New York, 1968), p. 65.
[3]J. Kestin, Pure Appl. Chem. **22**, 511 (1970).
[4]Reference 2, p. 192.
[5]Reference 2, p. 196.
[6]Defining the *dissipative effects* thus by *enumerating* all five known phenomena is didactically the simplest approach; although they could also be defined as those phenomena where their inclusion in the mechanical computation of work causes either loss of work done by the system or an increment in the work done on the system.
[7]We regard work done by a system as positive; that on a system, negative.
[8]Reference 2, pp. 199–211; or M. W. Zemansky, Am. J. Phys. **34**, 914 (1966).
[9]Reference 2, p. 233.
[10]L. Tisza, *Generalized Thermodynamics* (MIT, Cambridge, MA, 1966), pp.113–114; or H. B. Callen, *Thermodynamics* (Wiley, New York, 1960), pp. 24–25.
[11]H. A. Buchdahl, *The Concepts of Classical Thermodynamics* (Cambridge University, London, 1966), p. 77.
[12]See e.g.,Ref. 2, pp. 234–236. This would also include the original and more general form of the Carathéodory statement.
[13]J. Kestin, *A Course in Thermodynamics, Vol. I* (Blaisdell, Waltham, MA, 1966), p. 412.
[14]R. K. Allgeier, A Syndetic Argument for Second Law Controversy, M.S. thesis, Department of Mechanical Engineering, the University of Houston, Houston, TX, 1972 (unpublished).
[15]In contrast, Kestin[3] really meant (as clear from the context) by the clause "deformation variables retain constant values" that they do so at the end states.
[16]L. C. Woods, *Thermodynamics of Fluid Systems* (Oxford University, Oxford, 1975), p. 17.
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Relativistic time dilatation of bound muons and the Lorentz invariance of charge

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The relativistic lengthening of the lifetime of a decaying elementary particle bound in a stationary state of an exotic atom provides evidence that the particle is actually in motion even though such motion cannot be visualized classically. This application of special relativity to the particles within an atom helps illuminate an argument frequently used for the Lorentz invariance of electric charge.

One of the strongest arguments adduced for the Lorentz invariance of charge is the exact electrical neutrality of atoms and molecules. The essence of the argument is as follows: if the magnitude of the charge of two elementary particles at rest with respect to a stationary observer is equal, then, were the Lorentz invariance of charge not valid, the observer would measure a charge imbalance of the bound system within which these elementary particles are in relative motion. As eloquently expressed by Purcell[1]:

> We are so accustomed to taking this for granted that we seldom pause to think how remarkable and fundamental a fact it is....If *motion* has any effect on the amount of charge, we could not have exact cancellation of nuclear and electronic charge in *both* the hydrogen molecule and the helium atom.

In the above example, emphasis has been placed on the vastly different states of motion of two protons bound in a nucleus by strong forces compared with two protons bound in a molecule by electron exchange. Since the two systems, the helium atom and the hydrogen molecule, have been shown to be neutral to better than one part in 10^{20}, the evidence for charge invariance seems quite convincing.[2,3]

There is a subtle point in arguments of this kind which is connected with the concept of motion of bound elementary particles in stationary states. An electron orbiting a proton in a Bohr atom has a well-defined trajectory and velocity. An electron in a quantum-mechanical atomic bound state does not. Although it may have a nonvanishing expectation value of angular momentum or kinetic energy, there is no sequential connection between neighboring points in the probability distribution of an electron in a stationary state. To think of the electron as moving (classically) within its probability distribution is to encounter a host of well-known paradoxes. (For example, how does it jump the nodes?)

Quantum theory has altered substantially the classical concepts of motion and, notwithstanding semiclassical heuristic models (e.g., the vector model of the atom) or the pictorial representation of electron clouds,[4] has made the visualization of the motion of bound particles largely impossible. While this visualization is by no means necessary or even relevant to the consistency of quantum mechanics, it does play a certain important role in special relativity. In the latter one must be able to imagine the placement of clocks and rods in different inertial reference frames and the subsequent performance of physically realistic measurements. The phenomenon of length contraction or time dilatation follows from the relativity of simultaneity in the different inertial frames as deduced from the outcome of physically meaningful measurement procedures.

