

Rotational perturbations of high density matter in the brane cosmology

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Abstract. We consider the evolution of small rotational perturbations, with azimuthal symmetry, of the brane-world cosmological models. The equations describing the temporal, radial and angular dependence of the perturbations are derived by taking into account the effects of both scalar and tensor parts of the dark energy term on the brane. The time decay of the initial rotation is investigated for several types of equation of state of the ultra-high density cosmological matter. For an expanding Universe, rotation always decays in the case of the perfect dragging, for which the angular velocity of the matter on the brane equals the rotation metric tensor. For non-perfect dragging, the behaviour of the rotation is strongly equation of state dependent. For some classes of dense matter, like the stiff causal or the Chaplygin gas, the angular velocity of the matter on the brane is an increasing function of time. For other types of the ultra-dense matter, such as the Hagedorn fluid, rotation is smoothed out by the expansion of the Universe. Therefore the study of dynamics of rotational perturbations of brane-world models, as well as in general relativity, could provide some insights on the physical properties and equation of state of the cosmological fluid filling the very early Universe.

Keywords: Cosmological perturbation theory, cosmology with extra dimensions, extra dimensions

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1. Introduction

Most of the astronomical objects in the Universe (planets, stars or galaxies) have some form of rotation (differential or uniform). Hence the possibility that the Universe itself could be rotating has attracted a lot of attention. But even though observational evidence of cosmological rotation has been reported [1]–[4], it is still a controversial subject.

From the analysis of microwave background anisotropy Collins and Hawking [5], and Barrow *et al* [6] have found some very tight limits of the cosmological vorticity, $T_{obs} > 3 \times 10^5 T_H$, where T_{obs} is the actual rotation period of our Universe and $T_H = (1 \sim 2) \times 10^{10}$ years is the Hubble time, corresponding to an angular velocity of the order of 10^{-13} rad s⁻¹. Therefore our present day Universe is rotating very slowly, if at all. However, the existence of such a small rotation, when extrapolated to the early stages of the Universe, could have played a major role in the dynamics of the early Universe, and possibly also in the processes involving galaxy formation.

From a theoretical point of view, Gödel [7] gave in 1949 his famous example of a rotating cosmological solution to the Einstein gravitational field equations. Gödel also discussed the possibility of a cosmic explanation of the galactic rotation [7]. This rotating solution has attracted considerable interest because the corresponding Universes possess the property of closed time-like curves.

The investigation of rotating and rotating–expanding Universes generated a large amount of literature in the field of general relativity, the combination of rotation with expansion in realistic cosmological models being one of the most difficult tasks in cosmology (see [8] for a review of the expansion–rotation problem in general relativity). Hence rotating solutions of the gravitational field equations cannot be excluded *a priori*. But this raises the question of why the Universe rotates so slowly. This problem can also be naturally solved in the framework of the inflationary model. Ellis and Olive [9], and

Grøn and Soleng [10] pointed out that if the Universe came into being as a mini-Universe of Planck dimensions and went directly into an inflationary epoch driven by a scalar field with a flat potential, due to the non-rotation of the false vacuum and the exponential expansion during inflation the cosmic vorticity has decayed by a factor of about 10^{-145} .

The rotational perturbations of a spatially homogeneous and isotropic Universe in terms of a small variation of curvature have been investigated by Hawking [11]. He found that for pressureless dust and radiation the perturbations die away. Later Miyazaki [12] studied the perturbations by a spherical shell in a closed homogeneous Universe in the framework of the Brans–Dicke theory. The possibility of incorporating a slowly rotating Universe into the framework of Friedmann–Robertson–Walker (FRW) type metrics has been considered by Bayin and Cooperstock [13], who obtained the restrictions imposed by the field equations on the matter angular velocity. They also shown that uniform rotation is incompatible with the dust-filled (zero pressure) and with the radiation-dominated Universe. Moreover, Bayin [14] has obtained the solutions of the field equations for a special class of non-separable rotation functions of the matter distribution. The investigation of the first-order rotational perturbations of flat FRW-type Universes proved to be useful in the study of string cosmological models with dilaton and axion fields [15]. The form of the rotation equation imposes strong constraints on the form of the dilaton field potential U , restricting the allowed forms to two: the trivial case $U = 0$ and the exponential-type potential.

Recently, Randall and Sundrum [16, 17] have pointed out that a scenario with an infinite fifth dimension in the presence of a brane can generate a theory of gravity, which mimics purely four-dimensional gravity, both with respect to the classical gravitational potential and with respect to gravitational radiation. The gravitational self-couplings are not significantly modified in this model. This result has been obtained from the study of a single 3-brane embedded in five dimensions, with the 5D metric given by $ds^2 = e^{-f(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2$, which can produce a large hierarchy between the scale of particle physics and gravity, due to the appearance of the warp factor. Even if the fifth dimension is uncompactified, standard 4D gravity is reproduced on the brane. This model allows the presence of large or even infinite non-compact extra dimensions. Our brane is identified to a domain wall in a five-dimensional anti-de Sitter space-time.

The Randall–Sundrum (RS) model was inspired by superstring theory. The ten-dimensional $E_8 \times E_8$ heterotic string theory, which contains the standard model of elementary particle, could be a promising candidate for the description of the real Universe. This theory is connected with an eleven-dimensional theory compactified on the orbifold $R^{10} \times S^1/Z_2$ [18]. The static RS solution has been extended to time-dependent solutions and their cosmological properties have been extensively studied [19]–[26] (for a review of dynamics and geometry of brane universes see [27]).

The effective gravitational field equations on the brane world, in which all the matter forces except gravity are confined on the 3-brane, in a five-dimensional space-time with Z_2 -symmetry have been obtained, by using an elegant geometric approach, by Shiromizu, Maeda and Sasaki [28, 29]. The correct signature for gravity is provided by the brane with positive tension. If the bulk space-time is exactly anti-de Sitter, generically the matter on the brane is required to be spatially homogeneous. The electric part of the five-dimensional Weyl tensor E_{IJ} gives the leading-order corrections to the conventional Einstein equations on the brane. The effect of the dilaton field in the bulk can also be

taken into account in this approach [30].

In the brane-world model, the behaviour of an anisotropic Bianchi type I cosmology in the presence of inflationary scalar fields has been considered by Maartens *et al* [31]. By using dynamical systems techniques, the behavior of the FRW, Bianchi type I and V cosmological models in the RS brane world scenario, with matter on the brane obeying a barotropic equation of state, has been studied by Campos and Sopuerta [32, 33]. The general exact solution of the field equations for an anisotropic brane with Bianchi type I and V geometry, with perfect fluid and scalar fields as matter sources, has been found in [34]. In spatially homogeneous brane world cosmological models the initial singularity is isotropic, and hence the initial conditions problem is solved [35]. Consequently, these models do not exhibit Mixmaster or chaotic-like behaviour close to the initial singularity [36].

