Master of science thesis in physics

Classical models of charged relativistic particles

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Abstract

In this thesis various classical models of charged relativistic particles are described and discussed. We derive the Lorentz-Dirac equation for a radiating point-charge, which follows from the coupling of the Maxwell field equation and the Lorentz-force equation. A discussion of the runaway solution and the non-causal phenomenon that follows are made. Dirac's extended electron model and its stability is thereafter investigated. Finally a model describing a point-charge interacting with dynamical scalar and vector fields is elaborated. The inconsistencies in the Maxwell-Lorentz theory disappear in this model, when the requirement of stability is imposed. As a result the total mass becomes finitely computable in terms of field parameters.

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1

Introduction

The most successful physical theory that describes Nature in agreement with experiments at the small scale, is the Standard model of electroweak and strong interactions which is a quantum field theory. When obtaining this model infinite compensating terms has to be added to make it finite. These terms are often attributed to unknown physics at smaller distance scale. Thus, although we have quite a good descriptive understanding of the fundamental processes, we do not know the structure of the fundamental objects.

In order to understand the infinities and its relation to the structure of the particle we can go back to the realm of classical theory. Even here the problems of infinite self-energies exist, but in a different shape. Historically, one of the most important problems at the classical level has been to find a consistent model for relativistic particles interacting with relativistic fields, in particular charged particles in interaction with the electromagnetic field.

Even though classical physics is a limited part of a quantum theory, we can only establish a quantum mechanical theory by quantizing a classical one. In spite of the occurring infinities, the quantum electrodynamic theory (QED) is considered consistent. In the past one has speculated (and this could very well be true) that one would obtain a better quantum electrodynamic theory by finding a classical model that is free from infinities, and then quantize it. In this master thesis several different classical models of charged relativistic particles will be described and discussed. It will be shown how the classical point-particle description is strongly related to the infinities, but also how to overcome the problems.

To do this properly we begin in the ancient Greece. Greek philosophers had some suspicions that both electric and magnetic phenomenon were related to other natural phenomenon. For example, they saw that when amber[1] was rubbed it attracted light particles, and that certain iron ores affected other small iron particles. However, a connection between these two forces

[1] The Greek word for amber is electron.

was not found until more than two thousand years later, even though the forces seemed quite alike. It was not until 1820 that the Danish experimental physicist Ørsted found that charges in motion give rise to magnetic forces. This supported the idea that the electric and the magnetic phenomena were related. Later on Faraday made experimental tests that showed that the relation between these phenomena worked either way. He found that magnetic forces can induce electrical currents. What was left now was to combine these observations into a unified theory of electromagnetism. This task was solved by Faradays student James Clerk Maxwell.

Maxwell published his famous paper about the dynamics of electromagnetic fields in 1864. In 1895, Lorentz found the Lorentz-force equation that gives the equation of motion for a charged particle in the presence of an external field. Since we are dealing with classical physics these two concepts, particle and field, will be considered as different throughout this thesis, although we are aware of that they merge at the quantum level.

With the Lorentz-force equation and Maxwell's equations we know how a field exerts a force on a charged particle and how to calculate the field produced by sources of charges and currents. However, to get a satisfactory electrodynamic theory we must couple these equations. As the charged particle in an external force field is accelerating it radiates and this radiation carries off energy which must react back on its own motion. We have a so called radiation reaction.

In **Chapter 2** the Lorentz-force equation and Maxwell's field equation are deduced from an action principle. When solving the coupled equations for a point-particle, one gets into troubles with finiteness. The chapter also sets the relativistic notation used throughout the thesis.

To solve the infinity problems, Dirac suggested in 1938 to split the vector potential in a finite and an infinite part, where the infinite part can be taken care of by mass renormalization. In this way one ends up with an equation of motion for the charged point-like particle interacting with an electromagnetic field. This approach is discussed in **Chapter 3**.

Unfortunately a complete relativistic description of the radiation reaction has not yet been found. One gets into trouble with infinite self-energies if the particle is supposed to be point-like. These problems of point-like particles also occurs in the general theory of relativity and other field theories as well as in quantum theory.

The equation of motion, the so called Lorentz-Dirac equation, follows from the coupled Maxwell and Lorentz-force equation. It is a third order differential equation which causes some fatal difficulties. In **Chapter 4** the solutions to the Lorentz-Dirac equation are studied. To prevent runaway solutions (solutions that grows exponentially with time) one gets non-causal behavior for the point-charge. The difficulties associated with the coupling of the field equations with the equations of motion for the charged particle touch the fundamental question of the nature of an elementary particle.

An extended model for the charged particles seems inevitable. The relativistic string is one example. The string is a well understood concept and there is a lot of literature on this theory [1]. Instead I will here consider relativistic membranes. The first suggestion of a relativistic charged sphere was done by Dirac in 1962. He pictured the charged particle as a bubble in an external electromagnetic field. This quite simple and elegant theory was later on shown to be unstable at the classical level. The Dirac bubble and its stability problems are described in **Chapter 5**.

In **Chapter 6** we go back to the point-like particle but make a more general model of classical electrodynamics. When including both vector and scalar fields interacting with the charged relativistic point-particle and imposing a stability condition, we are able to get rid of the infinities and get a finite theory. For instance we find a finite expression for the total mass in terms of self-fields of the charged particle. It is also shown that this description of the interacting relativistic point-charge is connected to what we have in the Standard model.

A summary and some conclusions are given in **Chapter 7**.

2

Electromagnetic interaction of a point-charge

To describe the motion of a charged point-particle in an electromagnetic field we have to consider the interaction between the field and the particle. We know that an external electromagnetic field exerts a force on a particle and that a charged particle generates a field. But when a charged particle is accelerating it radiates and this loss of energy must affect its own motion. Therefore a complete theory of a charged particle have to include the reaction of the radiation from its own motion. From a mathematical point of view the Maxwell and Lorentz equations must be solved simultaneously as a set of coupled equations.

Starting from an action principle we define the Lagrangian and obtain the equations of motions by using the well-known Euler-Lagrange equation. Since the only existing field is the electromagnetic field we know from Maxwell equations and the Lorentz-force equation when the right choice of action has been made.

In this chapter we present our notations, the basic equations and their formal solutions. When solving these equations for a charged particle on the world line we run into troubles with finiteness which will be investigated further on.

2.1 Relativistic notations

Einstein's special theory of relativity [1] is based on two axioms: the velocity of light is a constant for all observers and the laws of physics are identical in all inertial frames. Throughout this thesis Lorentz tensors will be used in order to

[1] In [2] it is claimed that contrary to the general opinion Poincaré and Lorentz should have the credit for the special theory of relativity

easily find invariant physical equations. Lorentz tensors are formulated with respect to a four dimensional space-time. For instance, the Maxwell equations which usually are familiar to us in non-relativistic notation as

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \\ \nabla \cdot \mathbf{B} &= 0 \qquad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}, \end{aligned} \tag{2.1}$$

where $\mathbf{E}$, $\mathbf{B}$ are the electric and magnetic field, c the velocity of light and $\rho, \mathbf{J}$ are the charge density and current respectively. In the same notation the Lorentz-force equation for a point-charge q with velocity $\mathbf{v}$ and external force $\mathbf{F}$ is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{2.2}$$

To write these equations in a relativistic invariant form we use the electromagnetic field-strength tensor

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \tag{2.3}$$

and let ρ and $\mathbf{J}$ form a four-vector $J^\alpha = (c\rho, \mathbf{J})$, the Lorentz index α runs from 0 to 3. The standard Lorentz tensor notation used in this thesis is for example given in [3]). The Maxwell field equations are written in Lorentz covariant form as

$$\partial_\alpha F^{\alpha\beta} = \frac{1}{c} J^\beta \tag{2.4}$$

where $\partial_\alpha = \frac{\partial}{\partial z^\alpha}$, $z^\alpha = (ct, \mathbf{z})$. $z^\alpha(\tau)$ is a position-vector as a function of the invariant proper time τ on the world line. The Lorentz-force equation in covariant form is given by

$$m u^\alpha = \frac{q}{c} F^{\alpha\beta} u_\beta, \tag{2.5}$$

where the four-velocity u^α is defined by

$$u^\alpha(\tau) = \frac{dz^\alpha(\tau)}{dt} = \dot{z}^\alpha(\tau). \tag{2.6}$$

In the rest of the thesis natural units with $c = 1$ will be used. The metric $\eta_{\alpha\beta}$ is chosen to be the flat space-like Minkowski metric, defined as

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.7}$$

2.2 The Maxwell field equation

It is generally believed that Nature works according to an action principle. Thus we start by describing the interaction between the charged particle with mass m, charge q and velocity $\dot{z}(\tau) = u(\tau)$, and the electromagnetic field $F^{\alpha\beta}$ with the action

$$S = \int (-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + A_\alpha j^\alpha) \mathrm{d}^4 x - m \int \mathrm{d}\tau, \tag{2.8}$$

where the current density produced by the point-charge q is

$$j^\alpha(x) = q \int d\lambda \, u^\alpha \, \delta^4(x - z(\lambda)), \tag{2.9}$$

and where

$$d\tau = \left[-\eta_{\alpha\beta} \frac{dz^\alpha}{d\lambda} \frac{dz_\alpha}{d\lambda} \right]^{1/2} d\lambda. \tag{2.10}$$

λ is an arbitrary parameter on the world line and τ is the proper time. The metric $\eta_{\alpha\beta}$ is given in (2.7).

The first term in equation (2.8) describes the free electromagnetic field (massless vector field, from the quantum point of view), the second describes the interaction between the particle and field and the third describes the free particle. The electromagnetic field is an antisymmetric field tensor of the form

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{2.11}$$

where A_α is the vector potential.

With the Euler-Lagrange equations for fields the Maxwell's equations can be derived from the action (2.8). The Euler-Lagrange equations for fields are (see for example [4])

$$\frac{\partial \mathcal{L}(z, \dot{z})}{\partial A_\alpha(z)} - \partial_\beta \frac{\partial \mathcal{L}(z, \dot{z})}{\partial(\partial_\beta A_\alpha(z))} = 0. \tag{2.12}$$

The Lagrangian $\mathcal{L}$ may be extracted from the action (2.8) if the latter is split as follows

$$S = \int \mathcal{L}_{field} \mathrm{d}^4 x + \int \mathcal{L}_{part} \mathrm{d}\lambda \tag{2.13}$$

where

$$\mathcal{L}_{part} = -m \sqrt{-\frac{dz^\alpha}{d\lambda} \frac{dz_\alpha}{d\lambda}} \tag{2.14}$$

and where equation (2.11)

$$\mathcal{L}_{field} = -\frac{1}{4}\eta^{\lambda\mu}\eta^{\nu\sigma}(\partial_\mu A_\sigma - \partial_\sigma A_\mu)(\partial_\lambda A_\nu - \partial_\nu A_\lambda) + A_\lambda j^\lambda, \tag{2.15}$$

from equation (2.11). Hence,

$$\mathcal{L} = \mathcal{L}_{field} + \int \mathcal{L}_{part}\delta^4(x - z(\lambda))d\lambda. \tag{2.16}$$

Since $\mathcal{L}_{part}$ is independent of A_α we get from (2.12)

$$\begin{aligned} \frac{\partial \mathcal{L}_{field}}{\partial A_\alpha} &= j^\alpha, & (2.17)\\ \partial_\beta \frac{\partial \mathcal{L}_{field}}{\partial(\partial_\beta A_\alpha)} &= -\frac{1}{4}\eta^{\lambda\mu}\eta^{\nu\sigma}\{\delta^\beta_\mu \delta^\alpha_\sigma F_{\lambda\nu} - \delta^\beta_\sigma \delta^\alpha_\mu F_{\lambda\nu} \\ &\quad + \delta^\beta_\lambda \delta^\alpha_\nu F_{\mu\sigma} - \delta^\beta_\nu \delta^\alpha_\lambda F_{\mu\sigma}\} \\ &= F^{\alpha\beta}. & (2.18) \end{aligned}$$

Substituting this into the Euler-Lagrange equations (2.12) yield the Maxwell field equations

$$\partial_\beta F^{\alpha\beta} = j^\alpha \tag{2.19}$$

2.3 The Lorentz-force equation

The equation of motion for the particle can in the same way be computed from the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial z^\alpha} - \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{z}^\alpha} = 0. \tag{2.20}$$

If we write the last two terms in the action (2.8) as

$$S' = \int \mathcal{L}' d\lambda \tag{2.21}$$

where

$$\mathcal{L}' = qA_\alpha \frac{dz^\alpha}{d\lambda} - m\sqrt{-\eta_{\alpha\beta}\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}} \tag{2.22}$$

and $A_\alpha \equiv A_\alpha(z^\alpha(\lambda))$. It follows that

$$\left.\frac{\partial \mathcal{L}'}{\partial z^\alpha}\right|_{\lambda=\tau} = q\dot{z}^\beta \frac{\partial A_\beta}{\partial z^\alpha} = q\partial_\alpha A_\beta \frac{dz^\beta}{d\tau} \tag{2.23}$$

and

$$\begin{aligned} \left.\frac{d}{d\lambda}\frac{\partial \mathcal{L}'}{\partial \dot{z}^\alpha}\right|_{\lambda=\tau} &= \frac{d}{d\tau}qA_\alpha + \frac{d}{d\tau}\Big(m\dot{z}_\alpha \frac{1}{\sqrt{-\dot{z}^\alpha \dot{z}_\alpha}}\Big) \\ &= q\partial_\beta A_\alpha \dot{z}^\beta + m\ddot{z}_\alpha. & (2.24) \end{aligned}$$

In the calculations above we have used that $\dot{z}_\alpha(\tau)\dot{z}^\alpha(\tau) = -1$. This is due to our choice of metric, because the scalar product of two four-vectors is $A^\alpha B_\alpha = \eta_{\alpha\beta}A^\alpha B^\beta$.

As $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ and $a^\alpha = \ddot{z}^\alpha$, $u^\alpha = \dot{z}^\alpha$ it follows from (2.20) that

$$ma^\alpha = qF^\alpha_{\ \beta}(z)u^\beta. \tag{2.25}$$

This is just the well-known Lorentz-force equation.

