Interaction of circling relativistic charges and interference in their radiation

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The radiation emitted by two charges circling at opposite ends of a diameter at arbitrary uniform speed is considered, with special attention being paid to interference effects. The difference in the rate of radiation from the system and the sum of the powers emitted separately by each circling charge is shown to be equal to the work done by the particles on each other through their exact Liénard–Wiechert fields, in accordance with the Poynting theorem. Some peculiarities of the radiation at high and low speeds are noted and explained.

I. INTRODUCTION

The term interference is used when a joint local (mean) action of two or more waves by their superposition is not additive and depends on the phase difference between them. In particular, for similar coherent point sources (emitting harmonic waves of the same frequency in phase) constructive or destructive interference occurs at a point according to whether the path difference from it to the sources is $N\lambda$ or $(N+1/2)\lambda$ (where N is an integer, and λ is the wavelength), so that periodic space variations in intensity result.

Since energy admittedly is neither created nor destroyed by the interference process, it comes to just a spatial redistribution of energy (which supposedly remains the same in total), such that the average intensity over the interference pattern is the same as for incoherent sources. However, this is clearly not the case when a separation between coherent sources is less than a half wavelength, because the path difference of superposing waves for any point then also cannot exceed the value $\lambda/2$, and interference is nowhere entirely destructive (according to the general rule above). At the same time, for equidistant points (on the plane of symmetry midway between the sources) the condition for constructive interference is obviously fulfilled (with $N=0$), resulting in intensity which is twice that due to the same sources if interference did not happen. As a result, the total energy flow from a couple of nearby coherent sources exceeds that for incoherent (or distant) ones, and the surplus apparently should be ascribed to the mutual internal work done by the coherent sources on each other (increasing the rate of their exhaustion).

Although interference is usually treated as one of the basic subjects in any serious course of optics, the mutual influence of coherent sources seems not to be attended to properly in textbooks (probably because separations and sizes of luminous bodies are well above corresponding wavelengths). To fill the gap and corroborate the expectation based on energy conservation by an exact calculation, we consider electromagnetic waves emitted by accelerated charges, representing the simplest point sources of radiation. To simplify the consideration, we restrict it to the case of two charges q_1 and q_2 circling in an orbit of radius a at opposite ends of its diameter, which rotates with constant angular speed ω. Since the circling particles are always $2a$ apart, the source-to-source distance in such a system is less than $\lambda/2$ (at least for the fundamental frequency ω), because $\lambda=2\pi c/\omega=2\pi ca/v=2\pi a/\beta>6a$, in view of the relativistic restriction on the particles' speed $v:\beta=v/c<1$. This makes the model suitable for a display of the overall effect of interference.

To provide a uniform circling of charges, an external field $\mathbf{E}_0(\mathbf{x},t)$ is to be applied to the system, whose radial component would furnish centripetal acceleration, while the work done on the charges by its tangential component compensates for energy loss due to radiation. If the system is comprised of two similar particles of opposite charges (each of magnitude e), circling under their mutual attraction (a classical model for the ''positronium atom''), its stabilization against the ''fall to the center'' (cf. Problem 1 to Sec. 75 in Ref. 1) can be achieved by means of a rotating electric field, say, due to a circularly polarized electromagnetic wave. (In the general case of arbitrary charges it is not so simple, and a more sophisticated guiding field may be required.)

A proper tuning of the wave's frequency and amplitude allows for the compensation of radiative loss with the work done on the particles by the field of the wave, which renders the particles' acceleration just centripetal. Incidentally, if the stabilizing wave is standing and plane, and the particles' orbit lies in one of the antinodal planes for electric field, the magnetic counterpart of the wave's fields is ineffective, because the same planes prove to be nodal for it (see, e.g., Sec. 7.4 of Ref. 2). In particular, the wave then does not exert radiation pressure on the circling particles, since a force perpendicular to the plane of the orbit (in a transverse electromagnetic wave) might result from the magnetic field only (which is zero in the nodal planes).

One more observation is appropriate with respect to the system at hand. Since the Coulomb force between the charges is purely radial, it cannot be responsible for the work done by the circling particles on each other. Therefore, their interaction should be defined by the entire Liénard–Wiechert electromagnetic fields, and the whole treatment must be relativistic.

According to (6.110) of Ref. 3, the total work done on the particles per unit time is

$$\int_V (\mathbf{J}_1+\mathbf{J}_2)\cdot(\mathbf{E}_0+\mathbf{E}_1+\mathbf{E}_2)\,d^3x=dE_{\text{mech}}/dt=0, \tag{1}$$

since the uniform circling implies a constant mechanical energy E_{mech}. Here, $\mathbf{E}_0$ is the applied (guiding) field, and $\mathbf{E}_j$ is the Liénard–Wiechert electric field of the jth charge.