Realistic brane-world cosmological models require the consideration of more general matter sources to describe the evolution and dynamics of the very early Universe. The effects of the bulk viscosity of the matter on the brane have been analysed in [37]. Limits on the initial anisotropy induced by the five-dimensional Kaluza–Klein graviton stresses by using the CMB anisotropies have been obtained by Barrow and Maartens [38]. Anisotropic Bianchi type I brane-worlds with a pure magnetic field and a perfect fluid have also been analysed [39]. The effect of the bulk viscosity of the cosmological matter on the cosmological evolution on the brane for a Bianchi type I brane geometry was considered in [40].

The simplest way to investigate if brane world cosmologies are consistent with the observations is to investigate the behaviour of the perturbations in the model. Perturbations on the brane are associated with perturbations in the geometry of the bulk space-time. The linearized perturbation equations in the generalized RS model have been obtained, by using the covariant nonlinear dynamical equations for the gravitational and matter fields on the brane, by Maartens [41]. The gauge-invariant formalism for perturbations in the brane world was developed in [42, 43]. The equations governing the bulk perturbations in the case of a general warped Universe were computed by Langlois [44, 45]. A gauge-invariant formalism for metric perturbations in five-dimensional brane world theories, which also applies to models originating from heterotic M-theory, was obtained in [46]. Koyama and Soda [47] obtained a formalism for solving the coupled dynamics of the cosmological perturbations in the brane world and of the gravitational waves in the AdS bulk. A closed system of the perturbation equations on the brane, which is valid on a large scale and may be solved without solving for the bulk perturbations, was proposed in [48] and [49]. The perturbations of the brane worlds in conformally Minkowskian coordinates, which enable the disentanglement of the contributions of the bulk gravitons and of the motion of the brane, were considered in [50]. Dorca and van de Bruck [51] proposed a new gauge, in which the full five-dimensional problem of the perturbations is solvable. The second-order perturbations of the gravitational field induced on the 3-brane were analysed by Kudoh and Tanaka [52, 53]. The equations of motion for metric perturbations in the bulk and matter perturbations on the brane were presented, in an arbitrary gauge, in [54]. The evolution of density perturbations in brane world cosmological models with a bulk scalar field was considered by Brax *et al* [55]. The 1 + 3-covariant approach to cosmological perturbations in the brane-world models, and its application to CMB anisotropies, have been reviewed recently by Maartens [56].

In a previous paper [57], we studied first-order rotational perturbations of homogeneous and isotropic FRW brane world cosmological models. Assuming that the rotation is slow, and by keeping only the first-order rotational terms in the field equations, a rotation equation describing the space dependence and time evolution of the metric perturbations is obtained. However, in our previous consideration in [57] the bulk effects related to the tensorial part of the dark energy (coming from the five-dimensional Weyl tensor) have been neglected, and only the role played by the scalar part (via the unperturbed field equations) has been considered. The conservation of the angular momentum, following from the general energy–momentum conservation equation on the brane, has also not been included in the formalism. Moreover, the angular dependence of the angular velocity of the matter on the brane has also been overlooked.

It is the purpose of the present paper to consider the rotational perturbations of slowly rotating brane worlds, with azimuthal symmetry, by taking into account both the tensorial and scalar perturbations in the bulk, produced due to the matter perturbations on the brane. We assume that the background geometry of the unperturbed system is homogeneous and isotropic, of FRW type. Since the quadratic corrections terms are related, via a consistency condition, to the dark energy term, the contributions of the tensorial and scalar parts of the perturbation can be consistently included in the formalism. The conservation of the angular momentum of the matter on the brane gives a basic relation between the angular velocity ω of the matter and the metric rotation function Ω , in terms of the equation of state of the matter on the brane and the scale factor of the expanding Universe. In the case of perfect dragging, when $\omega = \Omega$, the decay of the rotational perturbations is independent of the equation of state of the matter and in the large-time limit rotation always vanishes. However, the behaviour of perturbations is very different for the general case $\omega \neq \Omega$. In this case the analysis of the long-time behaviour of the angular velocity for several proposed physical models of the high-density cosmological fluid, corresponding to different equations of state of the matter (Zeldovich causal model or Chaplygin gas) shows that in these cases the rotational perturbations increase in time. However, for the Hagedorn equation of state of dense matter, the late-time rotational perturbations tend, in the large-time limit, to zero. For other, more conventional equations of state, like the radiation fluid, pressureless dust, quark matter obeying the bag model equation of state and the Boltzmann gas, the rotational perturbations are wiped out by the expansion of the Universe. It is worth noting that *the results for the behaviour of perturbations are also specific to standard general relativity*.

The paper is organized as follows. The field equations for a slowly rotating brane-world are written down, and the basic rotation equation is derived in section 2. The case of perfect dragging is discussed in section 3. In section 4 the behaviour of rotational perturbations is analysed for several relevant equations of state of the cosmological fluid. We conclude our results in section 5.

2. Slowly rotating brane Universes

In 5D space-time the brane-world is located at $Y(X^I) = 0$, where X^I , $I = 0, 1, 2, 3, 4$, are five-dimensional coordinates. The effective action in five dimensions is [30]

$$S = \int d^5 X \sqrt{-g_5} \left(\frac{1}{2k_5^2} R_5 - \Lambda_5 \right) + \int_{Y=0} d^4 x \sqrt{-g} \left(\frac{1}{k_5^2} K^\pm - \lambda + L^{\text{matter}} \right), \quad (1)$$

with $k_5^2 = 8\pi G_5$ the five-dimensional gravitational coupling constant and where x^μ , $\mu = 0, 1, 2, 3$, are the induced four-dimensional brane world coordinates. R_5 is the 5D intrinsic curvature in the bulk and K^\pm is the extrinsic curvature on either side of the brane.

On the five-dimensional space-time (the bulk), with the negative vacuum energy Λ_5 source of the gravitational field the Einstein field equations are given by

$$G_{IJ} = k_5^2 T_{IJ}, \quad T_{IJ} = -\Lambda_5 g_{IJ} + \delta(Y) [-\lambda g_{IJ} + T_{IJ}^{\text{matter}}]. \quad (2)$$

In this space-time a brane is a fixed point of the Z_2 symmetry. In the following, upper-case Latin indices run in the range $0, \dots, 4$ while Greek indices take the values $0, \dots, 3$.