2.4 Solution of the field equation

In equation (2.25), $F_{\alpha\beta}(x)$ should be evaluated at the point $x^\alpha = z^\alpha(\tau)$. In order to see how the electromagnetic field behaves on the world line we have to solve the Maxwell equation (2.19).

To simplify the Maxwell equation without any restrictions we adopt the Lorentz gauge

$$\partial_\alpha A^\alpha = 0. \tag{2.26}$$

This can be done due to the fact that if K is a scalar function, the transformation

$$A_\alpha \rightarrow A_\alpha^{'} = A_\alpha + \frac{\partial K}{\partial x^\alpha} \tag{2.27}$$

does not affect the field, since

$$F_{\alpha\beta}^{'} = \partial_\alpha A_\beta + \partial_\alpha\partial_\beta K - \partial_\beta A_\alpha - \partial_\beta\partial_\alpha K = F_{\alpha\beta}. \tag{2.28}$$

The transformation (2.27) is called a gauge transformation and $F_{\alpha\beta}$ is gauge invariant, as well as all other observable quantities. This gauge transformation helps us to solve Maxwell's equations if we choose A_α such that $\partial_\alpha A^\alpha = 0$ (the Lorentz gauge).

Applying the Lorentz gauge on the Maxwell equations, they reduce to

$$\Box A^\alpha = -j^\alpha \tag{2.29}$$

where the wave operator is defined by

$$\Box \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta. \tag{2.30}$$

A solution to equation (2.29) is

$$A^\alpha(x) = A^\alpha_{hom.}(x) + A^\alpha_{inhom.}(x) \tag{2.31}$$

where $A^\alpha_{hom.}$ is the solution to the homogeneous equation $\Box A^\alpha_{hom.} = 0$. To solve the inhomogeneous equation we use Green functions $G(x-z)$, and begin by solving

$$\Box G(x-z) = -\delta^4(x-z). \tag{2.32}$$

The vector potential is then

$$A^{\alpha}(x) = A^{\alpha}_{hom}(x) + \int d^4z\, G(x-z) j^{\alpha}(x). \tag{2.33}$$

We transform to the wave vector space by Fourier transforms:

$$G(q) = \frac{1}{(2\pi)^4}\int d^4k\, \widetilde{G}(k) e^{-ik\cdot q} \tag{2.34}$$

$$\delta^4(q) = \frac{1}{(2\pi)^4}\int d^4k\, e^{-ik\cdot q} \tag{2.35}$$

where

$$q^{\alpha} = x^{\alpha} - z^{\alpha} \qquad \text{such that} \qquad \Box G(q) = -\delta^4(q). \tag{2.36}$$

Inserted into equation (2.36) we find

$$\widetilde{G}(q) = \frac{1}{k^2} \tag{2.37}$$

and therefore

$$G(q) = \frac{1}{(2\pi)^4}\int \frac{e^{-ik\cdot q}}{k^2}\, d^4k = \frac{1}{(2\pi)^4}\int e^{-i\mathbf{k}\cdot\mathbf{q}}\, d^3k \int_{-\infty}^{\infty} \frac{e^{ik_0 q_0}}{\kappa^2 - k_0^2}\, dk_0 \tag{2.38}$$

where $k\cdot q = -k_0 q_0 + \mathbf{k}\cdot\mathbf{q}$ and $\kappa = |\mathbf{k}|$.

The second integral has simple poles at $k_0 = \pm\kappa$ in the k_0-plane. For $q_0 > 0$, the factor $e^{ik_0 q_0}$ is only limited if we close our contour of integration in the lower half-plane. Thus we get two different Green functions $G_{ret.}$ and $G_{adv.}$ dependent on the choice of q_0. If $q_0 > 0$ we have that

$$\oint dk_0\, \frac{e^{ik_0 q_0}}{\kappa^2 - k_0^2} = 2\pi i \cdot Res\left\{\frac{e^{ik_0 q_0}}{\kappa^2 - k_0^2}; \pm\kappa\right\} = -\frac{2\pi}{\kappa}\sin(\kappa q_0). \tag{2.39}$$

If $q_0 < 0$ this integral with the above chosen contour vanish, due to its lack of singularities, according to Cauchy's residue theorem. To include this in our evaluation we have to insert the Heaviside step function

$$\theta(q_0) = \begin{cases} 1 & \text{iff } q_0 > 0 \\ 0 & \text{iff } q_0 < 0 \end{cases} \tag{2.40}$$

such that

$$G_{ret.}(q) = -\frac{\theta(q_0)}{(2\pi)^3}\int d^3k e^{-i\mathbf{k}\cdot\mathbf{q}} \frac{\sin(\kappa q_0)}{\kappa}. \tag{2.41}$$

Let $R = |\mathbf{q}| = |\mathbf{x} - \mathbf{z}|$,

$$\begin{aligned} G_{ret.}(q) &= -\frac{\theta(q_0)}{4\pi^2 R}\int d\kappa \sin(\kappa R)\sin(\kappa q_0) \\ &= -\frac{\theta(q_0)}{4\pi^2 R}\int_{-\infty}^{\infty} d\kappa \left(e^{i\kappa(q_0 - R)} - e^{i\kappa(q_o + R)}\right). \end{aligned} \tag{2.42}$$

But this integral is just two delta functions, where only one of them contributes due to our choice: $R > 0$ and $q_0 > 0$

$$G_{ret.}(x-z) = -\frac{\theta(q_0)}{4\pi R}\delta(q_0 - R) = -\frac{\theta(x_0 - z_0)}{4\pi R}\delta(x_0 - z_0 - R). \tag{2.43}$$

If we instead had chosen to evaluate $q_0 < 0$ in the upper half-plane we had found the advanced, anti-causal solution

$$G_{adv.}(x-z) = -\frac{\theta(z_0 - x_0)}{4\pi R}\delta(x_0 - z_0 + R). \tag{2.44}$$

$G_{adv.}$ is non-vanishing only if $(x-z)^\alpha$ is light-like and z^α lies on the forward light-cone. The field at $x^\alpha = z^\alpha(\tau)$ is therefore determined by a moving charge at the later time $z^\alpha(\tau_{adv.})$, $\tau_{adv.} > \tau$.

Thus, the equation (2.29) coupled to equation (2.25) states that as the charged particle is moving along its world line, it generates an electromagnetic field that propagates either on the future light cone or backwards. For the causal solution, the electromagnetic field at an event x^α is due to the moving charged particle at $z^\alpha(\tau_{ret.})$ $(\tau_{ret.} < \tau)$, and not elsewhere. These fields we call retarded, or causal because of its causality; the fields are determined by a source at an earlier time. The backward solution is called advanced (non-causal) if the field is produced by a source on the light cone at $z^\alpha(\tau_{adv.})$, i.e at a later time $(\tau_{adv.} > \tau)$. We have that $z^0(\tau_{ret.}) < z^0(\tau) = x^0 < z^0(\tau_{adv.})$.

To summarize we have two solutions to equation (2.29)

$$A^\alpha(x) = A^\alpha_{in}(x) + \int d^4k\, G_{ret.}(x-z) j^\alpha(z) \tag{2.45}$$

and

$$A^\alpha(x) = A^\alpha_{out}(x) + \int d^4k\, G_{adv.}(x-z) j^\alpha(z) \tag{2.46}$$

where A^α_{in} and A^α_{out} are the homogeneous solutions. Since the Green function includes a θ-function, the vector potential and corresponding field $F^{\alpha\beta}$ evaluated on the world line $x^\alpha = z^\alpha(\tau)$ are infinite. This renders the Lorentz-force equation (2.25) quite meaningless. So we have found that the motion of the charged particle gives rise to an electromagnetic field which yields a singularity in this basic theory. We have to make sense of the singularity. One way to do this is, as we shall see, to extract the radiation reaction part and substitute this into the Lorentz-force equation.

If we take the limit $x_0 \to -\infty$ in (2.45), only the term A^α_{in} survives and can be interpreted as an incoming potential, since the charged particle will emit radiation when accelerating. In (2.46) we let $x_0 \to +\infty$ so that A^α_{out} behaves like an outgoing potential, specified at $x_0 \to +\infty$.

Thus the radiated field can be seen as the difference between the incoming and the outgoing field

$$A^\alpha_{rad}(x) = A^\alpha_{out}(x) - A^\alpha_{in}(x) = \int d^4z\, G(x-z) j^\alpha(z) \tag{2.47}$$

where

$$
\begin{aligned}
G(x-z) &= G_{ret.} - G_{adv.} \\
&= -\frac{1}{4\pi R}\left[\theta(z_0 - x_0)\delta(x_0 - z_0 + R) - \theta(x_0 - z_0)\delta(x_0 - z_0 - R)\right].
\end{aligned}
\tag{2.48}
$$

3

Radiation and radiation reaction

We have not yet found an expression for the electromagnetic field in the Lorentz-force equation that is finite. In this chapter our main task is to find this expression and substitute it into the Lorentz-force equation and obtain the Lorentz-Dirac equation (LDE), which is

$$ma^{\alpha} = F^{\alpha}_{ext.} + \frac{2q^2}{3c^3}(\dot{a}^{\alpha} - \frac{a_{\beta}a^{\beta}}{c^2}u^{\alpha}) \tag{3.1}$$

where $a = \ddot{z}$, $u = \dot{z}$. This is an equation of motion for the charged particle under the influence of an external force and its own electromagnetic field.

There are two main derivations of the LDE. The one by Dirac himself in his classical paper in 1938 [5] and the one by Landau and Lifshitz [6]. These two derivations are also given in [7] and [8]. The derivation that will be considered here, is the one by Landau and Lifshitz.

Other very helpful books on this matter are the one written by Rohrlich [9], Jackson [10] and Barut [11].

3.1 The Lienard-Wiéchert potential

To calculate the corresponding field we have to evaluate the integral expression (2.47). If inserting the current density for the radiating field

$$j^{\alpha}(x) = q\int d\tau\, u^{\alpha}(\tau)\delta^4(x - z(\tau)) \tag{3.2}$$

we obtain

$$A^{\alpha}(x) = -2q\int d\tau\, \theta(x_0 - z_0(\tau))\delta([x - z(\tau)]^2)u^{\alpha}(\tau). \tag{3.3}$$

First we choose to evaluate the retarded solution with the light-cone condition

$$[x - z(\tau_{ret.})]^2 = 0 \qquad \text{and} \qquad z_0(\tau_{ret.}) < x_0. \tag{3.4}$$

Notice that

$$\frac{d}{d\tau}(x-z(\tau))^2\Big|_{\tau=\tau_{ret.}} = -2(x-z(\tau_{ret.}))^\alpha u_\alpha \tag{3.5}$$

and the formula for the delta function:

$$\delta[f(\zeta)] = \sum_i \frac{\delta(\zeta-\zeta_i)}{|f'(\zeta_i)|}. \tag{3.6}$$

When equation (3.3) is evaluated at $\tau = \tau_{ret.}$ the retarded solution is

$$A^\alpha_{ret.}(x) = -\frac{qu^\alpha(\tau)}{(x-z(\tau))^\alpha u_\alpha(\tau)}\Big|_{\tau=\tau_{ret.}}. \tag{3.7}$$

This is the well-known Lienard-Wiéchert potential, in covariant form. In the same way we find that the advanced Lienard-Wiéchert potential is

$$A^\alpha_{adv.}(x) = -\frac{qu^\alpha(\tau)}{(x-z(\tau))^\alpha u_\alpha(\tau)}\Big|_{\tau=\tau_{adv.}}. \tag{3.8}$$

3.2 The Lorentz-Dirac equation

Physically the field should be generated by a causal solution to the Maxwell equations (2.29). This retarded potential can now be written as

$$A^\alpha_{ret.} = \frac{1}{2}[A^\alpha_{ret.} - A^\alpha_{adv.}] + \frac{1}{2}[A^\alpha_{ret.} + A^\alpha_{adv.}]. \tag{3.9}$$

The radiation field in equation (2.47) is given by $A^\alpha_{rad.} = [A^\alpha_{ret.} - A^\alpha_{adv.}]$ and therefore we have

$$A^\alpha_{ret.} = \frac{1}{2}A^\alpha_{rad.} + \frac{1}{2}[A^\alpha_{ret.} + A^\alpha_{adv.}]. \tag{3.10}$$

Correspondingly we get

$$F^{\alpha\beta}_{ret.} = \frac{1}{2}F^{\alpha\beta}_{rad.} + \frac{1}{2}[F^{\alpha\beta}_{ret.} + F^{\alpha\beta}_{adv.}]. \tag{3.11}$$

As we shall see the radiation part $F^{\alpha\beta}_{rad.}$ is finite and alone responsible for the radiation reaction; the term $[F^{\alpha\beta}_{ret.} + F^{\alpha\beta}_{adv.}]$ describes the infinite part occurring in the Lorentz-force equation, and does not effect the motion of the particle. We ignore this last infinite term here, but it will be considered in the next section.

We begin by calculating the finite part $F^{\alpha\beta}_{rad.}$. The radiation reaction part of the vector potential is given by

$$\frac{1}{2}A^\alpha_{rad.} = \frac{1}{2}\Big(A^\alpha_{ret.} - A^\alpha_{adv.}\Big). \tag{3.12}$$

We write the Liénard-Wiechert potential (3.7) as

$$A^\alpha_{ret.}(x) = q\frac{u^\alpha(\tau_{ret.})}{r_{ret.}(x)}, \qquad r_{ret.}(x) = -(x - z(\tau_{ret.})^\alpha)u_\alpha(\tau_{ret.}). \tag{3.13}$$

Thus to find $A^\alpha_{rad.}$ we have to express the advanced potential

$$A^\alpha_{adv.}(x) = q\frac{u^\alpha(\tau_{adv.})}{r_{adv.}(x)}, \qquad r_{adv.}(x) = -(x - z(\tau_{adv.})^\alpha)u_\alpha(\tau_{adv.}). \tag{3.14}$$

in terms of retarded coordinates.

The invariant quantity in the Lienard-Wiéchert potential

$$r(x) = -(x - z(\tau))^\alpha u_\alpha, \tag{3.15}$$

is such that $(x - z(\tau_0))^\alpha$ is a null vector. The point $z^\alpha(\tau_0)$ is therefore the point where the world line cuts the light cone with vertex at x^α.

