# On the surface charge density of a moving sphere

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It is shown that the surface charge density on a spherical conductor (spherical in the *lab* frame) is uniform, regardless of its velocity.

## I. A FEATURE OF THE ELLIPSOIDAL ELECTROSTATIC CHARGE DISTRIBUTIONS

We first note a curious property of the charge density on ellipsoidal conductors. This property is closely related to a recent observation by Liu[1] who proved that the charge density on ellipsoidal conductors is proportional to the fourth root of the Gaussian curvature.[1,2]

All the ellipsoidal surfaces are related by linear transformations; that is, any ellipsoid can be obtained from any other by means of a suitable linear mapping. Specifically, the ellipsoid $S^*$ given by $(x^*/a^*)^2 + (y^*/b^*)^2 + (z^*/c^*)^2 = 1$ is obtained from the ellipsoid $S$ given by $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, by applying the transformation

$$x = (a/a^*)x^*, \quad y = (b/b^*)y^*, \quad z = (c/c^*)z^*, \tag{1}$$

where $a^*$, $b^*$, $c^*$, and $a$, $b$, $c$ are the semiaxes of the ellipsoids $S^*$ and $S$, respectively. If an ellipsoid is deformed under such a transformation, the Gaussian curvature of the points on its surface obviously changes.

As is well known,[3] the surface element $dS$ of a surface whose equation is given in parametric form as $\mathbf{r} = \mathbf{r}(u,v)$ can be written as $dS = (EG - F^2)^{1/2}\, du\, dv$, where $E = \mathbf{r}_u \cdot \mathbf{r}_u$, $F = \mathbf{r}_u \cdot \mathbf{r}_v$, and $G = \mathbf{r}_v \cdot \mathbf{r}_v$ (the subscripts denote differentiation with respect to the indicated coordinate). Taking as parameters $u = x$ and $v = y$, the parametric equations of the surface are now $x = x$, $y = y$, and $z = z(x,y)$, and the surface element $dS$ can be expressed as

$$dS = \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{1/2} dx\, dy. \tag{2}$$

In particular, for an ellipsoid

$$dS = \frac{c^2}{z}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{1/2} dx\, dy. \tag{3}$$

The Gaussian curvature for this surface is[1]

$$K = \left[abc\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\right]^{-2}. \tag{4}$$

Using (3) and (4) we obtain

$$\frac{K^{1/4}}{(abc)^{1/2}}\, dS = \frac{c}{ab}\, \frac{dx\, dy}{z}. \tag{5}$$

Finally, taking into account Eq. (5) and the transformation rule (1), we obtain the following relation when the ellipsoid $S$ is deformed into the ellipsoid $S^*$:

$$[K^{*1/4}/(a^*b^*c^*)^{1/2}]dS^* = [K^{1/4}/(abc)^{1/2}]dS. \tag{6}$$

This tells us how a patch of surface $dS$ on $S$ transforms into the corresponding patch $dS^*$ on $S^*$ (Fig. 1).

Suppose now that the ellipsoid $S$ is given an electric charge $Q$. This will distribute itself over the surface with a density given by Liu's formula[4]:

$$\sigma = (Q/4\pi)[K^{1/4}/(abc)^{1/2}]. \tag{7}$$

When this charged ellipsoidal conductor is deformed into the ellipsoid $S^*$, Eqs. (6) and (7) yield the remarkable conclusion

$$\sigma^*\, dS^* = \sigma\, dS. \tag{8}$$

Corresponding patches on the two ellipsoids carry the same electric charge. It is just as if the charge remained fixed on the surface while the deformation takes place.