Assuming a metric of the form $ds^2 = (n_I n_J + g_{IJ}) dx^I dx^J$, with $n_I dx^I = d\chi$ the unit normal to the $\chi = \text{constant}$ hypersurfaces and g_{IJ} the induced metric on $\chi = \text{constant}$ hypersurfaces, the effective four-dimensional gravitational equations on the brane take the form [28, 29]:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4^2 T_{\mu\nu} + k_5^4 S_{\mu\nu} - E_{\mu\nu}, \quad (3)$$

where

$$S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_\mu^\alpha T_{\nu\alpha} + \frac{1}{24} g_{\mu\nu} (3T^{\alpha\beta} T_{\alpha\beta} - T^2), \quad (4)$$

and $\Lambda = k_5^2(\Lambda_5 + k_5^2 \lambda^2/6)/2$, $k_4^2 = k_5^4 \lambda/6$ and $E_{IJ} = C_{IAJB} n^A n^B$. C_{IAJB} is the five-dimensional Weyl tensor in the bulk and λ is the vacuum energy on the brane. $T_{\mu\nu}$ is the matter energy-momentum tensor on the brane and $T = T^\mu_\mu$ is the trace of the energy-momentum tensor.

For any matter fields (scalar field, perfect fluids, kinetic gases, dissipative fluids etc) the general form of the brane energy-momentum tensor can be covariantly given as

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p h_{\mu\nu} + \pi_{\mu\nu} + 2q_{(\mu} u_{\nu)}. \quad (5)$$

The decomposition is irreducible for any chosen 4-velocity u^μ . Here ρ and p are the energy density and isotropic pressure, $\pi_{\mu\nu}$ are the anisotropic stresses on the brane, induced for example by the dissipative properties of the fluid (shear viscosity) and q_μ is the heat flux. $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ projects orthogonal to u^μ . The heat flux obeys the condition $q_\mu = q_{\langle\mu\rangle}$, while the anisotropic stress obeys $\pi_{\mu\nu} = \pi_{\langle\mu\nu\rangle}$, where angular brackets denote the projected, symmetric and trace-free part:

$$V_{\langle\mu\rangle} = h_{\mu}{}^\nu V_\nu, \quad W_{\langle\mu\nu\rangle} = \left[h_{(\mu}{}^\alpha h_{\nu)}{}^\beta - \frac{1}{3} h^{\alpha\beta} h_{\mu\nu} \right] W_{\alpha\beta}. \quad (6)$$

The symmetry properties of $E_{\mu\nu}$ imply that generally we can irreducibly decompose it with respect to a chosen 4-velocity field u^μ as

$$E_{\mu\nu} = -\frac{k_5^4}{k_4^4} \left[\frac{1}{3} \mathcal{U} (4u_\mu u_\nu + g_{\mu\nu}) + \mathcal{P}_{\mu\nu} + 2\mathcal{Q}_{(\mu} u_{\nu)} \right], \quad (7)$$

where \mathcal{U} is a scalar, \mathcal{Q}_μ a spatial vector and $\mathcal{P}_{\mu\nu}$ a spatial, symmetric and trace-free tensor. For a FRW model $\mathcal{Q}_\mu = 0$ and $\mathcal{P}_{\mu\nu} = 0$ [33]. Hence the only non-zero contribution from the five-dimensional Weyl tensor from the bulk is given by the scalar term \mathcal{U} . The ‘dark energy’ term, $E_{\mu\nu}$, is a pure bulk effect, therefore we cannot determine its expression without solving the complete system of field equations in five dimensions. However, its expression in the bulk is constrained by the relation

$$D_\mu E^{\mu\nu} = k_5^4 D_\mu S^{\mu\nu}. \quad (8)$$

The Einstein equation in the bulk also implies the conservation of the energy momentum tensor of the matter on the brane,

$$T_{\mu}{}^{\nu}{}_{;\nu} \big|_{\chi=0} = 0. \quad (9)$$

The rotationally perturbed metric can be expressed in terms of the usual coordinates in the form [58]

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] - 2\Omega(t, r, \theta) a^2(t) r^2 \sin^2 \theta dt d\varphi, \quad (10)$$

where $\Omega(t, r, \theta)$ is the metric rotation function. Although Ω plays a role in the ‘dragging’ of local inertial frames, it is not the angular velocity of these frames, except for the special case when it coincides with the angular velocity of the matter fields. $k = 1$ corresponds to closed Universes, with $0 \leq r \leq 1$. $k = -1$ corresponds to open Universes, while the case $k = 0$ describes a flat geometry, where the range of r is $0 \leq r < \infty$. In all models, the time-like variable t ranges from 0 to ∞ .

For the matter energy–momentum tensor on the brane we restrict our analysis to the case of the perfect fluid energy–momentum tensor,

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}. \quad (11)$$

The components of the four-velocity vector are $u^\mu = (1, 0, 0, \omega)$ and $\omega(t, r, \theta) = d\varphi/dt$ is the angular velocity of the matter distribution. Consequently, for the rotating brane the energy–momentum tensor has a supplementary component

$$T_{03} = \{[\Omega(t, r, \theta) - \omega(t, r, \theta)] \rho - \omega(t, r, \theta)p\} r^2 a^2(t) \sin^2 \theta. \quad (12)$$

As a first step to consider rotational perturbations on the brane, we should find a consistent assumption for $E_{\mu\nu}$ ‘induced’ by the small rotation. The reason that the small rotation changes the expression of $E_{\mu\nu}$ can be understood as follows. In the brane-world scenario an Z_2 symmetry is imposed. Once we introduce a small rotation effect on the brane, the bulk geometry should have a ‘self-tuning’ process in order to preserve the Z_2 symmetry. Therefore, some nontrivial components of $E_{\mu\nu}$ should be turned on to the order of perturbation. Hence, the first constraint we need to check for a consistent expression of $E_{\mu\nu}$ is equation (8).

For the perfect fluid source, two nontrivial equations of the energy–momentum conservation (9), corresponding to the t and ϕ components, are

$$\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0, \quad (13)$$

$$\partial_t [(\rho + p)(\Omega - \omega)a^5] = 0. \quad (14)$$

Imposing the above equations, the right-hand side of (8) has only one nonzero component as

$$D_\mu S^{\mu\phi} = -\frac{1}{6}\partial_t [\rho(\rho + p)(\Omega - \omega)a^5] a^{-5}. \quad (15)$$

Hence, the constraint (8) shows that the $E_{\mu\nu}$ should satisfy the following equations:

$$D_\mu E^{\mu t} = D_\mu E^{\mu r} = D_\mu E^{\mu\theta} = 0, \quad D_\mu E^{\mu\phi} = -\frac{k_5^4}{6}\partial_t [\rho(\rho + p)(\Omega - \omega)a^5] a^{-5}. \quad (16)$$

The last equation simply shows what we have already mentioned, that is the rotational perturbation does induce an effect on the bulk for consistency. Otherwise, the only possible perturbation is the so-called perfect dragging case, with $\Omega = \omega$.