For reasons that will become apparent later we re-scale the null vector by a factor r^{-1}:

$$k^\alpha(x) = \frac{1}{r}(x - z(\tau))^\alpha. \tag{3.16}$$

According to equation (3.15) and the null geodesic equation:

$$0 = \sigma(x, \tau) = (x - z(\tau))^\alpha(x - z(\tau))_\alpha \tag{3.17}$$

we see that at $\tau = \tau_{ret.}$

$$k_\alpha k^\alpha = 0 \quad \text{and} \quad k_\alpha u^\alpha = -1. \tag{3.18}$$

If we make an infinitesimal displacement from x to a new field point $(x+\delta x)$ the point where the particle trajectory intersect with the light cone changes to $z(\tau+\delta\tau)$. These points are of course still related by the null geodesic equation (3.17). Expanding to first order and using the above equations we obtain

$$\frac{\partial\tau}{\partial x^\alpha} = -k_\alpha. \tag{3.19}$$

Since, for a function $f(x) = F(x, \tau)$ the equation $df = \left(\frac{\partial F}{\partial x^\alpha}\right)dx^\alpha + \left(\frac{\partial F}{\partial\tau}\right)d\tau$ with (3.19) takes the form

$$\frac{\partial f}{\partial x^\alpha} = \left(\frac{\partial F}{\partial x^\alpha}\right)_\tau - k_\alpha\left(\frac{\partial F}{\partial\tau}\right)_x. \tag{3.20}$$

Then the derivation of the retarded distance becomes

$$\frac{\partial r}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}\Big(-(x - z(\tau))^\alpha u_\alpha\Big) = -u_\alpha + k_\alpha(1 + rk^\alpha a_\alpha) \tag{3.21}$$

where I've used equation (3.15), (3.18) and the relativistic fact that $u_\alpha u^\alpha = -1$. Later on these expressions will help us calculate the radiation-reaction field from the Liénard-Wiechert potential.

We are now almost ready to express the advanced potential in terms of the retarded coordinates. First we make a Taylor expansion of $\triangle\tau = \tau_{adv.} - \tau_{ret.}$ and $r_{adv.}$ in terms of $r_{ret.}$

$$z^\alpha(\tau_{adv.}) = z^\alpha + u^\alpha\triangle\tau + \frac{1}{2}a^\alpha\triangle\tau^2 + \frac{1}{6}\dot{a}^\alpha\triangle\tau^3 + \frac{1}{24}\ddot{a}^\alpha\triangle\tau^4 + O(\triangle\tau^5). \quad (3.22)$$

On the right-hand side all the quantities should be evaluated at the retarded time. Substitute this into the null geodesic equation (3.17) with equation (3.16), (3.18) and $a^2 = a_\alpha a^\alpha$

$$\begin{aligned} 0 &= \sigma(x,\tau) = (x - z(\tau_{adv.}))^\alpha (x - z(\tau_{adv.}))_\alpha \\ &= \Big[x^\alpha - z^\alpha + u^\alpha\triangle\tau + \frac{1}{2}a^\alpha\triangle\tau^2 + \frac{1}{6}\dot{a}^\alpha\triangle\tau^3 + \frac{1}{24}\ddot{a}^\alpha\triangle\tau^4 + O(\triangle\tau^5)\Big]^2 \\ &= -2r\triangle\tau + \triangle\tau^2 + 2ra_\alpha k^\alpha\triangle\tau^2 + \frac{1}{3}r\dot{a}_\alpha k^\alpha\triangle\tau^3 + \frac{1}{12}(r\ddot{a}_\alpha k^\alpha + a^2)\triangle\tau^4 \\ &\quad +O(\triangle\tau^5). \end{aligned} \quad (3.23)$$

The solution to this is

$$\triangle\tau = 2r\Big[1 - a_\alpha k^\alpha r + \Big((a_\alpha k^\alpha)^2 - \frac{1}{3}a^2 + \frac{2}{3}(\dot{a}_\alpha k^\alpha)\Big)r^2 + O(r^3)\Big]. \quad (3.24)$$

In a similar way we can expand $u^\alpha(\tau_{adv.})$:

$$u^\alpha(\tau_{adv.}) = u^\alpha + a^\alpha\triangle\tau + \frac{1}{2}\dot{a}^\alpha\triangle\tau^2 + \frac{1}{6}\ddot{a}^\alpha\triangle\tau^3 + O(\triangle\tau^4) \quad (3.25)$$

and with (3.24) we find an expression for

$$u^\alpha(\tau_{adv.}) = u^\alpha + 2a^\alpha + 2(\dot{a}^\alpha - a(a_\alpha k^\alpha)a^2)r^2 + O(r^3) \quad (3.26)$$

and

$$r_{adv.}(x) = r + \frac{2}{3}(a^2 + \dot{a}_\alpha k^\alpha)r^3 + O(r^4). \quad (3.27)$$

To obtain the advanced potential in terms of the retarded quantities we substitute (3.26) and (3.27) into (3.13)

$$A^\alpha_{adv.}(x) = q\frac{u^\alpha}{r} + 2qa^\alpha + 2q\Big[\dot{a}^\alpha - (a_\alpha k^\alpha)a^\alpha - \frac{1}{3}(a^2 + \dot{a}_\alpha k^\alpha)u^\alpha\Big]r + O(r^2). \quad (3.28)$$

Thus the radiation-reaction potential is

$$\begin{aligned} \frac{1}{2}A^\alpha_{rad.} &= \frac{1}{2}[A^\alpha_{ret.} - A^\alpha_{adv.}] \\ &= -qa^\alpha - q\Big[\dot{a}^\alpha - a_\alpha k^\alpha a^\alpha - \frac{1}{3}(a^2 + \dot{a}_\alpha k^\alpha)u^\alpha\Big]r + O(r^2) \end{aligned} \quad (3.29)$$

The $O(r^2)$-term will not contribute to the radiation-reaction field $F^{\alpha\beta}_{rad.}$, because after differentiating on the world line this term will vanish.
To calculate $F^{rad.}_{\alpha\beta} = \partial_\alpha A^{rad.}_\beta - \partial_\beta A^{rad.}_\alpha$ we use (3.20) so that:

$$\partial_\alpha A_\beta = \left(\frac{\partial A_\beta}{\partial x^\alpha}\right)_\tau - k_\alpha\left(\frac{\partial A_\beta}{\partial \tau}\right)_x. \tag{3.30}$$

After some algebra and using equation (3.21) it turns out that

$$\frac{1}{2}F^{rad.}_{\alpha\beta} = -\frac{2}{3}q(\dot{a}_\alpha u_\beta - u_\alpha \dot{a}_\beta). \tag{3.31}$$

Notice that this expression is finite.

Substituting into the Lorentz-force equation (2.25) yields

$$ma^\alpha = q(-\frac{2}{3}q(\dot{a}^\alpha u_\beta - u^\alpha \dot{a}_\beta))u^\beta = \frac{2}{3}q^2(\dot{a}^\alpha - a^2 u^\alpha). \tag{3.32}$$

Where we have used that $u^\alpha u_\alpha = -1$ which differentiated twice becomes $\dot{a}^\alpha u_\alpha = -a^2$.

With the right dimension and a possible external force this becomes the Lorentz-Dirac equation

$$ma^\alpha = F^\alpha_{ext.} + \frac{2q^2}{3c^3}(\dot{a}^\alpha - \frac{a_\beta a^\beta}{c^2}u^\alpha). \tag{3.33}$$

The first term on the right-hand side is the Lorentz-force from external electromagnetic forces only

$$F^\alpha_{ext.} = qF^{\alpha\beta}_{ext.}u_\beta. \tag{3.34}$$

The second one

$$\frac{2q^2}{3c^3}\dot{a}^\alpha \tag{3.35}$$

is called the Schott term. This term is, as we will see in the next chapter, responsible for non-local time dependence in the LDE. The third term is just the radiation reaction also appearing in the non-relativistic derivation by Lorentz and Abraham a hundred year ago. It concerns the energy loss due to radiation.

3.3 Mass renormalization

We now turn our attention to the infinite part of our invariant splitting of the retarded potential (3.9), namely $\frac{1}{2}(A^\alpha_{ret.} + A^\alpha_{adv.})$, which was excluded above.

If we use equation (3.13) and (3.28), the term $q\frac{u^\alpha}{(x-z)^\alpha}$ (that vanished in our calculation of the radiation-reaction (3.29)) will now be present and we get a singularity at $x = z(\tau_{ret.})$, or $x = z(\tau_{adv.})$. The corresponding field $\frac{1}{2}[F^{\alpha\beta}_{ret.} + F^{\alpha\beta}_{adv.}]$ is therefore infinite when evaluated on the world line.

In physical terms this represents the Coulomb field carried by the particle, which in quantum mechanics is described as a cloud of photons surrounding the particle that at all times are emitted and absorbed. This self-field is always present and is therefore included when the mass is measured in a laboratory. If we let δm be the Coulomb contribution to the "bare" mass m, the experimentally measured mass is

$$m_{exp.} = m + \delta m. \tag{3.36}$$

So if the mass term in the Lorentz-force equation (2.25) is $(m + \delta m)$, the Coulomb field (i.e the infinite part of expression (3.9)) is already included. In order to verify this we study the Lagrangian

$$\mathcal{L} = \frac{q}{2}\int d\tau u^\alpha \rho(x - z(\tau))(A^{ret.}_\alpha + A^{adv.}_\alpha). \tag{3.37}$$

The delta function is here replaced by a finite distribution function ρ to include the near Coulomb field. The potential can, after using equation (3.7) and (3.8), be written as

$$\frac{1}{2}(A^{ret.}_\alpha + A^{adv.}_\alpha) = -\frac{1}{2}\Big[\frac{qu_\alpha(\tau)}{(x - z(\tau))^\beta u_\beta(\tau)}\Big|_{\tau=\tau_{ret.}} + \frac{qu_\alpha(\tau)}{(x - z(\tau))^\beta u_\beta(\tau)}\Big|_{\tau=\tau_{adv.}}\Big]. \tag{3.38}$$

Thus,

$$\mathcal{L} = -\frac{q^2}{2}\int d\tau \rho(x - z(\tau_{ret.})) \quad \Big[\quad \frac{u_\alpha(\tau_{ret.})u^\alpha(\tau_{ret.})}{(x - z(\tau_{ret.}))^\beta u_\beta(\tau_{ret.})} + \frac{u_\alpha(\tau_{adv.})u^\alpha(\tau_{adv.})}{(x - z(\tau_{adv.}))^\beta u_\beta(\tau_{adv.})}\Big]. \tag{3.39}$$

Let $\tau = \tau_{ret.} + \epsilon$ (were ϵ is small) and Taylor expand $z(\tau)$ around $\tau_{ret.}$, and in the same way around $\tau_{adv.}$ such that

$$z^\alpha(\tau) = z^\alpha(\tau_{ret.}) + \epsilon\dot{z}^\alpha(\tau_{ret.}) + \frac{\epsilon^2}{2}\ddot{z}^\alpha(\tau_{ret.}) + \ldots \tag{3.40}$$

and

$$u^\alpha(\tau) = \dot{z}^\alpha(\tau) = \dot{z}^\alpha(\tau_{ret.}) + \epsilon\ddot{z}^\alpha(\tau_{ret.}) + \frac{\epsilon^2}{2}\dddot{z}^\alpha(\tau_{ret.}) + \ldots \tag{3.41}$$

If we only keep terms up to first order in ϵ and use that $\dot{z}_\alpha\dot{z}^\alpha = -1$, the Lagrangian is

$$\mathcal{L} = q^2\int d\tau\Big(\frac{\rho(x - z)}{\dot{z}_\beta(x - z - \epsilon\dot{z} + ...)^\beta} + \ldots\Big) \tag{3.42}$$

and for a point-particle (let $x \to z$) this mass term becomes an infinite constant

$$\delta m = q^2 \int \frac{d\epsilon}{\epsilon}. \tag{3.43}$$

The infinite self-field contribution δm may be turned into a finite term $m = m_{bare} + \delta m$, provided m_{bare} is infinite as well. This is called mass renormalization.

When Dirac derived the Lorentz-Dirac equation [5] he considered the charged relativistic particle as a tube in four-dimensional space-time with a small radius a and calculated the energy-momentum flow out from the surface. The mass renormalization were carried out by putting

$$m_{exp.} = \frac{q^2}{2a} - f(a), \tag{3.44}$$

where $f(a)$ is a function that depends on a. The terms on the right-hand side are both infinite for $a \to 0$, such that $m_{exp.}$ is finite.

The mass renormalization is certainly a little bit odd. On the other hand, the occurring infinity is due to the point-particle description which might not agree with our physical reality (probably it does not). In some sense we have assigned the occurring infinity to physics of smaller scale. Thus, the mass renormalization raises the fundamental question whether or not the charged particle should actually be an extended object. For example a string or a sphere instead of a point-particle. Extended charges, especially Dirac's bubble model of the electron are discussed in chapter 5.

It is interesting to see what happens if we assume the charged particle to have a purely electromagnetic origin. Let therefore the bare mass be zero, so that

$$m_{exp.} = m_{em.} = \delta m \tag{3.45}$$

which means that the particle, which initially is massless, get mass contribution from the electromagnetic field when it is moving. The mass is then of purely electromagnetic origin.

In a rest frame the expression for the mass, in terms of the total energy-momentum tensor (see section (6.6)) for a particle including the self-field is

$$m_{em.} = \int d^3x T^{00} = \frac{1}{2} \int E^2 d^3x = \frac{q^2}{8\pi} \int_0^\infty \frac{dr}{r^2}. \tag{3.46}$$

The electromagnetic mass is therefore infinite due to its boundary condition. If we think of this charged particle as a well localized charge distribution, the occurring infinity is not so strange when remembering that the electromagnetic field strength grows enormously as two charged particles approach each other.

If we assume a spherical extension of the particle with radius a we get the classical electron radius

$$mc^2 = q^2 \int_a^\infty \frac{dr}{r^2} \tag{3.47}$$

such that

$$a = \frac{q^2}{mc^2} = 2.8 \cdot 10^{-15} m. \tag{3.48}$$

3.4 Action-at-a-distance theory

As we have seen, it is the self-field of the particle which are causing us problems. Mass renormalization was necessary to take care of the infinite self-energy. One idea to make the self-field of the charged particle finite is to postulate that the field generated by a single particle does not act back on itself. Hence, the particle can not produce a field on its own. This was done by Wheeler and Feynman [12], inspired by Fokker [13] in the so called action-at-a-distance theory.