Inasmuch as the system is stationary, the energy supplied by the external field $\mathbf{E}_0$ to keep the charges' motion steady must be given off by the system at the same rate. In a sense, the circling charges represent an ''antenna,'' fed by the guiding field. Therefore, the total power radiated is

$$P=\int_V(\mathbf{J}_1+\mathbf{J}_2)\cdot\mathbf{E}_0 d^3x$$
$$=-\int_V[\mathbf{J}_1\cdot\mathbf{E}_1+\mathbf{J}_2\cdot\mathbf{E}_2+(\mathbf{J}_1\cdot\mathbf{E}_2+\mathbf{J}_2\cdot\mathbf{E}_1)]d^3x. \quad (2)$$

By Poynting's theorem the negative work done on each charge by its self-force,

$$P_j=-\int_V\mathbf{J}_j\cdot\mathbf{E}_j d^3x=-q_j\mathbf{v}_j\cdot\mathbf{E}_j(\mathbf{x}_j) \quad (3)$$

(where the δ-functional form of the current density $\mathbf{J}_j$ for a point particle has been used), describes its individual radiation in the absence of the partner. Thus the first two terms on the right-hand side of Eq. (2) correspond to incoherent sources; they are mutually independent and expressible via centripetal acceleration by means of the relativistic Larmor formula, see, e.g., (14.26) of Ref. 3.

There are two main difficulties associated with Eq. (3). One is an infinite value of the self-field at the site of a point particle; it can be handled by the mass-renormalization procedure[4] (see also Ref. 5). The other is the so-called Schott term (containing the second time-derivative of the particle's velocity), which appears in $\mathbf{E}_j(\mathbf{x}_j)$ but is absent in the Larmor power formula (see the discussion in the series of papers set forth in Ref. 6). Fortunately, for a uniform circular motion the term becomes ineffective, and the (relativistic) radiative reaction force (see, e.g., (4.18) of Ref. 7) simplifies to

$$q_j\mathbf{E}_j(\mathbf{x}_j)=\mathbf{F}_r=-2q_j^2\gamma^4\omega^2\boldsymbol{\beta}/3c^2, \quad (4)$$

where $\gamma=(1-\beta^2)^{-1/2}$.

Hence, Eq. (2) shows that the difference between the actual power P, radiated from the system, and the incoherent radiated powers P_1 and P_2:

$$P-P_1-P_2=-\int(\mathbf{J}_1\cdot\mathbf{E}_2+\mathbf{J}_2\cdot\mathbf{E}_1)d^3x$$
$$=-q_1\mathbf{v}_1\cdot\mathbf{E}_2(\mathbf{x}_1)-q_2\mathbf{v}_2\cdot\mathbf{E}_1(\mathbf{x}_2) \quad (5)$$

(which describes the interference effects) is defined by interaction of circling charges. The main objective of this paper is to corroborate Eq. (5) by independent direct calculations of its two sides, through a comparison of the power radiated by the circling charges with their mutual work on each other.

II. POWER RADIATED BY THE CHARGES

Since the uniform circling is a periodic motion, the energy flow emitted from the system consists of discrete spectral components whose frequencies are multiples of the angular velocity ω.

The intensity of radiation at the frequency $m\omega$ ($m=1,2,\ldots$) per unit solid angle at a distance R_0 from the center of the orbit is

$$dI_m/d\Omega=(c/2\pi)|\mathbf{k}\times\mathbf{A}_m|^2R_0^2, \quad (6)$$

where $|\mathbf{k}|=m\omega/c$. If the system of coordinates chosen corresponds to $k_x=0$, $k_y=k\sin\theta$, $k_z=k\cos\theta$, the mth Fourier component of the vector potential for one circling particle of charge q is defined by

$$\begin{Bmatrix}A_{xm}\\A_{ym}\end{Bmatrix}=\frac{q}{R_0}\exp(ikR_0)\begin{Bmatrix}-i\beta J_m'(m\beta\sin\theta)\\J_m(m\beta\sin\theta)/\sin\theta\end{Bmatrix},\quad A_{mz}=0, \quad (7)$$

where J_m and J_m' are the Bessel function (of order m) and its derivative, respectively; see, e.g., Eqs. (74.6) and (74.7) of Ref. 1. The fields of the second charge are given by similar expressions with opposite sign of $\boldsymbol{\beta}$, which can be allowed for by use of symmetry properties of the Bessel functions and their derivatives:

$$J_m(-x)=(-1)^mJ_m(x),\quad J_m'(-x)=-(-1)^mJ_m'(x). \quad (8)$$

Taking into account that

$$|\mathbf{k}\times\mathbf{A}_m|^2=k^2(|A_x|^2+|A_y|^2\cos^2\theta), \quad (9)$$

one arrives at the following expressions for the time-average power radiated per unit solid angle by a single circling particle:

$$\langle dP/d\Omega\rangle_1=q^2\sum_{m=1}^{\infty}P_m(m\beta,\sin\theta), \quad (10)$$

and for the radiation from a system of two circling charges q_1 and q_2

$$\langle dP/d\Omega\rangle_2=(q_1^2+q_2^2)\sum_{m=1}^{\infty}P_m(m\beta,\sin\theta)$$
$$+2q_1q_2\sum_{m=1}^{\infty}(-1)^mP_m(m\beta,\sin\theta). \quad (11)$$

Here, P_m is the power in the mth harmonic for one charge of unit strength:

$$P_m=(m^2\omega^2/2\pi c)[\beta^2J_m'^2(m\beta\sin\theta)$$
$$+\cot^2\theta J_m^2(m\beta\sin\theta)], \quad (12)$$

cf. (74.8) in Ref. 1 or Problem 14.8 of Ref. 3.