A possible solution of the constraint (16) is $E^{\mu\nu} = -(k_5^4/k_4^4)\mathcal{P}^{\mu\nu}$, which, for the perfect fluid energy–momentum tensor gives only one non-vanishing component

$$\mathcal{P}^{t\phi} = \mathcal{P}^{\phi t} = \mathcal{P} \sim O(\epsilon), \quad D_\mu E^{\mu\phi} = -\frac{k_5^4}{k_4^4} \partial_t (\mathcal{P} a^5) a^{-5}. \quad (17)$$

Hence the tensor component \mathcal{P} is of order of perturbation such that the quadratic terms, by itself or together with ω or Ω , can be neglected. Under this assumption, the solution for the equation (16) can be easily obtained

$$\mathcal{P} = \frac{k_4^4}{6} \rho(\rho + p)(\Omega - \omega), \quad (18)$$

which, as expected, is of the first order of perturbation. Straightforwardly the first order of the E -term is

$$E_{t\phi} = E_{\phi t} = \frac{k_5^4}{6} a^2 r^2 \sin^2 \theta \rho(\rho + p)(\Omega - \omega), \quad (19)$$

which is the consistently ‘induced’ effect by the rotational perturbation in the brane-world model.

Generally, the above result can be extended to the case including the scalar part of the E -correction, namely the \mathcal{U} term contribution. For the case, the first-order covariant derivatives are

$$\begin{aligned} D_\mu E^{\mu t} &= -\frac{k_5^4}{k_4^4} \left(\partial_t \mathcal{U} + 4 \frac{\dot{a}}{a} \mathcal{U} \right), & D_\mu E^{\mu r} &= D_\mu E^{\mu \theta} = 0, \\ D_\mu E^{\mu \phi} &= -\frac{k_5^4}{k_4^4} \partial_t \left[-\frac{4}{3} \mathcal{U} (\Omega - \omega) a^5 + \mathcal{P} a^5 \right] a^{-5} + \Omega D_\mu E^{\mu t}. \end{aligned} \quad (20)$$

Then the constraint (16) gives

$$\mathcal{U} = \mathcal{U}_0 a^{-4}, \quad \mathcal{P} = \left[\frac{4}{3} \mathcal{U} + \frac{k_4^4}{6} \rho(\rho + p) \right] (\Omega - \omega), \quad (21)$$

or

$$\begin{aligned} E_{tt} &= -\frac{k_5^4}{k_4^4} \mathcal{U}, & E_{ii} &= -\frac{1}{3} \frac{k_5^4}{k_4^4} \mathcal{U} g_{ii}, \\ E_{t\phi} = E_{\phi t} &= a^2 r^2 \sin^2 \theta \left[\frac{1}{3} \frac{k_5^4}{k_4^4} \mathcal{U} \Omega + \frac{k_5^4}{6} \rho(\rho + p)(\Omega - \omega) \right]. \end{aligned} \quad (22)$$

We assume that rotation is sufficiently slow, so that deviations from spherical symmetry can be neglected. Then to first order in Ω the gravitational and field equations (3) can be decomposed, by neglecting the $\mathcal{P}_{\mu\nu}$ contribution on the background geometry, into the following components:

$$3 \frac{\dot{a}^2}{a^2} + \frac{3k}{a^2} = \Lambda + k_4^2 \rho + \frac{k_5^4}{12} \rho^2 + \frac{k_5^4}{k_4^4} \mathcal{U}, \quad (23)$$

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \Lambda - k_4^2 p - \frac{k_5^4}{12} \rho(\rho + 2p) - \frac{1}{3} \frac{k_5^4}{k_4^4} \mathcal{U}, \quad (24)$$

$$3\frac{\dot{a}}{a}\frac{\partial\Omega}{\partial r} + \frac{\partial^2\Omega}{\partial t\partial r} = 0, \quad 3\frac{\dot{a}}{a}\frac{\partial\Omega}{\partial\theta} + \frac{\partial^2\Omega}{\partial t\partial\theta} = 0, \quad (25)$$

$$\begin{aligned} (1 - kr^2)\frac{\partial^2\Omega}{\partial r^2} + \left(\frac{4}{r} - 5kr\right)\frac{\partial\Omega}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Omega}{\partial\theta^2} + \frac{3\cot\theta}{r^2}\frac{\partial\Omega}{\partial\theta} + 2a^2\Omega\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) \\ = 2a^2\Omega\Lambda + 2k_4^2a^2[(\Omega - \omega)\rho - \omega p] + \frac{k_5^4}{6}a^2(\rho^2\Omega - 2\rho p\omega - 2\rho^2\omega) \\ - 2a^2\left[\frac{1}{3}\frac{k_5^4}{k_4^4}\mathcal{U}\Omega + \frac{k_5^4}{6}\rho(\rho + p)(\Omega - \omega)\right]. \end{aligned} \quad (26)$$

Equations (25) follow from the $(r\phi)$ and $(\theta\phi)$ components of the field equations and (26) is from the $(t\phi)$ -component. Moreover, the last equation can be simplified by applying (24) to be

$$(1 - kr^2)\frac{\partial^2\Omega}{\partial r^2} + \left(\frac{4}{r} - 5kr\right)\frac{\partial\Omega}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Omega}{\partial\theta^2} + \frac{3\cot\theta}{r^2}\frac{\partial\Omega}{\partial\theta} - 2k_4^2a^2(\rho + p)(\Omega - \omega) = 0. \quad (27)$$

In this equation the correction from the bulk effect exactly cancelled. Hence the form of this equation is the same as the version from the standard general relativity.

The field equations (23)–(27) must be solved together with the conservation equations (13), (14). If the equation of state of the cosmological matter, $p = p(\rho)$, is known, then equation (13) gives the time evolution of the energy density, $\rho = \rho(a)$. Once ρ is known, the general solution of the background (unperturbed) gravitational field equations on the brane can be obtained in the form

$$t - t_0 = \int \left[\frac{\Lambda}{3}a^2 + \frac{k_4^2}{3}\rho(a)a^2 + \frac{k_5^4}{36}\rho^2(a)a^2 + \frac{k_5^4}{3k_4^4}\frac{\mathcal{U}_0}{a^2} - k \right]^{-1/2} da, \quad (28)$$

where t_0 is a constant of integration.

