Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity

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Rapid interstellar travel by means of spacetime wormholes is described in a way that is useful for teaching elementary general relativity. The description touches base with Carl Sagan's novel *Contact*, which, unlike most science fiction novels, treats such travel in a manner that accords with the best 1986 knowledge of the laws of physics. Many objections are given against the use of black holes or Schwarzschild wormholes for rapid interstellar travel. A new class of solutions of the Einstein field equations is presented, which describe wormholes that, in principle, could be traversed by human beings. It is essential in these solutions that the wormhole possess a throat at which there is no horizon; and this property, together with the Einstein field equations, places an extreme constraint on the material that generates the wormhole's spacetime curvature: In the wormhole's throat that material must possess a radial tension τ_0 with the enormous magnitude $\tau_0 \sim$ (pressure at the center of the most massive of neutron stars) $\times$ (20 km)^2/(circumference of throat)^2. Moreover, this tension must exceed the material's density of mass-energy, $\rho_0 c^2$. No known material has this $\tau_0 > \rho_0 c^2$ property, and such material would violate all the "energy conditions" that underlie some deeply cherished theorems in general relativity. However, it is not possible today to rule out firmly the existence of such material; and quantum field theory gives tantalizing hints that such material might, in fact, be possible.

I. INTRODUCTION AND SUMMARY

A. Black holes not usable for interstellar travel

Science fiction stories, TV shows, and films often use black holes for rapid interstellar travel: Intrepid adventurers plunge into a black hole and find themselves almost immediately emerging at some distant location in our universe or even in some other universe—much to the annoyance of relativity afficionados who can marshal a long list of objections:

(1) A black hole horizon is the surface separating the interior, trapped regions of the hole (those regions that cannot communicate with the external universe) from the external universe in which we live. At the horizon of a black hole of mass M, tidal gravitational forces (inhomogeneities of gravity) produce enormous relative accelerations between the head and feet of an adventurer of height L, accelerations with magnitude $\sim L(2GM/c^3)^{-2} \sim$ (10 Earth gravities) $\times (L/1\ \mathrm{m}) \times (M/10^4$ solar masses$)^{-2}$ (see, e.g., Ref. 1, Sec. 32.6). Unless the hole is more massive than 10^4 suns and thus has a horizon with circumference, $4\pi GM/c^2$, which is larger than 10^5 km, the adventurer will be killed by tidal gravity before even reaching the horizon. A hole so massive and large will not fit into most science fiction scenarios.

(2) A black hole horizon is a one-way "membrane": Things can fall in, but nothing can emerge.[2,1] Thus, two-way travel (which is often invoked) is forbidden; and even in one-way travel the object at the other end, from which the adventurer emerges, cannot be a black hole. It must be some other, even more bizzare object—for example, a white hole.[3,4]

(3) All of the objects known as solutions to Einstein's equations that could exist at the other end (e.g., white holes[3,4]) possess "past event horizons" or "antihorizons," i.e., surfaces out of which things can emerge but down which nothing can go. Such antihorizons are known to be highly unstable against small perturbations.[5] If an antihorizon of mass M somehow were to form (e.g., in the big bang), a stray wave packet of light with arbitrarily small energy, falling toward it (but never able to reach it) would become more and more blueshifted and more and more energetic as it falls. By its exponentiating energy the wave packet would convert the antihorizon into a normal horizon in a time of (a few tens) $\times GM/c^3 \sim 1\ \mathrm{s} \times (M/10^4$ solar masses). This conversion, occurring within seconds after creation of the antihorizon, would seal off the antihorizon forever thereafter, preventing any adventurer from ever emerging through it.

