INTRODUCTION TO ENUMERATIVE ALGEBRAIC GEOMETRY

ROK GREGORIC

Abstract. These are the notes for the *Intersection Theory and Ennumerative Geometry* lectures given in July 2020, and part of the Univerity of Texas at Austin mathematics department gradate-student-run Summer minicourses. Due to contemporary realities, the lectures are to be given virtually.

We follow Eisenbud-Harris *3264 & All That, Intersection Theory in Algebraic Geometry*, and strongly recommend anyone whose interest is piqued to pick up that gorgeously-written text to learn more.

The goal of these notes is to showcase how techniques of Algebraic Geometry in general and Intersection Theory in particular may be applied to solve classical enumerative questions. The numbers obtained this way are largely unimportant[1], but the fact that they can be determined (and often via standard techniques at that) is truly fascinating.

The ground field. Throughout these notes, we implicitly work over a field k, which we require to be algebraically closed and have characteristic zero. Both of these assumptions could largely be compensated for by additional effort (e.g. by taking degrees of residue field extensions into account), which is why we choose not to.

The reader who wishes to assume that $k = \mathbf{C}$ is free to do so, but should be aware that they are doing so merely for psychological comfort. Said reader should also be warned that special complex-analytic or differential-geometric intuitions will not be used. In particular, we will think of algebraic curves as being 1-dimensional objects, and not a 2-dimensional, as their identification with Riemann surfaces might lead one think of them as. In short: please accept that we are in the realm of algebraic geometry, and try to make yourelf at home! As we hope to show, it's not such a scary place, and can be quite fun to live in!

The foundations of algebraic geometry. Some rudimentary knowledge of algebraic geometry on the reader's part would be beneficial, but we will try our best to minimize its importance. In particular, we will work throughout with varieties, which a reader well-versed in the contemporary language can take to be an integral (though sometimes we will want to relax this to assuming only reducedness, and as such allow reducible varieties as well) separated scheme of finite type over the base field.

But since the vast majority of the varieties that we will actually be thinking about will be projective, it is also perfectly acceptable if the reader wishes to imagine a subset of the projective space $\mathbf{P}^n$ for some n, cut out by a finite number of (homogeneous) algebraic equations. The high-flying technology of scheme theory, as crucial as it has proved over the last half-century, will largely play a back seat in our discussion.

Warning. These are informal notes, sure to be brimming with mistakes, all of them mine. Please take everything you read here with a hefty grain of salt, defer to Eisenbud-Harris whenever confused, and in general use at your own peril!

Date: August 1, 2020.
University of Texas at Austin.

[1]Albeit largely unimportant, the numbers are sometimes quite fascinating, however. For instance, Schubert computed in 1879 that there are 5,819,539,783,680 twisted cubics tangent to twelve quadric surfaces in general position in 3-space. That means that, if we were to evenly distribute these twisted cubics among all the people currently alive on planet Earth, each person would become the proud owner of almost 800 of them!

1. Framework of Intersection Theory

Fix a smooth algebraic variety X of dimension n. In this subsection we introduce a convenient setting for studying the intersection theory of subvarieties in X.

1.1. **Algebraic cycles and rational equivalence.** An *algebraic i-cycle* is a formal sum $\alpha = n_1 Z_1 + \cdots + n_k Z_k$ with $n_j \in \mathbf{Z}$ and $Z_j \subseteq X$ any i-dimensional subvarieties. Algebraic i-cycles form an abelian group $\mathcal{Z}_i(X)$.

We wish to consider algebraic cycles to be *rationally equivalent*, if we can deform one into another through a family of algebraic cycles. Informally, we define that $\alpha \simeq_{\text{rat}} \alpha'$ if there exists a family of i-cycles $\{\alpha_t \subseteq X\}_{t\in\mathbf{P}^1}$, depending algebraically on a parameter t ranging over the projective line $\mathbf{P}^1$, such that $\alpha_0 = \alpha$ and $\alpha_\infty = \alpha'$. Formally, this may be achieved by incarnating the family $\{\alpha_t \subseteq X\}_{t\in\mathbf{P}^1}$ as a cycle $\boldsymbol{\alpha} \in \mathcal{Z}_{i+1}(X \times \mathbf{P}^1)$, such that it is (or more precisely, its constituent subvarieties are) not fully contained in any of the fibers $X \times \{t\} \subset X \times \mathbf{P}^1$. Under this assumption, the restrictions to fibers $\alpha_t := \boldsymbol{\alpha}|_{X\times\{t\}}$ are i-cycles in X, and as such define elements of the family in question.

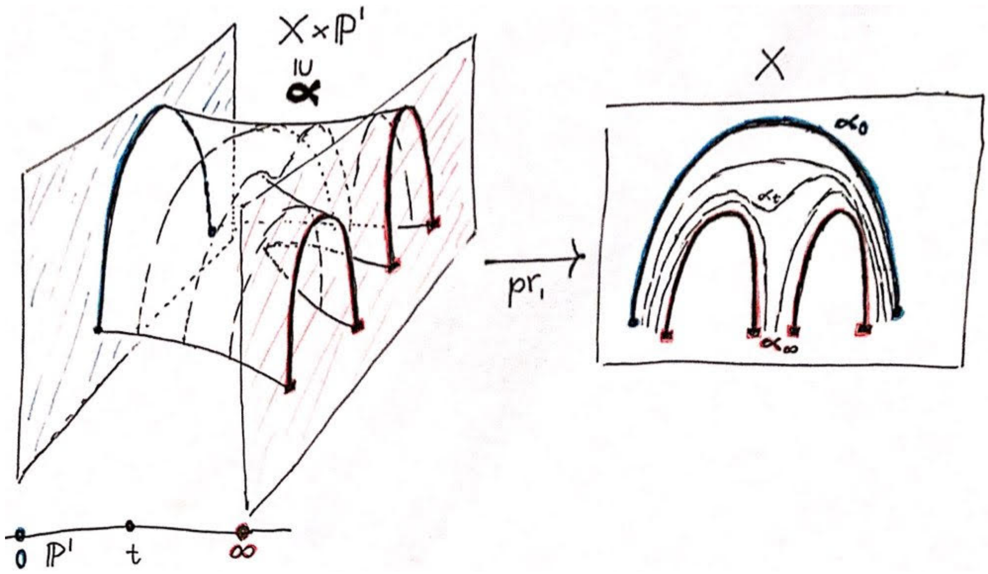

Figure 1. A family given by the subvariety $\boldsymbol{\alpha}$, exhibiting $\alpha_0 \simeq_{\text{rat}} \alpha_\infty$.

1.2. **The Chow groups.** By identifying i-cycles under rational equivalence, we define the *i-th Chow group of X* to be

$$A_i(X) := \mathcal{Z}_i(X)/\simeq_{\text{rat}} .$$

For any i-dimensional subvariety $Z \subseteq X$, we call the corresponding Chow group element $[Z] \in A_i(X)$ its *fundamental class.* It is often convenient to use an alternative grading $A^i(X) := A_{n-i}(X)$, under which the fundamental class of a subvariety $Z \subseteq X$ becomes indexed by its codimension.

Example 1.2.1. For those familiar with algebraic geometry, the codimension 1 cycles might be better known as *divisors* (or more precisely, Weil divisors). In that language, the group $\mathcal{Z}^1(X) = \mathrm{Div}(X)$ is the divisor group, rational equivalence is the same as linear equivalence, and $A^1(X) = \mathrm{Cl}(X)$ recovers the divisor class group. But if none of that means anything to you, that's alright - just remember the word "divisor" as shorthand.

The Chow groups may be viewed as an algebro-geometric analogue of the algebro-topological homology groups $\mathrm{H}_*(M;\mathbf{Z})$ for a compact oriented n-dimensional manifold

M. In that case, Poincaré duality identifies $\mathrm{H}^i(X;\mathbf{Z}) \simeq \mathrm{H}_{n-i}(X;\mathbf{Z})$, justifying the cohomological grading on the Chow groups. Recall that in algebraic topology, cohomology is often more useful than homology because it carries a ring structure.

1.3. **The intersection product and transversality.** In analogy with cohomology, we wish to equip the Chow groups with a ring structure, to make $A^*(X) := \bigoplus_i A^i(X)$ into the *Chow ring of* X. The multiplication should respect the grading, and as such be encoded by maps $A^i(X) \times A^j(X) \to A^{i+j}(X)$. By linearity, it suffices to define the product $[Z].[W] \in A^{i+j}(X)$ for a pair of subvarieties $Z, W \subseteq X$ of codimensions i and j respectively.

We wish to simply set $[Z].[W] = [Z \cap W]$. In order for the gradings to work out right, we wish $[Z].[W]$ to live in $A^{i+j}(X)$. The class $[Z \cap W]$ is an element of $A^k(X)$, where k is the codimension of $Z \cap W \subseteq X$. And while the *expected* codimension of $Z \cap W$ is indeed $i + j$, i.e. we expect that

$$\mathrm{codim}_X(Z \cap W) = \mathrm{codim}_X(Z) + \mathrm{codim}_X(W),$$

it could nonetheless happen that $Z \cap W$ has some components of a higher dimension.

This issue disappears if Z and W intersect *transversely*, which means that for every point $p \in Z \cap W$ the equality $T_pZ + T_pW = T_pX$ holds on the level of tangent spaces. In that case, we genuinely define $[Z].[W] = [Z \cap W]$. Note that if $\dim(Z) + \dim(W) < \dim(X)$, or equivalently $i + j > n$, then the interection can only be transvere if $Z \cap W = \varnothing$, in which case $[Z].[W] = 0$ (this is also sensible because $A^k(X) = 0$ for $k > n$). To deal with the remaining case of $i + j \leq n$, we call upon an infamous black box:

Theorem 1.3.1 (The Moving Lemma)**.** *Let X be a smooth quasi-projective variety. For any pair of cycles $Z \in \mathcal{Z}^i(X)$, $W \in \mathcal{Z}^j(X)$, there exist respectively rationally equivalent cycles $Z' \in \mathcal{Z}^i(X)$, $W' \in \mathcal{Z}^j(X)$ such that they (or more precisely, all of their component subvarieties) intersect transversely. The class $[Z'].[W'] \in A^{i+j}(X)$ is independent of the choice of transversal representatives Z', W'.*

Informally, the Moving Lemma (which we will not prove here) guarantees that any pair of cycles may be perturbed into intersecting transversely. Indeed, transverse intersection is a Zariski-open condition (non-transverseness may be expressed as the vanishing of certain determinants) and hence a generic representative of a rational equivalence class will intersect transversely with a given other (appropriately codimensional) subvariety.

Remark 1.3.2. The approach to defining the interection product by invoking the Moving Lemma is often viewed with some suspicion. And for good reason - this is the classical approach to interssection theory, but its hisotry is fraught with technical mistakes and confusion. To disspell a possible point misunderstanding: the Moving Lemma in the above version can be rigorously proved, see e.g. the account in the Stacks Project. It is however a lot more diffcult and technical than one might expect.

There exist other approaches to setting up the Chow ring however, a particularly powerful one due to Fulton-MacPhearson worked out in complete rigor in Fulton's *Intersection Theory* monograph, and another due to Serre partially worked out in him *Local Algebra*.

However, since we work informally, and the restriction of working only with smooth quasi-projective ambient varieties is perfectly acceptable for our purposes, we stick with the Moving Lemma approach, preferring it for its high intuitive appeal.

1.4. **Proper pushforward on Chow groups.** In analogy with homology, Chow groups admit covariant functoriality for proper morphisms. From the Poincare-duality perspective with de Rham cohomology of oriented differentiable manifolds, this functoriality corresponds to the theory of integration along the fibers, which might go some way towards motivating the properness requirement.

Let $f : X \to Y$ be a proper map of smooth varieties. We wish to define *pushforward* $f_* : A_i(X) \to A_i(Y)$ on Chow groups. By linearity, it suffices to define it on fundamental classes. For every subvariety $Z \subseteq X$ its image $f(Z) \subseteq Y$ is also a subvariety, of dimension $\dim(f(Z)) \leq \dim(Z)$. We define

$$f_*[Z] := \begin{cases} 0, & \dim(f(Z)) < \dim(Z) \\ \deg(f|_Z)[f(Z)], & \dim(f(Z)) = \dim(Z), \end{cases}$$

where the d*degree* of the retricted map $f|_Z : Z \to f(Z)$ is[2] the number of points in its generic fiber (or in an arbitrary fiber, if counted with multiplicity). It is true, albeit far from obvious, that this cycle-level definition respects rational equivalence, and as such descends to Chow groups.

Example 1.4.1. Let X be a proper smooth variety of dimension n. Properness of X, the algebro-geometric analogue of compactness, means that the morphism to the point $p : X \to \mathrm{pt}$ is proper (for instance, every projective variety is proper). Hence we get access to a pushforward map of Chow groups $p_* : A_i(X) \to A_i(\mathrm{pt})$, called the *degree map* and denote it $p_* = \deg$. Since clearly $A_0(\mathrm{pt}) = \mathbf{Z}$ and $A_i(\mathrm{pt}) = 0$ for all $i \geq 1$, the interesting part of this map is $\deg : A_0(X) \to \mathbf{Z}$. It may be described as follows: $A_0(X)$ is clearly spanned by fundamental classes $[x]$ of points $x \in X$, and the degree map is given by $\sum_i n_i[x_i] \mapsto \sum_i n_i$. In analogy with de Rham cohomology of differentiable manifolds, the degree map $A^n(X) \to \mathbf{Z}$ is sometimes also denoted by $\alpha \mapsto \int_X \alpha$, which has the advantage of including the variety X in the notation.

Example 1.4.2. To understand why properness is required for defining the degree map, let us consider what goes wrong in the case of the affine line $\mathbf{A}^1$, the poster boy of non-proper varieties. We claim that the class of a point $[t]$ for any $t \in \mathbf{A}^1$ is rationally equivalent to the empty subscheme, from which it follows that $A_0(\mathbf{A}^1) = 0$. Indeed, consider the "diagonal" subvariety $\boldsymbol{\alpha} \subseteq \mathbf{A}^1 \times \mathbf{P}^1$, in terms of the inclusion $\mathbf{A}^1 \subseteq \mathbf{P}^1$. It defined a family of subvarieties $\alpha_t := \boldsymbol{\alpha}|_{\mathbf{A}^1 \times \{t\}} \subseteq \mathbf{A}^1$ for all $t \in \mathbf{P}^1$, which are equal to

$$\alpha_t = \begin{cases} \{t\} & t \in \mathbf{A}^1 = \mathbf{P}^1 - \{\infty\} \\ \varnothing & t = \infty, \end{cases}$$

thus showing that $[\mathrm{pt}] = 0 \in A_0(\mathbf{A}^1)$. This is the stereotypical issue with the lack of properness: the "holes" in the variety allow us to push points into them through rational equivalence. Indeed, properness of a variety X may be characterized (this goes by the

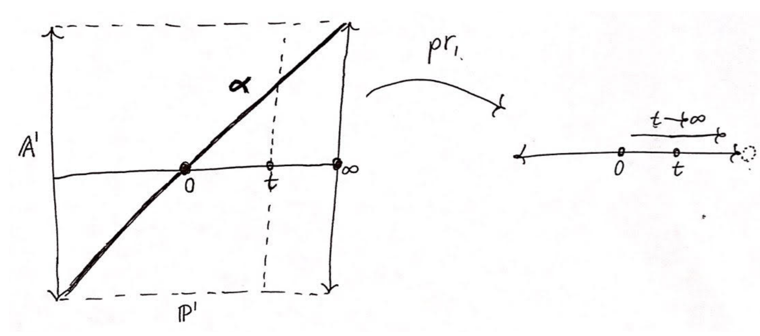

FIGURE 2. Visual illustration of the rational equivalence $\{t\} \simeq_{\mathrm{rat}} \varnothing$.

name of the Valuative Criterion) by only a slightly more refined property than demanding that any map $\mathbf{A}^1 \to X$ extends uniquely along the inclusion $\mathbf{A}^1 \subseteq \mathbf{P}^1$ to a map $\mathbf{P}^1 \to X$.

[2] Formally, it may be identified as $\deg(f|_Z) = [\mathcal{K}(Z) : \mathcal{K}(f(Z))]$, the degree of the field extension $\mathcal{K}(f(Z)) \to \mathcal{K}(Z)$ induced by f between the field of rational functions on $f(Z)$ and Z rsepetively.

1.5. **Pullback on Chow groups.** In analogy with cohomology, Chow groups also admit contravariant functoriality, but this time with respect to arbitrary morphisms.

Hence let $f : X \to Y$ be a morphim of smooth varieties, and we wish to defined *pullback* $f^* : A^i(Y) \to A^i(X)$ on Chow groups. By linearity, it once again suffices to define this map on fundamental classes. If a subvariety $Z \subseteq Y$ satisfies $\mathrm{codim}_X(f^{-1}(Z)) = \mathrm{codim}_Y(Z)$, we simply define[3] $f^*[Z] := [f^{-1}(Z)]$. If a subvariety does not satisfy this codimension estimate, that definition will clearly not work if we wish pullback to be compatible with the grading. However, a version of the Moving Lemma allows us to move any subvariety inside its rational equivalence class for f^{-1} to have correct codimension.

One salient feature of pullback, in contrast to pushforward, is that it is compatible with intersection products. That is to say, $f^* : A^*(Y) \to A^*(X)$ is a ring homomorphism.