The solution of equations (25) is

$$\Omega(t, r, \theta) = A(r, \theta)a^{-3}(t), \quad (29)$$

where $A(r, \theta)$ is an arbitrary integration function. Then the perturbed equations we need to solve are

$$(1 - kr^2)\frac{\partial^2 A}{\partial r^2} + \left(\frac{4}{r} - 5kr\right)\frac{\partial A}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A}{\partial\theta^2} + \frac{3\cot\theta}{r^2}\frac{\partial A}{\partial\theta} - 2k_4^2a^2(\rho + p)(A - \omega a^3) = 0 \quad (30)$$

and

$$\partial_t [(\rho + p)(A - \omega a^3)a^2] = 0. \quad (31)$$

The general solution of equation (31) is

$$A - \omega a^3 = \frac{F(r, \theta)}{(\rho + p)a^2}, \quad (32)$$

where $F(r, \theta)$ is an arbitrary integration function. With the use of equation (29) we obtain the following general relation between the metric perturbation function Ω and the angular velocity of the matter on the brane:

$$\omega(t, r, \theta) = \Omega(t, r, \theta) - \frac{F(r, \theta)}{(\rho + p)a^5} = \frac{A(r, \theta)}{a^3} - \frac{F(r, \theta)}{(\rho + p)a^5}. \quad (33)$$

Finally, with the use of equation (32) and of the unperturbed field equations, equation (30) reduces to

$$(1 - kr^2) \frac{\partial^2 A(r, \theta)}{\partial r^2} + \left(\frac{4}{r} - 5kr \right) \frac{\partial A(r, \theta)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A(r, \theta)}{\partial \theta^2} + \frac{3 \cot \theta}{r^2} \frac{\partial A(r, \theta)}{\partial \theta} - 2k_4^2 F(r, \theta) = 0. \quad (34)$$

3. The case of perfect dragging

As a first case for the study of the rotational perturbations in the brane world we consider the case of perfect dragging, corresponding to the choice $F(r, \theta) = 0$ of the arbitrary integration function. Therefore the angular velocity of the matter equals the metric perturbation function and is given by

$$\omega(t, r, \theta) = A(r, \theta) a^{-3}(t). \quad (35)$$

For an expanding Universe in the large-time limit the rotational perturbations always decay, ω tending to zero for $t \rightarrow \infty$.

In order to find the radial and angular dependence of the angular velocity, we consider two possible functional forms of the function $A(r, \theta)$. As a first case we take $A(r, \theta) = B(r)C(\theta)$. Then equation (34) can be separated into the following two independent equations:

$$\frac{d^2 C}{d\theta^2} + 3 \cot \theta \frac{dC}{d\theta} + nC = 0, \quad (36)$$

$$(1 - kr^2) r^2 \frac{d^2 B}{dr^2} + \left(\frac{4}{r} - 5kr \right) r^2 \frac{dB}{dr} - nB = 0, \quad (37)$$

where n is a separation constant. By introducing a new variable $x = \cos \theta$, equation (36) becomes

$$(1 - x^2) \frac{d^2 C}{dx^2} - 4x \frac{dC}{dx} + nC = 0. \quad (38)$$

Therefore the general solution of equation (36) is given by

$$C(\theta) = C_1 {}_2F_1(a_1, b_1; c_1; \cos^2 \theta) + C_2 \cos \theta {}_2F_1(a_2, b_2; c_2; \cos^2 \theta), \quad (39)$$

where $a_1 = (3 - \sqrt{9 + 4n})/4$, $b_1 = (3 + \sqrt{9 + 4n})/4$, $c_1 = 1/2$, $a_2 = (5 - \sqrt{9 + 4n})/4$, $b_2 = (5 + \sqrt{9 + 4n})/4$, $c_2 = 3/2$ and

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k x^k}{(c)_k k!}$$

is the hypergeometric function [59]. Hereafter in this section the parameters C_1 and C_2 are used to label the constants of integration in the expressions of exact solution.

In order to solve equation (37) we introduce first a new variable $\eta = r^2$. Hence the equation is transformed into

$$4\eta^2(1 - k\eta) \frac{d^2 B}{d\eta^2} + 2\eta(5 - 6k\eta) \frac{dB}{d\eta} - nB = 0, \quad (40)$$

with the general solution given by

$$B(r) = C_1 r^\alpha {}_2F_1(k_1, l_1; m_1; kr^2) + C_2 r^\beta {}_2F_1(k_2, l_2; m_2; kr^2), \quad \text{for } k = \pm 1, \quad (41)$$

where $\alpha = (-3 - \sqrt{9 + 4n})/2$, $\beta = (-3 + \sqrt{9 + 4n})/2$, $k_1 = -3/4 - \sqrt{9 + 4n}/4$, $l_1 = 5/4 - \sqrt{9 + 4n}/4$, $m_1 = 1 - \sqrt{9 + 4n}/2$, $k_2 = -3/4 + \sqrt{9 + 4n}/4$, $l_2 = 5/4 + \sqrt{9 + 4n}/4$, $m_2 = 1 + \sqrt{9 + 4n}/2$. For $k = 0$, the general solution of equation (37) is given by

$$B(r) = C_1 r^\alpha + C_2 r^\beta, \quad \text{for } k = 0. \quad (42)$$

A second class of solutions can be obtained by assuming that the function $A(r, \theta)$ can be represented as $A(r, \theta) = M(r) + N(\theta)$. Then equation (34) yields again two independent equations,

$$\frac{d^2 N}{d\theta^2} + 3 \cot \theta \frac{dN}{d\theta} = -m, \quad (43)$$

$$(1 - kr^2)r^2 \frac{d^2 M}{dr^2} + \left(\frac{4}{r} - 5kr\right)r^2 \frac{dM}{dr} = m, \quad (44)$$

where m is a separation constant. By introducing a new variables $N' = dN/d\theta$, equation (43) is transformed into

$$\frac{dN'}{d\theta} + 3 \cot \theta N' = -m. \quad (45)$$

Taking integration, equation (45) gives

$$N' = \frac{dN}{d\theta} = m \left(\frac{\cos \theta}{\sin^3 \theta} - \frac{1}{3} \cot^3 \theta \right) + \frac{C_1}{\sin^3 \theta}. \quad (46)$$

After further integration, we obtain the general solution of equation (43) in the form

$$N(\theta) = \frac{m}{3} \left(\ln \sin \theta - \frac{1}{\sin^2 \theta} \right) + \frac{C_1}{4} \left(\ln \frac{1 - \cos \theta}{1 + \cos \theta} - \frac{2 \cos \theta}{\sin^2 \theta} \right) + C_2. \quad (47)$$

The general solution of equation (44), corresponding to the three different background geometries is given by

$$M(r) = \begin{cases} -\frac{1}{6r^3} \left[mr + \sqrt{1+r^2}(2r^2-1)(m \sinh^{-1} r + C_1) - 2r^3(m \ln r + C_2) \right], & k = -1, \\ -\frac{1}{6r^3} \left[-mr + \sqrt{1-r^2}(2r^2+1)(m \sin^{-1} r + C_1) - 2r^3(m \ln r + C_2) \right], & k = 1, \\ C_1 + \frac{C_2}{r^3} + \frac{m}{3} \ln r, & k = 0. \end{cases} \quad (48)$$

Hence the general solution of the gravitational field equations for a slowly rotating brane with perfect dragging can be obtained in a closed form.