The Kapteyn series of Bessel functions in (10) can be summed up using the following relations:

$$\sum_{m=1}^{\infty}m^2J_m^2(mb)=b^2(b^2+4)/16(1-b^2)^{7/2},$$
$$\sum_{m=1}^{\infty}m^2J_m'^2(mb)=(3b^2+4)/16(1-b^2)^{5/2}, \quad (13)$$

which are derivable by averaging (over the period) squares of the Bessel's solution to the classical Kepler's problem or which can be found in Ref. 8. Incidentally, there is a misprint in Eq. (3), Sec. 17.6 on p. 573 of Ref. 8 (where 1/2 stands for the correct value of 7/2 in the bracket's exponent of our first formula), which is often inadvertently reproduced in reference books: cf. Ref. 9.

The summation in (10) yields [cf. (74.4) in Ref. 1]

$$\left\langle\frac{dP}{d\Omega}\right\rangle_1=(q\omega\beta)^2\,\frac{4(1+\cos^2\theta)-\beta^2(1+3\beta^2)\sin^4\theta}{32\pi c(1-\beta^2\sin^2\theta)^{7/2}}, \quad (14)$$

which, on integration over all possible directions, results in

$$P_1=\int\langle dP/d\Omega\rangle_1 d\Omega=2q^2\gamma^4\omega^2\beta^2/3c, \quad (15)$$

which is nothing but the relativistic Larmor power formula for a single particle:

$$P_1=(2q^2/3c)\gamma^6[\dot{\beta}^2-(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2], \tag{16}$$

since $\boldsymbol{\beta}$ is orthogonal to $\dot{\boldsymbol{\beta}}=d\boldsymbol{\beta}/dt$ for a circular motion, and $|\dot{\boldsymbol{\beta}}|=\omega^2 a/c=\omega\beta$ is just a magnitude of the centripetal acceleration.

Although an analogous summation cannot be performed in (11), the expression can be integrated over angles if one uses Schott's result, cited as (74.9) in Ref. 1:

$$\int P_m d\Omega=(q^2\omega^2/c)2m\beta\Bigg[J'_{2m}(2m\beta)-(2\beta^2\gamma^2)^{-1}\int_0^{2m\beta}J_{2m}(\xi)d\xi\Bigg]. \tag{17}$$

Thus the total power radiated jointly by both circling charges (which must be supplied, of course, by the stabilizing field) is found to be

$$\begin{aligned}P&=\int\left\langle\frac{dP}{d\Omega}\right\rangle_2 d\Omega=P_1+P_2+2q_1q_2\sum_{m=1}^{\infty}(-1)^m\int P_m d\Omega\\&=\frac{(q_1^2+q_2^2)2\gamma^4\omega^2\beta^2}{3c}+4\left(\frac{q_1q_2\omega^2}{c}\right)\beta\sum_{m=1}^{\infty}m(-1)^m\\&\quad\times\Bigg[J'_{2m}(2m\beta)-(2\beta^2\gamma^2)^{-1}\int_0^{2m\beta}J_{2m}(\xi)d\xi\Bigg],\end{aligned} \tag{18}$$

where (15) has been used in the first term.

Expression (18) consists of two parts: one, defined by the first term, which would be the only one if each particle were circling alone (incoherent sources), and the second one defined by the series, which describes the overall effect of interference and gives the desired value of the left-hand side of Eq. (5).

III. LIMITING CASES AND MULTIPOLE FIELDS

To see in more detail the behavior of the combined power of radiation P relative to the incoherent contributions of the single-charge powers P_1+P_2, it is helpful to consider the low-speed limit of the infinite sum in (18). In the case of a nonrelativistic motion of particles, $\beta\ll 1$, the series expansions of Bessel functions for small values of their argument yield

$$\sum_{m=1}^{\infty}m(-1)^m\Bigg[J'_{2m}(2m\beta)-(2\beta^2\gamma^2)^{-1}\int_0^{2m\beta}J_{2m}(\xi)d\xi\Bigg]\approx\frac{\beta}{3}\left(-1+\frac{14}{5}\beta^2+\cdots\right), \tag{19}$$

where only the first two terms (with $m=1,2$) survive in the sum up to order β^3.

In this approximation the interference term in (18) becomes

$$P-P_1-P_2=(4cq_1q_2\beta^4/3a^2)(-1+14\beta^2/5+\cdots). \tag{20}$$

On the other hand, for the low harmonics, the expansion in powers of β corresponds to expansion in powers of ka, which is just the multipole expansion of electromagnetic fields; see Secs. 9.1–9.3 in Ref. 3.

The lowest order multipole fields now (for $\omega\neq 0$) are the electric dipole and quadrupole, since the magnetic dipole moment of circling charges remains constant and is ineffective for radiation. The corresponding total powers radiated are

$$P_d=\frac{ck^4|\mathbf{p}|^2}{3},\quad P_Q=ck^6\sum_{\alpha,\beta}\frac{|Q_{\alpha\beta}|^2}{360}, \tag{21}$$

where $\mathbf{p}$ and $Q_{\alpha\beta}$ are the Fourier components of the electric dipole and quadrupole moments, respectively; see (9.24) and (9.49) in Ref. 3 or (67.11) and (71.5) in Ref. 1.