Remark 1.5.1. The pullback functoriality and the intersection multiplication are not only compatible, but also both defined similarly, by an application of the Moving Lemma. In fact, the intersection product may be recovered from the pullbacks functoriality of Chow groups. Indeed, the subvariety-level map $(Z, W) \mapsto Z \times Z$, which is easily seen to respect rational equivalence, thus defines a map of Chow groups $A^*(X) \otimes_{\mathbf{Z}} A^*(X) \to A^*(X \times X)$. Composing this map with the pullback $\Delta^* : A^*(X \times X) \to A^*(X)$, induced by the diagonal embedding $\Delta : X \to X \times X$, recovers the intersection product on the Chow ring.

The two functorialities of Chow groups are compatible through the *projection formula*, which says that for any proper morphism $f : X \to Y$ of smooth varieties, the equality

$$f_*(\alpha.f^*\beta) = f_*(\alpha).\beta$$

holds inside the Chow ring $A^*(Y)$ for any cycles $\alpha \in A^*(X)$ and $\beta \in A^*(Y)$.

1.6. **Chow ring of an affinely stratified variety.** For a general variety, the Chow ring is notoriously hard to compute. Most special cases in which we can determine it, and in particular all of the ones relevant for us in these notes, follow from a simple observation we encode in the following Proposition.

A *stratification* of a variety X consists of a disjoint union decomposition $X = \coprod_i U_i$ for a family of locally closed subvarieties $U_i \subseteq X$, such that for every i the closure $\overline{U_i}$ is the union of some of the subvarieties U_j. An *affine stratification* is a stratification in which each stratum U_i is isomorphic to an affine space $\mathbf{A}^{n_i}$ for some $n_i \geq 0$.

Proposition 1.6.1. *Let X be a variety with an affine stratification. Then the fundamental classes of the closed strata $[\overline{U}_i]$ generate the Chow groups $A^*(X)$.*

Proof. Let $Z \subseteq X$ be a subvariety, contained in some $\overline{U}_i$. Assume that $Z \neq \overline{U_j}$ and that $U_i \cap Z \neq \varnothing$. Then we claim that there exists a rationally equivalent subvariety $Z' \subseteq X$, which is fully contained in the bounadry ∂U_i. Iterating this process, we finally find that either $Z \simeq_{\mathrm{rat}} [\overline{U_j}]$ for some j, or else the dimensions reduces to 0.

To prove the claim, we choose an isomorphism $\mathbf{A}^n \cong U_i$ which maps the origin to some point in $U_i - Z$. Let the subvariety $\boldsymbol{\alpha} \subseteq \overline{U_i} \times \mathbf{P}^1$ be the closure of

$$\{(x, t) \in U_i \times (\mathbf{A}^1 - \{0\}) : tx \in Z\}.$$

This $\boldsymbol{\alpha}$ exhibits rational equivalence between $\alpha_1 = Z$ and the fiber α_0, obtained in the limit as $t \to 0$, which is centairly contatined purely inside ∂U_i. □

Remark 1.6.2. The technique we employed in the above proof is a version of projection away from the origin in $\mathbf{A}^n$ onto the "hyperplane at infinity". It is literally that when $\overline{U_i} \cong \mathbf{P}^n$ in which case $\partial U_i \cong \mathbf{P}^{n-1}$ is the literal hyperplane at infinity. In the context of the above proposition, it may be that $\overline{U_i}$ is a more involved compactification of $U_i \cong \mathbf{A}^n$, but as we have noted, the technique still applies.

[3] When f is a flat morphism, this formula works for an arbitrary subvariety $Z \subseteq Y$.

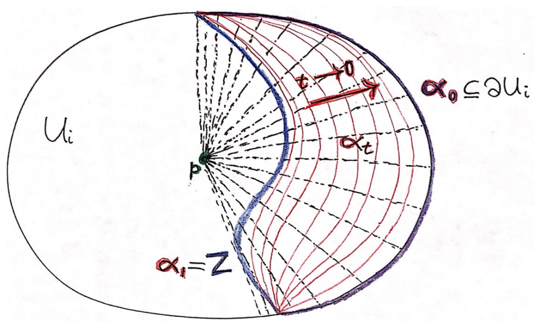

FIGURE 3. The "projection from a point to the hyperplane at infinity" argument, used in the proof of Proposition 1.6.1.

1.7. **The Chow ring of projective space.** A key application of the above Proposition is to computing the Chow ring of projective space $\mathbf{P}^r$. To obtain an affine stratification on $\mathbf{P}^r$, we conider the decompoition of it into the affine space $\mathbf{A}^r$, and the hyperplane at infinity $H \cong \mathbf{P}^{r-1}$. Iterating, we find an affine stratification

$$\mathbf{P}^r \cong \mathbf{A}^r \cup \mathbf{P}^{r-1} \cong \cdots \cong \mathbf{A}^r \cup \mathbf{A}^{r-1} \cup \cdots \cup \mathbf{A}^1 \cup \mathrm{pt}.$$

We have already identified the corresponding closed strata along the way, as the point, the line, the plane, and so on up to a hyperplane inside $\mathbf{P}^r$. Note that a point in $\mathbf{P}^r$ is the intersection of r generic hyperplanes, a line is the intersection of $r-1$ generic hyperplanes, etc. Thus in terms of the *hyperplane class* $\zeta = [H] \in A^1(\mathbf{P}^r)$, the fundamental classes of the closed strata are $\zeta^r, \zeta^{r-1}, \dots, \zeta^2, \zeta$ respectively. By the Proposition 1.6.1 of the preceeding section, these classes generate their respective Chow groups $A^r(\mathbf{P}^r), A^{r-1}(\mathbf{P}^r), \dots, A^2(\mathbf{P}^r), A^1(\mathbf{P}^r)$, while we already know from Example 1.4.1 that $A^0(\mathbf{P}^r)$ is generated by the fundamental class $\zeta^0 = [\mathbf{P}^r]$, which is the multiplicative unit in the Chow ring. Consequently we have just determined the Chow ring of projective space to be

$$A^*(\mathbf{P}^r) = \mathbf{Z}[\zeta]/(\zeta^{r+1}).$$

Exercise 1.7.1. Find an appropriate affine stratification on the product of projective spaces $\mathbf{P}^r \times \mathbf{P}^s$ (or if you wish more factors). Use it to compute the Chow ring to be

$$A^*(\mathbf{P}^r \times \mathbf{P}^s) = \mathbf{Z}[\alpha, \beta]/(\alpha^{r+1}, \beta^{s+1}),$$

where the $\alpha = \mathrm{pr}_1^*(\zeta)$ and $\beta = \mathrm{pr}_2^*(\xi)$ are the pullbacks of the hyperplane classes $\zeta \in A^1(\mathbf{P}^r)$ and $\xi \in A^1(\mathbf{P}^s)$. You should really think about doing this (easy) exercise - we will be using it indiscriminantly in the following Sections!

Remark 1.7.2. Familiarity with cohomology from algebraic topology and the rseult of the preceding Exercie may lead you astray to unfounded spectulation that a Künneth-like formula might hold for Chow rings. That is false, however, and the ring $A^*(X \times Y)$ is often much more complicated than $A^*(X) \otimes_{\mathbf{Z}} A^*(Y)$.

1.8. **Degree of a projective variety.** It follows from the determination of the Chow ring of projective space in the previous Subsection that any subvariety $X \subseteq \mathbf{P}^n$, or any algebraic cycle more generally, is classified uniquely up to rational equivalence (and thus completely for the purposes of intersection theory!) by two pieces of data: its *dimension* n, and its *degree* d. In that case, we have $[X] \simeq d\zeta^{r-n}$.

Noting that ζ^r corresponds to the class of a point, we have $\deg(\zeta^r) = 1$, and so we may extract the degree of an n-dimensional subvariety $X \subset \mathbf{P}^r$ through the degree map $\deg : A^0(\mathbf{P}^r) \to \mathbf{Z}$ of Example 1.4.1 as

$$d = \deg(\zeta^n[X]).$$

In words: the degree of an n-dimensional subvariety $X \subseteq \mathbf{P}^r$ is the number of points of intersection between X and n general hyperplanes in $\mathbf{P}^r$, or equivalently, between X and a general $(r-n)$-plane in $\mathbf{P}^r$.

Example 1.8.1. To clarify how this works, it is instructive to consider the case of a hyperfuface $X = V(F) \subseteq \mathbf{P}^r$, cut out by a (non-zero) degree d homogenous polynomial $F \in \Gamma(\mathbf{P}^r; \mathscr{O}(d)) = k[t_0, \dots, t_r]_d$. This gives rise to a class $[X] \in A^1(\mathbf{P}^r)$ of degree d, so $[X] = d\zeta$. That means that there exists a rational equivalence $X \simeq_{\mathrm{rat}} dH$ for a general hyperplane $H \subseteq \mathbf{P}^r$, counted with multiplicity d. Let us fix such an H and select the homogeneous coordinates on $\mathbf{P}^r$ so that $H = V(x_0)$, i.e. that this is the hyperplane at infinity. To obtain the desired rational equivalence explicitly, consider the rational the rational function $f := F/x_0^d$ on $\mathbf{P}^r$. It vanished precisely along X, and with the same multiplicity as F, while it has an order d pole along H. Thus viewing it as an algebraic map $f : \mathbf{P}^r \mapsto \mathbf{P}^1$, it satisfies $f^*(\{0\}) = X$ and $f^*(\{\infty\}) = dH$. The family of fibers $\alpha_t := f^*(\{t\}) \subseteq \mathbf{P}^n$ for $t \in \mathbf{P}^1$ is the family[4] of cycles which exhibits the rational equivalence.

Example 1.8.2. Another even more down-to-earth sanity check: consider an algebraic plane curve $C \subseteq \mathbf{P}^n$. In fact, consider a super classical one: the (projective closure of the affine) algebraic curve of solutions to the equation $y = F(x)$ for some degree d polynomial F. To determine the degree of C, we should count its points of intersection with a general line. If F has no iterated zeros, then we can take this line L to be the projective closure of the line $y = 0$, since the intersection of C and L will then be transverse. In that case, the degree of the curve C will be precisely the number of zeros of the polynomial C, which is equal to d by the assumption of algebraic closedness of our underlying field.

In fact, we see that if F had iterated zeros, then the degree would still be the same, recovering the fact that a degree d polynomial has precisely d roots, when counted with multiplicity. Great, our high school math teacher wasn't lying to us!

Exercise 1.8.3. Continuing with the setup of Exercise 1.7.1, let $H \subseteq \mathbf{P}^r \times \mathbf{P}^s$ be a hypersuface. We thus have $[H] = d\alpha + e\beta$, and (d, e) is called the *bidegree* of H. Give an interpretation of the bidegree in the $r = s = 1$ case in terms of the standard isomorphism of $\mathbf{P}^1 \times \mathbf{P}^1$ with a smooth quadric surface $Q \subseteq \mathbf{P}^3$, and the two rulings[5] on Q.

1.9. **Bezout's Theorem.** As an application of computing the Chow ring of $\mathbf{P}^r$, we can obtain one of the most classical results in all of Algebraic Geometry.

Proposition 1.9.1 (Bezout's Theorem)**.** *Let $C, C' \subseteq \mathbf{P}^2$ be a pair of algebraic curves of degrees d, d', which intersect transversely. Then the intersection $C \cap C'$ consists of dd' points. If we count the points of intersection with multiplicity, then the same remains true if C and C' do not intersect transversely.*

Proof. We have $[C] = d\zeta$ and $[C'] = d'\zeta$, and so $[C].[C'] = dd'\zeta^2$, with ζ^2 being the class of a point in $\mathbf{P}^2$. □

With our setup, generalizing Bezout's Theorem to higher dimensions is easy, with virtually the same proof: if $X_1, \dots, X_r \subseteq \mathbf{P}^r$ are hypersufaces of degrees $d_1, \dots, d_r$, then $\deg([X_1]\dots[X_r]) = d_1 \cdots d_r$, which equals $|X_1 \cap \cdots \cap X_r|$ if they intersect jointly transversely. Or another version: let $X, Y \subseteq \mathbf{P}^r$ be transversely-intersecting subvarieties of dimensions n, m and degrees d, e, with $n + m = r$. Then $|X \cap Y| = de$. Etc.

[4]Formally, in terms of the definition of rational equivalence that we have given, this family would be encoded by the graph of f, viewed as a subvariety $\boldsymbol{\alpha} := \Gamma_f \subseteq \mathbf{P}^n \times \mathbf{P}^1$.

[5]With the algebraic closure hypothesis we are working under, all smooth quadric surfaces are isomorphic. But to be able to visualize the two ruling, try thinking about the one-sheeted hyperboloid.

1.10. **Enumerative problems and transversality issues.** The versions of Bezout's Theorem we found in the previous Subsection are the first of many enumerative formulas that we shall extracted from intersection theory. In some sense, they are stereotipical of the endeavor: they are obtained by

- recognizing the cycles of interest inside the ambient variety,
- and then performing algebraic manipulation in the Chow ring of the ambient variety to obtain the degree of the relevant intersection product.

However, the number obtained this way will only be the "correct answer" to the problem when the cycles in question intersect transversely.

In good cases, "generic" choices of objects in question will lead to the relevant intersections being transverse. This technically needs to be verified on a case-by-case basis, tends to require some understanding of the algebraic geometry of objects in question, and while not usually hard can be tedious and technical. For this reason, we will almost exclusively omit this verification, but will of course point out when we encounter the odd example where that fails.

For those whose rigor-organ is incapable of sustaining such prolonged abuse, we offer a remedy. The following Theorem, whose proof we of course will not give (but is not exremely hard, and is easy to look up e.g. in Eisenbud-Harris), may be applied to affirmitively answer the "generic transversality" issue in the vast majority of question that we shall encounter. If that is the kind of thing that makes you happy, have fun figuring out which G applies in each case!

Theorem 1.10.1 (Kleiman)**.** *Let G be a linear algebraic group acting transitively on a variety X. For any two subvarieties $Z, W \subseteq X$ and a generic $g \in G$, we have $gZ \simeq_{\text{rat}} Z$, and the subvarieties gZ and W intersect transversely.*

2. First applications

After we have spent the previous Section setting up Intersection Theory, we spend this chapter eluminating the theory with some applications, and putting it to use to solve some enumerative problems.

2.1. **The Chow ring of a blowup of a surface at a point.** Let S be an algebraic surface, and $p \in S$ a point. A standard construction in algebraic geometry is the *blowup of S along p*, and we show in this section how the Chow rings $A^*(\widetilde{S})$ and $A^*(S)$ are related. That will turn out to be a nice exercise in getting to grips with the pullback and pushforward functoriality of the Chow groups. Just to have a leaving question though (and don't worry if you don't know what all the terms mean - we will explain it all):

Question 1. *What is the self-intersection number of the exceptional divisor inside the blowup of a proper algebraic surface at a point?*

Informally, the blowup amounts to leaving the rest of S unchanged, but inserting a copy of the projective line $\mathbf{P}^1$ in place of the point p, so that each direction along which the point p can be approached inside S determines a different point in $\widetilde{S}$. Let's quickly brush up about how blowups work in general though:

Review 2.1.1 (Crash course in blowups)**.** The *blowup* of an algebraic variety X along a subvariety $Y \subseteq X$ is an algebraic variety $\widetilde{X}$ together with a map $\pi : \widetilde{X} \to X$ for which the pre-image $\pi^{-1}(Y)$ is as an effective divisor, which is to say, a codimension 1 subvariety. The blowup is universal for this property of turning Y into a divisor, which can serve as a characterization (though more explicit constructions exist). The subvariety $E = \pi^{-1}(Y) \subseteq \widetilde{X}$ is called the *exceptional divisor*, while away from it, the map π gives an isomorphism $X - E \cong X - Y$. If the variety X is proper or projective, then so is its blowup $\widetilde{X}$. ∎

In the present case in question, we let $X = S$ be a proper algebraic surface, and $Y = \{p\}$ be a point on it. The blowup $\widetilde{S}$ is also a proper algebraic surface, and the exceptional divisor $E \subseteq \widetilde{S}$ is a curve on it, whose fundamental class we denote $e := [E] \in A^1(\widetilde{S})$. Note that the map $\pi : \widetilde{S} \to S$ is proper generically one-to-one, as such has degree 1. It induces by pullback a ring homomorphism $\pi^* : A^*(S) \to A^*(\widetilde{S})$, while puhforard along it induce an additive homomorphim $\pi_* : A^*(\widetilde{S}) \to A^*(S)$. Let us work degree-wise in the Chow ring to determine each Chow group of $A^*(\widetilde{S})$ in terms of those of $A^*(S)$.

For any class $\alpha \in A^2(S) = A_0(S)$, we may by the Moving Lemma find a representative cycle $Z \in \mathcal{Z}_0(S)$ which is disjoint from p. Thus we find that $\pi_*\pi^*\alpha = \pi_*[\pi^{-1}(Z)] = [Z] = \alpha$. Since we already know that $A^2(S) \cong \mathbf{Z}$ and $A^2(\widetilde{S}) \cong \mathbf{Z}$, we conclude that the maps π^* and π_* induce inverse isomorphisms $A^2(S) \cong A^2(\widetilde{S})$.