If the *Lorentz invariance* of charge is to be inferred from the state of motion of bound elementary particles, then (although the equation of motion must necessarily be quantum mechanical) it must still be meaningful to conceive of a *Lorentz transformation* relating the rest frame of a particle to the rest frame of the stationary observer. Given that a bound elementary particle cannot, in principle, be located without an alteration of its state of motion, is such a conception meaningful? Expressed somewhat differently, is there any phenomenon exhibited by a bound elementary particle that can be interpreted as indicative of motion in accordance with special relativistic implications of the concept of motion?

Despite the fact that the motion of a bound elementary particle cannot be pictured or followed, it will be seen that the above question can be answered affirmatively. The assertion that a particle is in motion relative to a stationary observer has an observable physical consequence: a clock moving *with* the particle must exhibit relativistic time dilatation.

Were the electron a classical charged particle orbiting a center of force, this effect would in principle be evident from the Doppler shift of radiation emitted at different points in the electron trajectory. As in a binary star system, there would in fact occur both a blue and a red shift for any angle of observation other than 90° to the plane of motion. The electron is, of course, *not* a classical charged particle. Radiation occurs only by transition out of a stationary state and at a frequency generally unequal to the frequency characteristic of the orbital motion. The Doppler shift of atomic spectral lines derives from the essentially classical motion of the atomic center of mass, not from the quantum-mechanical motion of the electron.

There is an alternative and peculiarly quantum-mechanical clock associated with an elementary particle, viz. its natural lifetime. Since the electron, so far as one knows, is a stable elementary particle, its lifetime is infinite and no motional effect on it is observable. Such an effect *is* observable with the muon, a particle characterizable as a heavy electron. Like the electron, the muon has spin 1/2 and is subject to the Pauli exclusion principle; muons can replace electrons in the shell structure of an atom although, as a result of the muon's greater mass ($m_\mu = 207\ m_e$), the average muon orbital radius is 207 times smaller than that of an electron with the same quantum numbers. A major distinction between the muon and the electron is that the muon can decay by a weak interaction process to yield an electron, electron antineutrino, and muon neutrino. The mean free muon lifetime (reciprocal of the decay rate in the muon rest frame) is currently known to be[5] $2.197120 \pm 0.000077 \times 10^{-6}$ sec. Since there is no lighter charged lepton to which an electron can decay, the electron remains stable. According to a Bohr model of muonic hydrogen, the ground-state muon with orbital speed $v_\mu = Z\alpha c$ (where $\alpha = e^2/\hbar c \sim 1/137$ is the fine structure constant and Z is the nuclear charge number) can complete about $10^{12} \times Z^2$ revolutions before undergoing weak decay. The muonic atom would therefore seem to be a reasonably stable system with well-defined ground state.

In addition to weak decay, the muon can disappear by a nuclear capture reaction in which a proton, interacting with a negative muon, gives rise to a neutron and muon neutrino. The probability for this process increases with nuclear charge. For $Z = 11$, the rates of weak decay and nuclear capture are approximately equal; for heavy nuclei, the mean lifetime of a bound muon is essentially determined by the nuclear capture process.[6] Nevertheless, since the end products of the two processes are different, one can experimentally study the bound muon weak decay rate alone by monitoring the rate of production of decay electrons.

Over forty years ago, the now classic measurement by Rossi and Hall[7] of the decay rate of free muons as a function of muon momentum provided a direct and rigorous confirmation of the time dilatation phenomenon. This investigation is described in much of the pedagogical literature relating to special relativity[8]; a filmed version of the experiment has even been made.[9] In contrast, it is *not* widely realized that the observed lifetime of a muon (or any other decaying elementary particle) in a stationary state of an atom should also be affected by time dilatation.

