{\delta r \ll \delta l \ll r_S}{=} 2 T_{\mathrm{H},r\to\infty} \frac{r_S}{\delta l} = 2 \frac{\hbar c}{4\pi k r_S} \frac{r_S}{\delta l} = \frac{\hbar c}{2\pi k \delta l}.
\end{aligned} \tag{13}$$

As noted in the paragraph containing Equations (8) and (9), even in principle (let alone in practice), δl can be no smaller than $(\delta l)_{\min} = l_{\mathrm{Planck}}$ [13–15]. Thus, minimizing δl at $(\delta l)_{\min} = l_{\mathrm{Planck}}$, by Equations (8), (9), and (13) we obtain

$$\begin{aligned}
T_{\mathrm{firewall}} &= T_{\mathrm{H},r_S+(\delta l)_{\min}} = T_{\mathrm{H},r_S+l_{\mathrm{Planck}}} = \frac{\hbar c}{2\pi k\,(\delta l)_{\min}} = \frac{\hbar c}{2\pi k l_{\mathrm{Planck}}} = \frac{\hbar c}{2\pi k \left(\frac{\hbar G}{c^3}\right)^{1/2}} \\
&= \frac{1}{2\pi k}\left(\frac{\hbar c^5}{G}\right)^{1/2} = \frac{1}{2\pi}\left[\frac{1}{k}\left(\frac{\hbar c^5}{G}\right)^{1/2}\right] = \frac{T_{\mathrm{Planck}}}{2\pi},
\end{aligned} \tag{14}$$

where

$$T_{\mathrm{Planck}} = \frac{1}{k}\left(\frac{\hbar c^5}{G}\right)^{1/2} = 1.416784\,(16) \times 10^{32}\,\mathrm{K} \tag{15}$$

is the Planck temperature [18]. (The standard uncertainty is 0.000016×10^{32} K [18].)

This result is *in*dependent of the mass M and hence also of the Schwarzschild radius $r_S = 2GM/c^2$ of a Schwarzschild black hole. As M and hence also r_S increases, by Equation (2) $T_{\mathrm{H},r\to\infty}$ decreases in inverse proportion. But $r_S/\delta l$ for any given $\delta l \ll r_S$ in general and hence $r_S/\,(\delta l)_{\min} = r_S/l_{\mathrm{Planck}}$ in particular increases in direct proportion. Hence in accordance with Equations (2) and (13)–(15) these two opposing factors cancel out. Because of quantum fluctuations in the metric at length scales of l_{Planck} [13–15], Equation (14) may be pushing the limit of accuracy of Equation (13), but we should expect Equation (14) to be valid at least in some average sense. Accordingly, perhaps we should not be too adamant about the small numerical factor of $1/2\pi$ in Equation (14), and hence recapitulate Equation (14) as

$$T_{\mathrm{firewall}} = T_{\mathrm{H},r_S+l_{\mathrm{Planck}}} \approx T_{\mathrm{Planck}}. \tag{16}$$

By Equation (13), recapitulated with the help of

$$\frac{\hbar c}{2\pi k} = 0.0003644464403\,\mathrm{m\,K} \tag{17}$$

as

$$T_{\mathrm{H},r_S+\delta l} \overset{\delta r \ll \delta l \ll r_S}{=} \frac{\hbar c}{2\pi k \delta l} = \frac{0.0003644464403\,\mathrm{m\,K}}{\delta l}, \tag{18}$$

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$T_{\mathrm{H},r_S+\delta l}$ still has high values in the region $l_{\mathrm{Planck}} \ll \delta l \ll r_S$, hence with quantum fluctuations in the metric of Equation (1) being negligible [13–15]. For example, the temperature of the Sun's core, 1.571×10^7 K [71], is equaled at $\delta l = 2.3198 \times 10^{-11}$ m, i.e., only slightly less than typical atomic dimensions; the (effective [67–71]) temperature of the Sun's photosphere, 5772 K [71], is equaled at $\delta l = 6.3140 \times 10^{-8}$ m, i.e., the dimensions of small microbes; and room temperature, 300 K, is equaled at $\delta l = 1.2148 \times 10^{-6}$ m, less than two orders of magnitude below the limit of naked-eye visibility $\approx 10^{-4}$ m.

Now let us consider the ruler-distance [12] wavelength of Hawking radiation in the region only slightly beyond r_S, i.e., where $\delta r \ll \delta l \ll r_S$. The ruler-distance [12] wavelength $\lambda^{\mathrm{Wien,max}}_{r_S+\delta l}$ of blackbody radiation in general and of Hawking radiation in particular at the Wien's-Displacement-Law maximum with respect to wavelength [59,75] corresponding to temperature T is [59,75]

$$\lambda^{\mathrm{Wien,max}} = \frac{0.002897771955}{T}\,\mathrm{m}\,. \tag{19}$$

(Since we are focusing on wavelength, we employ the Wien's-Displacement-Law maximum with respect to wavelength [59,75] as opposed to that with respect to frequency [59,75].) Hence by Equations (17) and (19) [59,75]:

$$\begin{aligned}
\lambda^{\mathrm{Wien,max}} &= \frac{\frac{0.002897771955}{T}\,\mathrm{m}}{\frac{\hbar c}{2\pi k}}\frac{\hbar c}{2\pi k} = \frac{\frac{0.002897771955}{T}\,\mathrm{m}}{0.0003644464403\,\mathrm{m\,K}}\frac{\hbar c}{2\pi k}\\
&= \frac{0.002897771955\,\mathrm{m\,K}}{0.0003644464403\,\mathrm{m\,K}}\frac{\hbar c}{2\pi k T} = 1.265466416\frac{\hbar c}{kT}\\
&\Longrightarrow \lambda^{\mathrm{Wien,max}}_{r_S+\delta l} \overset{\delta r \ll \delta l \ll r_S}{=} 1.265466416\frac{\hbar c}{kT_{\mathrm{H},r_S+\delta l}} = 1.265466416\frac{\hbar c}{k\frac{\hbar c}{2\pi k \delta l}}\\
&= 1.265466416 \times 2\pi\delta l = 7.951159991\delta l.
\end{aligned} \tag{20}$$

The numerical factor $1.265466416 \times 2\pi = 7.951159991$ is dimensionless and hence is valid in any self-consistent system of units. In the third line of Equation (20) we applied Equation (13). In accordance with the reasoning concerning quantum fluctuations in the metric in the paragraph ending with Equation (16) [13–15], perhaps we should not be too adamant about the small numerical factor of $1.265466416 \times 2\pi = 7.951159991$ in the last term of Equation (20), and hence recapitulate Equation (20) as

$$\lambda^{\mathrm{Wien,max}}_{r_S+\delta l} \overset{\delta r \ll \delta l \ll r_S}{\approx} \delta l. \tag{21}$$

Thus the ruler-distance [12] wavelength $\lambda^{\mathrm{Wien,max}}_{r_S+\delta l}$ in the region $\delta r \ll \delta l \ll r_S$ is on the order of the ruler distance [12] δl itself. In particular, at $(\delta l)_{\mathrm{min}} = l_{\mathrm{Planck}}$ [13–15]

$$\lambda^{\mathrm{Wien,max}}_{r_S+(\delta l)_{\mathrm{min}}} = \lambda^{\mathrm{Wien,max}}_{r_S+l_{\mathrm{Planck}}} \approx l_{\mathrm{Planck}}. \tag{22}$$

Hawking-radiation photons for which Equations (14)–(16) and (22) apply, and consequently for which $T_{\mathrm{firewall}} \approx T_{\mathrm{Planck}}$ and thus $E = h\nu \approx kT_{\mathrm{firewall}} \approx kT_{\mathrm{Planck}} \approx E_{\mathrm{Planck}} = m_{\mathrm{Planck}}c^2$ [13–15,76–79], are *themselves* Planck-mass black holes [13–15,80–82], specifically, Planck-mass geons [80–82], and thereby *themselves* contribute to the breakdown of spacetime as the Planck scale is approached, i.e., as $\delta l \to (\delta l)_{\mathrm{min}} = l_{\mathrm{Planck}}$ [13–15,80–82].