## II. A RELATIVISTIC CONSEQUENCE: ALL CHARGED CONDUCTING SPHERES EXHIBIT A UNIFORM DENSITY OF CHARGE

We now derive an interesting relativistic consequence of Eq. (8). Consider a conductor, moving at constant speed, that has a spherical shape *in the laboratory frame*. Due to Lorentz contraction, the conducting body must have the shape of a prolate spheroid in its own rest frame. Figure 2 represents this situation for $\gamma = (1 - v^2/c^2)^{-1/2} = 2$. The sphere $S$ (in the $OXYZ$ frame) is derived from the prolate spheroid $S^*$ (at rest in frame $O^*X^*Y^*Z^*$); it moves with velocity $v$ with respect to $OXYZ$, along $OX$. We shall assume that for $O = O^*$, $t = t^* = 0$. The relativistic invariance of electric charge[5,6] requires that the charges be equal on corresponding patches of the sphere in motion and the prolate spheroid at rest. But Eq. (8) relates the

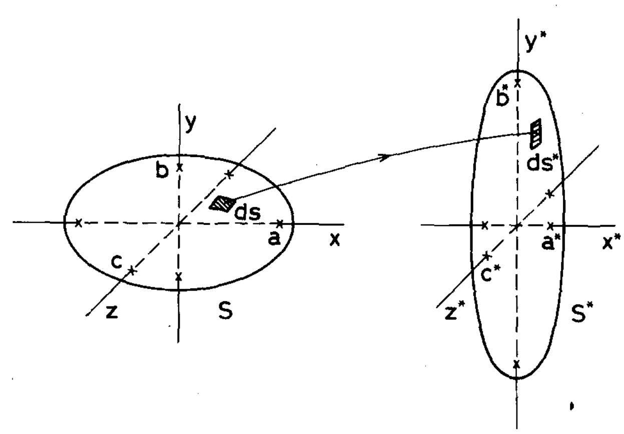


Fig. 1. Linear transformation between two ellipsoids and the corresponding patches of surface.

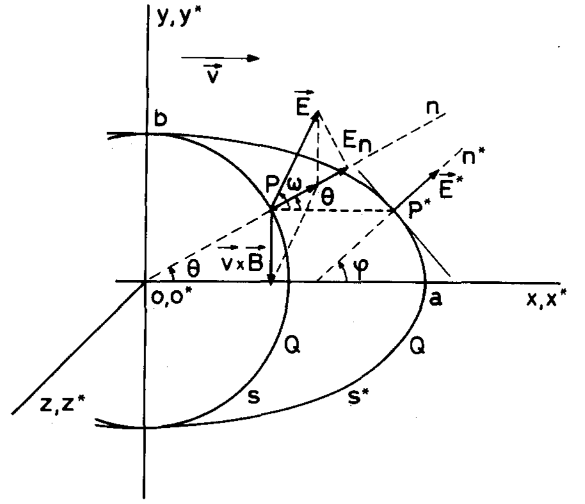


Fig. 2. Conducting prolate spheroid $S^*$ charged with an electric charge $Q$, at rest in the $O^*X^*Y^*Z^*$ frame, and the same body observed from the $OXYZ$ frame as a sphere $S$ in uniform motion. $O^*X^*Y^*Z^*$ moves with respect to $OXYZ$ along axis $OX$ with a velocity $v$ such that $\gamma = (1 - v^2/c^2)^{-1/2} = 2$.

densities of charge on a sphere and a prolate spheroid *both at rest.* Thus

$$(\sigma^* \, dS^*)_{\text{spheroid at rest}}$$
$$= (\sigma^{**} \, dS^{**})_{\text{sphere at rest}} \quad [\text{by Eq. (8)}]$$
$$= (\sigma \, dS)_{\text{sphere in motion}} \quad (\text{by invariance of charge}).$$

But $\sigma^{**}$ is uniform (obviously), and $dS^{**} = dS$ (since both derive from $dS^*$ by the same linear transformation). It follows that $\sigma$ is *also* uniform. We conclude that *every conductor with spherical shape* (in the laboratory frame) *exhibits a uniform density of charge, regardless of its velocity.* Incidentally, this argument can obviously be generalized: *Any* charged conducting surface observed in uniform motion with a relativistic ellipsoidal shape carries the same distribution of surface charge as a conducting ellipsoid at rest with the same total charge and the same geometric shape. For example, the charge density $\sigma$ of a moving conductor that is a sphere in its *rest* frame (and an oblate spheroid given by $\gamma^2 x^2 + y^2 + z^2 = R^2$ in the laboratory frame) is

$$\sigma = (\gamma Q / 4\pi R)(\gamma^4 x^2 + y^2 + z^2)^{-1/2}. \tag{9}$$

## III. DIRECT CONFIRMATION: ANALYSIS OF THE ELECTROMAGNETIC FORCES EQUILIBRIUM

However, in the case of an object in motion, the equilibrium of electric and magnetic interactions is more complex than in the trivial electrostatic case. Let us now see in detail how this electrodynamic equilibrium confirms our conclusion.