Next consider a class $\alpha \in A^1(S)$. Once again we may write $\alpha = [Z]$ for a cycle representative $Z \in \mathcal{Z}_1(S)$ disjoint from p, and get $\pi_*\pi^*\alpha = \alpha$. Alas this time, the map $\pi_* : A^1(\widetilde{S}) \to A^1(S)$ will have a kernel: recalling the definition of puhforward from Subsection 1.4, since $\pi(E) = p$ is zero-dimensional, we find that $\pi_* e = 0$. Since the map π is isomorphic away from E, it follows that this is also the only class in the kernel of π_*. Hence we obtain a short exact sequence

$$0 \to \langle e \rangle \to A^1(\widetilde{S}) \xrightarrow{\pi_*} A^1(S) \to 0$$

which further admits a splitting via the map $\pi^* : A^1(S) \to A^1(\widetilde{S})$. Hence $A^1(\widetilde{S})$ is generated by classes e and $\pi^*\alpha$ for $\alpha \in A^1(S)$, and it remain to determine what the subgroup $\langle e \rangle \subseteq A^1(\widetilde{S})$, generated by the exceptional class e, is like.

Given a cycle $\alpha \in A^1(S)$, the projection formula shows that

$$\pi_*(e.\pi^*(\alpha)) = \pi_*(e).\alpha = 0,$$

since $\pi_*(e) = 0$ as seen above. But $e.\pi^*(\alpha) \in A^2(\widetilde{S})$, and we saw above that $\pi_* : A^2(\widetilde{S}) \to A^2(S)$ is an isomorphism. It follows that

$$e.\pi^*(\alpha) = 0$$

for all $\alpha \in A^1(S)$.

It remain to determine the class e^2, for which we use a trick: fix an arbitrary curve $C \subseteq S$ through the point p (such a curve turns out to exist on any surface), which is smooth at the point p. Its preimage in the blowup is reducible into $\pi^{-1}(C) = E \cup \widetilde{C}$, the exceptional divisor and a curve $\widetilde{C} \subseteq S$ (usually called the *proper transform* of C) which meets E transversely at a single point (corresponding to the tangential direction of C at the point p).

Thus we have

$$\pi^*[C] = e + [\widetilde{C}]$$

in $A^1(\widetilde{C})$, and multuplying by the class e, we get

$$0 = e.\pi^*[C] = e^2 + [E \cap \widetilde{C}] = e^2 + [\mathrm{pt}].$$

In particular, applying the degree map $\deg : A^2(\widetilde{S}) \to \mathbf{Z}$, we recover the well-known fact that

$$\deg(e^2) = -1,$$

or in words:

Answer to Question 1. *The self-intersection of the exception divisor is* -1.

That may be interpreted as attesting that E is firmly lodged inside $\widetilde{S}$, and offer much resistence to being moved/deformed into any other curve on $\widetilde{S}$.

From the fact that $\deg(e^2) = -1$, we may deduce that the element $e \in A^2(\widetilde{S})$ is not torsion. Thus we may collect the conclusions of all of our work in this Subsection together as follows:

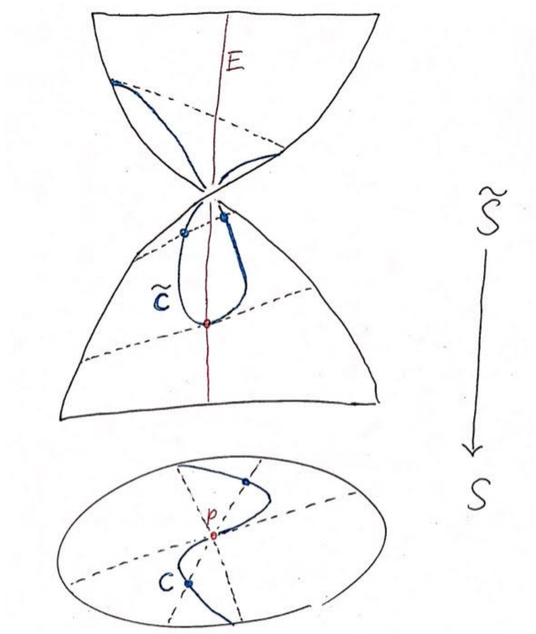

FIGURE 4. Blowup of a surface S at a point $p \in S$ (this picture is fully accurate for $S = \mathbf{P}^2$, but the blowup of any other smooth smooth surface looks the same locally), together with a proper transform $\widetilde{C} \subseteq S$ of a curve $C \subseteq S$. through p

Proposition 2.1.2. *The pullback map $\pi^* : A^*(S) \to A^*(\widetilde{S})$ exhibits the graded ring isomorphism*

$$A^*(\widetilde{S}) = (A^*(S) \oplus \mathbf{Z}e)/(e^2 + 1)$$

with the element $e = [E]$ in degreee one.

2.2. **Degree of the dual variety.** Next we address the following question:

Question 2. *Let $C \subset \mathbf{P}^2$ be a smooth cubic curve, and let $p \in \mathbf{P}^2$ be a general point. How many lines passing through the point p are tangent to X?*

Instead of working with a degree 3 curve inside $\mathbf{P}^2$, let us work in slightly greater generality of degree d smooth hypersurface $X \subseteq \mathbf{P}^n$. Since the above Question is about lines inside $\mathbf{P}^2$, it is useful to recall the general aparatus for dealing with those things.

Review 2.2.1 (Dual projective space)**.** The *dual projective space* $\mathbf{P}^{n*}$ has as its points hyperplanes $H \subseteq \mathbf{P}^n$. A choice of homogeneous coordinates on $\mathbf{P}^n$ induces an isomorphism $\mathbf{P}^{n*} \cong \mathbf{P}^n$ in the following way: it allows us to write any point in projective space as $[x_0 : \ldots : x_n] \in \mathbf{P}^n$, and therefore any hyperplane $H \subseteq \mathbf{P}^n$ is defined by a linear equation $a_0x_0 + \cdots + a_nx_n = 0$ for some scalars $a_0, \ldots, a_n \in k$. Sending $H \in \mathbf{P}^{n*}$ to the point $[a_0 : \ldots : a_n] \in \mathbf{P}^n$ defines the desired isomorphism. The passage from $\mathbf{P}^n$ to $\mathbf{P}^{n*}$ is called *projective duality*, and as we see, it interchanges points with hypersurfaces, hence also lines with $(n-1)$-planes, and in general i-planes with $(n-i)$-planes. ■

A smooth hypersurface $X \subseteq \mathbf{P}^n$ defines a corresponding *dual hypersufrace* $X^* \subseteq \mathbf{P}^{n*}$, defined to consist of all hyperplanes which are tangent to X. What we are looking for is the degree of X^*. Indeed, we know from Subsection 1.8 that, if $\eta \in A^1(\mathbf{P}^{n*})$ denotes the hyperplane class, then the degree of the dual hypersursace X^* is given by

$$\deg(X^*) = \deg([X^*].\eta^{n-1}).$$

In the $n = 2$ case, η is the line class inside $\mathbf{P}^{2*}$, which by projective duality coincides corresponds to a point in $\mathbf{P}^2$, and so then $\deg(C^*)$ will compute the number of points of intersection between C^*, the lines tangent to C, and η, lines passing through a general fixed point $p \in \mathbf{P}^2$.

Now that we have our work cut out for ourselves, we face the issue of how to approach the dual hypersurface. One way to do so it is to consider the *Gauss map* $\mathcal{G} : X \to \mathbf{P}^{n*}$ given by[6] $p \mapsto T_pX$, the image of which is clearly precisely X^*.

Remark 2.2.2. Note that the smoothness of X is essential for the Gauss map to make sense, as the tangent space at a non-smooth point would not be a hypersurface, and as such an element of the dual projective space $\mathbf{P}^{n*}$. Indeed, one characterization for an n-dimensional variety X to be smooth at a point $p \in X$ is that $\dim T_pX = n$.

A bit more work than we are willing to put in right now shows that the Gauss map $\mathcal{G}$ is birational (i.e. generically one-to-one). Thus we get $[X^*] = [\mathcal{G}(X)] = \mathcal{G}_*[X]$ from the definition of pushforward. Then the projection formula shows that

$$\begin{aligned} \deg(X^*) &= \deg([X^*].\eta^{n-1}) \\ &= \int_{\mathbf{P}^{n*}} \mathcal{G}_*([X]).\eta^{n-1} \\ &= \int_{\mathbf{P}^{n*}} \mathcal{G}_*([X].\mathcal{G}^*(\eta)^{n-1}) \\ &= \int_X \mathcal{G}^*(\eta)^{n-1} \end{aligned}$$

Thus it remains to determine the class $\mathcal{G}^*[\eta] = [\mathcal{G}^{-1}(H)]$ for a general hyperplane $H \subseteq \mathbf{P}^{n*}$.

To do that, we must get a more explicit grasp on the Gauss map. Fix a homogeneous polynomial $F \in \Gamma(\mathbf{P}^n; \mathscr{O}(d)) = k[x_0, \ldots, x_n]_d$ which cuts out the hypersurface X. Thus $X = V(F)$, which allows us to write down the definiting equation of the tangent space at a point $p \in X$ and so yield

$$\mathcal{G}(p) = \left\{[t_0 : \ldots : t_n] \in \mathbf{P}^n : \frac{\partial F}{\partial x_0}(p)t_0 + \cdots + \frac{\partial F}{\partial x_d}(p)t_n = 0\right\} \in \mathbf{P}^{n*}.$$

Under the isomorphism $\mathbf{P}^{n*} \cong \mathbf{P}^n$, determined by the choice of homogeneous coordinates, the Gauss map $\mathcal{G} : X \to \mathbf{P}^n$ is thus given by $p \mapsto [\frac{\partial F}{\partial x_0}(p) : \ldots : \frac{\partial F}{\partial x_d}(p)]$.

If the hyperplane $H \subseteq \mathbf{P}^{n*}$ is cut out in terms of the isomorphism $\mathbf{P}^{n*} \cong \mathbf{P}^n$ by the linear equation $ax_0^* + \cdots + a_0x_n^* = 0$, then $\mathcal{G}^{-1}(H)$ is the intersection in $\mathbf{P}^n$ of X with the hypersurface $V(G) \subseteq \mathbf{P}^n$ cut out by the polynomial $G = a_0\frac{\partial F}{\partial x_0} + \cdots + a_n\frac{\partial F}{\partial x_n}$. This defining polynomial is homogeneous of degree $d - 1$ (derivatives drop the degree of a polynomial down by one), and so $\deg(V(G)) = d - 1$. It follows that

$$\int_X \mathcal{G}^*(\eta)^{n-1} = \int_{\mathbf{P}^n} [X].[V(G)]^{n-1} = d(d-1)^{n-1}.$$

By our preceeding effort in this Subsection, we know this to be the degreee of the dual hypersurface X^*.

Answer to Question 2. *There are* $3(3-1)^{2-1} = 6$ *lines passing through a general point in* $\mathbf{P}^2$ *tangent to a given smooth cubic curve.*

Of course, computing the degree of the dual hypersurface bought us much more than what we bargained for. For instance, we find the classical fact that the degree of the dual curve $C^* \subseteq \mathbf{P}^{2*}$ to a smooth degree d plane curve $C \subseteq \mathbf{P}^2$ has degree $d(d-1)$.

Another enumerative problem that we are now in position to answer is:

Question 3. *Let* $S \subset \mathbf{P}^3$ *be a smooth cubic surface and* $L \subseteq \mathbf{P}^3$ *a general line. How many planes containing* L *are tangent to* S*?*

[6] We are being purpusefully a little sloppy here. What we mean by T_pX here is the *projective tangent space*, sometimes denoted $\mathbb{T}_pX$, which is canonically a projective subspace of $\mathbb{T}_p\mathbf{P}^n = \mathbf{P}^n$ via the inclusion $X \subseteq \mathbf{P}^n$. Alas, we will continue to be sloppy with our notation and simply use T_pX, and hope it is clear from context what we mean by that.

In this case, the degree of the dual is $\deg(S^*) = \deg([S^*]\eta^2)$. The class η correponds by projective duality to all plane in $\mathbf{P}^3$ passing through a general point, and since two general points determine a unique line, η^2 is the class of all plane in $\mathbf{P}^3$ containing a general line.

Answer to Question 3. *There are* $3(3-1)^{3-1} = 12$ *planes in* $\mathbf{P}^3$ *containing a general line and tangent to a given smooth cubic surface.*

Etc. :)

2.3. **Spaces of hypersurfaces.** Before we continue on our adventure, let us take a breif pause to think about how hypersurfaces in projective space work, since various enumerative questions regarding them will occupy us for much of the rest of these notes.

A degree d hypersurface $H \subseteq \mathbf{P}^n$ may always be written as $H = V(F)$, i.e. as the vanishing locus of some degree d polynomial F. Thus (non-zero) points of the space od degree d polynomials in $n+1$ variables $\Gamma(\mathbf{P}^n; \mathscr{O}(d))$ (which you might be more used to denoting $k[x_0, \ldots, x_n]_d$) give rise to degree d hypersufaces. But of course, the vanishing locus of F is invariant under scaling F, hence the actual *space of degree d hypersufcaces in $\mathbf{P}^n$* is the projectivization $\mathbf{P}(\Gamma(\mathbf{P}^n; \mathscr{O}(d)))$.

Determining the dimension of $\Gamma(\mathbf{P}^n; \mathscr{O}(d))$, i.e. the number of distinct degree d monomials in $n+1$ variables, is a basic combinatorics question, and yields the answer $\binom{n+d}{n}$. Therefore the space of degree d hypersufaces in $\mathbf{P}^n$ may be identified with the projective space $\mathbf{P}^N$ for $N = \binom{n+d}{n} - 1$.

Example 2.3.1. Let us consider how this works in low degrees:

- For $d = 1$, we have $N = \binom{n+1}{n} - 1 = n$, showing that hyperplanes in $\mathbf{P}^n$ are parametrized by another copy of $\mathbf{P}^n$. Indeed, this is the dual projective space $\mathbf{P}^{n*}$ discussed in the previous Subsection.
- For $d = 2$ and $n = 2$, we find that the space of plane conics has dimension $N = \binom{2+2}{2} - 1 = 5$. This identification is given explicity by sending

$$V(a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_0x_1 + a_4x_1x_2 + a_5x_2x_0) \mapsto [a_0 : a_1 : a_2 : a_3 : a_4 : a_5] \in \mathbf{P}^5.$$

- For $d = 3$ and $n = 2$, we get $N = \binom{2+3}{2} - 1 = 9$, identifying the space of plane cubics with the projective space $\mathbf{P}^9$.
- For $d = 2$ and $n = 3$ we get $N = \binom{3+2}{3} - 1 = 9$, identifying the space of quadric surfaces in $\mathbf{P}^3$ with $\mathbf{P}^9$ as well.

Determining these dimensions alone allows us to answer some very basic enumerative problems.

Question 4. *Let $p_1, \ldots, p_5 \in \mathbf{P}^2$ be points in the plane in general position. How many conics pass through all five?*

The condition that a point $[x_0, x_1, x_2] \in \mathbf{P}^2$ lies on a conic corresponding to the point $[a_0 : \ldots : a_5] \in \mathbf{P}^5$ amounts to asking that $a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_0x_1 + a_4x_1x_2 + a_5x_2x_0 = 0$. In particular, it is linear in the a_i, and therefore cuts out a hypersurface in the space of conics $\mathbf{P}^5$. A little work shows that if the points in $\mathbf{P}^5$ are in general position, the hypersurfaces they will cut out in $\mathbf{P}^5$ will meet transversely. Thus:

Answer to Question 4. *Five points in general position in the plane determine a unique conic passing through all five.*

Similarly, nine points in the plane determine a plane conic, nine points in 3-space determine a quadric surface, etc.

Remark 2.3.2. For the algebraic geometers reading this, you might notice that we've encountered a friend! Indeed, the map $\mathbf{P}^2 \to \mathbf{P}^{5*}$, sending a point in the plane to the hypersurface of plane cubics passing through that point, is precisely the Veronese map

$\nu : \mathbf{P}^2 \to \mathbf{P}^5$ sending $[x_0 : x_1 : x_2] \mapsto [x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_1x_2 : x_2x_3]$. More generally, the arbitrary Veronese embedding arises the same way: the map $\nu_d : \mathbf{P}^n \to \mathbf{P}^N$ for $N = \binom{n+d}{n} - 1$ may be described (via a fixed standard isomorphism $\mathbf{P}^{N*} \cong \mathbf{P}^N$) as sending a point $p \in \mathbf{P}^n$ to the hypersurface of degree d hypersurfaces in $\mathbf{P}^n$ that pass through p.

2.4. **Linear systems.** An *r-dimensional linear system of degree d hypersurfaces in* $\mathbf{P}^n$ is a family of subvarieties

$$\{H_t = V(t_0F_0 + \cdots + t_rF_r) \subseteq \mathbf{P}^n\}_{t \in \mathbf{P}^r}$$

for some collection of (linearly independent) $F_0, \ldots, F_r \in \Gamma(\mathbf{P}^n; \mathscr{O}(d))$. Thus for every value of $t = [t_0, \ldots, t_r] \in \mathbf{P}^r$, there is a degree d hypersuface $H_t \subseteq \mathbf{P}^n$ in the linear system, and as we see, its dependence on the parameter t is linear.

Linear systems therefore provide some of the simplest examples of families of varieties. As such, they have been studied extensively in the rich history of Algebraic Geometry, and have traditional names for low dimensions:

- A linear system of dimesnion 1 is called a *pencil*.
- A linear system of dimension 2 is called a *fan*.