In accordance with the three immediately preceding paragraphs, and for consistency with Equation (20) keeping the numerical factor 7.951159991 [59,72], by Equations (19) and (20) [59,72]

$$\lambda^{\mathrm{Wien,max}} = \frac{0.002897771955}{T}\,\mathrm{m} \overset{\delta r \ll \delta l \ll r_S}{=} 7.951159991\delta l. \tag{23}$$

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Thus $\lambda^{\text{Wien,max}}$ corresponding to values of T that are still high occurs in the region $l_{\text{Planck}} \ll \delta l \ll r_{\text{S}}$, hence with quantum fluctuations in the metric of Equation (1) being negligible [13–15]. For example, $\lambda^{\text{Wien,max}}$ corresponding to the temperature of the Sun's core, 1.571×10^7 K [71], is equaled at $\delta l = 2.3198 \times 10^{-11}$ m, i.e., only slightly less than typical atomic dimensions; $\lambda^{\text{Wien,max}}$ corresponding to the (effective [67–71]) temperature of the Sun's photosphere, 5772 K [71], is equaled at $\delta l = 6.3140 \times 10^{-8}$ m, i.e., the dimensions of small microbes; and $\lambda^{\text{Wien,max}}$ corresponding to room temperature, 300 K, is equaled at $\delta l = 1.2148 \times 10^{-6}$ m, less than two orders of magnitude below the limit of naked-eye visibility $\approx 10^{-4}$ m.

Of course, the last two lines of Equation (20), and Equations (21) and (23) [let alone Equation (22)], do *not* apply in the region $r \gg r_{\text{S}}$. For, as $r \to \infty$, $\delta l \to r - r_{\text{S}} + (r_{\text{S}}/2)\ln(r/r_{\text{S}})$ [83], whilst applying Equation (2) and the first two lines of Equation (20) [59,75]:

$$\begin{aligned} \lambda^{\text{Wien,max}}_{r\to\infty} &= 1.265466416 \frac{\hbar c}{kT_{\text{H},r\to\infty}} = 1.265466416 \frac{\hbar c}{k\frac{\hbar c}{4\pi k r_{\text{S}}}} = 1.265466416 \times 4\pi r_{\text{S}} \\ &= 15.90231998 r_{\text{S}} = \text{ constant.} \end{aligned} \tag{24}$$

The numerical factor $1.265466416 \times 4\pi = 15.90231998$ is dimensionless and hence is valid in any self-consistent system of units.

We have considered Schwarzschild black holes whose *only* energy source is their own Hawking radiation. This may eventually be the case for actual black holes if the Universe expands forever. But in the current Universe, black holes are bathed by photons emanating from $r \gg r_{\text{S}}$—effectively from $r \to \infty$—far more energetic than at temperature $T_{\text{H},r\to\infty}$ as per Equation (2): photons from the $T_{\text{CBR}} = 2.725$ K [44] cosmic background radiation [44], from starlight, etc. [44]. Radiation comprised of these far more energetic photons will be blueshifted to $T_{\text{firewall}} \approx T_{\text{Planck}}$ as given by Equations (14)–(16) at $r_{\text{S}} + \delta l$ with $\delta l \gg l_{\text{Planck}}$. But photons corresponding to $T_{\text{firewall}} \approx T_{\text{Planck}}$, i.e., for which $E = h\nu \approx kT_{\text{firewall}} \approx kT_{\text{Planck}} \approx E_{\text{Planck}} = m_{\text{Planck}}c^2$ [13–15,80–82], are *themselves* Planck-mass black holes [13–15,80–82], specifically, Planck-mass geons [80–82], and thereby *themselves* might contribute to the breakdown of spacetime at *this* $r_{\text{S}} + \delta l$, i.e., at *this* $\delta l \gg l_{\text{Planck}}$, *well before* $(\delta l)_{\text{min}} = l_{\text{Planck}}$ is approached [13–15,80–82]. Hence in the current Universe we should consider at least the possibility of the breakdown of spacetime at *this* $r_{\text{S}} + \delta l$, i.e., at *this* $\delta l \gg l_{\text{Planck}}$, *well before* $(\delta l)_{\text{min}} = l_{\text{Planck}}$ is approached [13–15,80–82]. But this is *not* what we mean by a Schwarzschild black hole's firewall. By a Schwarzschild black hole's firewall we mean that which is *intrinsic* to the black hole itself, i.e., owing *solely* to its own Hawking radiation.

5. The exponential nature of the gravitational frequency shift

In Section 5, as in Section 4, it may be helpful to envision a Schwarzschild black hole enclosed concentrically within an opaque thermally-insulating spherical shell at $r_{\text{shell}} \to \infty$ [57,58].[8] Hawking radiation at temperature $T_{\text{H},r\to\infty}$ as per Equation (2) is reradiated and/or reflected downwards[6] from the inner surface of this spherical shell, suffering increasing gravitational blueshift with decreasing r in accordance with Equation (3) [15–17,37–43,57,58][7,8]—which we re-emphasize is a special case of Equation (4) [8–10].

Expressed in terms of r, at all $r \geq r_{\text{S}}$ the relativistic gravitational scalar potential Φ of a Schwarzschild black hole and its magnitude $|\Phi|$ are [12,16,17]

$$\begin{aligned} &\Phi = \frac{c^2}{2}\ln\left(1 - \frac{2GM}{rc^2}\right) = \frac{c^2}{2}\ln\left(1 - \frac{r_{\text{S}}}{r}\right) \\ &\Longrightarrow |\Phi| = \frac{c^2}{2}\left|\ln\left(1 - \frac{2GM}{rc^2}\right)\right| = \frac{c^2}{2}\left|\ln\left(1 - \frac{r_{\text{S}}}{r}\right)\right|. \end{aligned} \tag{25}$$

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Applying Equations (2), (3), (10), (11), and (12) [especially the last two lines of Equation (12)], if $\delta r \ll \delta l \ll r_S$, expressing Φ and $|\Phi|$ in terms of δr and δl [12,16,17]:

$$\Phi \overset{\delta r \ll \delta l \ll r_S}{=} \frac{c^2}{2} \ln \frac{\delta r}{r_S} = \frac{c^2}{2} \ln \frac{(\delta l)^2}{4r_S^2} = \frac{c^2}{2} \ln \left(\frac{\delta l}{2r_S} \right)^2 = c^2 \ln \frac{\delta l}{2r_S}$$
$$\Longrightarrow |\Phi| \overset{\delta r \ll \delta l \ll r_S}{=} \frac{c^2}{2} \ln \frac{r_S}{\delta r} = \frac{c^2}{2} \ln \frac{4r_S^2}{(\delta l)^2} = \frac{c^2}{2} \ln \left(\frac{2r_S}{\delta l} \right)^2 = c^2 \ln \frac{2r_S}{\delta l}. \tag{26}$$