4. General time dependence of the angular velocity for $\omega \neq \Omega$

Generally, the angular velocity of the matter on the slowly rotating brane consists of two time-dependent terms, $\omega(t, r, \theta) = \omega_1(t, r, \theta) + \omega_2(t, r, \theta)$. The first term, $\omega_1(t, r, \theta) = A(r, \theta)a^{-3}(t)$ always tends to zero for an expanding Universe, with an increasing scale factor, so that in the large-time limit $\omega_1 \rightarrow 0$. However, the time evolution of the second term, $\omega_2(t, r, \theta) = -F(r, \theta)/(\rho + p)a^5$ essentially depends on the equation of state and the dynamic behaviour of the background cosmological fluid and geometry. In the following we shall investigate the time dependence of this term for a number types of equation of state which could be relevant for the description of the ultra-high density matter of which the Universe consisted in its very early stages.

4.1. Zeldovich stiff fluid

One of the most common equations of state, which has extensively been used to study the properties of the early Universe, is the linear barotropic equation of state, with $p = (\gamma - 1)\rho$, with $\gamma = \text{constant} \in [1, 2]$. For this equation of state the conservation equation (13) can be immediately integrated to give $\rho = \rho_0 a^{-3\gamma}$, $\rho_0 = \text{constant} \geq 0$. Therefore for the second term $\omega_2(t, r, \theta)$ of the angular velocity we obtain the following time dependence:

$$\omega_2(t, r, \theta) = -\frac{F(r, \theta)}{\gamma\rho_0} a^{3\gamma-5}. \quad (49)$$

In order to have decaying rotational perturbations the condition $\gamma < 5/3$ must be satisfied. It is satisfied for the case of the pressureless dust, with $\gamma = 1$, and also for the radiation fluid, having $\gamma = 4/3$. But this condition is not satisfied for a very important case of the so-called causal limit of the linear barotropic equation of state, corresponding to $\gamma = 2$, or the Zeldovich stiff fluid equation of state $p = \rho$. For the choice $\gamma = 2$ we obtain $\omega_2(t, r, \theta) = -F(r, \theta)a/2\rho_0$, showing that an initial rotational perturbation in the ultra-high density cosmological fluid do not decay and the angular velocity of the matter is linearly increasing with the scale factor of the expanding Universe.

The Zeldovich equation of state, valid for densities significantly higher than nuclear densities, $\rho > 10\rho_{\text{nuc}}$, with $\rho_{\text{nuc}} = 10^{14} \text{ g cm}^{-3}$ can be obtained by constructing a relativistic Lagrangian that allows bare nucleons to interact attractively via scalar meson exchange and repulsively via the exchange of a more massive vector meson [60]. In the non-relativistic limit both the quantum and classical theories yield Yukawa-type potentials. At the highest densities the vector meson exchange dominates and by using a mean field approximation one can show that in the extreme limit of infinite densities the pressure tends to the energy density, $p \rightarrow \rho$. In this limit the sound speed $c_s = \sqrt{dp/d\rho} \rightarrow 1$, and hence this equation of state satisfies the causality condition, with the speed of sound less than the speed of light [60].

The Zeldovich equation of state can also describe a non-interacting scalar field, with zero potential, with $\rho_\phi = p_\phi = \dot{\phi}^2/2$. Hence the rotational perturbations in a potential-free field do not decay. For a self-interacting field with potential $U(\phi)$, the energy density and pressure of the field are $\rho_\phi = \dot{\phi}^2/2 + U(\phi)$ and $p_\phi = \dot{\phi}^2/2 - U(\phi)$, respectively. The conservation equation (13) becomes in this case

$$\ddot{\phi} + 3\dot{\phi}\frac{\dot{a}}{a} + U'(\phi) = 0, \quad (50)$$

and has the first integral

$$\dot{\phi} = \frac{K - \int a^3 U'(\phi) dt}{a^3}, \quad (51)$$

with K a constant of integration. Therefore we obtain for ω_2

$$\omega_2 = -F(r, \theta) \frac{a}{K^2 - 2K \int a^3 U'(\phi) dt + [\int a^3 U'(\phi) dt]^2}. \quad (52)$$

Hence if the potential $U(\phi)$ is so that the quantity $a\{K^2 - 2K \int a^3 U'(\phi) dt + [\int a^3 U'(\phi) dt]^2\}^{-1}$ is a decreasing function of time, the initial rotational perturbation is rapidly decaying.

4.2. Hagedorn fluid

An alternative approach to the equation of state at ultra-high densities is based on the assumption that a whole host of baryonic resonant states arise at high density [60]. In the Hagedorn model the baryon resonance mass spectrum is given by $N(m)dm \sim m^a \exp(m/m_0)dm$, where $N(m)dm$ is the number of resonances between mass m and $m + dm$. The existing data on baryon resonances show that $m_0 = 160$ MeV and $-7/2 \leq a \leq -5/2$ [61]. For asymptotically large densities the particle number $n = \int_0^{\mu_n} N(m)dm \sim m_0 \mu_n^a \exp(\mu_n/m_0)$, where μ_n is the chemical potential of the nuclear matter. The density of the matter is $\rho \sim n \mu_n \sim m_0 \mu_n^{a+1} \exp(\mu_n/m_0)$. The pressure can be obtained from $p = n^2 d(\rho/n)/dn$ and is given by the Hagedorn equation of state,

$$p = \frac{\rho}{\ln(\rho/\rho_0)}, \quad (53)$$

where $\rho_0 = 2.5 \times 10^{12}$ g cm⁻³ [60].