Historically, the fact that positive and negative muons stopped in matter decayed at different rates provided the first evidence of the formation of muonic atoms and of the effect of binding on the muon decay rate.[10] A positive muon brought to rest in the presence of a positive nucleus cannot be captured into an atomic bound state; it subsequently decays at a rate characteristic of the free muon (either positive or negative). A negative muon in matter, however, is attracted by the Coulomb field of the nucleus and is generally captured into a high lying atomic state (Rydberg state); it then undergoes cascade transitions by Auger electron emission into deeper lying states, and thereafter preferentially makes radiative transitions until reaching the atomic ground-state. The time interval between atomic capture and formation of a ground-state muonic atom is on the order of 10^{-9} sec in a gas and less than 10^{-12} sec in condensed matter.[11] Fewer than about 10^{-2} muons decay from other than the ground state.

Although a detailed calculation employing a specific weak decay Hamiltonian is required to obtain the exact numerical value of the muon weak decay rate, the relativistic kinematical effect on this rate by time dilatation is independent of the particular structure of the decay Hamiltonian; it is deducible by constructing an appropriate operator $\hat{\gamma}$ whose expectation value in a stationary state gives directly the classical factor γ by which the observed rate is retarded. Classically, the dilatation factor is $\gamma = (1 - v^2/c^2)^{-1/2}$. The appropriate quantum-mechanical operator, obtained from the classical relativistic expression for kinetic energy, $T = (E - V) = mc^2\gamma$, is given by[12]

$$\hat{\gamma} = (H - V)/mc^2. \tag{1}$$

The expectation value of $\hat{\gamma}$ in the $1s_{1/2}$ ground state of a point Coulomb potential $V = Ze^2/r$ is

$$\begin{aligned}\langle\hat{\gamma}\rangle_{1s_{1/2}} &= \langle\Psi(1s_{1/2})|\hat{\gamma}|\Psi(1s_{1/2})\rangle\\ &= [1 - (Z\alpha)^2]^{-1/2},\end{aligned} \tag{2}$$

where $\Psi(1s_{1/2})$ is the four-component Dirac $1s_{1/2}$ wave function. Equation (2) follows readily from the expressions for the ground-state energy $E(1s_{1/2}) = mc^2[1 - (Z\alpha)^2]^{1/2}$ and the expectation value $\langle r^{-1}\rangle_{1s_{1/2}} = Z\alpha(mc/\hbar)[1 - (Z\alpha)^2]^{-1/2}$. It is interesting to note that this same dilatation factor is obtained for the $1s$ state in nonrelativistic quantum theory and for the circular orbits of the relativistic Bohr atom. From Eq. (2) one may infer that $(Z\alpha)^2$ in the Dirac theory is in some sense a measure of the mean square orbital velocity, as is the case in the nonrelativistic treatment of the point Coulomb potential.[13]

From Eq. (2) the relativistically retarded weak decay rate w of a muon bound to a point Coulomb potential is

$$w = [1 - (Z\alpha)^2]^{1/2} w_0, \tag{3}$$

where w_0 is the decay rate of a free, stationary muon. How does this prediction compare with the results of weak interaction theory and with experiment?

In the early 1960s a number of calculations of the bound-to-free muon decay rate ratio $R = w/w_0$ were made on the basis of a vector–axial vector (V–A) weak interaction. All calculations employed first-order perturbation theory in the weak interaction Hamiltonian (in the Fermi theory all higher-order terms are divergent) but differed slightly in the approximations used to evaluate the weak decay matrix element. Überall employed a relativistic muon wave function for a point nucleus and a Born expansion through Z^2 of the electron Coulomb wave function.[14] Gilinsky and Mathews employed a Sommerfeld–Maue wave function for the electron and a modified Coulomb wave function for the muon which took into account the influence of finite nuclear size.[15] Johnson *et al.* in studying the influence of the Coulomb field of a point nucleus, obtained analytical results to one power higher in $Z\alpha$ than the previous calculations.[16] The end result of this theoretical work has been to show that $R = 1 - (Z\alpha)^2/2$ to order $(Z\alpha)^3$. This result is equivalent to a Taylor expansion of Eq. (3) to the same order.