Our strategy will be to calculate the electrostatic field (for points just outside the surface) in the rest frame of the spheroid, use the relativistic transformations to deduce the fields in the frame of the moving sphere, and, from these, obtain the surface charge density on the sphere, as well as the electromagnetic force on the surface charge.

The surface density of charge $\sigma^*$ on the prolate conducting spheroid $S^*$ with charge $Q$, from Eqs. (7) and (4), is given by

$$\sigma^* = \frac{Q}{4\pi a b^2}\left(\frac{x^{*2}}{a^4} + \frac{y^{*2}}{b^4} + \frac{z^{*2}}{b^4}\right)^{-1/2}, \tag{10}$$

or, in terms of the angle $\theta$ (Fig. 2),

$$\sigma^* = (Q/4\pi b^2)(\cos^2\theta + \gamma^2 \sin^2\theta)^{-1/2}, \tag{11}$$

where $\gamma = (1 - v^2/c^2)^{-1/2} = a/b$. Now, the electric field $E^*$ at an external point arbitrarily close to $P^*$ is $E^* = \sigma^*/\epsilon_0$, and it points perpendicular to the surface. The angle $\varphi$ (Fig. 2) between the normal to the spheroid $n^*$ and the $O^*X^*$ axis is easily calculated: $\tan\varphi = \gamma \tan\theta$. Using the Lorentz transformation rules for electric and magnetic fields,[7] we obtain the components of the electromagnetic field at a point arbitrarily close to $P$:

$$E_x = E^*x^* = (Q/4\pi\epsilon_0 b^2) \times [\cos\theta/(\cos^2\theta + \gamma^2\sin^2\theta)], \tag{12}$$

$$E_y = \gamma E^*y^* = (Q/4\pi\epsilon_0 b^2) \times [\gamma^2 \sin\theta/(\cos^2\theta + \gamma^2\sin^2\theta)], \tag{13}$$

$$B_z = (v/c^2)\gamma E^*y^* = (v/c^2)E_y. \tag{14}$$

Evidently, the angle $\omega$ between $\mathbf{E}$ and the $OX$ axis is given by

$$\tan\omega = E_y/E_x = \gamma^2 \tan\theta. \tag{15}$$

The normal component of the electric field at a point arbitrarily close to the surface of the sphere $S$ is, therefore,

$$E_n = (E_x^2 + E_y^2)^{1/2}(\cos\omega\cos\theta + \sin\omega\sin\theta) = Q/4\pi\epsilon_0 b^2. \tag{16}$$

It follows that the charge density on the sphere is

$$\sigma = \epsilon_0 E_n = Q/4\pi b^2 \tag{17}$$

*independent of* $\theta$, confirming that the charge density on the moving sphere is uniform.

Now, with Eqs. (12)–(14), we can calculate the Lorentz force[8] on an element of surface charge $q$:

$$F_x = (q/2)(\mathbf{E} + \mathbf{v}\times\mathbf{B})_x = (q/2)E_x; \quad F_y = (q/2)(\mathbf{E} + \mathbf{v}\times\mathbf{B})_y = qE_y/2\gamma^2. \tag{18}$$

Thus the electromagnetic force acting on the surface charge of the relativistic sphere $S$ has a direction that is normal to the surface:

$$F_y/F_x = E_y/\gamma^2 E_x = \tan\theta. \tag{19}$$

Evidently the uniform charge density is sustained by a lucky conspiracy of electric and magnetic forces.