The velocity of sound in this type of matter is $c_s = [\ln(\rho/\rho_0)]^{-1/2} [1 - (\ln(\rho/\rho_0))^{-1}]^{1/2}$ [60, 61]. For the Hagedorn equation of state the speed of sound has the property $c_s \rightarrow 0$ for $\rho/\rho_0 \rightarrow \infty$, in striking contrast with the mean field theory approach in which $c_s \rightarrow 1$. The gravitational collapse of a high-density null charged matter fluid, satisfying the Hagedorn equation of state in the framework of the Vaidya geometry was considered in [62]. A collapsing Hagedorn fluid could end either as a black hole, or as a naked singularity. The collapse of Hagedorn fluid to a naked singularity is also a possible source of gamma-ray bursts [62].

In order to find the time behaviour of the rotational perturbations in a Hagedorn fluid, with the equation of state given by (53), we have to obtain first the scale factor dependence of the density. For the Hagedorn equation of state the conservation equation of the matter on the brane (13) takes the form

$$\dot{\rho} + 3\rho \left[1 + \left(\ln \frac{\rho}{\rho_0} \right)^{-1} \right] \frac{\dot{a}}{a} = 0, \quad (54)$$

which can be integrated to give

$$\frac{\rho}{\rho_0 (\ln(\rho/\rho_0) + 1)} = \frac{C}{a^3}, \quad (55)$$

with $C \geq 0$ a constant of integration. For an expanding Universe with an increasing scale factor the energy density of the Hagedorn cosmological fluid is decreasing in time. The time variation of the second component of the angular velocity of the matter can be written as a function of the time-dependent-only density as

$$\omega_2(t, r, \theta) = -\frac{F(r, \theta) (\rho/\rho_0)^{2/3} \ln(\rho/\rho_0)}{\rho_0 C^{5/3} (\ln(\rho/\rho_0) + 1)^{8/3}}. \quad (56)$$

The variation of the time-dependent part

$$f(\rho/\rho_0) = (\rho/\rho_0)^{2/3} \ln(\rho/\rho_0) / \left[(\ln(\rho/\rho_0) + 1)^{8/3} \right]$$

of ω_2 is represented in figure 1.

As one can see from the figure, once the Universe is expanding and its density decreases, the angular velocity of the matter is also decreasing, and in the large time limit

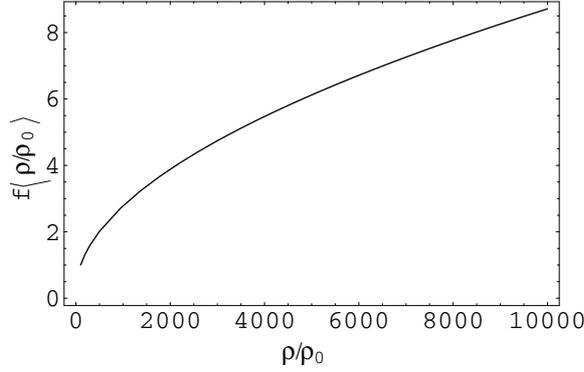


Figure 1. Variation, as a function of ρ/ρ_0 , of the time dependent part $f(\rho/\rho_0)$ of ω_2 .

it tends to zero. Therefore the cosmological behaviour of the rotational perturbations of the Hagedorn fluid is very different to that of the causal stiff (Zeldovich) cosmological matter, with the matter angular velocity increasing due to the cosmological expansion.

4.3. Chaplygin gas

A form of matter which has recently been invoked to account for the recent supernovae evaluations of the cosmological constant is the so-called Chaplygin gas [63]. The Chaplygin gas is a form of what is called ‘k-essence’, and satisfies an equation of state of the form $p = -1/\rho$ [64]. This equation of state comes from the Born–Infeld-type Lagrangian, $\int (1 - \sqrt{1 - \eta^{\mu\nu} \partial_\mu A \partial_\nu A}) d^n x$, with A the transverse coordinate of domain wall or a $(n-1)$ -brane. The energy–momentum tensor satisfies the Hooke law, and, by taking into account a cosmological constant λ_0 , it follows that the pressure and the energy density satisfy the equation of state $(\rho + \lambda_0)(p - \lambda_0) = -1$ [64]. In the following we shall take $\lambda_0 = 0$. Hence the energy conservation equation for a Chaplygin gas on the brane becomes

$$\dot{\rho} + 3 \left(\rho - \frac{1}{\rho} \right) \frac{\dot{a}}{a} = 0, \quad (57)$$

with the general solution given by

$$\rho = \sqrt{\frac{\rho_0}{a^6} + 1}, \quad (58)$$

with $\rho_0 \geq 0$ a constant of integration.

Therefore the second component of the angular velocity varies as

$$\omega_2(t, r, \theta) = -\frac{F(r, \theta)}{\rho_0} \frac{\sqrt{a^6 + \rho_0}}{a^2}. \quad (59)$$

In the limit of large times $a^6 \gg \rho_0$ and it follows that $\omega_2(t, r, \theta) \rightarrow -F(r, \theta)a/\rho_0$, that is, similar to the Zeldovich fluid, the angular velocity of the expanding Universe is an increasing function of time.

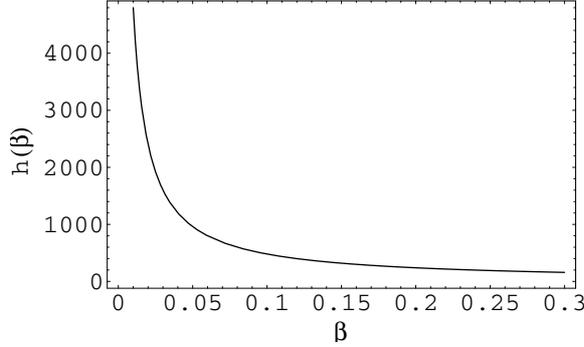


Figure 2. Variation, as a function of β , of the time-dependent part $h(\beta)$ of ω_2 .

4.4. Classical Boltzmann gas

Another important example of a rotationally perturbed cosmological fluid we consider next is the classical Boltzmann-gas-filled Universe. If the cosmological fluid is a collision-dominated classical gas in equilibrium, then the thermodynamic parameters of the gas are given by [65, 66]

$$p = nm\beta^{-1}, \quad nm = A_0 K_2(\beta)/\beta, \quad \rho = A_0 [\beta^{-1} K_1(\beta) + 3\beta^{-2} K_2(\beta)], \quad (60)$$

where n is the particle number density, $\beta = mc^2/k_B T$, with k_B the Boltzmann constant, T the temperature and $A_0 = m^4 g/2\pi^2 \hbar^3$, with g the spin weight of the fluid particles. K_n are the modified Bessel functions of the second kind.