(4) Although the Kerr metric that describes rotating black holes possesses, in its interior, pathways to other asymptotically flat regions of spacetime ("tunnels through hyperspace" to other "universes" or to other regions of our own universe),[6] those tunnels almost certainly do not occur in nature:

(a) The proof[7] that, once a newborn rotating hole settles down into a time-independent state it must have the Kerr form, applies only to the spacetime region at and outside the hole's horizon, not to the horizon's interior. The physical mechanisms that enforce the Kerr form (horizon as boundary condition on external physics; flow of gravitational radiation into horizon and off to infinity) do not operate in the hole's interior. Thus there is no reason whatsoever to expect a stellar collapse that forms a Kerr hole to form also a Kerr interior with tunnels to other regions of spacetime.

(b) Even if Kerr tunnels were to form, they could not live for long: The Kerr tunnels possess "Cauchy horizons" that are known to be highly unstable against small perturbations[8]: A wave packet of light, falling into a Kerr black hole with interior tunnel will become more and more blueshifted and more and more energetic as it nears the Cauchy horizon; and the wave packet's exponentiating energy presumably will create exponentially growing tidal gravitational fields that seal off the tunnel and convert it into a physical singularity. (We cannot be absolutely sure that this is the outcome, because this instability—by contrast with that of white hole antihorizons—has been analyzed only to first order in perturbation theory. However, all the evidence from first-order analyses and all physical intuition built up from other studies of nonlinearly strong gravity points toward a sealing off of the tunnel.) Thus it is almost certain that the interiors of black holes actually possess not tunnels to other regions of spacetime but, rather, singularities of near-infinitely strong tidal gravitational fields, singularities that would kill any human adventurer and that are described correctly not by general relativity but by the (as yet not fully understood) quantum theory of gravity.[9]

(c) If Kerr tunnels were somehow to form and were somehow stabilized to prevent infalling fields and particles from sealing them off, the tunnels would possess ring-shaped singularities. If physics were totally classical, and if the hole were sufficiently massive and sufficiently rapidly rotating, an adventurer would be able to pass unscathed through the center of such a ring singularity. However, quantum field theory predicts that these singularities by breaking down the vacuum should spew an intense flux of high-energy particles into the tunnel, almost certainly irradiating and killing any adventurer who tries to pass through and also almost certainly sealing off the tunnel against all passage.[10] (This objection, but only this, is avoided in a solution to Einstein's equations constructed by Bardeen.[11] Bardeen's solution describes a spacetime that almost certainly could not occur naturally, but that one might imagine an advanced civilization trying to build. In Bardeen's spacetime, a peculiar stress-energy threading the tunnels keeps them singularity-free, but all the other difficulties of Kerr tunnels remain.)

(d) If somehow an adventurer were to pass unscathed through a Kerr tunnel and emerge into some distant region of our own universe, by adjusting slightly her trajectory through the tunnel (we adopt the spirit of *Contact* in which the protagonist and wormhole traveler is Dr. Ellie Arroway) she could emerge *whenever* she wishes: late in the evolution of the universe or, more interestingly, early enough to return to Earth and kill her own newborn mother (causality violation).[12]

These objections make it seem exceedingly unlikely that black holes could ever be used by people or other intelligent beings for interstellar travel.

B. Schwarzschild wormholes: not traversible

Wormholes provide an alternative conceivable method for rapid interstellar travel. Figure 1(a) shows, by means of an embedding diagram (discussed below), a wormhole that connects two different universes; Fig. 1(b) shows a wormhole connecting two distant regions of our own universe. Both wormholes are described by the same solution of the Einstein field equations. Only their topologies differ, and the Einstein field equations do not constrain the topology of a solution.