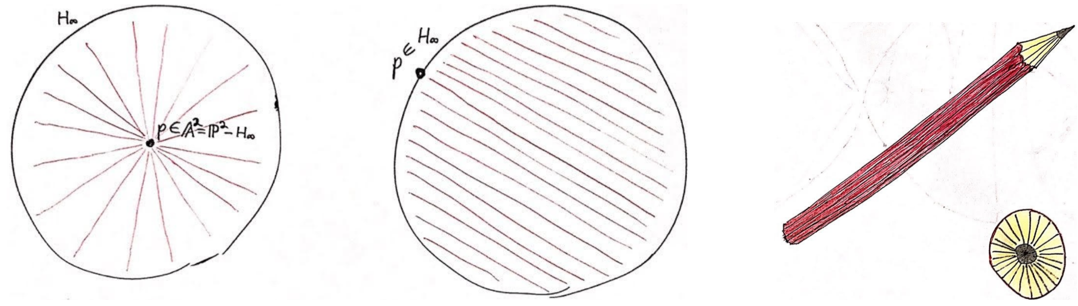

FIGURE 5. Pencils of lines in the plane, and an actual pencil for comparison.

Equivalently, an r-dimensional linear system is an r-plane inside the space of degree d hypersurfaces $\mathbf{P}^N$ for $N = \binom{n+d}{n} - 1$. This is the relevance to intersection theory: if $\zeta \in A^1(\mathbf{P}^N)$ denotes the hyperplane class, it follows that

- ζ^{N-1} is the class of a pencil,
- ζ^{N-2} is the class of a fan,
- ζ^r is the class of a general $(N-r)$-dimensional linear systems.

Thus if we are considering some condition on degree d hypersurfaces in $\mathbf{P}^n$, which corresponds to an r-codimensional subvariety $Z \subseteq \mathbf{P}^N$, the degree of Z, being given by $\deg(Z) = \deg([Z].\zeta^{N-r})$, has the geometric interpretation of counting the number of elements in general r-dimensional linear system which satisfy the condition given by Z. That is how local systems will usually enter into our discussion.

Remark 2.4.1. The archaic term "pencil" is incredibly thoroughly embedded in the English-laguage literature on geometry (algebraic and otherwise). According to Wikipedia, it seems to have been introduced into the English language at the strt of the 20th century by the American mathematician G. B. Halsted, employed at no other than the Univeristy of Texas at Austin! How appropriate then that we should discuss pencils here, in this UT Summer Minicourse![7]

[7]That said, Halsted was fired from UT Austin because of his staunch support of R. L. Moore, the staunch racist, the stain of whose name the Physics-Mathematics-Astronomoy building at UT succeded to shrug only a week ago! So perhaps it's OK that history seems to have to some extent forgotten Halsted.

2.5. **Singular hypersurfaces.**

Question 5. *How many elements in a general pencil of degree d hypersurfaces in $\mathbf{P}^n$ are singular?*

This is easy to answer by direct inspection in some cases for $n = 2$:

- Since no line can be singular, the answer for $d = 1$ is zero.
- For $d = 2$, we may note that a general pencil of conics $S_t = V(t_0F_0 + t_1F_1) \subseteq \mathbf{P}^2$ consists precisely of all the conics passing through the four points of intersection $S_0 \cap S_\infty = V(F_0) \cap V(F_1)$. Indeed, we saw above that a conic is uniquely specified by five general point it passes through, therefore there there is a single remaining parameter left determining the element of the pencil. Now determining the singular conics through four point is simple: there are three possible configurations of pairs of lines passing through the four point in general position. Thus a general pencil of plane conics contains three singular conics.

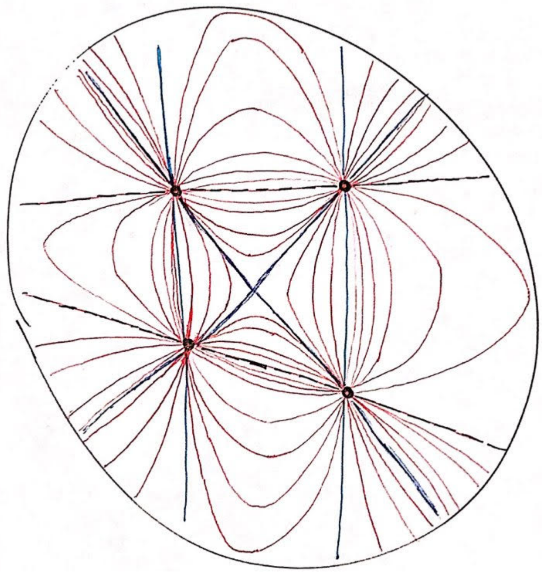

Figure 6. A general pencil of plane conics.

For higher values of n and d, it is harder to approach in such an *ad hoc* way. Instead, we consider the subvariety $\mathcal{D} \subseteq \mathbf{P}^N$ for $N = \binom{n+d}{n} - 1$ in the space of all degree d hypersurface $H \subseteq \mathbf{P}^n$ spanned by all the H which are singular. It turns out (as is surprisingly often the case) to be easier to study the incidence correspondence

$$\Sigma := \{(H, p) \in \mathbf{P}^N \times \mathbf{P}^n : p \in H_{\text{sing}}\}.$$

Under the projection map $p : \Sigma \to \mathbf{P}^n$, its image is precisely $p(\Sigma) = \mathcal{D}$. It can be shown (this is essentially the famous Bertini's Theorem) that the map p is generically one-to-one onto its image (geometrically: a general singular hypersurface only has a single singular point), from which it follows that

$$[\mathcal{D}] = [p(\Sigma)] = p_*[\Sigma].$$

On the other hand as a subvariety $\Sigma \subseteq \mathbf{P}^N \times \mathbf{P}^n$, the incidence correspondence consists of all $(V(F), p) \in \mathbf{P}^N \times \mathbf{P}^n$ which satisfy the equations $F(p) = 0$ and $\frac{\partial F}{\partial x_0}(p) = \cdots = \frac{\partial F}{\partial x_n}(p) = 0$. Thanks to our characteristic zero assumption, we have

$$F = \frac{1}{d}\left(x_0\frac{\partial F}{\partial x_0} + \cdots + x_n\frac{\partial F}{\partial x_n}\right),$$

for any degree d polynomial F, and so the first of the equations cutting out Σ is implied by the rest. Hence Σ is cut out by the $n+1$ equations $\frac{\partial F}{\partial x_i}(p) = 0$, each of which is linear with respect to the coefficients of F, and has degree $(d-1)$ with respect to the variables x_i. Each vanishing locus $H_i = V(\frac{\partial F}{\partial x_i}(p))$ thus satisfies

$$[H_i] = \alpha + (d-1)\beta \in A^*(\mathbf{P}^N \times \mathbf{P}^n)$$

for α and β pullbacks of the hyperplane classes from $\mathbf{P}^N$ and $\mathbf{P}^n$ respectively. For a general choice of F, the equations $\frac{\partial F}{\partial x_i}(p) = 0$ will be independent, and hence the hyperplane $H_i \subseteq \mathbf{P}^n \times \mathbf{P}^N$ transverse. Thus $[\Sigma] = (\alpha + (d-1)\beta)^{n+1}$ and so

$$[\mathcal{D}] = p_*[\Sigma] = p_*((\alpha + (d-1)\beta)^{n+1}).$$

From the definition of pushforward, it follows that the pushforward will kill any class containing a power β^i for $i < n$, as that would cause a dimension mismatch between the relevant subvariety and it image under the projection $p : \mathbf{P}^N \times \mathbf{P}^n \to \mathbf{P}^N$. For $i = n$, we instead get $p_*(\alpha\beta^n) = \zeta \in A^1(\mathbf{P}^N)$ be the hyperplane class. Recalling also that $\beta^{n+1} = 0$, we obtain through a binomial expansion

$$[\mathcal{D}] = p_*((\alpha + (d-1)\beta)^{n+1}) = (d-1)^n \binom{n+1}{n} \zeta = (n+1)(d-1)^n \zeta$$

In particular, $\mathcal{D} \subseteq \mathbf{P}^N$ is a hypersurface of degree $(n+1)(d-1)^n$. Indeed, it is traditionally called the *discriminant hypersurface*.

Answer to Question 5. *A general pencil of degree d hypersurfaces in* $\mathbf{P}^n$ *contains preciely* $(n+1)(d-1)^n$ *singular elements.*

For instance, a pencil of plane conics contains $(2+1)(3-1)^2 = 12$ singular conics. Linear systems of plane cubics are a good toy case to play around with - let us study them in detail in the next Subsection!

2.6. **Linear systems of plane cubics.** As we have noted in Subsection 2.3, the spaces of plane cubics may be identified with $\mathbf{P}^9$. By classifying the possible types of plane cubics, we obtain a stratification of this space as:

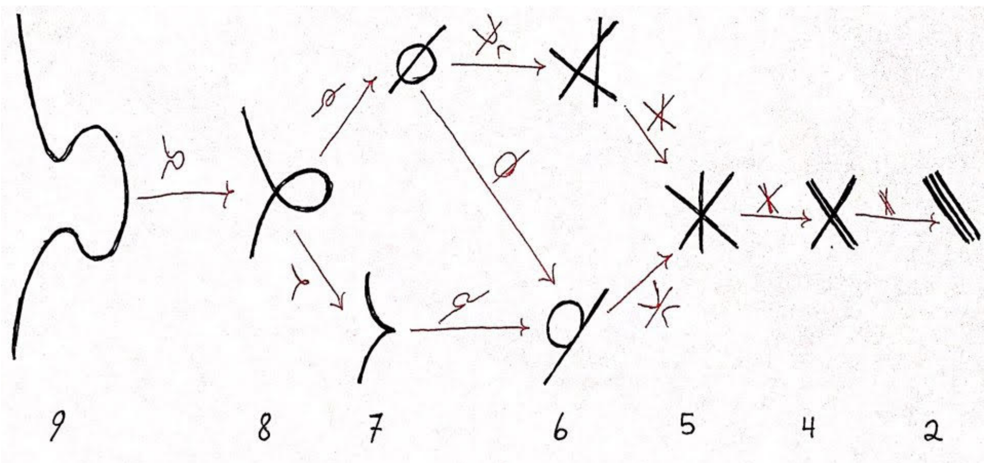

FIGURE 7. Stratification of plane cubics by type, indications how a cubic of one type can degenerate into one of another (i.e. which strata are in the closure of other ones), and dimensions of strata.

This raises a number of exciting enumerative question about local system of plane cubics, which we will now bite into.

Remark 2.6.1. The following questions (and in fact all such questions in these notes) all ask "the right questions", i.e. they are phrased in terms of linear systems of the precise correct dimension for the answers to turn out to be finite. That is not due to any arcane trickery: as we will see in the course of answering them, there is always only one such dimension available, and we will determine it as a byproduct of the solution process.

Question 6. *How many element of a general fan of plane cubics are reducible?*

A cubic curve is reducible if and only if its defining polynomial factors as $F = LQ$ into a linear factor L and a quadratic factor Q. That is to say, the locus $\Phi \subseteq \mathbf{P}^9$ of reducible cubics is the image of the embedding $\phi : \mathbf{P}^2 \times \mathbf{P}^5 \to \mathbf{P}^9$ sending the pair of a line and a conic into their union. In particular the subvariety $\Phi \subseteq \mathbf{P}^9$ is 7-dimensional, and so its degree will give precisely the answer to the desired question. The map ϕ is clearly generically one-to-one, and so $[\Phi] = \phi_*[\mathbf{P}^2 \times \mathbf{P}^5]$. The degree of Φ may thus be obtained by the projection formula as

$$\deg(\Phi) = \int_{\mathbf{P}^9} \phi_*([\mathbf{P}^2 \times \mathbf{P}^5]).\zeta^7 = \int_{\mathbf{P}^9} \phi_*(\phi^*\zeta^7) = \int_{\mathbf{P}^2\times\mathbf{P}^5} \phi^*\zeta^7.$$

If we denote by $\alpha, \beta \in A^1(\mathbf{P}^2 \times \mathbf{P}^5)$ the pullback of the hyperplane classes from $\mathbf{P}^2$ and $\mathbf{P}^5$ respectively, then we may observe that $\phi^*\zeta = \alpha + \beta$. Since $\alpha^3 = 0$ and $\beta^6 = 0$, and $\alpha^2\beta^5$ is the class of a point, we find the degree in question to be

$$\deg(\Phi) = \int_{\mathbf{P}^2\times\mathbf{P}^5} \phi^*\zeta^7 = \int_{\mathbf{P}^2\times\mathbf{P}^5} (\alpha + \beta)^7 = \binom{7}{2} = 21.$$

Answer to Question 6. *A general fan of plane cubics contains* 21 *reducible cubics.*

Another type of singular plane cubic is what we will call a *triangle*: a reduced cubic consisting of a union of three lines.

Question 7. *How many triangles does a 3-dimensional linear system of plane cubics contain?*

As before, we note that the locus of triangles $\Delta \subseteq \mathbf{P}^9$ may be obtained as the image of the map $\delta : \mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2 \to \mathbf{P}^9$, sending a triple of lines to their union. It follows that Δ is 6-dimensional, and so its degree will be obtained by intersecting it with a general 3-dimensional linear system, hence answering the question at hand.

But unlike the previous case, the map δ is not generically one-to-one. Indeed, permuting the three lines amongst themselves has no effect on their union. That is to say, δ kills the action of the symmetric group Σ_3, which is transitive on each of its generic fiber, showing that δ is has degree $|\Sigma_3| = 6$. Thus it follows from the definition of proper pushforward that $\delta_*[\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2] = 6[\delta(\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2)] = 6[\Delta]$.

Now we may procede as before with the projection formula to get

$$\deg(\Delta) = \tfrac{1}{6} \int_{\mathbf{P}^9} \delta_*([\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2]).\zeta^6 = \tfrac{1}{6} \int_{\mathbf{P}^2\times\mathbf{P}^2\times\mathbf{P}^2} \delta^*\zeta^6.$$

Denoting by $\alpha_1, \alpha_2, \alpha_3 \in A^1(\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2)$ the hyperplane class of each component, we have $\delta^*\zeta = \alpha_1 + \alpha_2 + \alpha_3$. Since $\alpha_i^3 = 0$ and $\alpha_1^2\alpha_2^2\alpha_3^2$ i, we find as before

$$\tfrac{1}{6} \int_{\mathbf{P}^2\times\mathbf{P}^2\times\mathbf{P}^2} \delta^*\zeta^6 = \tfrac{1}{6} \int_{\mathbf{P}^2\times\mathbf{P}^2\times\mathbf{P}^2} (\alpha_1 + \alpha_2 + \alpha_3)^6 = \frac{1}{6}\binom{6}{2,2,2} = \frac{6!}{6 \cdot 8} = 15.$$

Answer to Question 7. *A general* 3*-dimensional linear system of plane cubics contains* 15 *triangles.*

Yet another type of singular plane cubics are *asterisks*, which is to say, unions of three lines, all of which pass through the same point.

Question 8. *How many triangles does a general* 4*-dimensional linear system of plane cubics contain?*

The locus of asterisks $* \subseteq \mathbf{P}^9$ is a subvariety of Δ, and in particular coreesponds to a subvariety $I \subseteq \mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2$. If we are considering a triple of lines $L_i \subseteq \mathbf{P}^2$ cut out by linear equations $a_{i0}x_0 + a_{i1}x_1 + a_{i2}x_2 = 0$, then basic linear algebra tells us that the condition that $L_1 \cap L_2 \cap L_3$ is non-empty is equivalent to the equation $\det A = 0$ for the matrix $A = (a_{ij}) \in k^{3\times 3}$. From the form of the determinant $\det A$, as a polynomial in in the variables a_{ij}, we see that the locus of intersecting triples of lines $I \subseteq \mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2$ is a hypersurface with fundamental class $[I] = \alpha_1 + \alpha_2 + \alpha_3$. As before, we have $* = \delta(I)$ with the map $\delta : I \to *$ generically six-to-one, showing that the locus of asterisks $* \subseteq \mathbf{P}^9$ is 5-dimensional. Therefore the answer to Question 8 will be given by its degree, which we compute just like before to obtain

$$\deg(*) = \frac{1}{6}\int_{\mathbf{P}^9} \delta_*([I]).\zeta^5 = \frac{1}{6}\int_{\mathbf{P}^2\times\mathbf{P}^2\times\mathbf{P}^2} [I].\delta^*\zeta^5 = \frac{1}{6}\int_{\mathbf{P}^2\times\mathbf{P}^2\times\mathbf{P}^2} (\alpha_1+\alpha_2+\alpha_3)^6 = 15.$$

Answer to Question 8. *A general* 4*-dimensional linear system of plane cubics contains exactly* 15 *asterisks.*

Exercise 2.6.2. Perform a similar analysis of the loci þ, $\mathbb{X}$, III $\subseteq \mathbf{P}^9$ of the remaining types of singular cubics. That is, determine how-many-dimensional linear systems you need to take to find a finite number of such singular cubics in a generic one, and then compute those numbers.

Exercise 2.6.3. Do what we did in this Subsection for plane cubics for plane conics, and for quadric surfaces in $\mathbf{P}^3$.

3. Lines in 3-space

3.1. Grassmanians in general.

The *Grassman variety* or more commonly *Grassmanian* $G(k,n)$ parametrizes k-dimensinal linear subspaces of an n-dimensional vector space. That is to say, k-planes through the origin inside the affine space $\mathbf{A}^n$. An alternative indexing convention that we will find particularly useful is to denote $\mathbf{G}(k,n) := G(k+1,n+1)$, which thus parametrizes (projective) k-planes inside the projective space $\mathbf{P}^n$.