Actually, the bound muon decay rate is influenced by dynamical and statistical effects incurred through the binding as well as by the relativistic time dilation. The two principal effects, which work oppositely on the decay rate, are the electron Coulomb field effect and the phase space effect. In the former, the positively charged nucleus attracts the electron emitted in the muon decay and thereby produces a greater overlap of the muon and electron wave functions near the nucleus than would otherwise be the case for a plane-wave electron; the electron Coulomb field effect enhances the muon decay rate. In the latter, the volume of phase space accessible to the decay products of a bound muon is restricted in comparison to that of a free muon by the momentum distribution produced by the binding; the phase space effect reduces the decay rate. The expression for R shows that in the light and medium elements (for which the approximations leading to this expression are valid) there is a fortuitous cancellation of the Coulomb field and phase space effects; in these muonic elements, therefore, the departure of the bound muon weak decay rate from that of the free muon is principally a measure of the effect of time dilatation.

For the heavy muonic elements, the simple analytical expression for R is no longer necessarily justified. Huff, employing a general, local nonderivative decay interaction and relativistic electron and muon wave functions, has made extensive numerical calculations surveying elements throughout the periodic table.[17] His results are in basic agreement with the previous analytical estimates in their range of validity and indicate that the net bound muon decay rate is a smoothly decreasing function of Z. Direct comparison with Eq. (3), however, is no longer meaningful for heavy nuclei since the muon no longer sees a point coulomb potential. Moreover, the contributions of the various effects to the net decay rate ratio were not specified by Huff.

At the time of publication of the above-described theoretical work experimental results were consistent with theory for elements with $Z < 20$, but in the region $20 < Z < 30$ agreement was very poor.[18] In fact, for elements around iron the measured bound muon decay rate was even considerably greater than that of the free muon rate in complete disagreement with the trend predicted by Huff's numerical results. Except for the unlikely possibility that the basic structure of the weak decay interaction itself was in error, no theoretical explanation was found which could account for the anomalous results. An alternative possibility was that the anomalous decay rate enhancement around iron represented a contamination of the signal by low energy gamma rays associated with muon capture.[19] Fortunately, subsequent experimental work failed to reproduce the anomalous enhancement.[20,21] The experimental measurements reported in Ref. 21 of muonic elements ranging from vanadium ($Z = 23$) to lead ($Z = 82$) were in good accord with Huff's theory and provided evidence to support the belief that all important contributions to the bound muon decay process were reasonably well accounted for. A comparison of experimental results with the numerical calculations of Huff and with the expression for time dilation, Eq. (3), is given in Table I. A more comprehensive comparison of experiment and theory may be found in Fig. 1 of Ref. 20.

Table I. Experimental and theoretical values of $R(Z)$.

Z	$R(Z)^{a}_{\rm rel}$	$R(Z)^{b}_{\rm numer.}$	$R(Z)_{\rm expt.}$
6 C	1.00		1.00 ± 0.02 [c]
13 Al	0.99		0.99 ± 0.04 [c]
20 Ca	0.99		1.00 ± 0.03 [c]
23 V	0.99	0.98	1.00 ± 0.04 [d]
26 Fe	0.98	0.98	0.97 ± 0.04 [e]
			1.00 ± 0.04 [d]
27 Co	0.98	0.97	0.94 ± 0.04 [d]
28 Ni	0.98	0.97	0.96 ± 0.04 [d]
30 Zn	0.98	0.96	0.93 ± 0.04 [d]
50 Sn	0.93	0.92	0.87 ± 0.04 [d]
74 W	0.84	0.85	0.78 ± 0.04 [d]
82 Pb	0.80	0.84	0.86 ± 0.04 [d]

[a]Time dilatation; given by Eq. (3).
[b]Numerical results; taken from Refs. 17 and 21.
[c]Reference 18.
[d]Reference 21.
[e]Reference 20, cited in Ref. 21.

In conclusion, the experimental and theoretical evidence of the retardation of the bound muon decay rate—particularly for the light and medium elements ($Z < 30$) where this retardation can be clearly attributed to relativistic time dilation—may be taken as evidence that even within the interior of an atom where classical mechanics generally fails entirely to provide a quantitative or qualitative description of the motion of the constituent particles, the special relativistic *consequences* of motion still occur.[22] This is interesting and significant since, after all, special relativity is itself a classical theory predicated on measurement procedures that cannot be implemented in the interior of atoms. Although the Lorentz invariance of electic charge is a firmly established principle independent of the present demonstration, the recognition that a bound elementary particle manifests its motion in ways that one can appreciate classically helps illuminate the opening argument which is founded on the conviction that a particle in a stationary atomic state is in some sense moving.