A synthesis of the concepts fields and particles are made, considering the interaction of particles without fields.

The concept of self-field is eliminated and only action at a distance is considered. The relativistic interaction between particles is such that they simulate a field, which does not exist for a free particle. To get agreement with experiments which do measure radiation from a free particle, one have to include the detector as an absorber interacting with the particle.

It can be shown [14] that the equation of motion that follows from the action-at-a-distance theory is similar to the Lorentz-Dirac equation. They coincide if all radiation in the system is absorbed and thus we have to include all particles in the universe. This could imply some additional problems, but it is quite a fancy thought though.

4

Difficulties with the Lorentz-Dirac equation

We have now found an equation of motion including back radiation in the Lorentz-Dirac equation. This was done by coupling the Maxwell equations and the Lorentz-force equation together. However, a solution to the LDE has not yet been found. In this chapter we present a solution to the LDE and discuss the validity and the difficulties that follows from it. For a survey of the associated difficulties, see [15]. LDE solutions to particular problems are presented in [16].

4.1 Solution of the Lorentz-Dirac equation

To solve the Lorentz-Dirac equation (3.33), where the mass is the renormalized mass, we use the notation $u^\alpha = \dot{z}^\alpha$, $a^\alpha = \dot{u}^\alpha = \ddot{z}^\alpha$ and rewrite it as

$$m\ddot{z}^\alpha = F^\alpha_{ext.} + \frac{2q^2}{3}(\dddot{z}^\alpha - \ddot{z}_\beta \ddot{z}^\beta \dot{z}^\alpha). \tag{4.1}$$

Rearranging the terms this becomes

$$\ddot{z}^\alpha - t_0 \dddot{z}^\alpha = \frac{1}{m}F^\alpha_{ext.} - t_0 \ddot{z}_\beta \ddot{z}^\beta \dot{z}^\alpha \tag{4.2}$$

where

$$t_0 = \frac{2q^2}{3m}. \tag{4.3}$$

Equation (4.2) is a third order differential equation which implies that the motion of the particle is not determined when we know its position and velocity (which usually is the case). As we will see this demands an asymptotic condition on the acceleration $a^\alpha = \ddot{z}^\alpha(\tau)$

$$\lim_{|\tau|\to\infty} a^\alpha(\tau) = 0 \tag{4.4}$$

which turns the differential equation into an integro-differential equation.

Suppose that the external force is a function of time and use the integrating factor e^{-t/t_0} and let $\ddot{z}^2 = \ddot{z}_\beta \ddot{z}^\beta$,

$$-\frac{d}{dt}(t_0 e^{-t/t_0}\ddot{z}^\alpha) = e^{-t/t_0}\left(\frac{F^\alpha_{ext.}}{m} - t_0\ddot{z}^2\dot{z}^\alpha\right) \tag{4.5}$$

integrate,

$$t_0 e^{-t/t_0}\ddot{z}(t) = -\int_0^t dt' e^{-t'/t_0}\left[\frac{F^\alpha_{ext.}(t')}{m} - t_0\ddot{z}^2(t')\dot{z}^\alpha(t')\right] + t_0\ddot{z}^\alpha(0). \tag{4.6}$$

This integro-differential equation of motion involves a constant of integration which has to be determined on physical grounds. To avoid unphysical solutions that grows exponentially with time (so called runaway solutions) we have to let

$$t_0\ddot{z}^\alpha(0) = \int_0^\infty dt' e^{-t'/t_0}\left[\frac{F^\alpha_{ext.}(t')}{m} - t_0\ddot{z}^2(t')\dot{z}^\alpha(t')\right]. \tag{4.7}$$

With a change of variable we can write the integro-differential equation as

$$\ddot{z}^\alpha(t) = \int_0^\infty ds\, e^{-s}\left[\frac{1}{m}F^\alpha_{ext.}(t+t_0 s) - t_0\ddot{z}^2(t+t_0 s)\dot{z}^\alpha(t+t_0 s)\right] \tag{4.8}$$

where $s = \frac{1}{t_0}(t' - t)$. Without the asymptotic condition, the velocity of the charged particle will increase asymptotically to the speed of light, whether or not there is an applied force.

Now we see that the acceleration at time t depends on the force acting at times later then t. We have therefore a violation of causality, because the acceleration starts before the force begins to act. The responsible term in the LDE for the non-local effect in our integro-differential equation is the so called Schott term

$$\frac{2q^2}{3}\dddot{z}^\alpha. \tag{4.9}$$

Thus, instead of the problems of runaway solutions we have encountered the unphysical effect in the preacceleration. The time interval during which this effect occurs is of the order $t_0 \simeq \frac{q^2}{mc^3} \simeq 10^{-24}s$ for an electron. This is approximately the time it takes for light to travel through the "size" of an electron, which is impossible to measure by macroscopic experiments. So the non-causality is not possible for us to observe, but is it a real effect?

When we are dealing with these extremely small time and length scales (smaller then the Compton wave length) the validity of our classical description breaks down and are to be exchanged by a more accurate theory. The classical non-causal effect can therefore not be believed to be a real effect until quantum mechanics, which is valid in this region of time and space, is suggesting one.

We should also ask ourselves when this radiation reaction effect must be taken into account. Assume that a charged particle is affected by an external force to have an acceleration of order a during a time interval T. The energy radiated is given by Larmor's formula

$$E_{rad.} \sim \frac{2q^2}{3} a^2 T. \tag{4.10}$$

The relevant energy that this should be compared with is the kinetic energy of the charged particle, after the same period of time.

$$E_0 \sim m(aT)^2 \tag{4.11}$$

Thus, the reactive effect can be neglected if

$$\frac{2q^2}{3} a^2 T \ll m a^2 T^2$$

or

$$T \gg \frac{2q^2}{3m}, \tag{4.12}$$

which is just the characteristic time t_0 in (4.3). Radiative effects will therefore only be important if the external force is applied suddenly so that $T \sim t_0$. But when this is the case we have seen that non-causal behavior appear in our integro-differential equation (4.8).

Anyhow, the problems of preacceleration, the need for mass renormalization and runaway solutions gives rise to serious doubt of the validity of the LDE. Other peculiarities of the LDE includes a classical tunneling effect [17], [18] that occurs in the characteristic time interval t_0, i.e. in this time interval a classical particle can, according to the LDE, cross a potential barrier (which are not to be expected until considering quantum mechanics).

The fundamental problem of our outlined derivation, which ends in the difficulties of the LDE, is believed to be the idealization of a classical point-particle. Instead of assuming the particle to be point-like we might try to characterize it by a small extended charge distribution.

An electromagnetic field outside a charge distribution has the leading term q/r^2 followed by higher orders of r. If the distance r is large compared to the size of the charge distribution the field is well approximated by the electric mono pole term q/r^2, and the internal structure can be neglected. In this case a point-like description is therefore sufficient.

What if the distance is smaller so that we have to include finite size corrections to the LDE? Maybe an extended particle model better describes the interaction between the particle and the field. Is this extended model compatible with special relativity and does there exist any stable solution of it? Another quite obvious modification to the model used so far is to change the field equations by adding another field, for instance a scalar field which is included in the Standard model. Is a point-like description of the charged particle satisfactory if we include the possibility that the particle could be the source of both a vector and a scalar field? Will we then have a finite theory without unphysical solutions? These are questions that will be discussed in the following chapters.

4.2 Reduction of order

We will now investigate if it is possible to "rescue" the Lorenz-Dirac equation if we add finite size correction terms and see if it is possible to overcome our earlier difficulties.

In a frame where the particle is momentarily at rest, the LDE reduces to the non-relativistic Abraham-Lorentz equation:

$$\mathbf{a} = \frac{1}{m}\mathbf{F}_{ext.} + t_0\dot{\mathbf{a}}. \tag{4.13}$$

With finite-size correction terms (4.13) takes the form [9, 10]

$$\mathbf{a} = \frac{1}{m}\mathbf{F}_{ext.} + t_0\dot{\mathbf{a}} + O(t_0^2/t_a^2), \tag{4.14}$$

where t_a is a characteristic time interval over which the acceleration changes. For a point-like description we therefore have $t_0 \ll t_a$. In equation (4.14) the leading term is $\frac{1}{m}\mathbf{F}_{ext.}$ followed by $t_0\dot{\mathbf{a}}$, where $\dot{\mathbf{a}} \sim a_a/t_a$ and a_a is the characteristic acceleration in the time interval t_a.

$$\mathbf{a} = \frac{1}{m}\mathbf{F}_{ext.} + O(t_0/t_a) \tag{4.15}$$

Differentiate this equation and insert into (4.14).

$$\mathbf{a} = \frac{1}{m}\mathbf{F}_{ext.} + \frac{t_0}{m}\dot{\mathbf{F}}_{ext.} + O(t_0^2/t_a^2) \tag{4.16}$$

This equation does not involve third-order derivatives and the runaway solutions and preacceleration problems which comes with it, have therefore disappeared. But the equation is still only valid when $t_0 \ll t_a$. We have not gained any accuracy in this way but we have turned our equation into a second-order equation free of difficulties.

The technique just used is called "reduction of order" and can of course be applied to the relativistic LDE (3.33). First we rewrite the LDE as

$$a^\alpha = \frac{1}{m}F^\alpha_{ext.} + \frac{2}{3}q^2(\delta^\alpha_\beta + u^\alpha u_\beta)\dot{a}^\beta. \tag{4.17}$$

The leading term is $\frac{1}{m}F^{\alpha}_{ext.}$. Differentiating yields

$$\frac{1}{m}\frac{dF^{\alpha}_{ext.}}{d\tau} = \frac{1}{m}\frac{dz^{\lambda}}{d\tau}\partial_{\lambda}F^{\alpha} \tag{4.18}$$

so that

$$ma^{\alpha} = F^{\alpha}_{ext.} + t_0(\delta^{\alpha}_{\beta} + u^{\alpha}u_{\beta})\partial_{\lambda}F^{\alpha}u^{\lambda} \tag{4.19}$$

is the modified second-order LDE.

To solve the unphysical behavior of the LDE, such as preacceleration, Caldirola [19] introduced a fundamental interval of time, the chronon:

$$\Theta_0 = \frac{2q^2}{3m}, \tag{4.20}$$

which is exactly the characteristic time t_0 in equation (4.3). Time is considered as a continunm, in which events can take place only at discrete instants of time. This implies that when an external force acts on the charged particle its reaction is not continuous. In fact, continuity is not required by Lorentz invariance [20]. In macroscopic sense the particle behaves as if it was point-like. However, the internal motion of the particle is associated with a microscopic de Sitter space. This theory has also been considered recently by Yaghjian [21].

Another theory based on the fascinating idea of a quantized space-time, with relativistic invariant equations of motions for the electromagnetic field in a fully quantized space-time were done by Snyder already in 1947 [22]. The quantized space-time is related to the recent interest in so called non-commutative field theories.

5

Extended relativistic objects

As we have seen the point-like description of a charged relativistic particle interacting with an electromagnetic field is not satisfactory. An extended model has to be considered and the simplest example one may think of is a relativistic string.

A classical description of the open charged relativistic string was done in [23], where the interaction between the string and the electromagnetic field is supposed to act only at the endpoints. Thus only the endpoints of the string are charged.

Classically a more natural extended model when considering the electron is probably to view it as a sphere. This idea was proposed a long time ago.

In the beginning of the twentieth century Abraham and Lorentz made a purely electromagnetic model of a spherical symmetric charged sphere [24, 25]. Their model was a non-relativistic (Einstein proposed his special theory of relativity some years later, in 1905), which governed the infamous 4/3 problem. That is, the mass of a charged object gets contribution from the mechanical mass as well as its own electromagnetic field. Abraham and Lorentz found that the electromagnetic rest mass, with U as the self-energy, was $4/3(U/c^2)$. However, according to the Lorentz transformations on which special relativity is based on the electromagnetic rest mass is just (U/c^2). When considering a charged conducting sphere there is also a tendency for the sphere to explode as the charges repel each other.

To solve the 4/3 problem and to prevent the tendency of the charged sphere to explode, Poincaré [26] (or [27], with a more modern notation) introduced non-electromagnetic forces that prevented the spherical shell of charge to diverge by an inward pressure, though he was not at that time familiar with such forces. Together with a relativistic formulation the 4/3 problem was solved. The Poincaré stresses $P^{\mu\nu}$ adds to the electromagnetic stress-energy tensor $\Theta^{\mu\nu}$, so that the total stress-energy tensor is

$$S^{\mu\nu} = \Theta^{\mu\nu} + P^{\mu\nu}. \tag{5.1}$$

For a thorough discussion of the Abraham-Lorentz model including Poincaré stresses see [21].

The charged particle's total energy-momentum is then, according to equation (6.38)

$$P^\mu = \int d^3x S^{\mu 0}(x) \tag{5.2}$$

which in the the rest frame becomes the total mass

$$M = \int d^3x S^{00}(x). \tag{5.3}$$

If the right hand side of equation (5.2) also transforms as a four vector, then we have in the rest frame

$$\int d^3x S^{ij} = 0 \tag{5.4}$$

for $i, j = 1, 2, 3$. According to condition (5.4) the total stress-energy tensor vanish in the rest frame i.e. the sphere does not diverge acting on itself. This is just the stability condition.

One example of Poincaré stresses that we know about today is the gluon field that holds the three quarks in the proton together. The electromagnetic stress-energy tensor must be combined with a Poincaré stress-energy tensor i.e. the gluon field, to provide a stable entity of the three quarks.

5.1 Dirac's "bubble"

In 1962 Dirac published his article "An extensible model of the electron" [28] where the electron is seen as a charged bubble in the electromagnetic field. This was one of the first articles where the concept of relativistic branes were introduced and also one of the first relativistic theories including Poincaré stresses. For other models concerning extended charged objects, see [21, 29].

The model has also some resemblances to bag models of the hadrons [30, 31].

Consider the electron as having a charged conducted surface with some tension on it preventing it from diverging, i.e. some kind of non-electromagnetic Poincaré forces holding the bubble stable. The bubble is considered to be a perfect conductor such that there exists no field inside it. When stable oscillations about the equilibrium are made, Dirac suggests that the lowest exited state might be the muon.