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[3] D. J. Struik, *Lectures on Classical Differential Geometry* (Addison-Wesley, Reading, MA, 1966), Chaps. II and III.
[4] Note that Liu's Eq. (14) is missing a factor of $1/4\pi$.
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[6] A. M. Portis, *Electromagnetic Fields* (Wiley, New York, 1978), pp. 192–196.
[7] See, for instance, Ref. 5, p. 239 or W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, MA, 1969), 2nd ed., pp. 327–331.
[8] The factor of $\frac{1}{2}$ arises because the fields are discontinuous at the surface. See Ref. 5,,pp. 29–31.

# A computer-assisted experiment in single-slit diffraction and spatial filtering

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An intermediate-level experiment examines Fraunhofer diffraction and its implications for the image formation of a coherently illuminated object. A commercially available software package is used to provide an environment that allows interactive data acquisition and subsequent numerical and graphical analysis.

## I. INTRODUCTION

The subject of Fraunhofer diffraction is a standard component of both introductory and upper-level undergraduate courses devoted in part or entirely to the discussion of physical optics. Several authors have reported on experiments designed to investigate diffraction patterns resulting from the illumination of various apertures[1–11]; in this experiment we extend these considerations to include the formation of an image of the diffracting aperture by an objective lens. By limiting our attention to Fraunhofer (plane wave) illumination, we simplify the analysis, and the resulting formalism serves as an excellent introduction to Fourier analysis. A textbook treatment of Fourier optics is not a prerequisite for this experiment; as we shall see, the same Huygens–Fresnel approach that is used to develop diffraction at the introductory level may be usefully employed here as well. Utilizing a single-slit aperture gives an optical representation of analytical techniques that will be very important in the student's subsequent study of subjects such as quantum mechanics and circuit theory. In addition, the implications of this experiment relating to the ultimate resolution of a microscope provide a fascinating and important topic of discussion.

How best to integrate microcomputers into the undergraduate experimental curriculum has been a topic of considerable debate; we believe that this experiment satisfies several important criteria in this respect.[12] While this would be a very good experiment if analog strip-chart traces were simply compared to theoretical plots, our use of the computer in this case provides an enhancement to the students' understanding of the phenomena being studied in addition to providing an introduction to some common tasks performed by microcomputers in the laboratory. In particular, digitizing the data allows for a least-squares fit to a nonlinear theoretical expression. Since these expressions are in the form of Fourier transform integrals, numerical integration must precede the fitting procedure. Subsequent graphical display of the fitted data confirms experimental technique and analytical treatment. Implementation of each of these important computing tasks requires differing levels of familiarity with the computing system at the hardware and software level. To prevent practical details from overwhelming the desired physical insights, it is appropriate to provide the student with some level of software support; our use of a commercially available software product for this purpose will be discussed below.

## II. THEORY

Figure 1 illustrates the case where a single-slit aperture is illuminated with plane-parallel, coherent radiation. According to the Huygens–Fresnel principle,[13] we may think of each point on the diffracting aperture as contributing a secondary forward-traveling spherical wavelet. As long as we are either far away from the aperture or at the focal plane of an objective lens (Fraunhofer plane), the coherent superposition of these elemental wavelets at the Fraunhofer plane results in an optical disturbance that can be characterized by a Fourier transform integral[13]:

$$U(\nu) = C \int_{-\infty}^{+\infty} g(y') e^{i(k\nu/f)y'} \, dy' , \tag{1}$$

where $g$ is the aperture function, $C$ is a constant, $y'$ measures a point on the aperture, $\nu$ measures a point on the Fraunhofer plane, $f =$ focal length of the objective lens, $k = 2\pi/\lambda$, $\lambda =$ illumination wavelength, and where we have assumed that $\theta = \nu/f$ is small enough so that $\tan(\theta) \sim \sin(\theta) \sim \theta$. To complete the Fourier transform analogy, one may define the "spatial frequency" $\mu = k\nu/f$, giving

$$U(\mu) = C \int_{-\infty}^{+\infty} g(y') e^{i\mu y'} \, dy' . \tag{2}$$

  