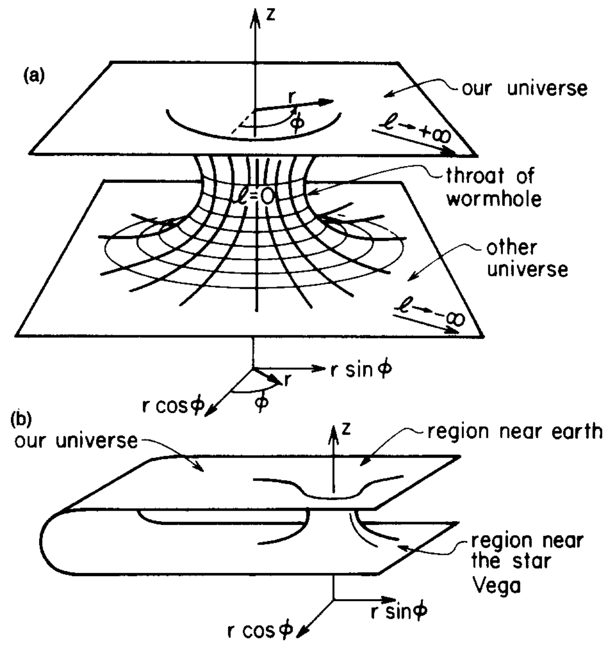

Fig. 1. (a) Embedding diagram for a wormhole that connects two different universes. (b) Embedding diagram for a wormhole that connects two distant regions of our own universe. Each diagram depicts the geometry of an equatorial ($\theta = \pi/2$) slice through space at a specific moment of time ($t = \text{const}$). These embedding diagrams are derived quickly in item (b) of Box 2, and—in a more leisurely fashion—in Sec. III C, where they are also discussed. This figure is adapted from Ref. 1, Fig. 31.5.

Remarkably, wormholes as objects of study in mathematical relativity predate black holes: Within one year after Einstein's final formulation of his field equations, the Viennese physicist Ludwig Flamm recognized that the Schwarzschild solution of Einstein's field equations represents a wormhole.[13] Possible roles of the Schwarzschild and other wormholes in physics were speculated upon in the 1920s by Herman Weyl,[14] in the 1930s by Einstein and Nathan Rosen,[15] and in the 1950s by John Wheeler.[16]

All hopes that Schwarzschild wormholes might exist in the real universe and be used for rapid interstellar travel are dashed by a series of major objections:

(1) Tidal gravitational forces at the throat of a Schwarzschild wormhole are of the same magnitude as at the horizon of a Schwarzschild black hole: They are so large that, unless the wormhole's mass exceeds 10^4 solar masses so its throat circumference exceeds 10^5 km, any adventurer would be killed trying to pass through the throat (cf. Ref. 1, Sec. 32.6).

(2) A Schwarzschild wormhole is actually dynamic, not static. As time passes, it expands from zero throat circumference (two disconnected universes) to a maximum circumference, and then recontracts to zero circumference (the universes disconnect).[17] This expansion and recontraction is so rapid that even moving at the speed of light one cannot pass all the way through the wormhole and into the other universe without being caught in the crunch of recontraction and killed by tidal gravity.[16]

(3) A Schwarzschild wormhole possesses a past horizon ("antihorizon") which, like that of a white hole, is unstable against small perturbations.[18,5] That instability enormously hastens the recontractive sealing off of the wormhole, making it even more impossible to get through.

C. Wormholes that are traversible

It turns out that there are very simple, exact solutions of the Einstein field equations, which describe wormholes that have none of the above problems. If, somehow, an advanced civilization could construct such wormholes, they could be used as a galactic or intergalactic transportation system and they might also be usable for backward time travel. As we shall see below, it is not clear today whether the laws of physics prohibit or actually permit the construction of such "traversible wormholes"; but nothing we know makes them seem nearly as impossible as black hole or Schwarzschild wormhole transportation systems.

The traversible wormhole solutions to Einstein's field equations are so simple that they can be used as a tool for teaching beginning relativity students how to solve the Einstein equations, how to interpret physically solutions they have obtained, and how to explore the properties of solutions. The purpose of this article is to derive and discuss the traversible wormhole solutions in such a way as to make them readily accessible to teachers of elementary relativity classes, and to beginning relativity students.

Because these wormhole solutions are so simple, it is hard for us to believe that they have not been derived and studied previously; however, we know of no previous studies. We were stimulated to find them in the summer of 1985, when Carl Sagan sent one of us a prepublication draft of his novel *Contact*[19] and requested assistance in making the gravitational physics in it as accurate as possible. Sagan, in response to our preliminary description of these solutions' properties, incorporated them into his novel at the galley proof stage. As a result, some teachers of relativity might want to use his novel as a stepping stone into these solutions.