In a relativistic four-dimensional picture the electron can be seen as a tube with a three-dimensional hyper surface. In curvilinear coordinates, the surface can be expressed by the equation $x^1 = 0$. Outside the bubble ($x^1 > 0$) we have Maxwell's equation in Minkowski space. Spin is not considered in this model.

To derive the equations of motions we start with a total action given by

$$S = S_0 + S_S \tag{5.5}$$

where the action for the Maxwell field and the surface of the electron is

$$\begin{aligned} S_0 &= -\frac{1}{4}\int_{x^1>0} J g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} d^4x \\ S_S &= -\sigma \int_{x^1=0} M dx^0 dx^2 dx^3, \end{aligned} \tag{5.6}$$

where $g_{\mu\nu}$ is the metric tensor and

$$J = \sqrt{-det(g_{\mu\nu})} \qquad \text{and} \qquad M = \sqrt{det(g_{ab})}. \tag{5.7}$$

The constant σ determines the mass and size of the electron at equilibrium and thus the strength of the surface tension. As boundary condition on the surface, Dirac demands

$$A_\mu(x) = 0 \qquad \text{at} \qquad x^1 = 0 \tag{5.8}$$

which lead to a condition on the field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,

$$F_{ab}(x) = 0 \qquad \text{at} \qquad x^1 = 0 \tag{5.9}$$

where a,b take only the values 0,2,3.

Introduce an orthogonal and rectilinear coordinate system $y^A(x)$, with $A = 0, 1, 2, 3$, so that the metric tensor can be written as

$$g_{\mu\nu} = \partial_\mu y^A \partial_\nu y^B \eta_{AB}. \tag{5.10}$$

Hamilton's principle tells us that a small variation in the action is zero, $\delta S = 0$. Applying this to our action by varying $y^A(x)$ and $A_\mu(x)$ we obtain

$$\delta S_0 = -\frac{1}{2}\int \{\frac{1}{2}\delta J g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + J g^{\mu\rho} \delta g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + J g^{\mu\rho} \delta g^{\nu\sigma} F_{\mu\nu} \delta F_{\rho\sigma}\}. \tag{5.11}$$

By means of

$$\begin{aligned} \delta J &= \frac{1}{2} J g_{\alpha\beta} \delta g^{\alpha\beta} \\ \delta g^{\nu\sigma} &= -g^{\nu\alpha} \delta g_{\alpha\beta} g^{\beta\sigma} \end{aligned} \tag{5.12}$$

together with the chain rule, equation (5.10) and the boundary condition $\delta A_\rho = 0$ equation (5.11) becomes

$$\begin{aligned} \delta S_0 &= \int \{-\partial_\sigma (J F^{\rho\sigma}) \delta A_\rho + J(F_\mu{}^\alpha F^{\mu\beta} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\alpha\beta}) \partial_\alpha y_A \partial_\beta \delta y^A\} d^4x \\ &= -\int \{\partial_\sigma (J F^{\rho\sigma}) \delta A_\rho + \partial_\beta [J(F_\mu{}^\alpha F^{\mu\beta} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\alpha\beta}) \partial_\alpha y_A] \delta y^\beta\} d^4x \\ &\quad + \int J(F_\mu{}^\alpha F^{\mu 1} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\alpha 1}) \partial_\alpha y_A \delta y^A dx^0 dx^2 dx^3. \end{aligned} \tag{5.13}$$

The three-dimensional integral is over the surface $x^1 = 0$ and the four-dimensional integral is over $x^1 > 0$. In addition we have that

$$\begin{aligned} \delta S_S &= -\sigma \int M c^{ab} \partial_a y^A \partial_b \delta y^A dx^0 dx^2 dx^3 \\ &= \sigma \int \partial_b (M c^{ab} \partial_a y^A) \delta y^A dx^0 dx^2 dx^3. \end{aligned} \tag{5.14}$$

M is defined by equation (5.7) and c^{ab} is the reciprocal matrix to g_{ab}, i.e.

$$g_{ab} c^{bc} = \delta_a^{\ c}. \tag{5.15}$$

As $\delta S = 0$, each coefficient of δA_ρ and δy^A must be zero. For δA_ρ in (5.13) we then arrive at the following Maxwellian equations in the region $x^1 > 0$

$$\partial_\sigma (J F^{\rho\sigma}) = 0. \tag{5.16}$$

For the coefficients of δy^A on the surface $x^1 = 0$ we have from (5.13) and (5.14)

$$J(F_\mu^{\ \alpha} F^{\mu 1} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\alpha 1}) \partial_\alpha y_A + \sigma \partial_b (M c^{ab} \partial_a y^A) = 0. \tag{5.17}$$

If we multiply with $\partial_\rho y^A$ and use equation (5.10) and

$$\partial_\rho M = \partial_\rho g_{ab} M c^{ab}, \tag{5.18}$$

we arrive at

$$J(F_{\mu\rho} F^{\mu 1} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^1_{\ \rho} + \sigma \partial_b (M c^{ab} g_{a\rho}) = \sigma \partial_\rho M. \tag{5.19}$$

We only get contributions for $\rho = 1$ due to our boundary values (5.9). Since $g_{a\mu} g^{\mu 1} = \delta_a^{\ 1} = 0$ and by means of the definition (5.15), we are able to write the reciprocal matrix as

$$c^{ab} = g^{ab} - \frac{g^{1a} g^{1b}}{g^{11}}. \tag{5.20}$$

We can now write down the equations of motion for the surface of the bubble in the electromagnetic field:

$$\frac{1}{2} F_{a1} F^{a1} = \frac{\sigma}{J} \partial_\mu \Big(\frac{M g^{1\mu}}{g^{11}} \Big). \tag{5.21}$$

5.1.1 A spherical symmetric solution

To see if it is possible that the muon can be an exited state of the electron Dirac makes some rough estimations in the calculation of the oscillations.

We start with ordinary spherical coordinates

$$\begin{aligned} y^0 &= t \\ y^1 &= r\sin\theta\cos\varphi \\ y^2 &= r\sin\theta\sin\varphi \\ y^3 &= r\cos\theta \end{aligned} \tag{5.22}$$

with ρ as the radius of the electron and $x^1 = r - \rho$ so that $x^1 = 0$ at the boundary as demanded. Since the invariant proper time interval

$$d\tau^2 = -g_{\mu\nu}dy^\mu dy^\nu \tag{5.23}$$

we differentiate and obtain the metric

$$d\tau^2 = (1-\dot{\rho}^2)dt^2 - 2\dot{\rho}dtdx^1 - (dx^1)^2 - (x^1+\rho)^2d\theta^2 - (x^1+\rho)^2\sin^2\theta d\varphi^2 \tag{5.24}$$

such that

$$g_{\mu\nu} = \begin{pmatrix} -(1-\rho^2) & \dot{\rho} & 0 & 0 \\ \dot{\rho} & 1 & 0 & 0 \\ 0 & 0 & (x^1+\rho)^2 & 0 \\ 0 & 0 & 0 & (x^1+\rho)^2\sin^2\theta \end{pmatrix}. \tag{5.25}$$

Using the formula for the inverse matrix

$$\left(A^{-1}\right)_{ij} = \frac{1}{detA_{ij}}A_{ji} \tag{5.26}$$

we also find that

$$g^{\mu\nu} = \begin{pmatrix} -1 & \dot{\rho} & 0 & 0 \\ \dot{\rho} & (1-\dot{\rho}^2) & 0 & 0 \\ 0 & 0 & (x^1+\rho)^{-2} & 0 \\ 0 & 0 & 0 & (x^1+\rho)^{-2}\sin^{-2}\theta \end{pmatrix}. \tag{5.27}$$

The right-hand side of the equation of motion (5.21) is then, on the surface $(x^1 = 0)$

$$\frac{1}{J}\partial_\mu\left(\frac{Mg^{1\mu}}{g^{11}}\right) = \frac{d}{dt}\frac{\dot{\rho}}{(1-\dot{\rho}^2)^{1/2}} + \frac{2}{\rho(1-\dot{\rho}^2)^{1/2}}. \tag{5.28}$$

Just outside the surface of the electron there exist a Coulomb field with $F_{a1}F^{a1} = e^2/\rho^4$, where e is the charge of the electron. From (5.21) and (5.28) we obtain the equation of motion

$$\frac{d}{dt}\frac{\dot{\rho}}{(1-\dot{\rho}^2)^{1/2}} = \frac{-2}{\rho(1-\dot{\rho}^2)^{1/2}} + \frac{e^2}{2\sigma\rho^4}. \tag{5.29}$$

If we set $\dot{\rho} = 0$ and let a be the equilibrium radius we have that

$$a^3 = \frac{e^2}{4\sigma}. \tag{5.30}$$

If the total energy gets contribution from the Coulomb field $(e^2/2\rho)$ and from the Poincaré stresses, e.g. $(A\rho^2)$ we can write it as

$$E = e^2/2\rho + A\rho^2. \tag{5.31}$$

After differentiating to get a minimum of the energy at $\rho = a$ we see that

$$a^3 = \frac{e^2}{4A}. \tag{5.32}$$

Therefore we must have that the constant $A = \sigma$.

As the total energy $E = mc^2$ (c=1), the radius can be expressed in terms of the mass and the charge

$$a = \frac{3e^2}{4m}. \tag{5.33}$$

For small oscillations with $\dot{\rho} = 0$ we can write equation (5.28) as

$$\ddot{\rho} = \frac{2a^3}{\rho^4} - \frac{2}{\rho} = \frac{2}{a}\left[(1 + \frac{\rho - a}{a})^{-4} - (1 + \frac{\rho - a}{a})^{-1}\right]. \tag{5.34}$$

Using that

$$(1 + z)^{-1} = 1 - z + z^2 - z^3 + \ldots \tag{5.35}$$

and only taking linear terms, this expression can be written as

$$\ddot{\rho} + \frac{6}{a^2}\rho = \frac{6}{a}. \tag{5.36}$$

We see that the frequency is

$$\omega = \frac{\sqrt{6}}{a}. \tag{5.37}$$

Thus the energy for this frequency of the electron is

$$\hbar\omega = \frac{4\sqrt{6}}{3}\frac{m\hbar}{e^2} \approx 448m. \tag{5.38}$$

which at least is of right order for the energy of the muon in this quite naive picture.

5.1.2 Problems with stability

Although Dirac's extended model seems to be a consistent relativistic model it has be shown by Gnadig et al. [33] that the bubble is not stable against non-spherical deformations. Any small perturbation turns the spherical electron into an infinitely long cigar-like object or an infinite string (with some thickness). Here we quote the essence of their analysis.

The equation of motion (5.29) can be interpreted as the Euler-Lagrange equation obtained from a Lagrangian of the form

$$\mathcal{L} = -4\pi\sigma\rho^2(1-\dot{\rho}^2)^{1/2} - \frac{e^2}{8\pi\rho}. \tag{5.39}$$

Since $\mathcal{L}$ has no explicit time dependence, the total energy is a conserved quantity. It is

$$E = \dot{\rho}\frac{\partial\mathcal{L}}{\partial\dot{\rho}} - \mathcal{L} = \sigma\frac{4\pi\rho^2}{(1-\dot{\rho}^2)^{1/2}} + \frac{e}{8\pi\rho}. \tag{5.40}$$

The first term is the surface energy including a relativistic factor $(1-\dot{\rho}^2)^{1/2}$ and the second is the Coulomb energy.

Now, consider the static energy of an oblate and prolate spherical surface.

$$E = \sigma S + \frac{e}{8\pi C} \tag{5.41}$$

If we denote the major and minor axes for the prolate spheroid as $2b$ and $2a$, with $b > a$, the surface energy becomes

$$S = 2\pi\left(a^2 + \frac{ab^2}{(b^2-a^2)^{1/2}}\cos^{-1}(a/b)\right) \tag{5.42}$$

and the capacity of the bubble is

$$C = \frac{(b^2-a^2)^{1/2}}{cosh^{-1}(b/a)}. \tag{5.43}$$

If we let $\gamma = b/a$, the minimum value of $E(\gamma)$ as a function of a, for every fixed γ, turns out to be finite as $\gamma \to 0$ (the spheroid becomes a disc). For $\gamma \to \infty$ it goes to zero as

$$E \sim \frac{(\log\gamma)^{2/3}}{\gamma^{1/3}} \to 0. \tag{5.44}$$

Since the energy function is positive semi-definite, the absolute minimum is for $\gamma \to \infty$. The so called zero-energy configuration therefore corresponds to an infinitely long cigar-like object. As $\gamma \to \infty$ the surface charge density goes faster to zero than the capacity per unit length. The Dirac bubble is unstable.

Consider a bubble that a pulse hits on one side. If the whole bubble starts to move at the same time, some sort of super luminal information are transported to the other side of the bubble. If it does not, some kind of deformation of the sphere must occur which demands internal motion. The bubble can not be a rigid body, it has to be dynamical. However, the question what its fundamental relativistic formulation, that allows stable bound states, remains.

Dirac, unaware of the instability of the bubble, even thought that he was on his way to find a theory for a finite size charged particle interacting with electromagnetic and gravitational fields [34].

Dirac's simple extended relativistic model have to be modified. Gnadig et al. proposes that an additional term to the action (5.5) involving a scalar field should make the bubble stable. According to J. Kuti there is indeed a stable solution with a scalar field inside the bubble. The incorporation of a scalar field for extended sources in classical physics raises the idea of doing the same for a point-charge.

6

Point-charge interacting with both vector and scalar fields

We have seen in the Maxwell-Lorentz theory that the interaction between the charged point-particle and its generated electromagnetic field yields an infinite self-force. When trying to overcome this divergence by splitting the vector potential and renormalizing the mass, we ended up with the Lorentz-Dirac equation. But the LDE turned out to include many difficulties and we can not be satisfied with this incomplete description of a moving charged particle in an electromagnetic field (massless vector field).

If the particle is supposed to only interact with a scalar field instead of a massless vector field, an equation of motion of the form of the Lorentz-Dirac equation with all its difficulties follows [35].