It may be interesting to note that corresponding to minimum-definable ruler distance [12] $\delta l_{\text{min}} = l_{\text{Planck}}$ [13–15] beyond r_S

$$|\Phi|_{r_S+(\delta l)_{\text{min}}} = |\Phi|_{r_S+l_{\text{Planck}}} = c^2 \ln \frac{2r_S}{\delta l_{\text{min}}} = c^2 \ln \frac{2r_S}{l_{\text{Planck}}} = c^2 \ln \frac{2r_S}{\left(\frac{\hbar G}{c^3}\right)^{1/2}} = c^2 \ln \left[r_S \left(\frac{4c^3}{hG} \right)^{1/2} \right]$$
$$= c^2 \ln \left(\frac{4r_S^2 c^3}{hG} \right)^{1/2} = \frac{c^2}{2} \ln \frac{4r_S^2 c^3}{hG} = \frac{c^2}{2} \ln \frac{Ac^3}{\pi hG} = \frac{c^2}{2} \ln \frac{4S}{\pi k}, \tag{27}$$

where A is the surface area of a black hole and S is its entropy [37–43].

We re-emphasize that a relativistic gravitational scalar potential Φ and hence also its magnitude $|\Phi|$ [16,17], and the relation thereof to gravitational potential energy [16,17], are valid concepts for *all* static, and even stationary, spacetimes [56] (not just Schwarzschild spacetime at $r \geq r_S$ [84–86]). And that the spacetime at *all* $r \geq r_S$ surrounding any Schwarzschild black hole is static [84–86], not merely stationary [56].

The blueshift of *any* photon (Hawking-radiation photon or otherwise) whose frequency, energy, and mass [76–79] at $r \to \infty$ are $\nu_{r\to\infty}$, $E_{r\to\infty} = h\nu_{r\to\infty}$, and $m_{r\to\infty} = E_{r\to\infty}/c^2 = h\nu_{r\to\infty}/c^2$ [76–79], respectively, upon falling radially inwards from $r \to \infty$, increases *exponentially* rather than merely linearly with increasing $|\Phi|$ (or, equivalently, with decreasing Φ) [16,17], in accordance with [16,17]

$$\nu(|\Phi|) = \nu_{r\to\infty} e^{|\Phi|/c^2}$$
$$\Longrightarrow E(|\Phi|) = h\nu(|\Phi|) = E_{r\to\infty} e^{|\Phi|/c^2} = h\nu_{r\to\infty} e^{|\Phi|/c^2}$$
$$\Longrightarrow m(|\Phi|) = \frac{E(|\Phi|)}{c^2} = \frac{h\nu(|\Phi|)}{c^2} = m_{r\to\infty} e^{|\Phi|/c^2} = \frac{E_{r\to\infty} e^{|\Phi|/c^2}}{c^2} = \frac{h\nu_{r\to\infty} e^{|\Phi|/c^2}}{c^2}. \tag{28}$$

This obtains because as a photon falls and gets blueshifted its mass [76–79] $m = E/c^2 = h\nu/c^2$ [which of course is solely its (kinetic energy)$/c^2$ [76–79], because a photon's rest mass is zero [76–79]] increases: the photon gets more massive as it falls. Thus as a photon falls through successive ruler-distance [12] increments dl, a Schwarzschild black hole's gravitational field at $r \geq r_S$—indeed, the gravitational field $g = -d\Phi/dl$ in *any* static, or even stationary, spacetime [56,84–86]—does successive increments of (positive) work [85,86]

$$dW = -md\Phi = -m\frac{d\Phi}{dl}dl = mgdl \tag{29}$$

not on a fixed mass m but on an *ever-increasing* mass m. [The minus sign in $g = -d\Phi/dl$ obtains because g acts *downwards*,[6] i.e., in the direction of *decreasing* l. dW in Equation (29) is positive, because g is negative, and both $d\Phi$ and dl are negative during infall.] $dW/d\Phi$ and thus the rate of increase of $m = E/c^2 = h\nu/c^2$ with decreasing Φ is proportional to $m = E/c^2 = h\nu/c^2$ itself: consequently the *exponential* form of Equation (28).

Hence also, in accordance with Equations (3), (13), (14), (25), (26), (28), and (29), the temperature T of any Planckian blackbody distribution of photons increases *exponentially* rather than merely linearly with increasing $|\Phi|$ (or, equivalently, with decreasing Φ) [15–17,37–43,57,58,74,76–79].[7] In this regard,

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let us recapitulate Tolman's [9] (identical) Equations (128.6) and (129.10), again rewritten as our Equation (4) but now with added terms [8–10,16,17,84–86] (recall Section 3):

$$T_{\mathrm{T},r} = T_\infty \left| g_{tt}\left(\vec{\mathbf{r}}\right)\right|^{-1/2} = \begin{cases} T_\infty e^{\left|\Phi\left(\vec{\mathbf{r}}\right)\right|/2c^2} = T_\infty e^{|\Phi(r,\theta,\phi)|/2c^2} \text{ in general} \\ T_{\mathrm{T},r\to\infty}\left(1-\frac{r_\mathrm{S}}{r}\right)^{-1/2} = T_{\mathrm{H},r\to\infty}\left(1-\frac{r_\mathrm{S}}{r}\right)^{-1/2} \text{ Schwarzschild} \end{cases}. \quad (30)$$

Of course, the same reasoning also applies in reverse: as a photon rises a Schwarzschild black hole's gravitational field at $r > r_\mathrm{S}$—indeed, the gravitational field $g = -d\Phi/dl$ in *any* static, or even stationary, spacetime [56,84–86]—does negative work on, or equivalently receives positive work from, *not* a fixed mass m but an *ever-decreasing* mass m. $dW/d\Phi$ and thus the rate of decrease of $m = E/c^2 = h\nu/c^2$ is proportional to $m = E/c^2 = h\nu/c^2$ itself: consequently as per Equations (28) and (29) a rising photon's mass [76–79] $m = E/c^2 = h\nu/c^2$ decreases *exponentially* rather than merely linearly with decreasing $|\Phi|$ (or, equivalently, with increasing Φ) [15–17,37–43,57,58,74,76–79].[7] Hence also, in accordance with Equations (3), (13), (14), (25), (26), (28), (29), and (30), the temperature T of any Planckian blackbody distribution of photons decreases *exponentially* rather than merely linearly with decreasing $|\Phi|$ (or, equivalently, with increasing Φ) [15–17,37–43,57,58,74,76–79].[7]

By contrast, for a slowly radially-moving (slow *physical*—*not* necessarily slow coordinate!—radial velocity $V_\mathrm{phys} = dl/d\tau \ll c$) nonzero-rest-mass particle, the increase of total mass in free fall (and its decrease in free rise from an upwards[6] flying start) is on a pro rata basis much smaller than for a photon—a linear rather than exponential function of $|\Phi|$ (or Φ). This obtains because its (kinetic energy)$/c^2 = V^2/2c^2$ is only a *negligibly small fraction* of its total mass—*not* the *entirety* [76–79] of its total mass as is the case for a photon (or other zero-rest-mass particle) [76–79].