Two relevant passages from Sagan's novel are reproduced in Box 1. The first passage describes the objections to black holes as means of interstellar travel, and a portion of the second passage describes the objections to Schwarzschild wormholes. Students, after reading the novel, might be referred to some of the literature cited above (Refs. 1–18) for the technical details of these objections. The remainder of the second passage describes some key features of the traversible wormhole solutions, including the wormholes' ability to remain always open, very small tidal forces,

Box 1. Excerpts from *Contact* by Carl Sagan.[19]

After traveling through some sort of "tunnel" that took them in less than an hour from Earth to an orbit around the star Vega, five of the characters in the novel speculate on the nature of the tunnel:

"You see," Eda explained softly, "if the tunnels are black holes there are real contradictions implied. There is an interior tunnel in the exact Kerr solution of the Einstein Field Equations, but it's unstable. The slightest perturbation would seal it off and convert the tunnel into a physical singularity through which nothing can pass. I have tried to imagine a superior civilization that would control the internal structure of a collapsing star to keep the interior tunnel stable. This is very difficult. The civilization would have to monitor and stabilize the tunnel forever. It would be especially difficult with something as large as the dodecahedron falling through."

"Even if Abonnema can discover how to keep the tunnel open, there are many other problems," Vaygay said. "Too many. Black holes collect problems faster than they collect matter. There are the tidal forces. We should have been torn apart in the black hole's gravitational field. We should have been stretched like people in the paintings of El Greco or the sculptures of . . . Giacometti. Then other problems: As measured from Earth it takes an infinite amount of time for us to pass through a black hole, and we could never, never return to Earth. Maybe this is what happened. Maybe we will never go home. Then, there should be an inferno of radiation near the singularity. This is a quantum mechanical instability. . . "

"And finally," Eda continued, "a Kerr-type tunnel can lead to grotesque causality violations. With a modest change of trajectory inside the tunnel, one could emerge from the other end as early in the history of the universe as you might like—a picosecond after the big bang, for example. That would be a very disorderly universe.

"Look, fellas," she said, "I'm no expert in General Relativity. But didn't we see black holes? Didn't we fall into them? Didn't we emerge out of them? Isn't a gram of observation worth a ton of theory?"

"I know, I know," Vaygay said in mild agony. "It has to be something else. Our understanding of physics can't be so far off. Can it?"

He addressed this last question, a little plaintively, to Eda, who only replied, "A naturally occurring black hole can't be a tunnel; they have impassible singularities at their centers."

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Eda was, considering the circumstances, very relaxed. She soon understood why. While she and Vaygay had been undergoing lengthy interrogations, he had been calculating.

"I think the tunnels are Einstein–Rosen bridges," he said. "General relativity admits a class of solutions, called wormholes, similar to black holes, but with no evolutionary connection—they cannot be generated, as black holes can, by the gravitational collapse of a star. But the usual sort of wormhole, once made, expands and contracts before anything can cross through; it exerts disastrous tidal forces, and it also requires—at least as seen by an observer left behind—an infinite amount of time to get through."

Ellie did not see how this represented much progress, and asked him to clarify. The key problem was holding the wormhole open. Eda had found a class of solutions to his field equations that suggested a new macroscopic field, a kind of tension that could be used to prevent a wormhole from contracting fully. Such a wormhole would pose none of the other problems of black holes; it would have much smaller tidal stresses, two-way access, quick transit times as measured by an exterior observer, and no devastating interior radiation field.

"I don't know whether the tunnel is stable against small perturbations," he said. "If not, they would have to build a very elaborate feedback system to monitor and correct the instabilities."

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Box 2. Simple example of a traversible wormhole (posed as an exercise for students who have never encountered wormholes but know how to interpret and work with solutions of the Einstein field equations).