In this chapter we let the charged point-particle interact with an arbitrary number of vector and scalar fields which both can be massive [1] or not, as was done by van Holten [36]. This assumption is not so strange as one may believe. According to the Standard model, in which classical physics is a part of, there exist one scalar field, the Higgs field which are supposed to endow all particles with mass. By including the static Coulomb- and Yukawa fields coupled to the particle, one finds the rest energy. These fields are infinite, thus to find a finite mass one has to add infinite compensating terms. These terms are supposed to come from physics at smaller distance scales.

Extended models such as bag models [31, 32], used in nuclear and particle physics, often include scalar fields that holds the charged sphere together (because like charges generated by scalar fields are attractive), and massive vector fields to prevent it from collapse. It has also been suggestions that the Dirac bubble could be stabilized by a scalar field [33].

The interaction between the particle and fields, with the condition that the

[1] Massive fields is only a valid concept in quantum physics, one might therefore consider the model as semi-classical.

charged particle is stable (the particle can not exert a force on its own) remove all inconsistencies in the Maxwell-Lorentz theory and gives rise to constraints that have connection to the unified electroweak standard model. Later on we calculate an expression for the point-particle mass that becomes finitely computable in terms of field parameters.

A similar mass calculation from the energy-momentum tensor (including scalar fields) were made by Stückelberg already in 1941 [37] and an attempt to do the same in quantum field theory were done by Pais in 1947 [38]. Until now there has not been any classical analogy for this model. The research has instead mostly been focused on solving the divergences in quantum field theory, mainly quantum gravity. However, if we find a classical theory without the infinite self-energies and then quantize it, a self-consistent finite quantum theory could be in reach. On the other hand, a classical theory is an approximation to a quantum theory, thus to find a finite classical theory one should start with a quantum theory and then investigate the classical limit and not the other way around. As far as I know one can not start from scratch in a quantum theory, you have to go from some sort of classical theory. It should be mentioned that string theory is at present the general approach for solving these problems.

6.1 The field equations

Following van Holten [36], we consider a charged point-particle coupled to a system of N_v vector fields and N_s scalar fields. The field part of the total action is

$$\begin{aligned} S_{field} = \int d^4x \bigg\{ &-\sum_{i=1}^{N_s} \Big[\frac{1}{2}(\partial_\mu \varphi_i)^2 + \frac{{\mu_i}^2}{2}(\varphi_i - f_i)^2 + \rho_i \varphi_i\Big] \\ &-\sum_{\alpha=1}^{N_v} \Big[\frac{1}{4}(F^\alpha_{\mu\nu})^2 + \frac{\mu_\alpha^2}{2}(A^\alpha_\mu)^2 - A^\alpha_\mu j^\mu_\alpha\Big] \bigg\}. \end{aligned} \tag{6.1}$$

Here μ_i, μ_α represents the masses of the scalar (φ_i) and vector fields ($F^\alpha_{\mu\nu}$), i.e. some ranges of the fields $l_{i,\alpha} = \mu^{-1}_{i,\alpha}$. Even though we are operating in classical physics we incorporate a concept used in quantum field theory, namely the possibility that the scalar field have a vacuum expectation value $\langle \varphi_i \rangle = f_i$. The first three terms describe the massive scalar fields and the last three the massive vector fields, also known as the Proca Lagrangian. The scalar charge density ρ_i and the current density j^μ_α are given by

$$\rho_i(x) = g_i \int d\lambda \sqrt{-\Big(\frac{dz^\mu}{d\lambda}\Big)^2}\, \delta^4(x - z(\lambda)) = g_i \delta^3\Big(\frac{\mathbf{x} - \mathbf{z}(t)}{\sqrt{1-v^2}}\Big) \tag{6.2}$$

$$j^\mu_\alpha(x) = q_\alpha \int d\lambda \frac{dz^\mu}{d\lambda} \delta^4(x - z(\lambda)) = q_\alpha u^\mu \delta^3\Big(\frac{\mathbf{x} - \mathbf{z}(t)}{\sqrt{1-v^2}}\Big). \tag{6.3}$$

The free particle is, as before, described by the action

$$S_{part} = -m \int d\lambda \sqrt{-\Big(\frac{dz^\mu}{d\lambda}\Big)^2}. \tag{6.4}$$

To get the equations of motion for the fields, one for the scalar and one for the vector part, we use the Euler-Lagrange equation (2.12). Our Lagrangian for the field, taken from (6.1), is

$$\begin{aligned}\mathcal{L}_{field} = &-\sum_{i=1}^{N_s}\Big[\frac{1}{2}(\partial_\mu\varphi_i)^2 + \frac{\mu_i^2}{2}(\varphi_i - f_i)^2 + \rho_i\varphi_i\Big] \\ &-\sum_{\alpha=1}^{N_v}\Big[\frac{1}{4}(F^\alpha_{\mu\nu})^2 + \frac{\mu_\alpha^2}{2}(A^\alpha_\mu)^2 - A^\alpha_\mu j^\mu_\alpha\Big].\end{aligned} \tag{6.5}$$

6.1.1 The scalar field equation

To obtain the field equation for the scalar fields we vary φ_i so that

$$\begin{aligned}\frac{\partial\mathcal{L}_{field}}{\partial\varphi_i} &= -\mu_i^2(\varphi_i - f_i) - \rho_i \\ \frac{\partial\mathcal{L}_{field}}{\partial(\partial_\mu\varphi_i)} &= -\partial^\mu\varphi_i.\end{aligned} \tag{6.6}$$

When this is inserted into the Euler-Lagrange equation for fields (2.12) we get the scalar field equation

$$(-\Box + \mu_i^2)(\varphi_i - f_i) = -\rho_i(x). \tag{6.7}$$

6.1.2 The vector field equation

By varying A^α_μ we get the equation of motion for the vector fields (see equation (2.17), (2.18))

$$\begin{aligned}\frac{\partial\mathcal{L}_{field}}{\partial A^\alpha_\mu} &= -\mu_\alpha^2\eta^{\mu\nu}A^\alpha_\nu + j^\mu\alpha \\ \frac{\partial\mathcal{L}_{field}}{\partial(\partial_\nu A^\alpha_\mu)} &= \partial^\mu A^{\alpha\nu} - \partial^\nu A^{\alpha\mu}\end{aligned} \tag{6.8}$$

and the vector field equation is

$$[(-\Box + \mu_\alpha^2)\eta^{\mu\nu} + \partial^\mu\partial^\nu]A^\alpha_\nu = j^\mu_\alpha. \tag{6.9}$$

6.2 Solution of field equations

For a particle moving with constant velocity we can completely solve the field equations.

The scalar field equation (6.7) can be written as

$$(-\nabla^2 + \frac{\partial^2}{\partial t^2} + \mu_i^2)(\varphi_i - f_i) = -\rho_i(x). \tag{6.10}$$

The source term is independent of time, so we solve the inhomogenius equation

$$(\nabla^2 - \mu_i^2)\Psi(\mathbf{x}) = \rho(\mathbf{x}) = g_i\delta^3(\mathbf{R}_{ret.}) \tag{6.11}$$

where $\Psi(x) = (\varphi_i - f_i)$. The homogeneous solution will correspond to the free waves and is here just denoted φ_i^{free}. The retarded distance parameter, for a particle in it's rest frame moving with velocity v in the z-direction of the lab. system, is

$$\mathbf{R}_{ret.} = \Big(x, y, \frac{z - vt}{\sqrt{1 - v^2}}\Big). \tag{6.12}$$

To solve the scalar field equation we Fourier transform

$$\begin{aligned}\Psi_i(\mathbf{r}) &= \frac{1}{(2\pi)^{(3/2)}}\int \widetilde{\Psi}_i(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}d^3k & (6.13)\\ \delta^3(\mathbf{r}) &= \frac{1}{(2\pi)^3}\int e^{-i\mathbf{k}\cdot\mathbf{r}}d^3k. & (6.14)\end{aligned}$$

Insert into (6.11):

$$(-k^2 - \mu_i^2)\widetilde{\Psi}_i(\mathbf{k}) = \frac{g_i}{(2\pi)^{3/2}} \tag{6.15}$$

and substitute into (6.13),

$$\Psi_i(\mathbf{r}) = \frac{-g_i}{(2\pi)^3}\int \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{(k^2 + \mu_i^2)}d^3k. \tag{6.16}$$

As $d^3k = k^2 sin\theta\, dk\, d\theta\, d\varphi$ and $\mathbf{k}\cdot\mathbf{r} = kR\, cos\theta$ we can go on calculating the scalar field.

$$\begin{aligned}\Psi_i(\mathbf{r}) &= \frac{-g_i}{(2\pi)^3}\int_0^\infty\int_0^{2\pi}\int_0^\pi \frac{k^2 sin\theta\, e^{-ikRcos\theta}}{(k^2+\mu_i^2)}dk\, d\varphi\, d\theta\\ &= \frac{-g_i}{4\pi^2 iR}\int_{-\infty}^\infty \frac{ke^{ikR}}{(k^2+\mu_i^2)}dk = \frac{-g_i}{4\pi^2 iR}\cdot 2\pi i\cdot Res\Big\{\frac{ke^{ikR}}{(k^2+\mu_i^2)}; +i\mu_i\Big\}\\ &= \frac{-g_i}{4\pi^2\cdot iR}\cdot 2\pi i\frac{(i\mu_i)e^{-\mu_i R}}{2i\mu_i} = \frac{-g_i}{4\pi}\frac{e^{-\mu_i}R_{ret.}}{R_{ret.}} & (6.17)\end{aligned}$$

So the retarded solution of the scalar field equation is

$$\varphi_i = \varphi_i^{free} + f_i - \frac{g_i}{4\pi}\frac{e^{-\mu_i R_{ret.}}}{R_{ret.}}. \tag{6.18}$$

In the same way we can obtain the retarded solution to the vector field equation (6.9) by using Lorentz gauge $\partial^\nu A^\alpha_\nu = 0$ such that

$$(\nabla^2 - \mu_i^2) A^\mu_\alpha = -j^\mu_\alpha. \tag{6.19}$$

With some Fourier transforms and a little calculus of residue the solution becomes

$$A^\alpha_\mu = A^{\alpha\, free}_\mu + \frac{q_\alpha u^\mu}{4\pi} \frac{e^{-\mu_i R_{ret.}}}{R_{ret.}}. \tag{6.20}$$

The term $A^{\alpha\, free}_\mu$ represents the free electromagnetic waves.

We notice that in equation (6.18) and (6.20) both the Yukawa and the Coulomb self-field are infinite at $R = 0$. The Yukawa and Coulomb self-fields always accompany the particle and are therefore contributing to its mass.

6.3 Equation of motion for the charged particle

The motion for the particle, dressed with it's Coulomb- and Yukawa field can be derived from the Euler-Lagrange equation, (previously mentioned in equation (2.20))

$$\frac{\partial \mathcal{L}}{\partial z^\mu} - \frac{\partial \mathcal{L}}{\partial \dot{z}^\mu} = 0. \tag{6.21}$$

From equation (6.1) and (6.4) we can evaluate the different terms of the Euler-Lagrange equation.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial z^\mu} &= -\sum_i g_i \partial_\mu \varphi_i + \sum_\alpha \partial_\mu A^\alpha_\nu \frac{dz^\nu}{d\tau} \\ \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{z}^\mu} &= \frac{d}{d\tau}\Big(m \frac{dz^\mu}{d\tau}\Big) + \frac{d}{d\tau}\Big(\sum_i \varphi_i g_i \frac{dz^\mu}{d\tau}\Big) + \sum_\alpha q_\alpha \partial_\nu A^\alpha_\mu \frac{dz^\nu}{d\tau} \end{aligned} \tag{6.22}$$

When these are inserted into (6.21) we obtain the equation of motion for the charged point-particle in interaction with dynamical scalar and vector fields:

$$\frac{d}{d\tau}\Big[(m + \sum_i g_i \varphi_i(z)) \frac{dz^\mu}{d\tau}\Big] = -\sum_i g_i \partial^\mu \varphi_i(z) + \sum_\alpha q_\alpha F^\mu_{\alpha\,\nu}(z) \frac{dz^\nu}{d\tau}. \tag{6.23}$$

Notice that this equation reduces to the familiar Lorentz-force equation (2.25) if the scalar field vanishes, $\varphi_i = 0$.

6.4 Imposing stability condition

The equation of motion should be evaluated on the world line of the particle. When we in the first chapter were dealing with only interaction between a particle and an electromagnetic field we saw that the field were singular on the world line. This is still the case, as can be seen in equation (6.18) and (6.20) when $R = 0$. However, now when a scalar field also is present this might be interesting to investigate further. As in the Maxwell-Lorentz theory we also here have field equations and an equation of motion for the particle that have to be solved as a set of coupled equations. This is done by inserting the solution of the field equations (6.18), (6.20) into (6.23).

Physics requires that if we do not have any external forces on the particle, it has to be stable and not exert a force acting on itself. This condition requires

$$\frac{d^2 z^\mu}{d\tau^2} = 0. \tag{6.24}$$

With this criterion of stability substituted into our equation of motion (6.23) we get

$$\frac{dz^\mu}{d\tau}\sum_i g_i \frac{d}{d\tau}\varphi_i(z) = -\sum_i g_i \partial^\mu \varphi_i(z) + \sum_\alpha q_\alpha F^\mu_{\alpha\,\nu}(z)\frac{dz^\nu}{d\tau}. \tag{6.25}$$

If we use that

$$\frac{d}{d\tau}\varphi_i(z) = \frac{dz^\nu}{d\tau}\partial_\nu \varphi_i \tag{6.26}$$

which turns the above equation into the form

$$\sum_i g_i \partial_\nu \varphi_i(z)\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau} = -\sum_i g_i \partial^\mu \varphi_i(z) + \sum_\alpha q_\alpha F^\mu_{\alpha\,\nu}(z)\frac{dz^\nu}{d\tau}. \tag{6.27}$$

In the particle's rest frame, where all fields are static this equation reduces, after summation over the indices μ and ν, to

$$0 = -\sum_i g_i \nabla \varphi_i(z) + \sum_\alpha q_\alpha F^j_{\alpha\,0}(z), \tag{6.28}$$

where the index $j = 1, 2, 3$. The electromagnetic field tensor is given by

$$F^\mu_{\ \nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ & 0 & B_3 & -B_2 \\ & & 0 & B_1 \\ & & & 0 \end{pmatrix}. \tag{6.29}$$

This implies

$$-\sum_i g_i \nabla \varphi_i(z) + \sum_\alpha q_\alpha \mathbf{E}_\alpha(z) = 0. \tag{6.30}$$

According to the solutions of the field equations (6.18), (6.20) both of the terms in (6.30) are by themselves singular as $R_{ret.} \to 0$. Now we let the free radiation field be equal to zero and use that $\mathbf{E}_\alpha \sim \nabla\varphi_\alpha \sim \nabla \frac{q_\alpha}{4\pi}\frac{e^{-\mu_\alpha R}}{R}$.