For the system at hand,

$$\mathbf{p}=a(q_1-q_2)\mathrm{Re}\begin{Bmatrix}1\\ i\\ 0\end{Bmatrix}e^{-i\omega t}, \tag{22}$$

$$\begin{Bmatrix}Q_{11}\\ Q_{12}\\ Q_{22}\\ Q_{33}\end{Bmatrix}=\frac{a^2(q_1+q_2)}{2}\left[\begin{Bmatrix}1\\ 0\\ 1\\ -2\end{Bmatrix}+3\,\mathrm{Re}\begin{Bmatrix}1\\ i\\ -1\\ 0\end{Bmatrix}e^{-2i\omega t}\right], \tag{23}$$

while all other components of the quadrupole moment tensor $Q_{\alpha\beta}$ vanish.

However, to order β^6 retained in (20) we must include a correction to the dipole moment, cf. (9.11)–(9.12) in Ref. 3. An exact expression for the radiated power is an incoherent sum of contributions from all multipoles:

$$P=\left(\frac{c}{8\pi k^2}\right)\sum_{l,m}[|a_E(l,m)|^2+|a_M(l,m)|^2]; \tag{24}$$

see (16.79) in Ref. 3. The ''amplitudes'' a_E and a_M (in the absence of intrinsic magnetization) are expressible in terms of the charge and current densities $\rho(\mathbf{x},t)$, $\mathbf{J}(\mathbf{x},t)$ as

$$\begin{Bmatrix}a_E\\ a_M\end{Bmatrix}=\frac{4\pi k^2}{i\sqrt{l(l+1)}}\int Y^*_{lm}\left[\begin{Bmatrix}\rho\dfrac{\partial}{\partial r}[rj_l(kr)]\\ 0\end{Bmatrix}+\begin{Bmatrix}ik(\mathbf{r}\cdot\mathbf{J})/c\\ \boldsymbol{\nabla}\cdot(\mathbf{r}\times\mathbf{J})/c\end{Bmatrix}j_l(kr)\right]d^3x, \tag{25}$$

where $j_l(kr)$ and $Y^*_{lm}(\theta,\phi)$ are spherical Bessel functions and (complex conjugate) spherical harmonics, respectively; see (16.91) and (16.92) in Ref. 3.

A Fourier series expansion for the charge density is suggested, e.g., in Problem 9.1(b) of Ref. 3:

$$\rho(\mathbf{x},t)=\rho_0+\sum_{n=1}^{\infty}\mathrm{Re}[2\rho_n(\mathbf{x})e^{-in\omega t}], \tag{26}$$

where for our system (in spherical coordinates)

$$\rho_n(\mathbf{x}) = (1/2\pi a^2)\,\delta(r-a)\,\delta(\theta-\pi/2)[q_1+(-1)^n q_2]e^{in\phi}. \tag{27}$$

Since, in this case,

$$\mathbf{r}\cdot\mathbf{J}=0, \quad \nabla\cdot(\mathbf{r}\times\mathbf{J}) = -\mathbf{r}\cdot(\mathbf{r}\times\mathbf{J}) = 0, \tag{28}$$

in the sum (24), all $a_M=0$, and only electric multipole terms survive. As a matter of fact, (24) provides an expression for P alternative to (18).

To find the necessary correction to the dipole moment, we need just $a_E(1,1)$, which is

$$a_E(1,1) = \tilde{a}_E(1,1)(3/2)[j_0(\beta) - j_1(\beta)/\beta], \tag{29}$$

where the long wavelength value $\tilde{a}_E$, corresponding to (22), is given by (16.93) of Ref. 3:

$$\tilde{a}_E(1,1) = (4\pi k^3\sqrt{2}/3i)\int Y_{11}^* r\rho \; d^3x \tag{30}$$

(the recursion formulas for the spherical Bessel functions also have been used to express j_1').

An expansion in powers of β yields

$$a_E(1,1) = \tilde{a}_E(1,1)\left[\frac{\cos\beta}{\beta^2} - \frac{\sin\beta}{\gamma^2\beta^3}\right] = \tilde{a}(1,1)(1-\beta^2/5+\dots), \tag{31}$$

and for powers of radiation, one obtains

$$\begin{aligned} P_d &\approx \frac{2c(q_1-q_2)^2\beta^4}{3a^2}\,(1-2\beta^2/5+\cdots), \\ P_Q &\approx \frac{8c(q_1+q_2)^2\beta^6}{5a^2}. \end{aligned} \tag{32}$$

Organized as a sum of incoherent and interference terms, they give

$$\begin{aligned} P = {} & (2c\beta^4/3a^2)[(q_1^2+q_2^2)(1+2\beta^2+\dots) \\ & +2q_1q_2(-1+14\beta^2/5+\dots)], \end{aligned} \tag{33}$$

which is to be compared with (20).

The following features of expression (33) should be noted.