A spacetime of special interest to certain people is one with the metric

$$ds^2 = -c^2\,dt^2 + dl^2 + (b_0^2 + l^2)(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{B2a}$$

where the coordinates have the ranges $-\infty < t < +\infty$, $-\infty < l < +\infty$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$, b_0 is a constant, and c is the speed of light.

(a) Give as complete a description as you can of the physical and geometrical properties of (i) the coordinates t, l, θ, ϕ, and (ii) the spacetime itself, including its symmetries, its asymptotically flat regions, if any, and its horizons, if any.

(b) Construct an embedding diagram for the "equatorial plane" $\theta = \pi/2$ at fixed "time" t. Discuss the physical interpretation of this embedding diagram.

(c) Let $\mathbf{e}_{\hat t}, \mathbf{e}_{\hat l}, \mathbf{e}_{\hat\theta}, \mathbf{e}_{\hat\phi}$ be *unit* basis vectors pointing along the t, l, θ, and ϕ directions. The only nonzero components of the Riemann curvature tensor of this spacetime, expressed in this basis, are $R_{\hat\theta\hat\phi\hat\theta\hat\phi} = -R_{\hat l\hat\theta\hat l\hat\theta} = -R_{\hat l\hat\phi\hat l\hat\phi} = b_0^2/(b_0^2 + l^2)^2$, and others related to these by the symmetries of the Riemann tensor $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = +R_{\gamma\delta\alpha\beta}$. What is the stress-energy tensor of the matter and fields that generate this spacetime? Discuss the possibility of constructing, in the laboratory, matter and fields with this stress-energy tensor.

(d) Show that an observer who falls freely and radially in this spacetime moves along the world line $l = vt$, $\theta = \text{const}$, $\phi = \text{const}$, where $v = \text{const} <$ (speed of light). Show that such an observer's local Lorentz frame has basis vectors $\mathbf{e}_{\hat 0'} = \gamma\mathbf{e}_{\hat t} + (v/c)\gamma\mathbf{e}_{\hat l}$, $\mathbf{e}_{\hat 1'} = \gamma\mathbf{e}_{\hat l} + (v/c)\gamma\mathbf{e}_{\hat t}$, $\mathbf{e}_{\hat 2'} = \mathbf{e}_{\hat\theta}$, $\mathbf{e}_{\hat 3'} = \mathbf{e}_{\hat\phi}$, where $\gamma = (1 - v^2/c^2)^{-1/2}$.

(e) Show that, if the observer falls sufficiently slowly, she feels arbitrarily small tidal gravitational force.

Solution:

(a) Here t is a time coordinate that measures proper time of a static observer; θ,ϕ are spherical polar coordinates; l is a radial coordinate measuring proper radial distance at fixed t. The spacetime is spherically symmetric and static; it has two asymptotically flat regions, $l \to +\infty$ and $l \to -\infty$; it has no horizons.

(b) Use cylindrical coordinates (z,r,ϕ) in the embedding space, so $ds^2 = dz^2 + dr^2 + r^2\,d\phi^2$. Then the two-surface $z(r) = \pm b_0 \ln(r/b_0 + \sqrt{(r/b_0)^2 - 1})$ with $l = \pm(r^2 - b_0^2)^{1/2}$ has the same geometry as the 2-surface $\theta = \pi/2$, $t = \text{const}$ in the spacetime of Eq. (1). This embedded surface is shown in Fig. 1(a).

(c) $-T^{\hat t\hat t} = -T^{\hat l\hat l} = T^{\hat\theta\hat\theta} = T^{\hat\phi\hat\phi} = (c^4/8\pi G)b_0^2/(b_0^2 + l^2)^2$. Note that the stress has the same form (radial tension; tangential pressure; equal in magnitude) as the stress tensor of a radial electric or magnetic field, but the energy density is negative. The negative energy density makes it hopeless to construct in the laboratory today material or fields with this $T^{\hat\mu\hat\nu}$. [Note: negative energy density is *not* a generic property of traversible wormholes; see Box 3 and Sec. III F]

(d) One may show constant free-fall velocity v by considering the geodesic equation for a test particle of mass μ falling on an equatorial, radial trajectory, i.e., a trajectory with four-momentum components $p^{\hat r} = \mu\gamma v$, $p^{\hat\theta} = p^{\hat\phi} = 0$, $p^{\hat t} = \mu\gamma$. The resulting equation of motion tells us that $dl/dt = v = \text{const}$. The basis vectors of the freely falling observer can be gotten from the orthonormal basis of the static observers by a special relativity Lorentz transformation.