Determining the dimension of the Grassmanian is a matter of basic linear algebra: to specify a k-dimensional linear subspace in an n-dimensional vector space requires, after picking a basis, to specify an $n\times k$-matrix, but we must discount for all the different ways we could have selected a basis for the k-dimensional subspace. In total, there are thus $nk - k = (n-k)k$ parameters. That is to say, $\dim(\mathbf{G}(k,n)) = (n-k)(k+1)$.

Remark 3.1.1. The classical way of verifying that Grassmanians are objects of algebraic geometry is by exhibiting them as projective varieties. This may be achieved through the Plücker embedding $G(k,n) \to \mathbf{P}^N$ for $N = \binom{n}{k} - 1$, sending a k-dimensional linear subspace $V \subseteq \mathbf{A}^n$ to the 1-dimensional linear subspace $\Lambda^k V \subseteq \Lambda^k \mathbf{A}^n \cong \mathbf{A}^{N+1}$. The image of the Plücker emebedding may be seen to be cut out by certain determinantial equations, called the Plücker relations, which thus confirms the algebraic nature of the Grassmanian. A modern approach is to instead focus on the fact that Grassmanians parametrize linear subspaces, and make that as a definition in terms of a universal property, e.g. folloing the so-called functor of points approach.

For our purposes, it suffices to know that Grassmanians are proper (quasi-)projective varieties, and as such a good ambient space for Intersection Theory.

3.2. Schubert cycles in the Grassmanian of lines in $\mathbf{P}^3$.

From now on we specialize to considering the Grassmanian $\mathbf{G}(1,3)$, whose points are lines $L \subseteq \mathbf{P}^3$. As such, it will help us answer a rich variety of enumerative questions involving lines in 3-space. We know from the dimension counting in the previous Subsection that $\mathbf{G}(1,3)$ is a $(3-1)(1+1) = 4$-dimensional. Let us determine its Chow ring!

We approach this the same way as we computed the Chow ring of projective space: by finding an affine stratification. Fix a complete flat, which is to say a nested sequence

$p \in \Lambda \subseteq H \subseteq \mathbf{P}^3$ of a point p, a line Λ passing through that point, and a plane H containing that line. Using it, we define the following subvatrieties of $\mathbf{G}(1,3)$:

$$\begin{aligned}
\Sigma_0 &:= \{L \subseteq \mathbf{P}^3\} = \mathbf{G}(1,3)\\
\Sigma_1 &:= \{L \subseteq \mathbf{P}^3 : L \cap \Lambda \neq \varnothing\}\\
\Sigma_2 &:= \{L \subseteq \mathbf{P}^3 : p \in L\}\\
\Sigma_{1,1} &:= \{L \subseteq \mathbf{P}^3 : L \subseteq H\}\\
\Sigma_{2,1} &:= \{L \subseteq \mathbf{P}^3 : p \in L \subseteq H\}\\
\Sigma_{2,2} &:= \{L \subseteq \mathbf{P}^3 : L = \Lambda\} = \{\Lambda\}
\end{aligned}$$

called the *Schubert cycles*. Counting degrees of freedom of the lines they contain, we find that they have dimensions $4, 3, 2, 2, 1, 0$ respectively.

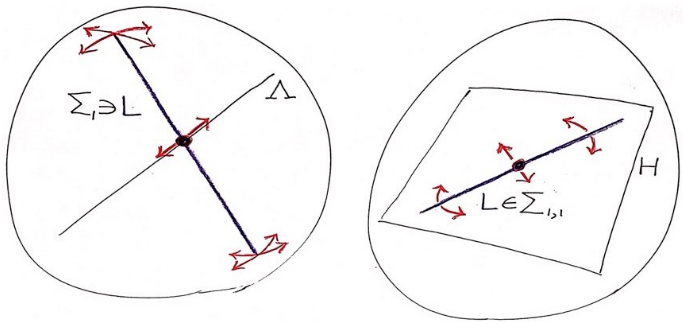

FIGURE 8. Determining that an element $L \in \Sigma_1$ has 3 degrees of freedom, while $L \in \Sigma_{1,1}$ has 2 degrees of freedom.

Thus, while we may complain about the unintuitive indexing convention for Schubert cycles (which only truly shines for higher-dimensional Grassmanians), we should appreciate the convenience of having

$$\operatorname{codim}_{\mathbf{G}(1,3)} \Sigma_{i,j} = i + j.$$

To get a feeling for them, it might be helpful to draw out the Schubert cycles:

We note that the Schubert cycles are related by inclusions as

$$\begin{array}{ccccccc}
 & & & \Sigma_{1,1} & & & \\
 & & \nearrow & & \searrow & & \\
\{\Lambda\} = \Sigma_{2,2} & \hookrightarrow & \Sigma_{2,1} & & & \Sigma_1 & \hookrightarrow \Sigma_0 = \mathbf{G}(1,3),\\
 & & \searrow & & \nearrow & & \\
 & & & \Sigma_2 & & &
\end{array}$$

and leave the verification that they form an affine stratification to the reader. We do so since proving that, while essential for allowing us to apply Proposition 1.6.1 and determine generators for the Chow groups, will play no further role in our discussion.

Exercise 3.2.1. Verify that the Schubert cycles define an affine stratification on $\mathbf{G}(1,3)$.

3.3. **The Chow ring of $\mathbf{G}(1,3)$.** The fundamental classes of Schuber cycles are denoted

$$\sigma_{i,j} := [\Sigma_{i,j}] \in A^{i+j}(\mathbf{G}(1,3))$$

and called *Schubert classes*. By Proposition 1.6.1, each Chow group $A^k(\mathbf{G}(1,3))$ is generated by the Schubert classes $\sigma_{i,j}$ for $i + j = k$. For $k = 0$ and $k = 4$, we find σ_0 and $\sigma_{2,2}$ to just be the fundamental class of the whole Grassmanian and the point class respectively. The "interesting" Schubert classes are $\sigma_{1,1}, \sigma_2 \in A^2(\mathbf{G}(1,3))$ and $\sigma_{2,1} \in A^3(\mathbf{G}(1,3))$.

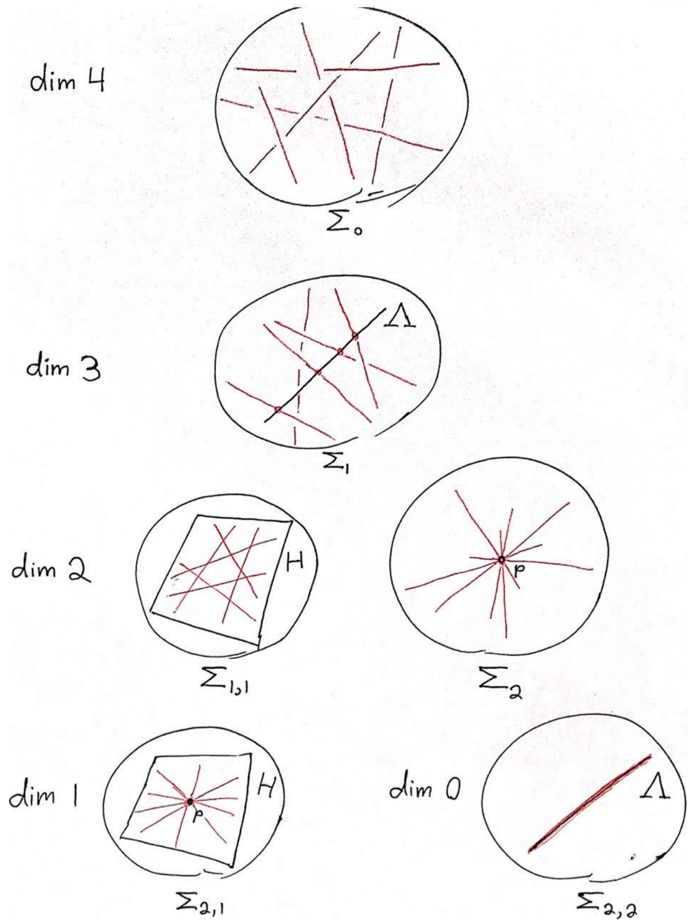

FIGURE 9. Stratification of plane cubics by type, indications how a cubic of one type can degenerate into one of another (i.e. which strata are in the closure of other ones), and dimensions of strata.

Now that we know the generators of the Chow groups, we must determine the multiplication rules between them. Some of that is easy: the square $\sigma_{2,1}^2$ vanishes for dimension reasons (it would live in $A^6(\mathbf{G}(1,3)) = 0$), the product of anything with the point class $\sigma_{2,2}$ is again $\sigma_{2,2}$, and finally σ_0 is the multiplicative unit in the Chow ring $A^*(\mathbf{G}(1,3))$. The more interesting products, we may determine by rephrasing them as basic geometry problems:

- The product $\sigma_{1,1}\sigma_1$ is the class of the subvariety in $\mathbf{G}(1,3)$ consisting of lines $L \subseteq \mathbf{P}^3$ intersecting a general line $\Lambda \subseteq \mathbf{P}^3$ and lying inside a general plane $H \subseteq \mathbf{P}^3$. Since Λ and H are generic, their intersection is a point $p \in \Lambda \cap H$, and the condition is equivalent to asking for lines $L \subseteq \mathbf{P}^3$ passing through p and lying on H. That is preicisely the Schubert cycle $\Sigma_{2,1}$, and so
$$\sigma_{1,1}\sigma_1 = \sigma_{2,1}.$$
- The product $\sigma_2\sigma_1$ is the class of lines in $\mathbf{P}^3$ that pass through a general point $p \in \mathbf{P}^3$ and intersect a general line $\Lambda \subseteq \mathbf{P}^3$. The poin p and line Λ determine a unique plane $H \subseteq \mathbf{P}^3$ on which they both lie, and so the condition in question is now equivalent to the Schubert cycle $\Sigma_{2,1}$. This shows that
$$\sigma_2\sigma_1 = \sigma_{2,1}.$$
- The product $\sigma_{1,1}\sigma_2$ corresponds to lines in $\mathbf{P}^3$ which lie on a general plane $H \subseteq \mathbf{P}^3$ and pass through a general point $p \in \mathbf{P}^3$. Alas, a general point in 3-space will not

lie on a general surface. Thus no lines satisfy this condition, and

$$\sigma_{1,1}\sigma_2 = 0.$$

- The product $\sigma_{1,1}^2$ is the class of lines in $\mathbf{P}^3$ which lie inside a general pair of planes $H, H' \subseteq \mathbf{P}^3$. Since $H \cap H' = L$ is a single line, we find that this is just the Schubert cycle $\Sigma_{2,2}$, from which we deduce

$$\sigma_{1,1}^2 = \sigma_{2,2}.$$

- Similarly, the product σ_2^2 corrsponds to lines containing a general pair of points $p, p' \in \mathbf{P}^3$. But a pair of points determine a unique line, so we once more get $\Sigma_{2,2}$ and therefore

$$\sigma_2^2 = \sigma_{2,2}.$$

- Finally the product $\sigma_1\sigma_{2,1}$ is the class of lines in $\mathbf{P}^2$ which intersect a general line $L \subseteq \mathbf{P}^3$, are contained in a general plane $H \subseteq \mathbf{P}^3$, and pass through a general point $p \in H$. The intersection $L \cap H$ is another point $p' \in H$, and so the condition is equivalent to demanding that a line passes through the points p and p'. But once again, a pair of lines determines a unique line on which they both lie, so only a single line will satisfy this condition. Hence the cycle in question is (rationally equivalent to) $\Sigma_{2,2}$ and the product in the Chow ring is

$$\sigma_1\sigma_{2,1} = \sigma_{2,2}.$$

The one remaining product that we have not yet determined is σ_1^2, and that is because it is a touch more difficult. Playing the same game as above, $\sigma_1^2 \in A^2(\mathbf{G}(1,3))$ is the class of lines in $\mathbf{P}^3$ which intersect two general lines $L, L' \in \mathbf{P}^3$. Unlike the geometry problems encountered so far in this Subsection, this is less obvious. In fact, by noting that σ_1^4 will be in $A^4(\mathbf{G}(1,3))$, and as such a multiple of the point class $\sigma_{2,2}$, we can turn this into an enumerative problem:

Question 9. *How many lines in $\mathbf{P}^3$ meet four general lines?*

To answer this, we observe that, since σ_1^2 is an element of the Chow group $A^2(\mathbf{G}(1,3))$, which we know to be generated by the Schubert classes $\sigma_{1,1}$ and σ_2, we have $\sigma_1^2 = a\sigma_{1,1} + b\sigma_2$ for the as-of-yet-undetermined coefficients $a, b \in \mathbf{Z}$. We determine these coefficients by using the multiplication rules we derived so far: since $\sigma_{1,1}^2 = \sigma_{2,2}$ is the point class while $\sigma_2\sigma_{1,1} = 0$, we get

$$b\sigma_{2,2} = \sigma_1^2\sigma_{1,1}.$$

The right-hand side corresponds to a more tractible geometric problem: the lines in $\mathbf{P}^2$ which intersect two general lines $L, L' \subseteq \mathbf{P}^3$ and also lie on a general plane $H \subseteq \mathbf{P}^3$. Since the intersections $L \cap H$ and $L \cap H'$ determine two points through which any such curve must pass, we find the cycle in question to be $\Sigma_{2,2}$. That is to say, $\sigma_1^2\sigma_{1,1} = \sigma_{2,2}$, which shows that $b = 1$. Playing a similar game, we find that

$$a\sigma_{2,2} = \sigma_1^2\sigma_2,$$

which corresponds to lines in $\mathbf{P}^3$ meeting two general lines $L, L' \in \mathbf{P}^3$ and passing through a point $p \in \mathbf{P}^3$. The point p and the lines L resp. L' determine planes $H, H' \subseteq \mathbf{P}^3$, whose intersection is the only line satisfying the requirement. That once again means that $\sigma_1^2\sigma_2 = \sigma_{2,2}$ and hence $a = 1$. Combining these computations, we find the last remaining product of Schubert classes to be

$$\sigma_1^2 = \sigma_{1,1} + \sigma_2.$$

Consequently we may answer Question 9 by computing

$$\deg(\sigma_1^4) = \deg(\sigma_{1,1}^2 + \sigma_2^2) = \deg(2\sigma_{2,2}) = 2.$$

Answer to Question 9. *There are precisely two lines in $\mathbf{P}^3$ meeting four general lines.*

With this, the Chow ring $A^*(\mathbf{G}(1,3))$ is fully determined. Now that we have it, let us use it to do some more enumerative computations!

3.4. **Lines meeting curves.** The leading question for this Subsection is

Question 10. *How many lines in* $\mathbf{P}^3$ *meet four general twisted cubics?*

Once again, we work slightly more generally than necessary, to prove a slightly cooler result. Let $C \subseteq \mathbf{P}^3$ be a curve of degeee d (twisted cubics would correspond to $d = 3$) and consider the subvariety

$$\Gamma_C := \{L \subseteq \mathbf{P}^3 : L \cap C \neq \varnothing\} \subseteq \mathbf{G}(1,3)$$

of the Grassmanian, consisting of all lines which meet the curve C. To determine the dimension of Γ_C, we note that there are 3 degree of freedom for such lines: one for the point at which they meet C, and two for degrees of freedom for a line passing through a fixed point in $\mathbf{P}^3$ (as that is just that the Schubert cycle Σ_2). The correspondig cycle $\gamma_C := [\Gamma_C]$ is thus an element of $A^1(\mathbf{G}(1,3))$, and as such of the form $\gamma_C = a\sigma_1$ for some coefficient $a \in Z$ that we must now determine.

Since $\sigma_1\sigma_{2,1} = \sigma_2$, we get $a = \deg(\gamma_C\sigma_{2,1})$. This is asking for the number of lines inside a general plane $H \subseteq \mathbf{P}^3$, passing through a fixed point $p \in H$, and intersecting the curve C. Thus all such lines must meet the intersection $C \cap H$, which by Bezout's theorem (and the plane H being generic, and hence intersecting C transversely) consists of precisely d distinct points $p_1, \ldots, p_d$. The lines in question are now precisely the n lines determined by p and each one of the points $p_1, \ldots, p_d$ respectively. Hence $a = d$ and so $\gamma_C = d\sigma_1$.

For any four general curves $C_1, \ldots, C_4 \subseteq \mathbf{P}^3$ of degrees $d_1, \ldots, d_4$, the class $\gamma_{C_1} \cdots \gamma_{C_4}$ belongs to $A^4(\mathbf{G}(1,3))$, showing that the number $\deg(\gamma_{C_1} \cdots \gamma_{C_4})$ of lines which meet all four of the curves, is finite and equal to

$$\deg(\gamma_{C_1} \cdots \gamma_{C_4}) = \deg(d_1 \cdots d_4 \sigma_1^4) = 2d_1 \cdots d_4.$$

In particular, by choosing $d_1 = \cdots = d_4 = 3$, we find the answer to the leading question of this Subsection.

Answer to Question 10. *Precisely* $2 \cdot 3^4 = 164$ *linear in* $\mathbf{P}^3$ *meet four general twisted cubics.*

Of course, we may use the fact that $\gamma_C = d\sigma_1$ for a degree d-curve $C \subseteq \mathbf{P}^3$ to solve other enumerative problems as well. For instance, we find that the numer of lines meeting a general curve C of degree d, a general curve C' of degree d', and a general point, is $\deg(\gamma_C\gamma_{C'}\sigma_2) = dd'$.