(a) If $q_1=q_2$, there is no dipole radiation, only a quadrupole one occurs. The sum of the powers is proportional to β^6. This is also seen in the exact result (11) as the absence of power radiated in the fundamental, because the terms with $m=1$ are then canceled out.

(b) If $q_1=-q_2$, there is only dipole, but no quadrupole radiation. To order β^6 inclusive, the power is given, due to interference, by 4 times (not twice!) the power of a single charge; cf. Problem 1 to Sec. 67 of Ref. 1.

(c) The incoherent term has a correction factor $(1+2\beta^2)$ which is just the expansion of the factor γ^4 in the first term of (18).

(d) The interference term is just the order β^6 approximation (20) of the exact result (18).

It is to be mentioned that one-to-one correspondence between multipoles and harmonics extends here only to the fundamental and the first harmonic. Incidentally, the exact expression for $a_E(1,1)$ is not to be used in our approximation, because the radiation into the fundamental frequency receives contributions from higher multipoles; e.g., the third moment contributes to frequencies ω and 3ω, and so on.

Quite another interplay between the incoherent and interference contributions is observed in the high-speed limit, when terms of higher order in β become important. In the case of ultrarelativistic motion, the effect of interference relatively lessens, because the first member of (18) contains the factor γ^4, which becomes increasingly large since $\gamma\to\infty$ as $\beta\to 1$. This is understandable in view of the confinement of the fast particle's radiation to a narrow pencil parallel to its velocity. Since the velocities of the circling particles are always opposite each other, their radiations at high speeds take place mainly in mutually opposite directions, so that their fields just do not overlap; such overlap is necessary for interference. Thus, as $\beta\to 1$, the particles radiate effectively as incoherent sources. This is in accord with another observation, that for fast particles the higher harmonics become important, for which the intersource distance $2a$ may considerably exceed the wavelength $\lambda_m = 2\pi c/m\omega = \lambda/m$, when m is large enough. Therefore the same circling particles are to be considered, for the fundamental mode ω, as adjacent sources of the dipole radiation whose mutual work is important in the energy balance, but for spectral components of higher frequencies $m\omega$, $m \gg 1$, they should be treated as distant ones (with a negligible inter-influence).

IV. A DYNAMICAL APPROACH

To calculate the work done by the particles on each other (and thereby justify our main inference), we should find the force of their interaction, which is defined by the Liénard–Wiechert electromagnetic fields:

$$\mathbf{E}_i = \mathbf{E}_v + \mathbf{E}_a, \quad \mathbf{B}_i = \mathbf{n}'\times\mathbf{E}_i, \quad \mathbf{E}_v = q(\mathbf{n}'-\beta')/\gamma^2 R'^2 s^3, \tag{34}$$

$$\mathbf{E}_a = q\mathbf{n}'\times[(\mathbf{n}'-\beta')\times\boldsymbol{\beta}']/cR's^3, \quad s = 1-\mathbf{n}'\cdot\boldsymbol{\beta}.$$

The prime by a symbol indicates that the corresponding function of time is to be taken at a previous moment $t'=t-R(t')/c$, where $R(t')$ is the distance from the previous position of the particle generating the fields, $\mathbf{x}_q(t')$, to an observation point, $\mathbf{x}$, i.e., $R'=R(t')=|\mathbf{R}(t')|=|\mathbf{x}-\mathbf{x}_q(t')|$, $\mathbf{n}'=\mathbf{R}'/R'$.

Focusing for definiteness on the charge q_1, we calculate the fields in its close vicinity due to the second particle. Since the observation point, $\mathbf{x}$, and the source of the field, $\mathbf{x}_q$, are both assumed to move in the same circle with equal angular velocities, this implies the following time dependence of their coordinates:

$$\begin{aligned} x &= -a\cos\omega t, \quad x_q = a\cos\omega t, \\ y &= -a\sin\omega t, \quad y_q = a\sin\omega t, \end{aligned} \quad z = z_q = 0. \tag{35}$$

Here, the counterclockwise rotation in the XOY plane is supposed, with the initial condition corresponding to the symmetric arrangement of the particles on the OX axis at $t=0$ (q_1 being on the left and q_2 on the right of the origin, which is placed at the center of the orbit).

For the retarded separation of the particles, we have

$$R'=[(-a\cos\omega t-a\cos\omega t')^2+(-a\sin\omega t-a\sin\omega t')^2]^{1/2}=2a\cos\frac{\omega(t-t')}{2},\qquad(36)$$

whence the retardation relation defining t' in (34) becomes

$$t-t'=(2a/c)\cos[\omega(t-t')/2],$$

or

$$\varphi=\beta\cos\varphi,\qquad(37)$$

with $\varphi=\omega(t-t')/2$ (that can also be seen from a simple geometry or found in Ref. 10).