Of course, the First Law of Thermodynamics (energy conservation) always obtains. The kinetic energy that any entity gains (loses) by falling (rising) in a gravitational field is exactly offset by the energy of the gravitational field itself becoming more (less) strongly negative. This point will be discussed more thoroughly in Section 6.

6. Negative gravitational mass-energy and Birkhoff's Theorem versus massiveness of the firewall

We have taken for granted in our calculations that a firewall does not contribute (at a maximum, not more than negligibly) to the mass M of a Schwarzschild black hole. But this has been seriously questioned [3]. It has been averred that this can*not* be even approximately true for any Schwarzschild black hole whose mass M appreciably exceeds the Planck mass $m_\mathrm{Planck} = (\hbar c/G)^{1/2}$ [3]—a minimum-possible-mass $M = M_\mathrm{min} = m_\mathrm{Planck} = (\hbar c/G)^{1/2}$ Schwarzschild black hole—which would Hawking-evaporate on a time scale on the order of the Planck time $t_\mathrm{Planck} = \left(\hbar G/c^5\right)^{1/2}$ [3]. And that at best this can just barely be even approximately true *even if* $M = M_\mathrm{min} = m_\mathrm{Planck} = (\hbar c/G)^{1/2}$ [3]. This is the firewall-mass problem [3].

There is not universal agreement concerning the firewall-mass problem [3]. Counter-arguments resolving this problem have been proposed [4].

In Section 6, we do not make any assumption about what the mass of a firewall might be: small, large, or perhaps annulled to zero [3,4]. However, we consider the firewall-mass problem [3], and provide an at least tentative resolution that at least *prima facie* seems to be valid irrespective of what its mass might be. Our at least tentative resolution is based on: (i) the mass of a firewall (whatever it might be, if not annulled to zero [4]) being *exactly* canceled by the *negative* gravitational mass [16,17] = (*negative* gravitational energy)$/c^2$ [16,17] accompanying its formation, (ii) the *un*changed observations of a distant observer upon formation of a firewall, and (iii) Birkhoff's Theorem [19–22]—actually first discovered by Jørg Tofte Jebsen [22]. This is in addition to, and perhaps may complement, other lines of reasoning [4] disputing massiveness [3] of firewalls. (There is a caveat [23–25][1] with respect to Birkhoff's Theorem [19–22], but it [23–25] is *not* relevant with respect to our considerations.[1])

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The viewpoint [3] that formation of a firewall imparts a *huge net* increase to the mass of a Schwarzschild black hole [3] seems to overlook the *negative* gravitational mass [16,17] = (*negative* gravitational energy)$/c^2$ [16,17] contribution to the black-hole/firewall system. The *negativity* of gravitational energy [16,17] is the perhaps the central aspect of our at least tentative resolution of the firewall-mass problem [3]. We hope to show that the *negative* gravitational energy [16,17] accompanying *formation* of a firewall *exactly*—not merely approximately—cancels the firewall mass, so that the mass M of a black hole remains *exactly*—not merely approximately—*un*changed if a firewall forms. Is this, at least *prima facie*, what Ref. [3] overlooks? Reference [3] derives the mass of a firewall of an *already-collapsed* black hole, but seems to overlook the increase in negative gravitational mass-energy accompanying *formation* of the firewall *during collapse*.

We note that the negative gravitational mass-energy accompanying formation of a firewall should *not* be confused with considerations regarding negative energy states of the firewall *itself* [3]. While we do not make any assumption about what the mass of a firewall might be, in accordance with, and in agreement with, the first two paragraphs of *Discussion* in Ref. [3], we *always* construe its mass (if not annulled to zero [4]) to be positive—*even if* there exist negative energy states: the squares of both positive and negative numbers are positive: see the term E_F^2 in Equation (10) of Ref. [3]. We show that, whatever the mass of a firewall might be, the *negative* gravitational mass [16,17] = (*negative* gravitational energy)$/c^2$ [16,17] accompanying its formation annuls it (*even if* it is not otherwise annulled [4])—effecting *zero net change* in the mass of a Schwarzschild black hole.

Consider a spherically-symmetrical non-rotating gravitator of mass M but of sufficiently low average density that it is by a very wide margin a *non*-black hole ($r \gg r_S = 2GM/c^2 \iff M \ll r_S c^2/2G$), surrounded by a vacuum at absolute zero (0 K). As shown in Section 3, this gravitator will eventually completely *Tolman*-radiation [8,9] evaporate (see also Garrod [10]), yielding energy $E = Mc^2$ to a distant observer at $r_{obs} \gg r \gg r_S$. We re-emphasize that this has been corroborated by recent research [11].

Now instead consider another *identical* non-rotating gravitator of mass M. But this time let the structural strength of the gravitator be annulled, so that it gravitationally collapses *radially* to a black hole. *This* gravitator will then eventually completely *Hawking*-radiation evaporate, *also* yielding the *same* energy $E = Mc^2$ to a distant observer at $r_{obs} \gg r \gg r_S$. Indeed, this is required not only by the First Law of Thermodynamics (energy conservation), but also by Birkhoff's Theorem [19–22]. (There is a caveat [23–25][1] with respect to Birkhoff's Theorem [19–22], but it [23–25] is *not* relevant with respect to our considerations.[1]) For Birkhoff's Theorem [19–22] states that *any* purely radial gravitational collapse (and *any* purely radial dispersion against gravity from a flying start) of a spherically-symmetric gravitator can*not* cause *any* observable change by a distant observer [not even gravitational waves, because radial collapse (or radial dispersion) does not generate them [19–22]: Birkhoff's Theorem [19–22] *authorizes no exception* for gravitational collapse of the innermost shell of the star's Tolman-Hawking [8–10] radiation atmosphere to a firewall. This is possible *if and only if* the mass of the gravitator does *not* change during collapse—*even if* a firewall forms. And *this*, in turn, is possible *if and only if* the mass of the firewall is *exactly counterbalanced* by the increased negativity of gravitational mass-energy accompanying its formation.

Thus there *must* be *zero* net change in mass of the gravitator. Any increase in mass—whether due to formation of a firewall and/or otherwise—accompanying collapse *must be exactly counterbalanced by a negative contribution*. Gravitational mass = (gravitational energy)$/c^2$ is *always* a *negative* contribution to mass. And the only possible counterbalancing negative contribution is the gravitational mass-energy of the gravitator becoming *more strongly* negative during collapse. This must be true whether or not a firewall forms. If a firewall does *not* form, the increase in mass of the collapsing gravitator's Tolman-Hawking [8–10] radiation atmosphere will be less than if one *does* form—but so will the increase in the negativity of the entity's gravitational mass-energy.