3.5. **Secant lines to cubic curves.** Recall that a *secant line* to a curve $C \subseteq \mathbf{P}^3$ (also called *chord spanned by* C) is a line which intersects it in at least two points.

Question 11. *How many common secant lines do two general twisted cubics have?*

To solve this problem, we consider the subvariety $\Psi_C \subseteq \mathbf{G}(1,3)$ of secant lines to a cubic curve $C \subseteq \mathbf{P}^3$. Note in particular that any tangent curve, which is to say, a line that intersects C at a point with degree ≥ 2, is counted as a secant here, albeit the space of "genuine secants" is a dense open subset of Ψ_C, and so a general secant will not be a tangent line.

There are two degrees of freedom available to a secant line to the cubic curve C: the location of the two points of intersection. Thus $\dim(\Psi_C) = 2$ and so its fundamental class $\psi_C = [\Psi_C] \in A^2(\mathbf{G}(1,3))$ may be written in the form $\psi_C = a\sigma_{1,1} + b\sigma_2$. We find the coefficients as $a = \deg(\psi_C\sigma_{1,1})$ and $b = \deg(\psi_C\sigma_2)$.

Determining the first one is simple: $\sigma_C\sigma_{1,1}$ is the class of all the lines lying on a general plane $H \subseteq \mathbf{P}^3$, and which meet C in at least 3 points. that is equivalent to saying that they must meet the locus $H \cap C$, which consists of d distinct points by Bezout's Theorem, in at

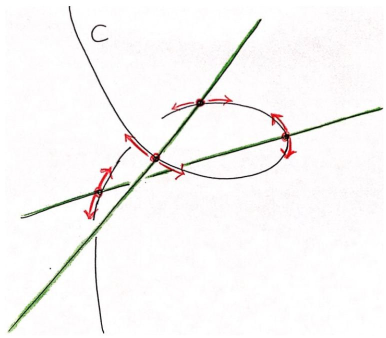

Figure 10. Degrees of freedom of a secant line to C.

least two points. Of course, any two points determine a unique line that passes through them, thus there are precisely $\binom{3}{2} = 3$ such lines. That is to say, $a = 3$.

For the coefficient b, we need to be a little trickier. The class $\sigma_C \sigma_2$ consists of all those secants to C which also pass through a fixed general point $p \in \mathbf{P}^3$. To determine their number, we choose a general plane $H \subseteq \mathbf{P}^3$ and consider projection from the point p to H. That is the map $C \to H$ given by sending any point $q \in C$ to the point of intersection $H \cap L$ beteen the plane H and the unique line L passing through the points p and q. The image of this map will be a plane curve $C' \subseteq H \cong \mathbf{P}^2$, and we see that the map $C \to C'$ is generically $1:1$, and as such C' will be generically smooth amd have the same degree as C. But at the ramification points of the projection map, the curve C' will have a double point, and as such, a singularity. With the generic choice of H, this will be an ordinary double point. Harkening back to our analysis of the space of plane cubics, we note that the generic plane cubic with an ordinary double point is the nodal cubic. Hence C' is a nodal cubic, and as it has only a single singular point, it follows that $b = 1$.

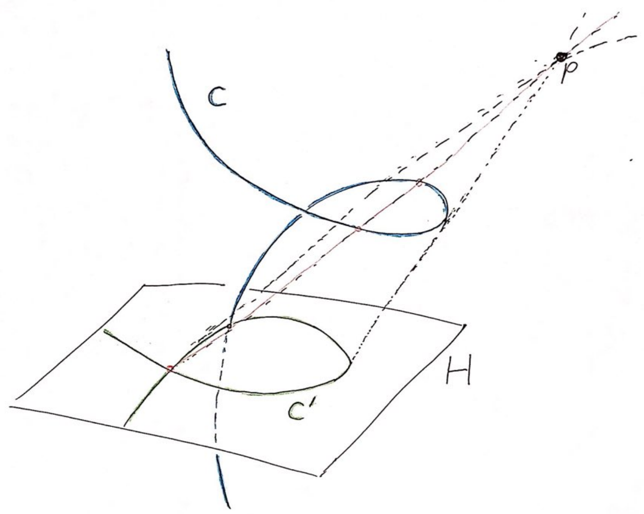

Figure 11. The projection from a point argument illustrated.

Remark 3.5.1. The preceding argument is the reason why we chose to restrict our attention to cubic curves in this Subsection. For a general degree d-curve $C \subseteq \mathbf{P}^3$, we would still have $\psi_C = a\sigma_{1,1} + b\sigma_2$, and the same argument as above would show that $a = \binom{d}{2}$. But the argument to determine b relied on special knowledge of cubics we have. Nonetheless, as you can find in the Eisenbud-Harris book, it can be made to work for a general degree d curve as well. The projection from a point trick is still used, but determining the number of double points of the projected curve $C' \subseteq \mathbf{P}^3$ requires a bit more work. It turns out that $b = \binom{d-1}{2} - g$ where g is the genus of C. For a twisted cubic we have $d = 3$ and $g = 0$, recovering $d = 1$ as above.

Thus we have $\psi_C = 3\sigma_{1,1} + \sigma_2$ and so

$$\deg(\psi_C^2) = \deg(9\sigma_{1,1}^2 + \sigma_2^2) = 10.$$

Answer to Question 11. *Two general twisted cubics have precisely 10 common secants.*

3.6. **Tangents to surfaces.** Yet another enumerative problem we are now equipped to tackle:

Question 12. *How many lines in $\mathbf{P}^3$ are tangent to four general quadric surfaces?*

Let $S \subseteq \mathbf{P}^3$ be a smooth algebraic surface of degree d. We wish to consider the subvariety $T_S \subseteq \mathbf{G}(1,3)$ of lines which are at some point tangent to S. Using once again the informal degree-of-freedom-counting, we find that a tangent line $L \subseteq T_pS$ for $p \in S$ has three degrees of freedom: two for moving around the point p along the surface S, and one for rotating the line around inside the fixed tangent space T_pS. Thus $\dim T_S = 3$ and so its class $\tau_S := [T_C] \in A^1(\mathbf{G}(1,3))$ is of the form $\tau_S = a\sigma_1$ for some coefficient $a \in \mathbf{Z}$.

Said coefficient is given by $a = \deg(\tau_S\sigma_{2,1})$, which is the answer to the following enumerative question: for a general plane $H \subseteq \mathbf{P}^3$ and poin $p \in H$, how many lines $L \subseteq H$ which pass through the point p are tangent to S? Of course, that is equivalent to counting the number of tangent lines in $H \cong \mathbf{P}^2$ to the algebraic curve $C := S \cap H \subseteq H \cong \mathbf{P}^2$ through a general point. The plane curve $C \cap H$ of course has the same degree d as the surface S, and so this is precisely the question addressed in Subsection 2.2. I.e. the sought coefficient a is the degree of the dual curve $C^* \subseteq \mathbf{P}^{2*}$, which we know from Subsection 2.2 to be equal to $d(d-1)$.

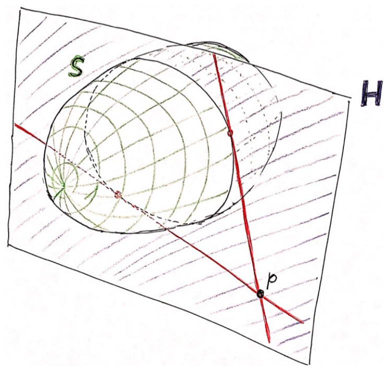

FIGURE 12. The two tangent lines to a quadric surface S that lie on a general plane H and pass through a general point $p \in H$.

Hence we have found that $\tau_S = d(d-1)\sigma_1$. This allows us to solve several enumerative question, for instance the computation

$$\deg(\tau_S^4) = d^4(d-1)^4 \deg(\sigma_1^4) = 2d^4(d-1)^4$$

solves by setting $d = 2$ the leading question of this Subsection:

Answer to Question 12. *There are $2 \cdot 2^4 \cdot (2-1)^4 = 32$ lines in $\mathbf{P}^3$ tangent to four general quadric surfaces.*

Of course, we can combine the Chow classe computations we have made in this Section. For instance, since $\gamma_C = d\sigma_1$, $\psi_{C'} = 3\sigma_{1,1} + \sigma_2$. and $\tau_S = e\sigma_1$ for a curve $C \subseteq \mathbf{P}^3$ of degree d, a twisted cubic $C' \subseteq \mathbf{P}^3$, and a degree e surface $S \subseteq \mathbf{P}^3$, we get

$$\deg(\sigma_C\psi_{C'}\tau_S) = de\deg((\sigma_{1,1} + \sigma_2)(3\sigma_{1,1} + \sigma_2)) = de\deg(3\sigma_{1,1}^2 + \sigma_2^2) = 4de.$$

That means for instance that there are precisely $8{\cdot}5{\cdot}2 = 80$ lines in $\mathbf{P}^3$ which simultaneously intersect a general quintic curve, are secant to a twisted cubic, and are tangent to a quadric surface. Enjoy deriving more such conseqneces for yourself! :)

4. Chern classes and the 27 lines on a cubic

Now that we have seen how to do several fun computations with lines in 3-space, let us spend this Section to perhaps the most famous one: the existence of 27 lines on a smooth cubic surface. Thus without furter ado, the leading question for this whole Subsection:

Question 13. *How many lines does a smooth cubic surface in $\mathbf{P}^3$ contain?*

4.1. **Game plan.** Any cubic surface $S \subseteq \mathbf{P}^3$ is cut out by some cubic polynomial $F \in \Gamma(\mathbf{P}^3; \mathscr{O}(3))$. A line $L \subseteq \mathbf{P}^3$ lies on X if any only if $F|_L = 0$. In principle though, $F|_L$ is a cubic polynomial on the line $L \cong \mathbf{P}^1$ itself. That is, for every $L \in \mathbf{G}(1,3)$, we are interested in the element $F|_L$ of the vector space $\Gamma(L; \mathscr{O}(3))$. There is a name for a family of vector spaces, parametrized by the points of a base space: a *vector bundle*.

Thus the lines on cubic surfaces problem leads us to consider the vector bundle $\mathscr{E}$ on $\mathbf{G}(1,3)$, which we define somewhat informally be specifying its fiber over every point $L \in \mathbf{G}(1,3)$ as $\mathscr{E}_L := \Gamma(L; \mathscr{O}(3))$. This is to say, the vector bundle of cubic forms on lines in $\mathbf{P}^3$.

Any polynomial $F \in \Gamma(\mathbf{P}^3; \mathscr{O}(3))$ defines a global section $s_F \in \Gamma(\mathbf{G}(1,3); \mathscr{E})$ fiber-wise by $s_F(L) := F|_L \in \Gamma(L; \mathscr{O}(3)) = \mathscr{E}_L$, and if F is non-zero, then so is s_F. A line $L \subseteq \mathbf{P}^3$ lies on a cubic $X = V(F)$ if and only if the section s_F vanishes at the point $L \in \mathbf{G}(1,3)$. That is to say, what we wish to study is the vanishing locus $V(s_F)$.

4.2. **The top Chern class.** Let $\mathscr{E}$ be a rank r vector bundle on a variety X. Suppose that $s \in \Gamma(X; \mathscr{E})$ is a non-zero global section. Then we define the *r-th Chern class of $\mathscr{E}$* to be the class of the vanishing locus

$$c_r(\mathscr{E}) := [V(s)].$$

Since the vector bundle $\mathscr{E}$ has rank r, the locus $V(s)$ is locally (for instance, working on a small enough open $U \subseteq X$ on which $\mathscr{E}|_U \cong \mathscr{O}_U^{\oplus r}$) cut out by r equations. It thus has codimension r, and so we have $c_r(\mathscr{E}) \in A^r(X)$.

In our case of interest, we have $X = \mathbf{G}(1,3)$, and $\mathscr{E}$ is the vector bundle of cubic forms on lines. Since its fiber is $\mathscr{E}_L = \Gamma(L; \mathscr{O}(3)) \cong \Gamma(\mathbf{P}^1; \mathscr{O}(3))$ is $\binom{3+1}{1} = 4$-dimensional (in terms of the homogeneous coordinates x_0, x_1, its elements look like $a_0x_0^3 + a_1x_0^2x_1 + x_2x_0x_1^2 + a_4x_1^3$), it follows that the rank of $\mathscr{E}$ is $r = 4$. Thus we are looking for the Chern class

$$c_4(\mathscr{E}) = [V(s_F)] \in A^4(\mathbf{G}(1,3)).$$

Remark 4.2.1. We may read off from this that cubic surfaces are really the only ones among algebraic surfaces in $\mathbf{P}^3$ that have a chance to generically contain finitely many lines. That is because an analogous analysis as above would show that the locus of lines on a generic degree d hypersurface in $\mathbf{P}^3$ would be the degree $\dim \Gamma(\mathbf{P}^1; \mathscr{O}(d)) = \binom{d+1}{1} = d+1$-th Chern class of the appropriate vector bundle on $\mathbf{G}(1,3)$, and as such an element of $A^{d+1}(\mathbf{G}(1,3))$. But since the Grassmanian is 4-dimensional, this class must vanish for all degrees $d + 1 \gneq 4$.

Thus the only possibilities for a surface with finitely many lines are degrees $d = 1,2,3$. A plane clearly contains infinitely many lines, and a quadric surface turns out to as well - it contains two pencils of lines, comprising the famous double ruling on the one-sheeted hyperboloid. Thus only cubic surfaces are left as an option.

So far we have done nothing but repackaged our ignorance: we slapped a fancy name onto our object of study. Before we can reap the benefits of doing so, we need to drop further down the deep end of the theory of Chern classes. But when we come back out on the other side, it will be what we will have learned in Wonderland that will allow us to do the Futterwacken and the famous 27-lines computation alike.

4.3. **Chern classes and their properties.** Let us return to the general setting from the start of the previous Subsection. That is, X is a general variety, and $\mathscr{E}$ a rank r vector bundle on X.

As the name suggests, $c_r(\mathscr{E})$ is not the only Chern class that a vector bundle possesses - in fact, it is only the top-degree one! To define the one of next-lowest-degree, let us suppose $s, s' \in \Gamma(X; \mathscr{E})$ are two linearly independent global sections of $\mathscr{E}$. We define the $(r-1)$*-st Chern class of* $\mathscr{E}$ as the class $c_{r-1}(\mathscr{E})$ of the locus of linear dependence of s and s', i.e. the locus of those points $x \in X$ for which the values $s(x)$ and $s'(x)$ are linearly dependent in the fiber $\mathscr{E}_x$. Said differently, it is the class of the vanishing locus

$$c_{r-1}(\mathscr{E}) = [V(s \wedge s')].$$

By a similar codimension consideration to the one we performed above, we see that $c_{r-1}(\mathscr{E})$ lives in the Chow group $A^{r-1}(X)$.

In general, for linearly independent sections $s_1, \dots, s_{r-i} \in \Gamma(X; \mathscr{E})$, we may compute the *$i$-th Chern class of* $\mathscr{E}$ as

$$c_i(\mathscr{E}) := [V(s_1 \wedge \cdots \wedge s_{r-i})] \in A^i(X),$$

or equivalently, the locus of linear independence of the sections $s_1, \dots, s_{r-i}$. It is sometimes convenient to arrange all the Chern classes together into the *total Chern class*

$$c(\mathscr{E}) := 1 + c_1(\mathscr{E}) + \cdots + c_r(\mathscr{E}) \in A^*(X)$$

(where the role of the 0-th Chern class is played by the multiplicative unit $1 = [X]$, i.e. the fundamental class of the whole variety).

Remark 4.3.1. This approach to Chern classes only works for vector bundles with sufficiently many global sections. Unlike in differential geometry or algebraic topology, that is a steep restriction in the realm of algebraic geometry (for instance (the Serre twisting sheaves $\mathscr{O}(n)$ have no non-zero global sections for any $n < 0$).

The main properties of Chern classes may be summarized in the following Proposition:

Proposition 4.3.2. *There exists a unique theory of Chern classes*

$$c(\mathscr{E}) = 1 + c_1(\mathscr{E}) + \cdots + c_r(\mathscr{E}) \in A^*(X)$$

for any degree r vector bundle $\mathscr{E}$ on a variety X, so that

1. *(Functoriality) For any map $f : X \to Y$, there is a canonical isomorphism*

$$f^* c(\mathscr{E}) = c(f^*\mathscr{E})$$

between the Chow ring pullback of the total Chern class of $\mathscr{E}$ on Y, and the total Chern class of the vector bundle pullback $f^*\mathscr{E}$ on X.

2. *(Additivity) For any short exact sequence $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ of vector bundles, the total Chern classes satisfy*

$$c(\mathscr{E}) = c(\mathscr{E}')c(\mathscr{E}').$$

3. *(Compatibility) If a vector bundle $\mathscr{E}$ is generated by global sections, then $c(\mathscr{E})$ agrees with its definition in the previous Subsection.*

Remark 4.3.3. This is a continuation of Example 1.2.1. The first Chern class c_1 lands inside the Chow group $A^1(X) = \mathrm{Cl}(X)$, i.e. the divisor class group. If we restrict c_1 only to line bundles (rank 1 vector bundles), and pass to isomorphism classes, then the first Chern class defines the famous bijection $c_1 : \mathrm{Pic}(X) \cong \mathrm{Cl}(X)$ between the Picard group: the group of line bundles up to isomorphisms, and the divisor class group.