The interaction fields (34) are expressible in terms of the same angle φ. Since the circling is supposed to be uniform, it suffices to consider it at an arbitrary moment of time, say, for $t=0$ when $\varphi=-\omega t'/2$. Then the corresponding retarded vectors in (34) have the following components:

$$\mathbf{R}'=\{-a(1+\cos 2\varphi),a\sin 2\varphi\},\ \boldsymbol{\beta}'=\beta\{\sin 2\varphi,\cos 2\varphi\},$$
$$\mathbf{n}'=\mathbf{R}'/R'=\{-\cos\varphi,\sin\varphi\},\ \dot{\boldsymbol{\beta}}'=(c\beta^2/a)\{\cos 2\varphi,\sin 2\varphi\},\qquad(38)$$

whence $\mathbf{n}'\cdot\boldsymbol{\beta}'=-\beta\sin\varphi$, so that $s=1+\beta\sin\varphi$ $(\beta=a\omega/c)$. Substituting these expressions into (34) yields

$$E_{ix}=q(\beta^2\cos^2\varphi-s^2)/4a^2s^3\cos\varphi,$$
$$E_{iy}=\frac{q}{4a^2}\left(\frac{\beta+\sin\varphi}{\gamma^2s^3\cos^2\varphi}-\frac{2\beta}{s^2}\right),\qquad(39)$$
$$B_{iz}=-(E_{ix}\sin\varphi+E_{iy}\cos\varphi),$$

while all the other components of the fields vanish.

The expressions (39), though useful, suffer from a shortcoming because they contain an as yet unknown function $\varphi(\beta)$, implicitly defined by the retardation relation (37). It can be found explicitly as Lagrange's expansion in terms of Bessel's functions (cf. No. 829 in Ref. 11):

$$\varphi(\beta)=2\sum_{n=0}^{\infty}(-1)^n\{J_{2n+1}[(2n+1)\beta]\}/(2n+1).\qquad(40)$$

Some details of derivation can be found in the Appendix, as well as the results of a numerical solution of (37), illustrating the convergence of the series. Another useful series for $\sin\varphi$ can be also obtained in a similar way (or found in Sec. 17.21 of Ref. 8):

$$\sin\varphi=\beta/2-2\sum_{n=1}^{\infty}(-1)^n[J'_{2n}(2n\beta)]/2n,\qquad(41)$$

where the prime again denotes a derivative of the corresponding Bessel functions (not retardation).

Differentiating these uniformly convergent series with respect to β, using the relation $d\varphi/d\beta=(\cos\varphi)/s$ immediately obtainable from (37), and substituting finally for $J''_\nu(\nu\beta)$ from Bessel's differential equation, one can derive a family of expansions for several useful functions of the "angular retardation" φ:

$$\frac{1}{s}=(1+\beta\sin\varphi)^{-1}=1+2\sum_{n=1}^{\infty}(-1)^nJ_{2n}(2n\beta),$$
$$\frac{1}{s\cos\varphi}=1-2\sum_{n=0}^{\infty}(-1)^n\int_0^{(2n+1)\beta}J_{2n+1}(\xi)d\xi,\qquad(42)$$
$$\frac{d(1/s\cos^2\varphi)}{d\beta}=\frac{\sin\varphi}{s^2\cos^2\varphi}-\frac{\beta}{s^3}=-2\sum_{n=1}^{\infty}(-1)^n2n\int_0^{2n\beta}J_{2n}(\xi)d\xi.$$

Incidentally, these relations are similar to those in Sec. 17.22 on p. 556 of the book by Watson (Ref. 8).

To find the corresponding expressions for the Liénard–Wiechert fields, it is enough now to transform the right-hand sides of (39) to combinations of the left-hand sides of (42). Fortunately, this can be accomplished after some manipulations, and the results have the following form:

$$E_{ix}=-\frac{q}{4a^2}\frac{d(\beta/s\cos\varphi)}{d\beta},$$
$$E_{iy}=\frac{q}{4a^2}\left[(1-\beta^2)\frac{d(1/s\cos^2\varphi)}{d\beta}+2\beta^2\frac{d(1/s)}{d\beta}\right],\qquad(43)$$
$$B_{iz}=-\frac{q\beta^3}{4a^2}\frac{d(1/\beta s\cos\varphi)}{d\beta}.$$

Thus the Lorentz force of interaction $\mathbf{F}=q(\mathbf{E}_i+\boldsymbol{\beta}\times\mathbf{B}_i)$ experienced by q_1 is found to have the following components:

$$F_x=\frac{q_1q_2}{4a^2\gamma^4}\frac{d}{d\beta}\left(\frac{\gamma^2\beta}{s\cos\varphi}\right),$$
$$F_y=\frac{q_1q_2}{4a^2}\left[(1-\beta^2)\frac{d(1/s\cos^2\varphi)}{d\beta}+2\beta^2\frac{d(1/s)}{d\beta}\right],\qquad(44)$$

where $(\boldsymbol{\beta}\times\mathbf{B}_i)_y=0$ since $\beta_x=0$ at $t=0$. Notice that both the components in (44) are just proportional to the Coulomb force $q_1q_2/4a^2$.

At the moment under consideration, $t=0$, q_1 finds itself at the point $\{-a,0\}$ on the OX axis with its velocity vector just opposite to the OY direction, $\beta_1=\{0,-\beta\}$ (counterclockwise circling). Therefore, the x component of the force (44) corresponds to the radial direction, thus being responsible (in conjunction with the x component of the guiding field $\mathbf{E}_0$) for the centripetal acceleration. It controls the size of the orbit a, but being orthogonal to trajectory it does not contribute to the work done on the particles, which is determined by the tangential (at $t=0$) component of the force F_y. It is worthy to note that the first term in the square brackets of the second expression (44) comes from $\mathbf{E}_v$ in (34), while the second one represents the acceleration field $\mathbf{E}_a$ of q_2, so that (44) incorporates the whole electromagnetic interaction of the particles including their radiation field. To provide a uniform circling, the y component of the stabilizing field $\mathbf{E}_0$ should compensate F_y of (44) as well as for radiation damping.