It may be helpful to briefly discuss Tolman-Hawking [8–10] radiation atmospheres. Consider a spherically-symmetrical non-rotating entity (black hole or non-black hole) enclosed concentrically

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within an opaque thermally-insulating spherical shell at $r_{\text{shell}} \to \infty$ [57,58].[8] Such an entity is enveloped by a Tolman-Hawking [8–10] radiation atmosphere. Because the entity is enclosed within an opaque thermally-insulating spherical shell, its Tolman-Hawking [8–10] radiation atmosphere is at thermodynamic equilibrium throughout. Hence photons of radiation emanate from *anywhere* in this radiation atmosphere. To be specific, if this entity *is* a black hole, it is equally valid to construe photons emanating *not* (i) from $r_S + l_{\text{Planck}}$ and then suffering gravitational redshift upon streaming outwards towards the inner surface of our spherical shell at $r \to \infty$, but instead emanating (ii) from the inner surface of our spherical shell at $r \to \infty$ and then suffering gravitational blueshift upon falling inwards. Thus instead of construing Hawking radiation as suffering maximal gravitational redshift at $r \to \infty$ and no gravitational redshift at $r_S + l_{\text{Planck}}$, we can construe it as suffering no gravitational blueshift at $r \to \infty$ and maximal gravitational blueshift at $r_S + l_{\text{Planck}}$. This viewpoint is valid because: (a) the *entire region* within our spherical shell is at *thermodynamic equilibrium throughout*. And at thermodynamic equilibrium, the principles of microscopic reversibility and detailed balance obtain [87]: hence it is equally valid to consider microscopic processes occurring in either the "forward" or "reverse" direction [87]. Indeed, at *thermodynamic equilibrium*, which direction (i) or (ii) immediately above is taken as "forward" or "reverse" is arbitrary [87]. (There are caveats [88,89],[10] but they are *not* relevant with respect to our considerations.[10]) (b) Curved spacetime is hot [8–10]. Thus—if the gravitational frequency shift and hence temperature increasing downwards[6] in gravitational fields is taken into account [8–10,74]—it is equally valid to construe Tolman-Hawking [8–10] radiation as emanating from *any* $r > r_S$ [8–10,74,87]. A Tolman-Hawking [8–10] radiation photon of mass $m_{r\to\infty} = E_{r\to\infty}/c^2 = h\nu_{r\to\infty}/c^2 \approx kT_{\text{H},r\to\infty}/c^2$ [76–79] at the inner surface of our spherical shell at $r \to \infty$ does indeed gain mass $m_{\text{Planck}} - m_{r\to\infty}$ during its infall to $r_S + l_{\text{Planck}}$, i.e., to $(\delta l)_{\text{min}} \approx l_{\text{Planck}}$ [13–15,76–79], attaining mass $\approx m_{\text{Planck}} = E_{\text{Planck}}/c^2 = h\nu_{r_S+l_{\text{Planck}}}/c^2 \approx kT_{\text{firewall}}/c^2 \approx kT_{\text{Planck}}/c^2$ after having fallen to $r_S + l_{\text{Planck}}$, i.e., to $(\delta l)_{\text{min}} \approx l_{\text{Planck}}$ [13–15,76–79]. But the increase $m_{\text{Planck}} - m_{r\to\infty}$ in the photon's mass [13–15,76–79] that occurs during its infall is *exactly counterbalanced* by the decrease in the gravitational mass-energy [16,17] of the black-hole/photon system [76–79] that, by the First Law of Thermodynamics (energy conservation), *also* occurs during the photon's infall. Thus the *net* contribution to the mass of the black-hole/photon system [76–79] continues to be *only* $m_{r\to\infty}$—it does *not* increase to $m_{\text{Planck}} - m_{r\to\infty}$—*exactly as if the photon had not suffered infall!*

If this is true with respect to any *one* infalling photon, then it must also be true with respect to *all* of the infalling photons combined required to produce a spherical shell of equilibrium blackbody radiation with inner boundary at r_S, of ruler-distance [12] radial thickness l_{Planck}, and at temperature T_{Planck}—i.e., to produce a firewall. Hence at least *prima facie* it seems that the large increase in the mass [3]—indeed *any* increase in mass at all—of the black hole attributable to firewall formation [3] is *exactly canceled out to zero*.

We re-emphasize that the downwards[6] increase in the temperature of Tolman-Hawking [8–10] radiation in the gravitational fields of Schwarzschild black holes is a special case of the general result of relativistic thermodynamics that at thermodynamic equilibrium temperature increases downwards[6] in *any* gravitational field [8–10] (at least, in *any* static, or even stationary, one [56,84–86]). Tolman-Hawking [8–10] radiation should be construed as emanating *not only* from $r_S + l_{\text{Planck}}$—indeed *not only* from *any* $r \geq r_S + l_{\text{Planck}}$—in the gravitational field of a Schwarzschild black hole—but from *anywhere* in *any* gravitational field whatsoever. This was very well conveyed by a seminar given by Dr. James H. Cooke at the Department of Physics at the University of North Texas in the 1980s—and most succinctly expressed by the title of this seminar: "Curved spacetime is hot"—confirming Tolman [8,9] (see also Garrod [10], and recall our Sections 3, 4, and 5). Of course, by "hot" it is meant hotter than absolute zero (0 K)—in even the weakest gravitational fields. Tolman-Hawking [8–10] radiation emanates from *every* location in *any* gravitational field however weak *in general*—not only from black holes, but also from *non*-black holes: Curved spacetime is hot [at least, hotter than absolute zero (0 K)] *in general*. This is *required* for consistency with temperature increasing downwards[6] given thermodynamic equilibrium in *any* gravitational field, however weak [8–10]. In this regard we

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re-emphasize, as Dr. James H. Cooke pointed out, that not only black holes, but also *non*-black holes, Tolman-Hawking [8–10]) radiate: In this regard, it may at this point be worthwhile to again recall Section 3 and the paragraph containing Equation (30) in Section 5. Indeed, as we noted in Section 3, when Tolman [8,9] anticipated Hawking radiation (see also Garrod [10]), if that anticipation had borne fruit in 1930 (or shortly thereafter), it would have (i) initially been construed with respect to *non*-black holes and (ii) dubbed Tolman radiation instead of Hawking radiation!

Generalizing, the free fall of *any* entity in *any* gravitational field can*not* result in *any* change in the mass of the gravitator/entity system, because by the First Law of Thermodynamics (energy conservation) the gain in the falling entity's kinetic energy [via increased frequency if it is a photon, or via increased *physical* downwards[6] velocity $V_{\text{phys}} = dl/d\tau$ (*not* necessarily *coordinate* downwards[6] velocity) if it is of nonzero rest mass] *must be exactly counterbalanced* by the gravitational mass-energy [16,17] of the gravitator/entity system becoming more strongly negative. And likewise the free rise (from an upwards[6] flying start) of *any* entity in *any* gravitational field can*not* result in *any* change in the mass of the gravitator/entity system, because by the First Law of Thermodynamics (energy conservation) the loss in the rising entity's kinetic energy [via decreased frequency if it is a photon, or via decreased *physical* upwards[6] velocity $V_{\text{phys}} = dl/d\tau$ (*not* necessarily *coordinate* upwards[6] velocity) if it is of nonzero rest mass] *must be exactly counterbalanced* by the gravitational mass-energy [16,17] of the gravitator/entity system becoming less strongly negative. Furthermore this remains true even if the fall or rise is *not* free but retarded by friction [20], because friction merely thermalizes the entity's kinetic energy within the gravitator/entity system. For example, a landslide on Earth (whether or not retarded by friction) does *not* change Earth's total mass-energy $E_{\text{Earth}} = M_{\text{Earth}}c^2$ (which includes the negative contribution from Earth's gravitational energy), because the kinetic energy of the landslide (whether or not thermalized by friction [20]) is *exactly counterbalanced* by the gravitational mass-energy [16,17] of the Earth/landslide system becoming more strongly negative.