To examine the singularity we insert the solutions to the field equations and take the limit $R \to 0$ (from here on the subscript $ret.$ is left out).

$$\lim_{R\to 0} \nabla \left[\sum_i \frac{g_i^2}{4\pi} \frac{e^{-\mu_i R}}{R} - \sum_\alpha \frac{q_\alpha^2}{4\pi} \frac{e^{-\mu_\alpha R}}{R} \right] = 0 \tag{6.31}$$

If we insert the power expansion of exponentials, i.e.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{6.32}$$

which has infinite radius of convergence, then we find

$$\begin{aligned} \lim_{R\to 0} \nabla \quad \Bigg[\quad & \sum_i \left(\frac{g_i^2}{R} - g_i^2 \mu_i + \frac{g_i^2 \mu_i^2 R}{2} + O(R^2) \right) \\ & - \sum_\alpha \left(\frac{q_\alpha^2}{R} - q_\alpha^2 \mu_\alpha + \frac{q_\alpha^2 \mu_\alpha^2 R}{2} + O(R^2) \right) \Bigg] = 0. \end{aligned} \tag{6.33}$$

Then we differentiate, so that

$$\begin{aligned} \lim_{R\to 0} \quad \Bigg[\quad & \sum_i \left(- \frac{g_i^2}{R^2} + \frac{g_i^2 \mu_i^2}{2} + O(R) \right) \\ & - \sum_\alpha \left(- \frac{q_\alpha^2}{R^2} + \frac{q_\alpha^2 \mu_\alpha^2}{2} + O(R) \right) \Bigg] = 0. \end{aligned} \tag{6.34}$$

As we now let $R \to 0$, the residue terms $1/R^2$ and the constant terms only vanish if the scalar and vector components cancel. For the infinite $1/R^2$-terms, we find the condition:

$$\sum_i^{N_s} g_i^2 = \sum_\alpha^{N_v} q_\alpha^2 \tag{6.35}$$

which also was found by Stückelberg [37]. The vanishing of the constant terms requires:

$$\sum_i^{N_s} g_i^2 \mu_i^2 = \sum_\alpha^{N_v} q_\alpha^2 \mu_\alpha^2. \tag{6.36}$$

Due to the stability condition (6.24) we have two constraints on the scalar and vector charges g_i, q_α and the scalar and vector masses μ_i, μ_α.

From these constraints it follows that if any scalar field is present, there must at least be one vector field present. If all vector fields are massless, all scalar fields are massless. If one scalar field has mass, the particle must couple to at least one massive vector field.

6.5 Comparison with the Standard model

Dealing with classical electrodynamics we ought to know that this theory is a part of a greater unified description of electromagnetic, strong and weak interaction based on the fundamental particles, quarks and leptons. This description is given by the Standard model. The Standard model is a quantum field theoretical model based on symmetrical properties of the particles and gauge invariance of the forces. The gauge group is $SU(3) \otimes SU(2) \otimes U(1)$. We have several force carriers corresponding to the different interactions. The electromagnetic interaction with the photon, γ, as massless force carrier, the weak interaction with three massive vector bosons $Z^0, W^\pm$ and the strong interaction with eight gluons as massless force carrier. All of these particles are bosons and have spin one and are therefore described by vector field equations.

The Higgs particle, which behaves more as a regulator is supposed to endow the gauge bosons with mass. This is a consequence of spontaneous symmetry breaking, which turns the massless force carriers into massive bosons $Z^0, W^\pm$ and the eight gluons, but leaves the photon massless. The Higgs particle has spin zero and is represented by a scalar field operator. The Higgs scalar is yet to be observed, but analysis of experimental data and the Standard model suggests that it is a massive particle. At the moment much effort are made to find it. In fact, when this thesis was just about finished there were rumors that the Higgs scalar had been found at CERN, with a mass of 114 GeV.

In our model described in this chapter we do not consider self-interaction of the fields. Thus our vector fields are of abelian type and does not fully apply to the Standard model. More precisely, our model can only be applied to a version of the electroweak standard model based on gauge group $U(1) \otimes U(1) \otimes U(1) \otimes U(1)$ where our charged scalar particle only couples to photons and Z^0, and the massive, charged vector bosons $W^\pm$.

When considering the electroweak standard model it follows from the constraints (6.35) and (6.36) that the Higgs field must have mass. Because if the neutrino couples to the massive vector boson Z^0, then it couples to some massive scalar field, and as the only scalar field in the Standard model is the Higgs field it must have mass.

6.6 Finite mass of a charged point-particle

One of Einstein's cornerstones in his general theory of relativity is the equivalence principle. It is based on the fact that the kinematic mass (entering in Newton's second law, $\mathbf{F} = m_k \mathbf{a}$) is the the same as the gravitational one ($\mathbf{F} = m_g \mathbf{g}$). This equality was in fact already suggested by Newton and later observed by Eötvös in 1889. In order to calculate the mass of a charged point-particle fully interacting with dynamical scalar and vector fields we can use this equivalence. We first set up the masshell condition for the kinematic mass

$$p_\mu^2 + m^2 = -E^2 + \mathbf{p} + m^2 = 0. \tag{6.37}$$

The gravitational mass of the charged particle is defined by its energy-momentum tensor. In flat Minkowski space the conserved four-momentum of equation (6.37) over the space-like 3-dimensional hyper surface $\sum$ is

$$p^\mu = \int_\sum d^3x T^{\mu 0}. \tag{6.38}$$

In the rest frame ($\mathbf{p} = 0$) this expression reduces to

$$m = \int_\sum d^3x T^{00}. \tag{6.39}$$

6.6.1 The stress-energy tensor

To calculate the total mass we have to find an expression for the total stress-energy tensor. We begin by splitting the stress-energy tensor of the particle and field into different parts.

$$T_{\mu\nu} = T^{part.}_{\mu\nu} + T^{scalar}_{\mu\nu} + T^{vector}_{\mu\nu} + \Lambda\eta_{\mu\nu} \tag{6.40}$$

where the last term is an arbitrary cosmological constant. The stress-energy tensor is a real and symmetric matrix and can be written in terms of eigenvectors $n_{(\kappa)}, \kappa = 0, 1, 2, 3$ and eigenvalues $\alpha_{(\kappa)}$.

$$T^\mu_\nu n^\nu_{(\kappa)} = \alpha_{(\kappa)} n^\mu_{(\kappa)} \qquad \text{so that} \qquad \eta_{\mu\nu} n^\mu_{(\kappa)} n^\nu_{(\kappa')} = n_{\kappa\kappa'} \tag{6.41}$$

For a particle moving in the z-direction, with velocity v, the following eigenvectors can be chosen:

$$\begin{aligned}
n^\mu_{(0)} &= \Big(\frac{1}{\sqrt{1-v^2}}, 0, 0, \frac{v}{\sqrt{1-v^2}}\Big) \\
n^\mu_{(1)} &= \Big(\frac{(z-vt)}{R\sqrt{1-v^2}}v, \frac{x}{R}, \frac{y}{R}, \frac{z-vt}{R\sqrt{1-v^2}}\Big) \\
n^\mu_{(2)} &= \Big(\frac{-\sqrt{x^2+y^2}}{R\sqrt{1-v^2}}v, \frac{x}{\sqrt{x^2+y^2}}\frac{(z-vt)}{R\sqrt{1-v^2}}, \\
&\quad , \frac{y}{\sqrt{x^2+y^2}}\frac{(z-vt)}{R\sqrt{1-v^2}}, \frac{-\sqrt{x^2+y^2}}{R\sqrt{1-v^2}}\Big) \\
n^\mu_{(3)} &= \Big(0, \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}, 0\Big).
\end{aligned} \tag{6.42}$$

Notice that the time-like eigenvector is the four-velocity

$$n^\mu_{(0)} = u^\mu. \tag{6.43}$$

For the retarded distance parameter we have

$$R = R_{ret.} = \sqrt{x^2 + y^2 + \frac{(z-vt)^2}{(1-v^2)}}. \tag{6.44}$$

With spherical coordinates and the particle taken to be in it's rest frame, these eigenvectors simplifies to

$$\begin{aligned} n_{(0)} &= (1,0,0,0) \\ n_{(1)} &= (0,\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta) \\ n_{(2)} &= (0,\cos\theta\cos\phi,\cos\theta\sin\phi,-\sin\theta) \\ n_{(3)} &= (0,-\sin\theta,\cos\phi,0). \end{aligned} \tag{6.45}$$

In terms of the just named basis, the stress-energy tensor is

$$T_{\mu\nu} = \sum_{\kappa} \alpha_{(\kappa)} n_{(\kappa)\mu} n_{(\kappa)\nu}. \tag{6.46}$$

and $\alpha_{(\kappa)}$ is Lorentz invariant. From Noether's theorem for fields we find that the conserved canonical stress-energy tensor is

$$\widetilde{T}^{\mu\nu}(x) = \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi^s(x))}\partial^\nu\phi^s(x) - \eta^{\mu\nu}\mathcal{L}(x). \tag{6.47}$$

With the relation between the canonical and the non-canonical stress-energy tensor:

$$T^{\mu\nu} = \widetilde{T}^{\mu\nu} - \partial_\sigma f^{\mu\sigma\nu}. \tag{6.48}$$

The term $\partial_\sigma f^{\mu\sigma\nu}$ comes from the transformation properties of the underlying fields and is zero for scalar fields.

For the first part of the total stress-energy tensor (6.40) we have that

$$\begin{aligned} T^{part.}_{\mu\nu} &= \int d\tau \Big(m + \sum g_i\varphi_i(z)\Big)\frac{dz_\mu}{d\tau}\frac{dz_\nu}{d\tau}\delta^4(x - z(\tau)) \\ &= \Big(m + \sum g_i\varphi_i(z)\Big)\delta\Big(\frac{\mathbf{x} - \mathbf{z}(t)}{\sqrt{1-v^2}}\Big) n_{(0)\mu}n_{(0)\nu}. \end{aligned} \tag{6.49}$$

With the scalar field solution (6.18) and $\varphi_i^{free} = 0$ and (6.46) we get the eigenvalue

$$\alpha^{part.}_{(0)} = \Big(m + \sum_i g_i\varphi_i(z)\Big)\delta^3(\mathbf{R}) = \Big(m + \sum_i g_i f_i - \sum_i g_i^2\frac{e^{-\mu_i R}}{4\pi R}\Big)\delta^3(\mathbf{R}). \tag{6.50}$$

To compute the scalar field contribution we first express our scalar Lagrangian we started with (6.1).

$$\mathcal{L}^{scalar} = -\sum_{i=1}^{N_s}\Big[\frac{1}{2}(\partial_\mu\varphi_i)^2 + \frac{{\mu_i}^2}{2}(\varphi_i - f_i)^2 + \rho_i\varphi_i\Big] \tag{6.51}$$

and notice from (6.47) that

$$\frac{\partial \mathcal{L}(x)}{\partial(\partial^\mu \varphi_i)}\partial_\nu \varphi_i = (\partial_\mu \varphi_i)(\partial_\nu \varphi_i). \tag{6.52}$$

We have now obtained the scalar part of the stress-energy tensor.

$$T_{\mu\nu}^{scalar} = \widetilde{T}_{\mu\nu}^{scalar} = \sum_i (\partial_\mu \varphi_i)(\partial_\nu \varphi_i) - \frac{1}{2}\eta_{\mu\nu}\Big[(\partial_\lambda \varphi_i)^2 + \frac{\mu_i^2}{2}(\varphi_i - f_i)^2\Big] \tag{6.53}$$

If we substitute the solution (6.18) and let $\varphi_i^{free} = 0$ and use equation (6.41) we get

$$\begin{aligned} T_{\mu\nu}^{scalar} = & \sum_i \frac{g_i^2}{32\pi^2}\frac{e^{-2\mu_i R}}{R^4}\Big\{2 + 2\mu_i^2 R^2 + 4\mu_i R \\ & -\eta_{\kappa\kappa'} n_{(\kappa)\mu} n_{(\kappa)\nu}[1 + 2\mu_i^2 R + 2\mu_i R]\Big\}. \end{aligned} \tag{6.54}$$

Here $\alpha_{(0)}^{sc.} = -\alpha_{(2)}^{sc.} = -\alpha_{(3)}^{sc.}$ so that

$$T_{\mu\nu}^{scalar} = \alpha_{(0)}^{sc.}(n_{(0)\mu}n_{(0)\nu} - n_{(2)\mu}n_{(2)\nu} - n_{(3)\mu}n_{(3)\nu}) + \alpha_{(1)}^{sc.} n_{(1)\mu}n_{(1)\nu} \tag{6.55}$$

and

$$\begin{cases} \alpha_{(0)}^{sc.} = \sum_i \frac{g_i^2}{32\pi^2}\frac{e^{-2\mu_i R}}{R^4}\Big[1 + 2\mu_i^2 R^2 + 2\mu_i R\Big] \\ \\ \alpha_{(1)}^{sc.} = \sum_i \frac{g_i^2}{32\pi^2}\frac{e^{-2\mu_i R}}{R^4}\Big[1 + 2\mu_i R\Big] \end{cases} \tag{6.56}$$

For the vector part we have from (6.1)

$$\mathcal{L}^{vector} = -\sum_{\alpha=1}^{N_v}\Big[\frac{1}{4}(F_{\mu\nu}^\alpha)^2 + \frac{\mu_\alpha^2}{2}(A_\mu^\alpha)^2 - A_\mu^\alpha j_\alpha^\mu\Big]. \tag{6.57}$$

We also notice that

$$\frac{\partial \mathcal{L}^{vector}}{\partial(\partial_\nu A_\mu)} = F^{\mu\nu} \tag{6.58}$$

so that

$$\widetilde{T}_{\mu\nu}^{vector} = -\sum_\alpha F_{\sigma\mu}^\alpha \partial_\nu A^\sigma - \eta_{\mu\nu}\mathcal{L}. \tag{6.59}$$