Multiplying F_y of (44) by $v=c\beta$, we get an expression for the work done (per unit time) on each particle by the interaction force. Substituting there for s^{-1} and $d(s\ \cos^2\varphi)^{-1}/d\beta$ from (42) (and doubling the result to incorporate the contributions from both particles) shows that it is identical to the series in (18). On the other hand, the first term in (18) is also obtainable as the work due to the radiative reaction force (4).

Thus we have accomplished a proof that the work done by the tangential component of the interaction force between two circling charges is equal to the interference contribution to the total power radiated from the system. This enables one to treat the component as a dynamical effect of interference analogous to the radiation damping force. The result is certainly a corollary of energy conservation (5), but our rather involved calculation seems to fill a gap between a mere inference and a rigorous treatment (cf. Ref. 6 to see that going from energy conservation to dynamics might not be trivial).

To conclude the section, it is worthwhile to consider the low-speed approximation for (44). An iterative solution of (37) through $\varphi_{n+1}=\beta\cos\varphi_n$ (with $\varphi_0=0$) to fifth power in β yields

$$\begin{aligned}
&\cos\varphi\approx 1-\beta^2/2+13\beta^4/24-(541\beta^6/720)+\cdots,\\
&\sin\varphi=\sqrt{1-\cos^2\varphi}\approx\beta(1-2\beta^2/3+4\beta^4/5-\cdots),\qquad (45)\\
&s=1+\beta\sin\varphi\approx 1+\beta^2-2\beta^4/3.
\end{aligned}$$

Substituting into the second expression of (44), one gets for the tangential force due to ''velocity interaction field'' $\mathbf{E}_v$ in (34)

$$\left(\frac{q_1q_2}{4\gamma^2a^2}\right)\frac{d(1/s\ \cos^2\varphi)}{d\beta}\approx\left(\frac{q_1q_2\beta^3}{3a^2}\right)\left(1-\frac{22}{5}\beta^2\right),\qquad (46)$$

while the action of the ''acceleration field'' $\mathbf{E}_a$ is defined by

$$(q_1q_2\beta^2/2a^2)\ \frac{d(1/s)}{d\beta}\approx-(q_1q_2\beta^3/a^2)(1-10\beta^2/3).\qquad (47)$$

The results are also obtainable by use of series expansions for Bessel functions in (42).

A sum of these two expressions, multiplied by the particles' speed $c\beta$ and doubled (because the same work is done on the other charge), coincides with the corresponding expressions for the electric dipole–quadrupole radiated power (20) and (33) in Sec. III. Noteworthy are the opposite signs of the expressions (46) and (47) showing the competitive contributions from the velocity and acceleration interaction fields of (34).

V. CONCLUDING REMARKS

The results of this study may be summed up as follows.

(1) The total power radiated from a system of nearby coherent sources, exemplified by two circling charges, is shown (by a rigorous calculation) to differ from the sum of their individual radiations due to interference. (As another extreme example of the overall effect of interference, the fact that there is no emission of radiation from a steady current in a loop can be mentioned, notwithstanding centripetal acceleration of the charges constituting the current, see Problems 14.12–14.13 in Ref. 3). This means that interference is not always to be treated as just a redistribution of energy in space around coherent sources, contrary to the widespread assertion in courses of optics.

(2) In the low speed limit, the expression obtained reduces (in the case of opposite charges) to the known result for dipole radiation, whose power is four times that for a single charge. On the other hand, with enhancement of speed the interference contribution becomes of relatively little consequence since superposition of fast particles' fields (which is necessary for interference) is rendered insignificant due to a narrow cone concentration of radiation in the direction of their velocities (mutually opposite for the circling particles).

(3) A calculation of the interaction force using the entire Liénard–Wiechert fields (with no approximations whatsoever) discloses that the excess of radiation due to interference is just attributable to the work done by the opposite charges on each other. This reveals a general mechanism of mutual influence of the coherent sources (in close proximity), stimulating their excessive radiation. On the other hand, for the case of charges of the same sign, the tangential component of the interaction force changes its direction to opposite, so that the mutual work becomes negative and suppresses the radiation from the system (the situation then corresponds to coherent sources with a phase difference of π).

(4) The preceding conclusion is so natural that the results obtained appear to be rather modest to repay for the tedious manipulation of Bessel functions, but this is not quite so. The point is that, strictly speaking, in Eq. (5) (based on the Poynting theorem) a term is missing, which is to describe (in general) the rate of change of the total energy inside the boundary surface. This is justified if (5) is averaged over the period of motion. In our calculation of the radiated power (18) we have considered indeed its average value; however, treating the right-hand side of (5) in Sec. IV, we never employ any averaging procedure. To justify (5) by a direct calculation of instantaneous power of radiation in its left-hand side (bypassing the Fourier expansion) would be a rather difficult problem.