We briefly remark that Earth's negative gravitational mass-energy [16,17] reduces Earth's mass by a fraction on the order of $V_{\text{escape}}^2/c^2 \sim 10^{-9}$, where V_{escape} is the escape velocity from Earth's surface ($\approx 1.1 \times 10^4$ m / s). While this fraction is small in relative terms, in absolute terms it is a substantial negative contribution to Earth's mass, on the order of the mass of an asteroid ~ 10 km in diameter ($\sim 10^{-3}$ of Earth's diameter)—e.g., the K–T boundary asteroid [90] that was the major factor (even if not the only one) that ended the dinosaurs' reign [90].[11]

We close Section 6 with this speculative paragraph. It has been speculated [3] that owing to a firewall perhaps an infalling particle "burns up at the horizon [3]". So we are steered in the direction of asking the following four admittedly speculative questions: (i) Might the particle be saved from falling through the horizon, i.e., through the Schwarzschild radius r_S of a black hole, by burning up? (ii) If so, does this at least *prima facie* seem to suggest the possibility that a collapsing *near*-black hole might be saved from falling through *its own* Schwarzschild radius r_S by beginning to burn up mass as soon as its surface approaches a ruler distance [12] of one Planck length beyond r_S: that black holes can thus come within a gnat's eyelash of forming, but can*not completely* form? This gnat's eyelash would of course *not* be sufficient to result in any *measurable or observable* astronomical or astrophysical dissimilarity from a *completely*-formed black hole. (iii) And, for example, given (ii) immediately above, that as Hawking evaporation of a gnat's-eyelash *near*-black hole proceeds into a vacuum whose temperature is at (or at least sufficiently close to) absolute zero (0 K), its surface always remains a ruler distance [12] of one Planck length beyond r_S, this being maintained until Hawking evaporation is complete? (iv) Might this be relevant, for example, with respect to solving the black-hole information paradox? For, if black holes *can* come within a gnat's eyelash of fully forming but can*not* fully form, no information can ever fall into a fully-formed black hole and hence there is no need for it to be retrieved from one. Of course, various (hopefully, mutually compatible) resolutions of the black-hole information paradox have been discussed [91–98][12]. We note that if black holes can thus come within a gnat's eyelash—but no further—of forming, the *maximum* possible depth $|\Phi|_{\max}$ of their gravitational wells is *finite*. For then, applying Equations (8) and (27) yields [37–43]

$$|\Phi|_{\max} = |\Phi|_{r_S+(\delta l)_{\min}} = |\Phi|_{r_S+l_{\text{Planck}}} = c^2 \ln \frac{2r_S}{\delta l_{\min}} = c^2 \ln \frac{2r_S}{l_{\text{Planck}}} = c^2 \ln \frac{2r_S}{\left(\frac{\hbar G}{c^3}\right)^{1/2}}$$
$$= c^2 \ln \left[r_S \left(\frac{4c^3}{hG}\right)^{1/2} \right] = c^2 \ln \left(\frac{4r_S^2 c^3}{hG}\right)^{1/2} = \frac{c^2}{2} \ln \frac{4r_S^2 c^3}{hG} = \frac{c^2}{2} \ln \frac{Ac^3}{\pi hG} = \frac{c^2}{2} \ln \frac{4S}{\pi k}. \tag{31}$$

7. Concluding thermodynamic remarks

In concluding, we note that equilibrium vertical gravitational temperature gradients that exist [8–10]—indeed that are *required* [8–10]—by relativistic thermodynamics [8–10] can*not* be exploited to violate the Second Law of Thermodynamics.

First, consider a gravitator enclosed concentrically within an opaque thermally-insulating spherical shell. Now consider a heat engine trying to so exploit an equilibrium relativistic gravitational temperature gradient, via a hot reservoir at a lower altitude at temperature T_{hot} and a cold reservoir at a higher altitude at temperature T_{cold}.

Macroscopic consideration: Thermodynamic equilibrium [8–10,99,100] exists within the shell, and thermodynamic equilibrium [8–10,99,100] necessarily implies hydrostatic equilibrium [100–105] (but not necessarily vice versa [8–10,99–105]). Owing to hydrostatic equilibrium [100–105] that thermodynamic equilibrium [8–10,99,100] necessarily implies, the weight Eg/c^2 of a parcel of heat energy E where the gravitational acceleration is g [8–10] exactly counterbalances its tendency to flow from higher temperatures at lower altitudes to lower temperatures at higher altitudes, so there is no flow of heat that a heat engine can utilize. [Likewise at hydrostatic equilibrium—even without, let alone with, thermodynamic equilibrium—the weight mg of a parcel of fluid (gas or liquid) of mass m where the gravitational acceleration is g [8–10] exactly counterbalances its tendency to flow from higher pressures at lower altitudes to lower pressures at higher altitudes, so there is no flow of fluid that a pneumatic engine can utilize.]

Microscopic consideration: While *macro*scopically there is no flow at thermodynamic equilibrium, by contrast, *micro*scopically, thermodynamic equilibrium is *dynamic*. At thermodynamic equilibrium, individual blackbody-radiation photons move up and down in any gravitational field. But: *Even without* our engine trying to convert any heat whatsoever from the hot reservoir into work, the gravitational redshift diminishes the temperature of equilibrium blackbody photons radiated at T_{hot} from the lower altitude of the hot reservoir to T_{cold} upon them reaching the higher altitude of the cold reservoir—thus diminishing the Carnot efficiency $\epsilon_{\text{Carnot}} = 1 - (T_{\text{cold}}/T_{\text{hot}})$ to $\epsilon_{\text{Carnot}} = 1 - (T_{\text{cold}}/T_{\text{cold}}) = 0$. What the gravitational temperature gradient giveth, the gravitational redshift taketh away [8–10]: after the gravitational redshift has taken its cut, there is *nothing* left over to be converted into work [8–10]. (Similarly, in accordance even with *non*relativistic hydrodynamics and thermodynamics [99–105], even though *micro*scopically at thermodynamic equilibrium individual fluid molecules comprising a gas or liquid move up and down in any gravitational field, gravitational pressure gradients can*not* be exploited by a pneumatic engine: at hydrostatic equilibrium *even without*—let alone with—thermodynamic equilibrium [99–105], what the gravitational pressure gradient giveth, the weight taketh away [100–105].)

Next, consider a gravitator *not* enclosed concentrically within an opaque thermally-insulating spherical shell, but instead surrounded by a vacuum at (or at least sufficiently close to) absolute zero (0 K). Because the depth of gravitational wells at least of non-black holes and [recall Equation (27)] at all $r \geq r_S + (\delta l)_{\min} = r_S + l_{\text{Planck}}$ of black holes—and, as per the last paragraph of Section 6, perhaps of *all* gravitational wells—is *finite*, the gravitator's equilibrium blackbody radiation will seep into the surrounding vacuum. Hence with *no* enclosure within such a shell, a heat engine *can* operate—but only at the expense of the increase in entropy accompanying the dispersal of the radiation into this vacuum.