(e) In the observer's local Lorentz frame the only nonzero tide-producing components of the Riemann tensor, $R_{\hat j\hat 0'\hat k'\hat 0'}$ (obtained by Lorentz transforming $R_{\hat\alpha\hat\beta\hat\gamma\hat\delta}$), are $R_{\hat 2'\hat 0'\hat 2'\hat 0'} = R_{\hat 3'\hat 0'\hat 3'\hat 0'} = -(v/c)^2\gamma^2 b_0^2/(b_0^2 + l^2)^2$. Thus the tidal accelerations (which are proportional to $R_{\hat j\hat 0'\hat k'\hat 0'}$) vanish in the limit $v \to 0$.

two-way travel (no horizons), rapid transit times as seen by both travelers and external observers, lack of intense, singularity-produced radiation fluxes, and also the requirement that some sort of matter or field with radial tension thread the wormhole. A teacher of very talented students might want to point out this passage and, with no further hint except to assume spherical symmetry and time independence, might challenge the students to discover for themselves the traversible wormhole solutions of the Einstein equations. Some readers of this article might want to stop at this point and derive the new solutions without further hints.

A simple special case of the traversible wormhole solutions is the metric described in Box 2. This Box is adapted, with only small changes, from the final exam in a 6-week beginning course on general relativity, which one of us taught at Caltech in autumn 1985. (The students were not taught anything about wormholes in the course; but they *were* taught how to explore the physical meanings of spacetime metrics, and their ability to do so was tested by the questions in the Box. It was startling to see how hidebound were the students' imaginations: Most could decipher detailed properties of the metric, but very few actually recognized that it represents a traversible wormhole connecting two different universes.)

Box 3 describes, in brief, the properties of the traversible wormhole solutions; and the remainder of this article presents them in considerable detail, in a manner and at a level of technicality appropriate for a person who has had only a very quick, elementary introduction to general relativity—e.g., somebody who has studied only Price's beautiful "General Relativity Primer."[20]

Section II of this article lists and discusses the properties that one would like a wormhole to have if it is to be a viable route for interstellar travel. This list of desired properties is used as a guide for Sec. III's detailed derivation of the traversible wormhole solutions and its detailed exploration of their properties—including the possibility that they might be usable for backward time travel (Sec. III I). The mathematical details of several specific wormhole solutions are presented in an Appendix. We conclude with a summary discussion in Sec. IV.

Throughout we shall use Price's notational conventions,[20] which are the same as those of MTW,[1] except for using cgs units with c (speed of light) and G (Newton's gravitation constant) not set to unity.

Box 3. Overview of traversible wormhole solutions (for details see Secs. II and III of text).

(1) Radial coordinates: r such that $2\pi r =$ circumference; $l = \pm$ (proper radial distance from wormhole throat) with $+$ in upper universe, $-$ in lower; see Fig. 1(a).

(2) Solution is determined by two freely specifiable functions of r: $b(r) =$ shape function, defined by $dl/dr = \pm(1-b/r)^{-1/2}$, $\Phi(r) =$ redshift function, defined by $g_{tt} = -e^{2\Phi}$.

(3) Spacetime metric: $ds^2 = -e^{2\Phi}c^2\,dt^2 + (1-b/r)^{-1}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$.