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[6]E. Segrè, *Nuclei and Particles* (Benjamin, New York, 1965), p. 604.
[7]B. Rossi and D. B. Hall, Phys. Rev. **59**, 223 (1941).
[8]See, for example, G. Holton, Am. J. Phys. **30**, 462 (1962); a fairly extensive discussion is given in A. P. French, *Special Relativity* (Norton, New York, 1968), p. 97.
[9]See D. H. Frisch and J. H. Smith, Am. J. Phys. **31**, 342 (1963); the film "Time Dilation—an Experiment with Mu-Mesons" has been produced by the authors for the Education Development Center, Newton, MA.
[10]For a historical review of muonic atoms, see D. West, Rep. Prog. Phys. **XXI**, 272 (1958).
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[12]H. C. von Baeyer and D. Leiter, Phys. Rev. A **19**, 1371 (1979).
[13]In the nonrelativistic treatment of the hydrogen atom the expectation value of the velocity operator (squared) $v^2/c^2 = p^2/m^2c^2$ in a state with quantum numbers n,l, is $(Z\alpha/n)^2$. In the Dirac theory there are certain difficulties attendant to the definition of a suitable orbital velocity or velocity squared operator. [See, for example, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw–Hill, New York, 1964), pp. 11 and 37; and L. F. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).] It is for this reason that $\hat{\gamma}$ is represented by Eq. (1) and not by an expression of the form $(1 - \hat{v}\cdot\hat{v}/c^2)^{-1/2}$ where $\hat{v}$ is a velocity operator. It is also for this reason that the expression "in some sense" was appended to the characterization of $(Z\alpha)^2$ as a mean square orbital velocity in the Dirac theory. A fuller discussion of the mean square orbital velocity of a bound Dirac particle will be published elsewhere.
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[22]There are, of course, other relativistic consequences that come to mind as, for example, the Thomas precession that contributes to the atomic fine structure interaction. The author's concentration on the dilatation of the muon lifetime is motivated in part by the view that it is a particularly direct *kinematical* demonstration of bound particle motion, whereas the manifestation of the Thomas precession must be inferred from an analysis of the energy level structure.

Gyrotron or electron cyclotron resonance maser: An introduction

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The gyrotron or electron cyclotron resonance maser is a high-power microwave source in which the microwave fields gain energy from a beam of relativistic electrons. In this article, an elementary account of the gain mechanism based on a linear, small-signal analysis is presented.

I. INTRODUCTION

The gyrotron or electron cyclotron resonance maser is a new high-power microwave source operating from centimeter to submillimeter wavelengths. At the present time, its development is being stimulated by the demand for high-power microwave radiation to heat tokamak plasmas towards fusion temperatures and by new radar applications.

The heart of the device is a resonant cavity in a steady magnetic field (Fig. 1). A hollow cylindrical beam of electrons gyrating about the magnetic field direction with relativistic energies enters the cavity and interacts with the microwave fields there. The microwave fields gain energy from the electrons and a fraction of this energy leaves the cavity via the output waveguide.

The possibility that, under some conditions, wave amplification can occur when radiation traverses a magnetized, astrophysical plasma was first pointed out by Twiss[1] in 1958. Then, in 1959, two distinct accounts of a gain mechanism for relativistic monoenergetic electrons in a magnetic field were published. Schneider[2] used quantum mechanics while Gaponov[3] used classical physics. The first experiment that definitely confirmed the existence of the mechanism was reported by Hirshfield and Wachtel[4] in 1964.

Since then, gyrotrons have been built that deliver kilowatts of continuous power over a broad range of wavelengths (Table I). The initial development of these devices took place in the Soviet Union although now the work is quite widespread. The subject has recently been reviewed by Hirshfield.[8]

The purpose of this article is to present what may be called an elementary theory of the gyrotron. An expression is derived that shows how much energy is transferred from the electrons to the microwave fields.