[Similarly a gravitator's gas (liquid) atmosphere (hydrosphere) will evaporate into a surrounding vacuum. Hence with *no* enclosure within such a shell, a pneumatic engine *can* also operate—but only at the expense of the increase in entropy accompanying the dispersal of the atmosphere (hydrosphere) into the vacuum.]

Hence both with and without enclosure by an opaque thermally-insulating spherical shell, the Second Law of Thermodynamics is obeyed.

To re-emphasize, thermodynamic equilibrium [8–10,99,100] necessarily implies hydrostatic equilibrium [100–105], but not necessarily vice versa [8–10,99–105]. The terms "hydrostatic equation [100–103]" or "barometric equation [105]" are sometimes employed to denote hydrostatic equilibrium [100–103] but not necessarily thermodynamic equilibrium [8–10,99–105]. Earth's atmosphere and oceans are typically at hydrostatic equilibrium (or at least very nearly so). But, of course, because they are impelled by the large temperature difference between the hot solar disk and the cold rest of the sky, they are not at thermodynamic equilibrium.

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Notes

[1] Based on Birkhoff's Theorem (see Refs. [19–22]), it is usually averred that in General Relativity—as in Newtonian gravitational theory—the gravitational field vanishes and the gravitational potential is negative and *constant* within an evacuated non-rotating spherical shell. This implies that spacetime is Minkowskian within the shell. (See, for example, Ref. [12], Section 12.2B.) But, to the contrary, it has also been averred that, in General Relativity—*un*like in Newtonian gravitational theory—the gravitational field does *not* vanish within an evacuated non-rotating spherical shell, but instead that the field within the shell is directed radially outwards from the center [23,24]. This implies that the gravitational potential within the shell is negative but *not* constant, being least strongly negative at the center of the shell and most strongly negative at the inner surface of the shell. Moreover, contrary to the corresponding usual inference based Birkhoff's Theorem (see Ref. [12], Section 11.2B and Refs. [19–22]), this *further* implies that spacetime is *not* Minkowskian within the shell [23,24]. This would resolve a clock paradox in General Relativity [25]: If the gravitational *field* vanishes, the gravitational potential is negative and *constant*, and hence spacetime is Minkowskian within an evacuated non-rotating spherical shell, how is a clock within the shell to *know* that it is within the shell and thus at a *negative* gravitational *potential*, and hence that it must tick more slowly than a clock at $r/r_{\text{shell}} \to \infty$ and hence

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at *zero* gravitational *potential*? For, like a clock at $r/r_{\text{shell}} \to \infty$, it would then see *zero* gravitational *field* and hence Minkowski spacetime. And according to General Relativity, a clock, like any other entity, interacts *locally* with a gravitational *field—no action at a distance*. A *non*-vanishing gravitational *field* within an evacuated non-rotating spherical shell, which a clock therein can interact with *locally*, thus resolves this clock paradox [23–25]: via *local* interaction with a *non*-vanishing gravitational *field* a clock at the center of the shell *knows* that it must tick more slowly than a clock at $r/r_{\text{shell}} \to \infty$ [23–25], and a clock at the inner surface of the shell *knows* that it must tick more slowly yet [23–25]. Nonetheless a non-vanishing gravitational *field* within an evacuated non-rotating spherical shell does *not* alter any *other* inferences based on Birkhoff's Theorem. If, on the contrary, the gravitational field *does* vanish, the gravitational potential is negative and *constant*, and hence spacetime *is* Minkowskian within an evacuated non-rotating spherical shell, then resolution of this clock paradox would seem to require either (i) *local* interaction of *the clock's gravitational field*—which extends *beyond* the shell—with the *shell's gravitational field* somehow being communicated to the clock itself [24] or (ii) *local* interaction of the clock with the *shell's gravitational potential* [25]. The Aharonov-Bohm-effect counterpart of Option (i) is interpreting the Aharonov-Bohm effect as due to *local* interaction of an electron's *magnetic field* with the *magnetic field* within a tightly-wound solenoid—the electron's *magnetic field* penetrates *into* the solenoid—even though the electron *itself* sees *only* the solenoid's magnetic vector potential and *not* the solenoid's magnetic field [25]. (The electron *must be moving* relative to the solenoid in order for the Aharonov-Bohm-effect to occur and hence *must* generate a *magnetic* field in the reference frame of the solenoid. If the solenoid is tightly wound, the electron's *electric* field can*not* penetrate into it.) The Aharonov-Bohm-effect counterpart of Option (ii) is the standard interpretation of the Aharonov-Bohm effect: *local* interaction of the electron with the solenoid's *magnetic vector potential*, which does *not* vanish *outside* of the solenoid [25].

[2] Note, however, that while the *angular* term of the Schwarzschild metric [the last term in all three lines of Equation (1)] is of *identical Euclidean* form at *all* $r \geq 0$, by contrast *radial* ruler distance is $dl = \left(\frac{r_S}{r} - 1\right)^{1/2} dt$ at $r < r_S$, as opposed to $dl = \left(1 - \frac{r_S}{r}\right)^{-1/2} dr$ at $r \geq r_S$. This obtains because the dt and dr terms of Schwarzschild metric [Equation (1)] switch sign as r_S is crossed. See Ref. [12], Sections 11.1 and 12.C–12.1E (especially Sections 12.1D and 12.1E). Yet also note that in the line immediately following Equation (12.15) in Section 12.1E: At $r < r_S$: r is referred to as a 'time'—quotation marks in the original text—recognizing that while r is time*like*, r is *not* time itself.

[3] The special case discussed in Ref. [28]—the excess (extra-Euclidean) vertical radial ruler distance of $GM/3c^2$ from the center to the surface of a non-rotating sphere of mass M and uniform density (in the weak-field limit, i.e., $M \ll r_S c^2/2G$)— may help to clarify the vertical stretching of space from the Euclidean by gravity in general. It is a special case of the more general result discussed in Section 11.5 of Ref. [12]. By *vertical* it is of course meant perpendicular to the equipotential surface. The vertical direction does not in general coincide with the geometric center of a gravitator [see Ref. [12], Section 9.6 (especially the last two paragraphs)], but it does so coincide in the special case of a non-rotating spherical gravitator whose density varies at most only radially.

[4] English translations of Ref. [29] are provided in Refs. [30–32]. See also the Editor's Note (Ref. [33]) and Ref. [34], which synopsize and discuss Ref. [29].

[5] Even if the classical vacuum might be construed as nothingness, the quantum-mechanical vacuum—space as it actually exists—certainly can*not*. (See Ref. [13], pp. 418–419 and 480, Section 21.4, and Chapters 43–44; and Refs. [14,15].) If gravity stretches space, can space sustain tension? Since a medium capable of sustaining tension is required for the transmission of transverse waves [by contrast, longitudinal waves, e.g., sound, can travel through any (material, i.e., non-vacuum) medium], and since electromagnetic radiation is comprised of transverse waves, might space be construed as a latter-20th-century and 21st-century interpretation of the ether [sometimes spelled aether (the a is silent)] postulated in 19th-century physics? [See Ref. [12], Chapter 1 (especially Sections 1.6–1.10); Ref. [34], Chapter 1 (especially pp. 8–20), and p. 66; and Ref. [35], pp. 495–496.] The conventional viewpoint is, of course, that electromagnetic waves serve as their own medium—their own ether—via

the continual handoff of energy from transverse electric field to transverse magnetic field to transverse electric field ... . See Ref. [35], pp. 450–458 (especially pp. 452–453).