In this case

$$f_{\mu\sigma\nu} = F_{\sigma\mu}A_\nu \qquad \Longrightarrow \qquad \partial^\sigma f_{\mu\sigma\nu} = \partial^\sigma A_\nu F_{\sigma\mu}. \tag{6.60}$$

Using the solution (6.20) with $A^{\alpha}_{\mu} = 0$ and (6.47), (6.48) we find

$$T^{vector}_{\mu\nu} = \sum_{\alpha} \frac{g_i^2}{32\pi^2} \frac{e^{-2\mu_i R}}{R^4} \Big\{ -(2 + 2\mu_\alpha^2 R^2 + 4\mu_\alpha R) \\ -\eta_{\kappa\kappa'} n_{(\kappa)\mu} n_{(\kappa')\nu} [1 + 2\mu_\alpha^2 R^2 + 2\mu_\alpha R] \Big\}. \tag{6.61}$$

In eigenfunctions with $\alpha^{vec.}_{(0)} = \alpha^{vec.}_{(2)} = \alpha^{vec.}_{(3)}$ this expression simplifies to

$$T^{vector}_{\mu\nu} = \alpha^{vect.}_{(0)} (n_{(0)\mu} n_{(0)\nu} - n_{(2)\mu} n_{(2)\nu} - n_{(3)\mu} n_{(3)\nu}) + \alpha^{vect.}_{(1)} n_{(1)\mu} n_{(1)\nu} \tag{6.62}$$

where

$$\begin{cases} \alpha_{(0)} = -\sum_\alpha \frac{q_\alpha^2}{32\pi^2} \frac{e^{-2\mu_\alpha R}}{R^4} \Big[1 + 2\mu_\alpha^2 R^2 + 2\mu_\alpha R\Big] \\ \alpha_{(1)} = -\sum_\alpha \frac{q_\alpha^2}{32\pi^2} \frac{e^{-2\mu_\alpha R}}{R^4} \Big[1 + 2\mu_\alpha R\Big] \end{cases} \tag{6.63}$$

Since we will integrate the stress-energy tensor over the whole space to get the total mass in the rest frame, the contribution of the cosmological constant $\Lambda\eta_{\mu\nu}$ in expression (6.40) will be infinite. To get some kind of useful information from this calculation we therefore put it to zero. However, the cosmological constant is nowaday assumed to be different from zero. To find a term that could compensate for this is another interesting task.

6.6.2 Mass

In analogy with our previously mentioned equation (6.38) we have that the total conserved energy-momentum is

$$P^\mu = \int d^3x T^{0\mu}(x) = \sum_\kappa \int d^3x \alpha_{(\kappa)} n^0_{(\kappa)} n^\mu_{(\kappa)} \tag{6.64}$$

and in the rest frame

$$P^\mu = (M, 0, 0, 0) \tag{6.65}$$

this becomes, when using (6.45)

$$M = \int d^3x \, \alpha_{(0)}. \tag{6.66}$$

M thus denotes the total mass of the particle, including contributions from the vector and scalar fields. As we see the components $\alpha_{(j)}$, $j = 1, 2, 3$ must cancel each other. This happens if and only if $\sum_i g_i^2 = \sum_\alpha q_\alpha^2$, which is exactly the condition (6.35) we found earlier.

We can now compute the mass:

$$
\begin{aligned}
M &= \int d^3x(\alpha_{(0)}^{part.} + \alpha_{(0)}^{sc.} + \alpha_{(0)}^{vect.}) \\
&= m + \sum_i g_i f_i + \int\int\int R^2 \sin\theta \, dR \, d\theta \, d\varphi (\alpha_{(0)}^{sc.} + \alpha_{(0)}^{vect.}) \\
&= m + \sum_i g_i f_i + \frac{1}{8\pi}\int_0^\infty g_i^2 e^{-2\mu_i R}\Big(\frac{1}{R^2} + \frac{2\mu_i}{R} + 2\mu_i^2\Big) dR \\
&\quad - \frac{1}{8\pi}\int_0^\infty q_\alpha^2 e^{-2\mu_\alpha R}\Big(\frac{1}{R^2} + \frac{2\mu_\alpha}{R} + 2\mu_\alpha^2\Big) dR. \qquad (6.67)
\end{aligned}
$$

Use integration by part so that

$$
\int e^{-2\mu R}\Big(\frac{1}{R^2} + \frac{2\mu}{R}\Big) dR = -\frac{e^{-2\mu R}}{R}. \qquad (6.68)
$$

Substituting into (6.67) yields

$$
\begin{aligned}
M &= m + \sum_i g_i f_i + \frac{1}{8\pi}\Big\{ g_i^2 \lim_{R'\to\infty} \Big[-\frac{e^{-2\mu_i R}}{R} - \mu_i e^{-2\mu_i R}\Big]_0^{R'} \\
&\quad - q_\alpha^2 \lim_{R'\to\infty} \Big[-\frac{e^{-2\mu_\alpha R}}{R} - \mu_\alpha e^{-2\mu_\alpha R}\Big]_0^{R'} \Big\}. \qquad (6.69)
\end{aligned}
$$

When this is evaluated we get the finite total mass of the point-particle

$$
M = m + \sum_i g_i f_i + \frac{1}{8\pi}\left(\sum_i g_i^2 \mu_i - \sum_\alpha q_\alpha^2 \mu_\alpha\right). \qquad (6.70)
$$

Thus we have found that the mass is finite and determined by some bare mass m, the scalar vacuum expectation value and the coupling to the scalar and vector field, which is the Yukawa and Coulomb self-energy. The attractive nature of the scalar force between the particle and its own scalar field seems to cancel the infinities in the Yukawa and Coulomb fields.

It is believed (among field theoretician) that the field masses are built up by the coupling to the scalar vacuum expectation value f_i. In a linear approximation we can therefore wright the self-energies as

$$
\mu_i = \sum_j A_{ij}(g) f_j \qquad\qquad \mu_\alpha = \sum_j B_{\alpha j}(q) f_j. \qquad (6.71)
$$

Now all terms that determines the physical mass M are proportional to the vacuum expectation value of the scalar fields.

To obtain the familiar classical mass we take the lowest-order terms and get the condition

$$
\sum_i g_i^2 \mu_i = \sum_\alpha q_\alpha^2 \mu_\alpha. \qquad (6.72)
$$

Combined with our earlier relations (6.35) and (6.36) these can be reduced to one, namely

$$\sum_i g_i^2 = \sum_\alpha q_\alpha^2 \tag{6.73}$$

iff for all i and α

$$\mu_i = \mu_\alpha. \tag{6.74}$$

It suggests that the scalar and vector fields are parts of a multiplet perhaps indicating an underlying supersymmetric theory, if the particles are replaced by fermions.

Supersymmetry is a symmetry between fermions (quarks and leptons) and bosons (exchange particles). In flat Minkowski space-time the proper framework into which a supersymmetric model should be applied is quantum field theory. From this it follows that for unbroken supersymmetry, all states in a multiplet have the same mass. This means that every elementary particle has a superpartner. However, experiments do not show any evidence of these superpartners, thus it is believe that supersymmetry only occurs in broken form, if occurring at all.

6.7 Non-linear field theories

Can Nature really be described by linear field theories? Is Nature that simple so that linear approximations are valid in the macroscopic domain as well as in the microscopic? Probably not, non-linear effects have to be taken into account. One example of non-linear field theories is the general theory of relativity.

The Maxwell equations are linear in the **E** and **B** fields, which agree satisfactory with experiments for macroscopic objects. But what is the case in the subatomic domain? If we think of the electron as a sphere with localized charge distribution and then shrink it to a point, the energy grows enormously. Therefore it is not to drastic too believe that there will be some kind of saturation for the field strength, as non-linear effects could dominate.

6.7.1 Born-Infeld dynamics

One example of non-linear models is the Born-Infeld electrodynamics [39]. A new Lagrangian

$$\mathcal{L} = b^2\Big(\sqrt{1 - \frac{1}{2b^2}F_{\alpha\beta}F^{\alpha\beta}} - 1\Big) \tag{6.75}$$

is applied instead of the usual electromagnetic one $\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$. In this new field theory the point-charge, as source of the field, is seen as a singularity of the electromagnetic field. The new Lagrangian (6.75) turns approximately into the usual one if the constant b^{-1} is small.

A serious drawback of the Born-Infeld theory is its non-linearity which causes mathematical problems and makes it hard to split the field of the charge and the external field. This was an important ingredient in our derivation of the LDE. The square root also makes it almost impossible to quantize the theory.

It should be pointed out that the Born-Infeld action builds up the D-branes in string theory [40].

6.7.2 Non-linear Dirac electrodynamics

Dirac noticed the problems of the infinite self-energy in his point-particle model [5]. Quantum mechanics did not seem to overcome these problems without mass renormalization. To get a satisfactory finite quantum theory for point-particles Dirac tried to improve his old model before quantization.

In [41], Dirac imposes a non-linear gauge on the vector potential for a free electromagnetic field.

$$A_\mu A^\mu = k^2 \tag{6.76}$$

where $k = m/q$ is a universal constant. This break of gauge transformation introduce a beam of charges into the theory. Nothing specific can be said about the individual charges. From the theory it follows that for a velocity v_μ

$$k^{-1} A_\mu = v_\mu . \tag{6.77}$$

The physical meaning of this model is that the current flows with velocity v_μ. If no charge is present, v_μ is the velocity that it would have if it where added. The velocity seems to fill up all space-time and Dirac regards it to be the velocity of an aether [42]. The aether was a popular topic before the special theory of relativity ruled it out [2]. Dirac introduced it again on basis of his new relativistic invariant theory and quantum mechanics. It should be pointed out that nowadays the aether is not considered to be a real physical thing, however the re-introduction by Dirac is quite interesting from a historical point of view. The discussion nowadays is, in analogy with the aether, about the vacuum which is the "starting point" for a quantum field theory.

Another nonlinear field theory is the one by Finkelstein, Mie, Rosen [43], after the ideas of Pauli and Heisenberg, in which the particle are considered as being the charge and energy of a field, concentrated in a small volume. The equation of motion for the particle follows from the field equations which are nonlinear such that the particles could influence each other. A spectrum of masses follows as soon as the charge of the particle is fixed. More references on non-linear and linear field theories of the first decades in the twentieth century may be found in the book by Schweber [44].

7

Conclusions and Summary

In the beginning of this thesis we studied the interaction between a charged relativistic particle and an electromagnetic field. From an action principle we ended up with the Maxwell field equations coupled to the Lorentz-force equation.

When trying to solve these equations for the charged particle on its world line, we ended up with serious troubles with finiteness. By splitting the field potential into a finite and an infinite part, for which we made a mass renormalization and turned it into a finite one, we found the Lorentz-Dirac equation of motion. This equation turned out to be a third order differential equation to which the initial position and velocity are insufficient to determine the motion of the particle. (These conditions are sufficient to determine ordinary equations of motions, since they are of second order.) When applying an asymptotic condition to the acceleration we excluded the runaway solutions and made the LDE solvable.

The solution was unfortunately not without problems. It seemed that the acceleration could begin shortly before an external force was applied. This preacceleration occurred in a time region which is supposed not to be in the classical domain of viability. However, it is only in this region of time that radiation reaction effects are considered to be important. If we believe that Nature does not allow non-causal behavior, our conclusion must be that the Lorentz-Dirac equation is not satisfactory.

The difficulties associated with the Maxwell-Lorentz theory were believed to be associated with the point-like particle description of the charges. An extended model seemed to be a natural way out. We studied the relativistic invariant bubble of Dirac including Poincaré stresses, which unfortunately turned out to be unstable. Dirac's simple and beautiful bubble was the first example of a relativistic brane which are of interest in string theory. The so called p-branes (where p stands for the space dimension) as higher dimensionally extended objects, $p \geq 3$, are even more fragile then the Dirac bubble, but a compactification of some of its spatial dimensions are supposed to give stabilizing forces [45]. The interaction would effectively involve several fields which should stabilize them. Branes are ingredients of string theory in the search for a unified theory of everything, the “mysterious” M-theory.

As discussed in chapter 5, the extended object must have some dynamics, but in the case of the Dirac bubble no stable bound states were found. The charged relativistic sphere seemed to lack a fundamental relativistic formulation. However, if a scalar field is included inside the bubble there might exist one. This idea have to be investigated further, but that is beyond the scope of this thesis.

After considering the Dirac bubble we turned to a more general model for point-charges, in which the particle interact with both scalar and vector field of arbitrary masses. Also here we started from an action principle and deduced the equations of motion for the fields and the particle. When applying the stability condition and inserting the solutions of the field equation to the equation of motion for the charged particle, we still found that the fields where singular on the world line. However, now these singularities cancelled each other out, leaving us with two conditions on the coupling constants and masses, i.e. ranges of the fields (6.35), (6.36). Comparing these conditions with the unified electroweak theory we found that the Higgs field is massive as is generally believed. In fact, when this thesis was just about finished there were rumors that the Higgs scalar had been found at CERN, with a mass of 114 GeV.

In contrast to the usual Maxwell-Lorentz theory, it turned out that the total mass of a charged particle became finitely computable in terms of the field parameters, as was shown in (6.70). This was done without renormalizing the mass. When obtaining the familiar classical mass, the constraints reduces to one if the masses are equal for all scalar and vector fields. This suggests that we should have a multiplet of fields, perhaps indicating a supersymmetric theory. Multiplets appear naturally in string theory. Now, the considered model does not take into account the possible self-interactions of the scalar and vector fields (this was done in order to find analytical solutions), neither does it consider charged particles with spin. By adding spin to the particle it would certainly improve the model. There are of course still more about this model that has to be investigated further. For example, there is no description of the radiation reaction, when an external force field is present. We must also incorporate quantum mechanics since a point-like description in classical physics is kind of awkward because it does not apply to physics of short length scales. A classical point-like model is therefore a kind of oxymoron, in itself. Instead we regard the classical approach as an approximation where quantum correction terms are supposed to be added. A possible many-body theory or possible incorporations of gravity remains to be investigated.

Hopefully the physical and computational problems discussed in this thesis have been interesting to the reader, because they surely have to me.

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