For our purposes, the relevant part of this remark is the observation that the formula $c_1(\mathscr{L} \otimes \mathscr{L}') = c_1(\mathscr{L}) + c_1(\mathscr{L}')$ holds in $A^1(X)$ for any pair of line bundles $\mathscr{L}, \mathscr{L}'$ on X. When the two line bundles admit non-zero global sections, this is quite easy to see from the explicit formula for the top Chern class that we have given above.

4.4. **Chern classes of the tautological bundle on $\mathbf{G}(1,3)$.** As an example of how Chern classes work, and with the foresight that this will be useful to us in a little bit, we work out in detail the computation of Chern classes of a particularly simple vector bundle.

The *tautological bundle* $\mathscr{V}$ on $\mathbf{G}(1,3)$ is defined fiber-wise as $\mathscr{V}_L := \Gamma(L; \mathscr{O}(1))$. That is to say, its fiber over a point $L \in \mathbf{G}(1,3)$ is the vector space of linear forms on the line L. Since $\dim \Gamma(L; \mathscr{O}(1)) = \dim \Gamma(\mathbf{P}^1; \mathscr{O}(1)) = \binom{1+1}{1} = 2$, this gives rise to a rank 2 vector bundle on the Grassmanian $\mathbf{G}(1,3)$.

Remark 4.4.1. In terms of the alternative perspective on the Grassmanian $\mathbf{G}(1,3) = G(2,4)$ as parametrizing 2-dimensional linear subspaces $V \subseteq \mathbf{A}^4$, the dual to the tautological vector bundle $\mathscr{V}^*$ is given fiber-wise as having fiber at a point $V \in G(2,4)$ equal to $\mathscr{V}^*_V = V$. Equivalently, its total space can be identified with the incidence correspondence

$$\{(V,p) \in G(2,4) \times \mathbf{A}^4 : p \in V\}$$

viewed as a vector bundle over $G(2,4)$ through the projection map $(V,p) \mapsto V$. We hope that somewhat clarifies our choice of the adjective "tautological" as the name.

Having rank 2, the tautological bundle possesses two Chern classes: $c_2(\mathscr{V}) \in A^1(\mathbf{G}(1,3))$ and $c_1(\mathscr{V}) \in A^2(\mathbf{G}(1,3))$. To compute them, let us choose two global sections. As with $\mathscr{E}$ before, we exhibit them by restriction of two linear polynomials $F, G \in \Gamma(\mathbf{P}^3; \mathscr{O}(1))$, that is, we define $s_F, s_G \in \Gamma(\mathbf{G}(1,3); \mathscr{V})$ fiber-wise as $s_F(L) = F|_L$ and $s_G(L) = G|_L$. If F and G are linearly independent, the same will hold for s_F and s_G.

- Now the top Chern class is $c_2(\mathscr{V}) = [V(s_F)]$ consists of those lines $L \subseteq \mathbf{P}^3$ which are contained in the plane $H := V(F) \subseteq \mathbf{P}^3$. Recalling the computation of the Chow ring of the Grassmanian from the previous Section, we can recognize that as the Schubert class $\sigma_{1,1}$. Thus

$$c_2(\mathscr{V}) = \sigma_{1,1} \in A^2(\mathbf{G}(1,3)).$$

- The first Chern class is conversely $c_1(\mathscr{V}) = [V(s_F \wedge s_G)]$, the locus of those lines $L \subseteq \mathbf{P}^3$ for which $s_F(L)$ and $s_G(L)$ are linearly dependent. That means that there exist some non-zero coefficients t_0, t_1 for which $0 = t_0 F|_L + t_1 G|_L = (t_0 F + t_1 G)|_L$. That is to say, there exists an element on the pencil of planes $\{H_t = V(t_0 F + t_1 G) \subseteq \mathbf{P}^3\}_{t \in \mathbf{P}^1}$ which contains L.

  Any plane H_t certainly passes through the line $\Lambda := V(F) \cap V(G)$, and since lines planes satisfying this property are a 1-dimensional family, it follows that that is precisely this pencil. That is to say, the family $\{H_t \subseteq \mathbf{P}^3\}_{t \in \mathbf{P}^1}$ consists of all the planes which contain the line Λ. Thus for the line L to be contained in one of the planes H_t, it is necessary and sufficient that L and Λ intersect.

  Therefore the locus $V(s_F \cap s_G) \subseteq \mathbf{P}^3$ consists of all lines $L \subseteq \mathbf{P}^3$ which intersect a general line $\Lambda \subseteq \mathbf{P}^3$. That is precisely the Schubert cycle Σ_1, and so

$$c_1(\mathscr{V}) = \sigma_1 \in A^1(\mathbf{G}(1,3)).$$

In particular, the total Chern class of the tautological bundle on the Grassmanian is

$$c(\mathscr{V}) = 1 + \sigma_1 + \sigma_{1,1} \in A^*(\mathbf{G}(1,3)).$$

To relate this to our end-goal of computing the Chern class $c_4(\mathscr{E})$, let us explain how the vector bundles $\mathscr{E}$ and $\mathscr{V}$ are related. Since any cubic form is comprised of (a sum of) triple products of linear forms, we have $\mathscr{E} = \mathrm{Sym}^3(\mathscr{V})$.

4.5. **The Splitting Principle, in theory.** One of the most powerful techniques for performing Chern clas computations is the so-called *Splitting Principle*. Said somewhat informally, it states that:

Splitting Principle. *Any statement about the Chern classes $c_i(\mathscr{E})$ that can be made for a vector bundle $\mathscr{E}$ under the assumption (false as it may be!) that its splits as $\mathscr{E} = \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_r$ into a sum of line bundles, will be true if re-expressed from the Chern classes $c_1(\mathscr{L}_j)$ into the Chern classes $c_i(\mathscr{E})$ of $\mathscr{E}$.*

That sounds incredibly false, but the more in-depth reasn for why it holds is that for any vector bundle $\mathscr{E}$ on a variety X, there exists some map $f : F \to X$ for which

- the vector bundle pullback $f^*\mathscr{E}$ admits a filtration
$$f^*\mathscr{E} = \mathscr{E}_0 \supseteq \mathscr{E}_1 \supseteq \cdots \supseteq \mathscr{E}_r \supseteq 0$$
with each successive quotient $\mathscr{L}_i := \mathscr{E}_{i-1}/\mathscr{E}_i$ (the *associated graded* of the filtration, as the jargon goes) being a line bundle on F.
- the Chow ring pullback $f^* : A^*(X) \to A^*(F)$ is injective.

Note that from the perspective of the Chern classes, a filtration as above is just as good as a splitting $f^*\mathscr{E} = \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_r$. That is to say, Chern classes are by their additivity property blind to the difference between split and non-split short exact sequences[8].

Thus we may manipulate the Chern classes $c_i(\mathscr{E})$ as if $\mathscr{E}$ split into a sum of line bundles, by actually looking at the classes $f^*c_i(\mathscr{E}) = c_i(f^*\mathscr{E})$ inside $A^*(F)$, which are equal to $c_i(\mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_r)$, and then using the injectivity of f^* to pull the obtained formulas back to $A^*(X)$.

Proof sketch. To construct $f : F \to X$, it suffices to construct a map $p : P \to X$ such that $p^* : A^*(X) \to A^*(P)$ is injective, and the pullback bundle $p^*\mathscr{E}$ filts into a short exact sequence
$$0 \to \mathscr{E}' \to p^*\mathscr{E} \to \mathscr{L} \to 0$$
for a line bundle $\mathscr{L}$ on P and a rank $r-1$ vector bundle $\mathscr{E}'$. The desired $f : F \to X$ is then obtained by iteratively applying the same proceedure to the vector bundle $\mathscr{E}'$ on P, and so on, until the rank drops to zero.

We claim that the (fiber-wise) projectivization $P := \mathbf{P}_X(\mathscr{E}^\vee)$ of the dual bundle $\mathscr{E}^\vee$ of $\mathscr{E}$ does the job. To see why that is, let us briefly recall how projectivizations of vector bundles work[9].

Informally, the points of the projectivization $\mathbf{P}_X(\mathscr{E})$ of a vector bundle $\mathscr{E}$ on X consist of pairs (x, V) of a point $x \in X$ and a 1-dimensional linear subspace $V \subseteq \mathscr{E}_x$, and the projection map $p : \mathbf{P}_X(\mathscr{E}) \to X$ sends $(x, V) \mapsto x$. It follows that the pullback vector bundle $p^*\mathscr{E}$ has fibers $(p^*\mathscr{E})_{(x,V)} = \mathscr{E}_x$. On the other hand, $\mathbf{P}_X(\mathscr{E})$ also admits the tautological line bundle $\mathscr{O}(-1)$, given fiber-wise by $\mathscr{O}(-1)_{(x,V)} = V$. Through the inclusion, this defines a vector subbundle $\mathscr{O}(-1) \subseteq p^*\mathscr{E}$ is a line subbundle.

Applying this to the dual bundle $\mathscr{E}^\vee$, and since $\mathscr{E}$ has finite rank r, the rank 1 subbundle $\mathscr{O}(-1) \subseteq p^*\mathscr{E}^\vee$ corresponds to a rank $(r-1)$-subbundle $\mathscr{O}(-1)^\vee \subseteq p^*\mathscr{E}$. This is the desired vector subbundle $\mathscr{E}' := \mathscr{O}(-1)^\vee$ of $p^*\mathscr{E}$ on $P = \mathbf{P}_X(\mathscr{E}^\vee)$. ☐

[8]Lest one think this is a shortcoming and not a perk, we should point out that this observation served at the key motivation for Grothendieck to introduce the "Grothendieck group" K_0 in his work on the Riemann-Roch Theorem, beginning the road to algebraic K-theory, a huge and highly active area of modern math. In fact, K-theory boasts the honor of having a spot among the keywords in the standard math index classification used say by the arXiv. One of these days someone will explain to me why poor old homotopy theory is in contrast denied this basic privilige.

[9]For the scheme theorists among us, of course the conciseness of defining it as $\mathbf{P}_X(\mathscr{E}) := \mathrm{Proj}_X(\mathrm{Spec}_X(\mathscr{E}^\vee))$ can not be beat. The universal property hinted at in the informal description we give below is then a fun exercise to prove starting from the relative Proj approach.

Remark 4.5.1. The space $f : F \to X$ constructed above is sometimes called the *flag variety of $\mathscr{E}$*. Indeed, we may see from its iterative construction that it parametrizes complete flags in $\mathscr{E}$. This can be made rigorous from the functor of points perspective, but informally, it amounts to saying that the points of F may be identified with tuples $(x; V_1, \dots, V_{r-1})$ of a point $x \in X$ and a chain of proper inclusions of linear subspaces $\mathscr{E}_x \supsetneq V_{r-1} \supsetneq V_{r-2} \supsetneq \cdots V_1 \supsetneq 0$, i.e. a complete flag on the vector space $\mathscr{E}_x$.

4.6. **The Splitting Principle, in action.** As an example of how using the Splitting Principle works in practice, let us determine how the Chern classes of a symmetric power $c_i(\mathrm{Sym}^3(\mathscr{V}))$ may be expressed in terms of the Chern classes $c_i(\mathscr{V})$ of a rank 2 vector bundle $\mathscr{V}$.

(Wait! That's precisely the same situation as the one that we were describing at the end of the previous Subsection. What a wild coincidence!)

Let us suppose (in all likelihood falsely, but by the Splitting Principle, we know does not matter) that $\mathscr{V} = \mathscr{L} \oplus \mathscr{L}'$ and let us denote by $\alpha = c_1(\mathscr{L})$ and $\beta = c_1(\mathscr{L}')$ the Chern classes of the line bundle summands. The classes α and β are usually called the *Chern roots* of $\mathscr{V}$. From the additivity of the total Chern class, we obtain

$$c(\mathscr{V}) = c(\mathscr{L})c(\mathscr{L}') = (1+\alpha)(1+\beta) = 1 + \alpha + \beta + \alpha\beta.$$

Reading off each degree, we obtain the expressions[10]

$$c_1(\mathscr{V}) = \alpha + \beta, \qquad c_2(\mathscr{V}) = \alpha\beta$$

for the Chern classes of $\mathscr{V}$ in terms of its Chern roots α and β.

On the other hand, by using basic linear algebra to determine how the third symmetric power of the sum of two 1-dimensional vector spaces works, we find that

$$\mathrm{Sym}^3(\mathscr{V}) = \mathrm{Sym}^3(\mathscr{L} \oplus \mathscr{L}') \cong \mathscr{L}^{\otimes 3} \oplus \left(\mathscr{L}^{\otimes 2} \otimes \mathscr{L}'\right) \oplus \left(\mathscr{L} \otimes \mathscr{L}'^{\otimes 2}\right) \oplus \mathscr{L}'^{\otimes 3}.$$

Passing to the total Chern class, and recalling that $c_1(\mathscr{L} \otimes \mathscr{L}') = c_1(\mathscr{L}) + c_1(\mathscr{L}')$ holds for any pair of line bundles, we get

$$c(\mathrm{Sym}^3(\mathscr{V})) = (1+3\alpha)(1+2\alpha+\beta)(1+\alpha+2\beta)(1+3\beta).$$

By focusing on each degree separately, we obtain formulas for the individual Chern classes. It remains to massage the formulas until we can replace all the occurences of α and β inside them by $\alpha+\beta$ and $\alpha\beta$, which we saw above are the Chern classes of $\mathscr{V}$. In degree 1, that is

$$\begin{aligned} c_1(\mathrm{Sym}^3(\mathscr{V})) &= (3+2+1)\alpha + (1+2+3)\beta \\ &= 6(\alpha+\beta) \\ &= 6c_1(\mathscr{V}), \end{aligned}$$

in degree 2 we get the much messier

$$\begin{aligned} c_2(\mathrm{Sym}^3(\mathscr{V})) &= (6+3+2)\alpha^2 + (3+6+9+4+6+1)\alpha\beta + (2+3+6)\beta^2 \\ &= 11\alpha^2 + 29\alpha\beta + 11\beta^2 \\ &= 11(\alpha+\beta)^2 + 18\alpha\beta, \\ &= 11c_1(\mathscr{V})^2 + 18c_2(\mathscr{V}), \end{aligned}$$

[10] As the reader experienced with polynomials might recognize, we would be obtaining elementary symmetric polynomials in r variables if we were assuming splitting into r factors, which is to say, if $\mathscr{V}$ had rank r.

in degree 3 things are not much better, but at least not much worse either

$$\begin{aligned} c_3(\mathrm{Sym}^3(\mathscr{V})) &= 6\alpha^3 + (12+18+3+6)\alpha^2\beta + (6+3+18+12)\alpha\beta^2 + 6\beta^3 \\ &= 6(\alpha^3+\beta^3) + (18+21)(\alpha^2\beta+\alpha\beta^2) \\ &= 6(\alpha+\beta)^3 + 21(\alpha+\beta)\alpha\beta \\ &= 6c_1(\mathscr{V})^3 + 21c_1(\mathscr{V})c_2(\mathscr{V}), \end{aligned}$$

and finally in degree 4 we find

$$\begin{aligned} c_4(\mathrm{Sym}^3(\mathscr{V})) &= 3\alpha(2\alpha+\beta)(\alpha+2\beta)3\alpha \\ &= 3\alpha(2\alpha^2+5\alpha\beta+2\beta^2)3\beta \\ &= 9\alpha\beta(2(\alpha+\beta)^2+\alpha\beta) \\ &= 9c_2(\mathscr{V})(2c_1(\mathscr{V})^2+c_2(\mathscr{V})). \end{aligned}$$

Hopefully the reader has found this fully worked-out computation highly illuminating, because to be frank, it was not the most fun to type out.

4.7. **The finishline.** Now we have assembled all the moving pieces, and it remains just to assemble them! In Subsection 4.2 we found that the class of lines which lie on a general cubic surface is encoded as the Chern class $c_4(\mathscr{E})$ of the vector bundle of cubic forms on lines $\mathscr{E}$ over $\mathbf{G}(1,3)$. In Subection 4.4, we recognized that $\mathscr{E} = \mathrm{Sym}^3(\mathscr{V})$ of the tautological bundle $\mathscr{V}$ on $\mathbf{G}(1,3)$, whose Chern classes we also computed there in terms of Schubert classes to be

$$c_1(\mathscr{V}) = \sigma_1, \qquad c_2(\mathscr{V}) = \sigma_{1,1}.$$

Finally in Subection 4.6, we used the Splitting Principle to find c_4 of a symmetric power. Putting all this togehter, and using the Schubert calculus rules for $\sigma_{1,1}^2 = \sigma_{2,2}$ and $\sigma_{1,1}\sigma_2 = 0$ that we learned in the last Section, we get

$$c_4(\mathscr{E}) = 9\sigma_{1,1}(2\sigma_1^2 + \sigma_{1,1}) = 9\sigma_{1,1}(3\sigma_{1,1} + 2\sigma_2) = 27\sigma_{2,2}.$$

Voila, we have obtained (most of) the famous Cayley-Salmon Theorem:

Answer to Question 13. *There are precisely* 27 *lines on any smooth cubic surface.*

To be clear, we have not done all the work we claim here: what we have shown was only that a general cubic surface will contain 27 lines. It takes some additional work to determine that the dense open subset inside the space of conic surfaces $\mathbf{P}^8$ for which this will be achieved is coincides with the locus of smooth conic surfaces.

Exercise 4.7.1. [11] Determine general algebraic 3-folds in $\mathbf{P}^4$ of which degree contain finitely many lines, and compute how many.

4.8. **More bang for our buck.** In order to answer the leading question of this section, we only needed have computed $c_4(\mathscr{E})$. But since we put in the work to determine all the lower Chern classes as well, let us reap some fun conseqneces.