[6] By *downwards* it is of course meant perpendicular to the equipotential surface and towards a gravitator. Downwards is not in general towards the geometric center of a gravitator [see Ref. [12], Section 9.6 (especially the last two paragraphs)], but it is so in the special case of a non-rotating spherical gravitator whose density varies at most only radially. (Of course, upwards is in the opposite direction, i.e., perpendicular to the equipotential surface and away from the gravitator.)

[7] Equations (2), (3), (4), and (30) also follow from considerations of Unruh radiation and the equivalence principle (see Refs. [65,66]). An object undergoing acceleration a in Minkowski spacetime experiences Unruh radiation at temperature $T_U = \frac{\hbar a}{2\pi ck}$. In Schwarzschild spacetime, force $f = \frac{mc^4}{4GM}\left(1-\frac{r_S}{r}\right)^{-1/2}$ is required to dangle a mass m at r_S from a higher altitude $r > r_S$ (with a massless string): see Ref. [12], Section 12.2 [especially Equation (12.17)]. The corresponding acceleration is $a = \frac{f}{m} = \frac{c^4}{4GM}\left(1-\frac{r_S}{r}\right)^{-1/2}$ and hence the corresponding Unruh-radiation temperature is $T_{U,r} = \frac{\hbar a}{2\pi ck} = \frac{\hbar}{2\pi ck} \times \frac{c^4}{4GM}\left(1-\frac{r_S}{r}\right)^{-1/2} = \left(1-\frac{r_S}{r}\right)^{-1/2}\frac{\hbar c^3}{8\pi GkM} = \left(1-\frac{r_S}{r}\right)^{-1/2} T_{H,r\to\infty} = T_{H,r}$. In accordance with the equivalence principle, $T_{U,r}$ is equal to $T_{H,r}$ as per Equation (3). In the limit $r \to \infty$, in accordance with the equivalence principle, $T_{U,r\to\infty} = \frac{\hbar c^3}{8\pi GkM} = T_{H,r\to\infty}$ as per Equation (2). {See Ref. [12], Section 12.2 [especially Equation (12.17)] and Section 12.6.} But a caveat: It is *important* to note that: Hawking-radiation temperature T_H is by Equations (2), (3), (4), (28)–(30) a function of the gravitational *potential* Φ . By contrast, Unruh-radiation temperature T_U is a function of the motional *acceleration* a in Minkowski spacetime, so *prima facie* the equivalence principle might seem to suggest that it be the *same function* of the magnitude $|g| = |d\Phi/dl|$ of the gravitational *acceleration*, i.e., the *same function* of the magnitude of the *gradient* of the potential rather than a function of the potential itself (whether of a black hole or a non-black hole). But this *in*correctly implies that T_U need *not* in general be equal to T_H : e.g., at large enough r or for a weak enough (non-black-hole) gravitator that the Newtonian approximation $a = f/m = GM/r^2$ is valid with negligible error, this *in*correctly implies $T_{U,r} = \frac{\hbar a}{2\pi ck} = \frac{\hbar GM}{2\pi ckr^2}$, and in the limit $r \to \infty$, $T_{U,r\to\infty} = 0$ —in *dis*agreement with $T_{H,r\to\infty} > 0$ as per Equation (2), $T_{H,r}$ as per Equation (3), and Tolman's [8,9] generalization (see also Garrod [10]) as per Equations (4) and (30). The *correct* correlation between T_H and T_U is that obtained as per Ref. [12], Section 12.2, especially Equation (12.17): via dangling a mass at r_S [if the Schwarzschild gravitator is a *non*-black hole, *as if* it was a black hole (of the same mass but smaller radius)].

[8] Because matter is not a continuum but is comprised of atoms, our opaque thermally-insulating spherical shell cannot be arbitrarily thin and therefore cannot have an arbitrarily small surface mass density ρ_{shell} . Even to exist at all, it must be at least one atom thick. To be thermally-insulating, it must be opaque, and to be opaque it must be many atoms thick. (Opacity is a necessary but not sufficient condition for thermal insulation.) Hence (ignoring our speculations as per the last paragraph of Section 6) its M_{shell}/r_{shell} ratio must be within a finite upper limit if our spherical shell is not to be a black hole itself and suffer gravitational collapse: we must require the inequality $M_{shell}/r_{shell} = 4\pi\rho_{shell}r^2_{shell}/r_{shell} = 4\pi\rho_{shell}r_{shell} < c^2/2G \implies r_{shell} < c^2/8\pi G\rho_{shell}$. But it certainly is feasible for r_{shell} to greatly exceed $\lambda^{Wien,max}_{r\to\infty} = 15.90231998 r_S$ [see the paragraph containing Equation (24)] while still meeting this inequality and hence without risk of its gravitational collapse: the strong inequality $\lambda^{Wien,max}_{r\to\infty} \ll r_{shell} < c^2/8\pi G\rho_{shell}$, indeed, even the double strong inequality $\lambda^{Wien,max}_{r\to\infty} \ll r_{shell} \ll c^2/8\pi G\rho_{shell}$, is very easily met. This is sufficient is for $r_{shell} \to \infty$ to *effectively* obtain for all practical purposes.

[9] Usually it is assumed that electromagnetic radiation can be tracked to a source, but Maxwell's equations do *not* require this. This was pointed out to me by Dr. James H. Cooke in a private communication in the 1980s.

[10] The concepts of microscopic reversibility and detailed balance require modifications in cases of (i) time-symmetry-violating dynamics and (ii) collisions between *un*symmetrical molecules even given *non*-time-symmetry-violating dynamics. See, for example, Ref. [88] concerning (i) and Ref. [89] concerning (ii). But these modifications do *not* apply with respect to electromagnetic radiation in

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general and hence with respect to equilibrium blackbody radiation in particular. Hence the analyses provided in Refs. [8–10,57,58] concerning equilibrium blackbody radiation are *completely valid*.

[11] Auxiliary phenomena that might have contributed to the end of the dinosaurs' reign could have been a surge in volcanic activity, the impact of a secondary asteroid if the primary impactor had a satellite, etc.

[12] Reference [96] states that owing to quantum gravitational corrections, Hawking radiation is *not exactly* Planckian, i.e., *not exactly* blackbody, and thus *not exactly* maximum-entropy and hence a carrier of information. But this assumes that a black hole radiates into empty space. What happens if, instead, a black hole is enclosed concentrically by an opaque thermally-insulating spherical shell? *Initially* upon emission from the black hole, Hawking radiation emanating from the black hole would still carry information. But the Hawking radiation emanating from the black hole would then be thermalized to an *exactly* Planckian distribution within the spherical shell. Would its information then be lost? Or would the information be preserved, even if only in latent form, even after thermalization? Also, a few caveats concerning Ref. [96] are quoted in Ref. [97].

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