(5) By the way, a misprint in the comprehensive book by Watson[8] in a sum of Kapteyn series of Bessel functions has been noticed.

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APPENDIX

To find explicit expressions for $\varphi(\beta)$ from (37) and for some functions of this angle, we can use the following Lagrange's result (see, e.g., p. 133 of the course by Whittaker and Watson[12]).

If $f(z)$ and $g(z)$ are analytic on and inside a contour C surrounding a point a, and if w is such that $|wg(z)|<|z-a|$ at all points on C, then $z=a+wg(z)$ has one root in the interior of C, and

$$f(z)=f(a)+\sum_{n=1}^{\infty}\frac{w^n}{n!}\frac{d^{n-1}}{da^{n-1}}\{f'(a)[g(a)]^n\}.$$

Using this expansion with Eq. (37) for $a=0$, $f(z)=z=\varphi$, $w=\beta$, $g(\varphi)=\cos\varphi$, we obtain

$$\varphi(\beta)=\sum_{n=1}^{\infty}\frac{\beta^n}{n!}\left[\frac{d^{n-1}\cos^n\xi}{d\xi^{n-1}}\right]_{\xi=0} \tag{A1}$$

The derivatives in (A1) can be found if one writes $\cos\varphi$ as a sum of two exponentials and employs the binomial expansion:

$$\frac{d^j}{d\xi^j}\left[\frac{\cos^n\xi}{n!}\right]=2^{-n}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\frac{d^j}{d\xi^j}e^{-i(n-2k)\xi}.$$

Splitting the sum for $n=2m+1$ into two parts, $0\leqslant k\leqslant m$ and $m+1\leqslant k\leqslant 2m+1$ (for $n=2m$ the splitting can be $0\leqslant k\leqslant m$ and $m\leqslant k\leqslant 2m$, since the term with $k=m$ is zero because of $n-2k=2m-2k$ in the exponential), it is easy to see that the odd derivatives with $j=2i+1$ vanish, while the even ones are

$$\frac{1}{n!}\left[\frac{d^{2j}}{d\xi^{2j}}\cos^n\xi\right]_{\xi=0}=\frac{(-1)^j}{2^{n-1}}\sum_{k=0}^{E(n/2)}\frac{(n-2k)^{2j}}{k!(n-k)!}.$$

Substitution of these into (A1) yields

$$\varphi=\sum_{m=0}^{\infty}(-1)^m\frac{\beta^{2m+1}}{2^{2m}}\sum_{k=0}^{m}\frac{(2m+1-2k)^{2m}}{k!(2m+1-k)!}. \tag{A2}$$

Although the series in (A2) converges only for $0<\beta<0.66$ (see, e.g., Ref. 11), an analytic continuation is possible in the following way (if to change the order of summations):

$$\begin{aligned}\varphi&=\sum_{m,k=0}^{\infty}\frac{(-1)^{m+k}\beta^{2m+1+2k}(2m+1)^{2m+2k}}{2^{2m+2k}k!(2m+1+k)!}\\&=2\sum_{m=0}^{\infty}\frac{(-1)^m J_{2m+1}[(2m+1)\beta]}{2m+1}.\end{aligned} \tag{A3}$$

This Kapteyn series converges even for $\beta=1$ (see p. 553 of Watson's book[8]).

The expansion (A3) coincides with the Bessel's solution to the classical Kepler's problem, $\psi-\epsilon\sin\psi=\tau$. It is readily seen that for $\tau=\pi/2$, $\epsilon=\beta$, this is nothing but (37) if $\varphi=\psi-\pi/2$. Using the Lagrange's expansion with other $f(z)$ and $g(z)$, or drawing the corresponding formulas from Chap. XVII of Ref. 8 and differentiating them with respect to β, one can derive all necessary expressions. For example, from

$$\frac{d\psi}{d\tau}=(1-\epsilon\cos\psi)^{-1}=1+2\sum_{n=1}^{\infty}J_n(n\epsilon)\cos n\tau,$$

the first of the relations (42) follows at once.

Finally, to estimate the convergence of the series, it is worthwhile to compare the results of calculations by means of (A3) using, say, the first four terms, $0\leqslant m\leqslant 3$, with a numerical solution of (37):

β	$\gamma=\beta\cos\gamma$	(A3)	β	$\varphi=\beta\cos\varphi$	(A3)
0.1	0.099 505	0.0995	0.6	0.520 533	0.5190
0.2	0.196 164	0.1962	0.7	0.583 989	0.5796
0.3	0.287 672	0.2877	0.8	0.641 134	0.6314
0.4	0.372 559	0.3725	0.9	0.692 619	0.6749
0.5	0.450 183	0.4498	1.0	0.739 085	0.7118

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VOUS VS TU

After I passed the *theorminimum,* Landau told me that I could now use the singular (familiar) form of the second-person pronoun with him. It was like a medieval ritual: When the apprentice passes some threshold, the master craftsman permits the familiar mode of discourse.

Alexander I. Akhiezer, ''Recollections of Lev Davidovich Landau,'' Physics Today, June 1994, pp. 35–42.