Question 14. *Let $L \subseteq \mathbf{P}^3$ be a line and $\{S_t \subseteq \mathbf{P}^3\}_{t\in\mathbf{P}^1}$ be a general pencil of cubic surfaces. How many of the lines inside the surfaces S_t interect the line L?*

To answer this, we interpret the third Chern class of the bundle of cubic forms on lines $\mathscr{E}$. We choose two general cubic polynomials $F, G \in \Gamma(\mathbf{P}^3; \mathscr{O}(3))$, which give rise to global sections s_F, s_G of $\mathscr{E}$. Then $c_3(\mathscr{E})$ is the locus of all those lines $L \subseteq \mathbf{P}^3$ for which there exist some coefficients t_0, t_1 for which $0 = t_0 s_F(L) + t_1 s_G(L) = s_{t_0F+t_1G}(L)$, which is equivalent to saying that $L \subseteq V(t_0F + t_1G) = S_t$. That is, $c_3(\mathscr{E})$ is the given by the locus of lines in $\mathbf{P}^3$ which are contained in some element of a general pencil of cubic surfaces.

[11] You are going to have a lot of work with this one - along the way, you might need to compute the Chow ring of the Grassmanian $\mathbf{G}(1,4)$. Luckily, the Schubert calculus approach still works, albeit you may need to think a bit harder to figure out precisely what the Schubert cycles should be.

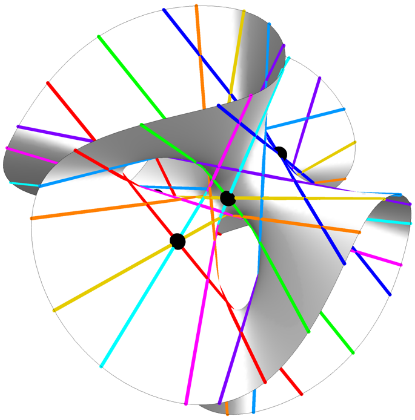

FIGURE 13. The illustration by Greg Egan of the 27 lines on the *Clebsch cubic*, cut out by the equation $(x_0 + x_1 + x_2 + x_3)^3 = x_0^3 + x_1^3 + x_2^3 + x_3 = 0$. Check out the AMS Blogs post by John Baez for more about it, including a gorgeous animation of the Clebsch surface rotating.

It follows from this interpretation of the third Chern class of $\mathscr{E}$ that Question 14 may be rephrased in terms of intersection theory as asking us to compute the product $\sigma_1 c_3(\mathscr{E})$. Thankss to our work from Subection 4.6, we can express the Chern class in question in terms of Schubert cycles as

$$c_3(\mathscr{E}) = 6\sigma_1^3 + 21\sigma_1\sigma_{1,1} = (12 + 21)\sigma_{1,2} = 33\sigma_{1,2},$$

and so we get as the answer to the enumerative problem

$$\deg(\sigma_1 c_3(\mathscr{E})) = 33.$$

Answer to Question 14. *There are* 33 *lines contained in elements of a general pencil of cubic surfaces intersecting a given line* $L \subseteq \mathbf{P}^3$.

Another one:

Question 15. *Let* $p \in \mathbf{P}^3$ *and* $H \subseteq \mathbf{P}^3$ *be a point and a line. How many lines contained in elements of a general fan of cubic surfases* $\{S_t \subseteq \mathbf{P}^3\}_{t\in\mathbf{P}^2}$

(1) pass through the point p*?*
(2) lie on the plane H*?*

Just as above, wwe may interpret the second Chern class of $\mathscr{E}$ as the locus of lines $L \subseteq \mathbf{P}^3$ which are contained in some element of a general fan of cubic surfaces. Then

$$c_2(\mathscr{E}) = 11\sigma_1^2 + 18\sigma_{1,1} = 29\sigma_{1,1} + 11\sigma_2.$$

Recalling the rules $\sigma_{1,1}^2 = \sigma_2^2 = \sigma_{2,2}$, $\sigma_{1,1}\sigma_2 = 0$ of Schubert calculus, the answers to the two enumerative questions are now computed as

$$\deg(\sigma_2 c_2(E)) = 11, \qquad \deg(\sigma_{1,1} c_2(E)) = 29.$$

Answer to Question 15. *Of the lines on the elements of a general fan of cubic surfaces,* 11 *of them pass through any fixed point* $p \in \mathbf{P}^3$*, and* 29 *of them lie on any given plane* $H \subseteq \mathbf{P}^3$.

Exercise 4.8.1. Give an interpretation of $c_1(\mathscr{E})$ in terms of local systems of cubic surfaces, and find the relevant finite number the way we did for c_2 and c_3 in this subsection.

Exercise 4.8.2. How many elements in a general pencil of quartic surfaces contain a line?

5. Afterthoughts: the Adjunction Formula

This Section is an appendix of sorts, having previous little to do with intersection theory. We will spend most of it discussing the Adjunction Formula, and will derive some enumerative consequences from it, tying back to the previously discussed matrial. Though we will try to keep up the minimal-prerequisite & thorough-explanation style of the rest of the notes, this Section will inadvertently require more machinery. Thus we hope the reader is familiar with Kähler differentials (the algebro-geometric theory of differential forms), and some light homological algebra of quasi-coherent sheaves. But if these things are not your mathematical friends yet, then don't fret - firstly, they are much less scary than they seem, and secondly, you will miss little or nothing by skipping this Section altogether.

5.1. **The canonical divisor.** The *canonical divisor* of a smooth variety X may be most succinctly described as $K_X := c_1(\Omega^1_X) \in A^1(X)$, which is to say, the first Chern class of the bundle[12] of Kähler differentials Ω^1_X.

If X is n-dimensional, then this is equivalent to $c_1(\Omega^n_X)$, the first Chern class of the line bundle $\Omega^n_X := \bigwedge^n \Omega^1_X$ of (algebraic) differential n-forms on X. That is to say, if $\omega \in \Omega^n(X)$ is a generic non-zero differential volume form, then the canonical divisor may be viewed as the class of the vanishing locus $K_X = [V(\omega)]$.

Example 5.1.1. Let $X = \mathbf{P}^n$ be the projective space. Any differential 1-form ω on X (i.e. local section of Ω^1_X) may be written in terms of homogeneous coordinates as $\omega = \sum f_i\, dx_i$. In principle, the functions f_i might be taken to be any rational functions in the variables $x_0, \dots, x_n$, but since ω has to be invariant under re-scaleing, and $d(\lambda x)_i = \lambda\, dx_i$, it follows that f_i must be homogeneous of degree -1. Thus sending $\omega \mapsto (f_0, \dots, f_n)$ defines a map of vector bundles $\Omega^1_{\mathbf{P}^n} \to \mathscr{O}(-1)^{\oplus(n+1)}$ on $\mathbf{P}^n$, which turns out to be injective, and fit into the so-called *Euler short exact sequence*

$$0 \to \Omega^1_{\mathbf{P}^n} \to \mathscr{O}(-1)^{\oplus(n+1)} \to \mathscr{O}_{\mathbf{P}^n} \to 0.$$

Applying the first Chern class to this sequence, we find that

$$-(n+1)\zeta = c_1(\mathscr{O}(-1)^{\oplus(n+1)}) = c_1(\Omega^1_{\mathbf{P}^n}) + c_1(\mathscr{O}_{\mathbf{P}^n}) = K_{\mathbf{P}^n} + 0.$$

Thus we have expressed the canonical class of projective space in terms of the hyperplane class $\zeta = c_1(\mathscr{O}(1)) \in A^1(\mathbf{P}^n)$ to be $K_{\mathbf{P}^n} = -(n+1)\zeta$. In particular for $n = 3$, we get $K_{\mathbf{P}^3} = -4\zeta$.

Example 5.1.2. When X is a proper smooth algebraic curve, its genus may be defined as the dimension $g := \dim \Omega^1(X)$. With that, an easy application of the Riemann-Roch Theorem shows that $\deg(K_X) = 2g - 2$.

5.2. **The adjunction formula.** The adjunction formula is a classical and frequently useful recipe for expressing the canonical class of an effective divisor (e.g. a hypersurface, if the ambient variety is a projective space).

Let $Y \subseteq X$ be a subvariety, and let $\mathscr{I} \subseteq \mathscr{O}_X$ be the sheaf of ideals cutting out Y inside X. This allows us to express the kernel of the canonical map $\Omega^1_X|_Y \to \Omega^1_Y$, given by restricting a 1-form on X to Y, and find the short exact sequence

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega^1_X|_Y \to \Omega^1_Y \to 0$$

of quasi-coherent sheaves[13] on Y.

Now let us specialize to the context where $Y = D$ is an effective divisor. In that case, the ideal sheaf $\mathscr{I}$ may be identified with the sheaf $\mathscr{L}(-D)$ associated to the divisor $-D$

[12] We restricted our attention only to smooth varieties here, so that the sheaf of Kähler differentials is a vector bundle (an equivalent condition to smoothness of X). However, had we the full theory of Chern classes of coherent sheaves at our disposal, this restriction would not be necessary.

[13] The quotient sheaf $\mathscr{I}/\mathscr{I}^2$ is here viewed as a quasi-coherent $\mathscr{O}_Y$-module through the usual identification $\mathscr{O}_Y = \mathscr{O}_X/\mathscr{I}$.

- indeed, the way that the proceedure of associating sheaves to divisors works, $\mathscr{L}(-D)$ is precisely the sheaf of all those functions on X which vanish at D. In particular, the ideal sheaf $\mathscr{I}$ is in that case an invertible sheaf, i.e. a line bundle[14] which allows for the following manipulation

$$\begin{aligned}\mathscr{I}/\mathscr{I}^2 &= \mathscr{I}\otimes_{\mathscr{O}_X}\mathscr{O}_X/\mathscr{I}\\ &= \mathscr{I}\otimes_{\mathscr{O}_X}\mathscr{O}_D\\ &= \mathscr{L}(-D)\otimes_{\mathscr{O}_X}\mathscr{O}_D\\ &= \mathscr{L}(-D)|_D.\end{aligned}$$

In conclusion, the short exact sequence from before takes on the form

$$0\to\mathscr{L}(-D)|_D\to\Omega^1_X|_D\to\Omega^1_D\to 0.$$

We apply the first Chern class and get

$$K_X|_D = c_1(\Omega^1_X) = c_1(\mathscr{L}(-D)|_D) + c_1(\Omega^1_D) = -D^2 + K_D.$$

Using that $K_X|_D = K_X.D$, we find the usual form of the *adjunction formula*

$$K_D = K_X.D + D^2 = (K_X + D).D.$$

5.3. **Applications of the adjunction formula.** The adjunction formula is extremely versitile, as we hope the following examples adequately portray:

Example 5.3.1. Let $X\subseteq\mathbf{P}^n$ be a degree d hypersurface. The adjunction formula tells us that

$$K_X = (K_{\mathbf{P}^n} + [X]).[X].$$

By the definition of degree, the fundamental class of X is expressable as $[X] = d\zeta$ in terms of the hyperplane class on $\mathbf{P}^n$, and we have computed above that $K_{\mathbf{P}^n} = -(n+1)\zeta$. Thus the canonical class of X is given by

$$K_X = d(d-n-1)\zeta^2.$$

Example 5.3.2. Specializing the previous Example further, let $C\subseteq\mathbf{P}^2$ be a smooth algebraic plane curve of degree d. By the previous Example, its fundamental class is $K_C = d(d-3)\zeta^2$, but since we are working in $\mathbf{P}^2$, the square ζ^2 is the point class. It follows that

$$2g-2 = \deg(K_C) = d(d-3),$$

where g is the genus of C, and if we express it, we get the relation

$$g = \frac{d(d-3)-2}{2} = \frac{(d-1)(d-2)}{2} = \binom{d-1}{2},$$

known as the *genus-degree formula*. This hallmark of the theory of algebraic plain curves shows that the genus of a plane curve is always entirely determined by its degree. Note that this is false for space curves: the twisted cubic in $\mathbf{P}^3$ has degree 3, while a line has degree 1, but both are rational, and as such of genus 0.

Example 5.3.3. Let $S\subseteq\mathbf{P}^3$ be an algebraic surface of degree d and $L\subseteq S$ a line inside S. Then

$C\subseteq S$ a smooth degree e genus g curve. That is to say, we have $[S] = d\zeta$ and $[C] = e\zeta^2$. We saw above that $K_S = d(d-4)\zeta^2$,

Using the adjunction formula for the inclusion $C\subseteq S$, we get the equality

$$K_L = (K_S + [L]).[L]$$

in the Chow ring $A^*(S)$. Since ζ^2 is the line class in $A^*(\mathbf{P}^3)$, we have $[L] = \zeta^2|_S$. Above we saw that $K_S = (d-4)\zeta|_S$, and since $\zeta^2|_S.\zeta|_S = \zeta^3|_S$ is the point class on S due to

[14] In fact, that is precisely one possible characterization of effective Cartier divisors - closed subschemes the ideal sheaves of which are invertible.

restriction (and pullbacks in general) being ring homomorphisms between Chow rings, the adjunction formula takes on the form

$$K_L = (d-4)[\mathrm{pt}] + [L]^2.$$

When we pass to degrees, we get (since the genus of a line is always 0)

$$-2 = d - 4 + \deg(L^2),$$

expressing the self-intersection number of a line L inside the surface S as $2-d$.

Testing the formula $\deg(L^2) = 2-d$ out for low degrees:

- For $d=1$, we find that $\deg(L^2) = 2-1 = 1$. Indeed, any line in the plane has a self-intersection number 1, since any perturbation of the line will make it intersect the original line in one and precisely one point by Bezout's theorem.
- For $d=2$, we find that $\deg(L^2) = 2-2 = 0$. This makes sense the quadric surface S admits a ruling (two in fact), and a perturbation of a line will simply move it along the ruling, making it disjoint with the original one. Hence the self-intersection number of any line inside a quadric should indeed be zero.
- For $d=3$, we find that $\deg(L^2) = 2-3 = -1$.

5.4. **What this means the 27 lines in our friend, the smooth cubic surface.** According to the Adjunction Formula computation of the last Subsection, each of the 27 lines inside a smooth cubic surface S intersects itself -1 times. That is not as farfetch'd as it might seem: there are only finitely many lines inside S, and so we can hardly expect to be able to deform them very much.

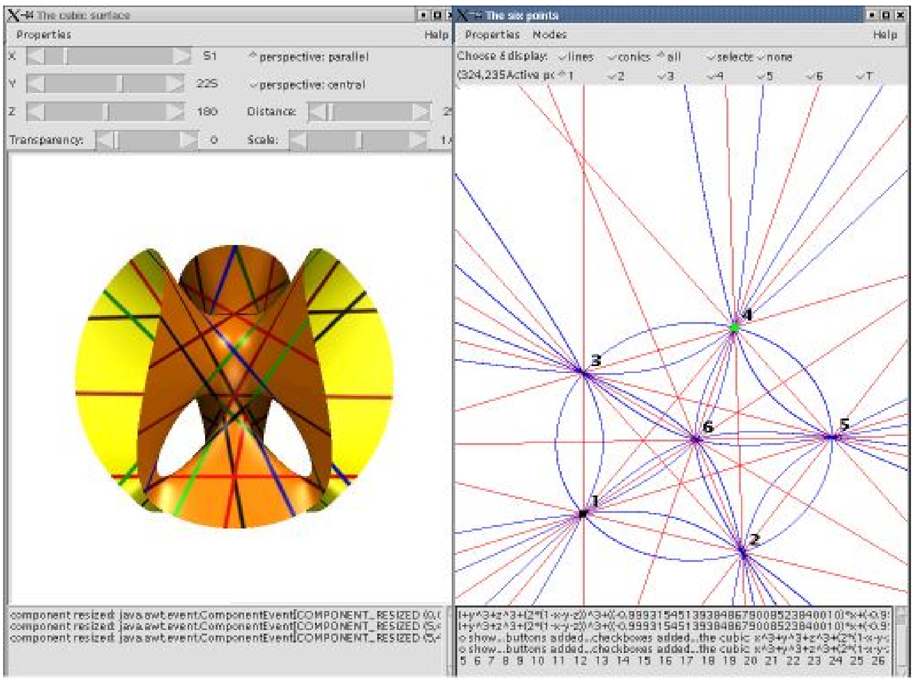

FIGURE 14. The 27 lines in terms of the blowup of $\mathbf{P}^2$ at six points, screenshot from a computer program of Claus Michael Ringel, from his website.

To bring these notes full-circle, let us tie this back to the first (non-projective space) application of intersection theory that we looked in Subsection 2.1 Indeed, we met there another smooth algebraic surface containing a line of self-intersection -1: the blowup at a point. It turns out that any smooth cubic surface inside $\mathbf{P}^3$ may be obtained by blowing up the plane $\mathbf{P}^2$ at six distinct points. This creates five exceptional divisors in the blow-up, accounting for 6 of the 27 lines of self-intersection -1. Where are the rest hiding?

Well, the proper transform of any line connecting two of the six base points in $\mathbf{P}^2$ at which we are blowing up will give rise to a line inside the blowup. Perhaps slightly less obviously, the proper transform any conic passing through five of the six base points (and remember, a plane conic is determined uniquely by any five points it passes through) is also a line in the blowup. Fittingly, we thus conclude these notes on enumerative geometry with the following computation:

$$6 + \binom{6}{2} + \binom{6}{5} = 6 + 15 + 6 = 27.$$