With the use of equations (60) the energy conservation equation of the cosmological fluid can be written as

$$\dot{\beta} - g(\beta) \frac{\dot{a}}{a} = 0, \quad (61)$$

where

$$g(\beta) = \frac{6\beta[\beta K_1(\beta) + 4K_2(\beta)]}{\beta^2 K_0(\beta) + 5\beta K_1(\beta) + (12 + \beta^2)K_2(\beta) + 3\beta K_3(\beta)}. \quad (62)$$

In order to obtain equation (62) we have used the recurrence relation $K'_n(x) = -[K_{n-1}(x) + K_{n+1}(x)]/2$ [59]. Therefore the scale factor can be expressed as a function of the temperature as

$$a(\beta) = a_0 \exp\left(\int \frac{d\beta}{g(\beta)}\right), \quad (63)$$

with $a_0 \geq 0$ a constant of integration.

The second component of the angular velocity of the matter on the brane is given by

$$\omega_2(\beta, r, \theta) = -\frac{F(r, \theta) \beta^2 \exp\left(-5 \int g^{-1}(\beta) d\beta\right)}{A_0 a_0^5 [\beta K_1(\beta) + 4K_2(\beta)]} = -\frac{F(r, \theta)}{A_0 a_0^5} h(\beta), \quad (64)$$

where we denoted $h(\beta) = \beta^2 \exp\left[-5 \int g^{-1}(\beta) d\beta\right] / [\beta K_1(\beta) + 4K_2(\beta)]$. The variation of $h(\beta)$ is presented in figure 2.

The angular velocity of the Boltzmann gas decreases with increasing β , so that for small temperatures (large β) $\omega_2 \rightarrow 0$.

4.5. Quark matter

The rotational perturbations always decay in the important case of quark matter, satisfying the bag model equation of state $p = (\rho - 4B)/3$ [67], where $B = 10^{14}$ g cm⁻³ is the bag constant [68]. Quark matter is assumed to have played an important role in the early evolution of the Universe, and a first-order phase transition from quark to hadronic matter took place when the temperature was around 100 MeV. It is also possible that inhomogeneities in the baryon number density were produced during this transition, and perhaps they persisted even to the time of nucleosynthesis, which could alter the abundance of light elements [67]. For the bag model equation of state of the quark matter the density-scale factor dependence is given by

$$\rho = B + \frac{\rho_0}{a^4}, \quad (65)$$

and consequently ω_2 becomes

$$\omega_2(t, r, \theta) = -\frac{3F(r, \theta)}{4\rho_0} \frac{1}{a}. \quad (66)$$

Therefore in the large-time limit, for $a \rightarrow \infty$, the rotational perturbations in the cosmological quark matter tend to zero.

In the general case, due to the dependence of the rotation equation on the arbitrary function $F(r, \theta)$, the solution of the rotation equation cannot be obtained. The particular choice $F(r, \theta) = A(r, \theta)$ leads again to a radial and angular distribution of the matter which is very similar to the perfect dragging case. Moreover, if A is represented as a product of two r and θ only dependent functions, $A(r, \theta) = X(r)Y(\theta)$, then the angular velocity ω is also a separable function of all variables. But if A is the sum of two independent functions, $A(r, \theta) = X(r) + Y(\theta)$, the angular velocity is a non-separable function. In these cases the functions $X(r)$ and $Y(\theta)$ can be obtained in terms of the hypergeometric function.

5. Discussion and final remarks

In this paper we have considered the evolution of small rotational perturbations in the brane-world cosmological model. The evolution of the angular velocity of matter is related to the metric perturbation function via an equation derived from the energy-momentum conservation on the brane. This equation is in fact the same in both standard general relativity and brane-world cosmology. Once the evolution of the background non-perturbed geometry is known, the time dynamics of the metric rotation function $\Omega(t, r, \theta)$ and of the angular velocity $\omega(t, r, \theta)$ of the matter on the brane is uniquely determined by the unperturbed field equations and, from a physical point of view, by the equation of state of the ultra-high density cosmological fluid. The time behaviours of $\omega(t, r, \theta)$ and $\Omega(t, r, \theta)$ are independent of their spatial distribution. For $\omega = \Omega$, in the large-time limit the angular velocity of the matter tends to zero, showing that the initial small rotational perturbations of the brane world are smoothed out, due to the expansion of the Universe. This result is also independent of the geometry (flat, open or closed) of the space-time. However, despite the fact that the presence of the dark matter term $\mathcal{U} = \mathcal{U}_0/a^4$ is not essential for the rapid decay of the rotational perturbations of the matter on the brane, the effects induced by the five-dimensional bulk still contribute to the decay (or amplification)

of the first-order rotational perturbations via their effect on the unperturbed background geometry.

A very different situation occurs for $\omega \neq \Omega$. In this case the dependence of the angular velocity on the equation of state leads to two distinct classes of behaviours. For some equations of state (radiation fluid or quark matter), the rotational perturbations decay and this behaviour is consistent with the observational constraint that our present day Universe is rotating very slowly. However, some equations of state which try to model the thermodynamical properties of the super-dense matter at the very early stages of the evolution of our Universe, like the Zeldovich or Chaplygin gas equations of state, do not lead to an observationally consistent description of the early Universe. In particular, the causal Zeldovich equation of state with $\gamma = 2$, which has been used, for example, to determine the maximum mass of the neutron stars [60], does not lead, once applied to a cosmological framework, to a description of the rotational perturbations consistent with the observations, that is, it does not satisfy the basic observational requirement of a very small rotation in the large-time limit. Hence, if still one assumes these equations of state for the early Universe, a physical mechanism supplementary to the expansion is also needed to suppress rotation.

However, if the brane Universe experiences an inflationary period, than any rotational perturbation is reduced exponentially to a very low level, the decrease of cosmological vorticity being of the order of 10^{-145} [10]. Hence, any growth of the rotational perturbations during a maximally stiff phase in a post-inflationary period is unlikely to increase them back up to observationally significant levels.

The equation of state of the Hagedorn fluid is consistent with the requirement of a non-rotating Universe. Therefore the study of the perturbations of cosmological models could also provide some insights into the equation of state of the ultra-high density nuclear or sub-nuclear matter. The actual very tight limit on the rotation of the Universe imposes very strong constraints on the initial equation of state of the cosmological fluid.

The spatial and angular distribution of the angular velocity depend on two arbitrary integration constants $C_i, i = 1, 2$. In principle the values of these constants could be determined by fixing the value of $A(r, \theta)$ at the centre, and from the actual value of the angular velocity of the Universe.

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