(4) Orthonormal basis of reference frame of static observers: $\mathbf{e}_{\hat{t}} = e^{-\Phi}\mathbf{e}_t$, $\mathbf{e}_{\hat{r}} = (1-b/r)^{1/2}\mathbf{e}_r$, $\mathbf{e}_{\hat{\theta}} = r^{-1}\mathbf{e}_\theta$, $\mathbf{e}_{\hat{\phi}} = (r\sin\theta)^{-1}\mathbf{e}_\phi$.

(5) Contraints on $b(r)$ and $\Phi(r)$ to produce a traversible wormhole:

(a) General constraints:

· Spatial geometry must have wormhole shape. Throat is at minimum of r; there $r = b = b_0$; throughout spacetime $1 - b/r \geqslant 0$; as $l \to \pm\infty$ (asymptotically flat regions of two universes) $b/r \to 0$ so $r \cong |l|$; see Fig. 1.

· No horizons or singularities $\Rightarrow \Phi$ is everywhere finite; t measures proper time in asymptotically flat regions $\Leftrightarrow \Phi \to 0$ as $l \to \pm\infty$.

(b) Description of a trip through wormhole, and constraints that follow from it:

· Radial velocity of traveler as measured by static observers: $v(r)$; $\gamma \equiv [1-(v/c)^2]^{-1/2}$.

· Trip begins at $l = -l_1$, ends at $l = +l_2$; $v = 0$ at $-l_1$ and $+l_2$; gravity is weak at $-l_1$ and $+l_2$: $b/r \ll 1$, $|\Phi| \ll 1$, $|\Phi' c^2| \lesssim g_\oplus =$ (Earth gravity). Here and below $' \equiv d/dr = (1-b/r)^{-1/2} d/dl$.

· Trip takes less than 1 year as seen by traveler, $\Delta\tau = \int_{-l_1}^{+l_2} (v\gamma)^{-1} dl \lesssim 1$ yr., and also as seen by static observers at $-l_1$ and $+l_2$, $\Delta t = \int_{-l_1}^{+l_2} (ve^{\Phi})^{-1} dl \lesssim 1$ yr.

· Traveler feels less than $g_\oplus$ acceleration, $|e^{-\Phi} d(\gamma e^{\Phi})/dl| \lesssim g_\oplus/c^2 = 1/(0.97\ 1\ \text{yr.})$.

· Traveler feels tidal-gravity accelerations between different parts of her body with magnitude $\lesssim g_\oplus$:

$$|(1-b/r)[-\Phi'' + \tfrac{1}{2}(b' - b/r)\Phi'/(r-b) - (\Phi')^2]| \lesssim 1/(10^{10}\ \text{cm})^2;$$

$$|\tfrac{1}{2}(\gamma^2/r^2)[(v/c)^2(b' - b/r) + 2(r-b)\Phi']| \lesssim 1/(10^{10}\ \text{cm})^2.$$

· Traveler must not couple strongly to material that generates wormhole curvature (wormhole must be threaded by a vacuum tube through which she moves, or wormhole material must be of type that couples weakly to ordinary matter).

(6) The material that generates the wormhole's spacetime curvature:

(a) Stress-energy tensor as measured by static observers:

$T_{\hat{t}\hat{t}} = \rho c^2 =$ (density of mass-energy), $T_{\hat{r}\hat{r}} = -\tau = -$ (radial tension), $T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = p =$ (lateral pressure).

(b) Einstein field equations: With $' \equiv d/dr = (1-b/r)^{-1/2} d/dl$

$$\rho = \frac{b'}{8\pi G c^{-2} r^2}, \quad \tau = \frac{b/r - 2(r-b)\Phi'}{8\pi G c^{-4} r^2}, \quad p = \frac{r}{2}[(\rho c^2 - \tau)\Phi' - \tau'] - \tau.$$

Correspondingly, in the throat (at $r = b = b_0$), ρ and p depend on the throat shape, while

$$\tau = (8\pi G c^{-4} b_0^2)^{-1} \simeq 5 \times 10^{41}\ \text{dyn cm}^{-2}(10\ \text{m}/b_0)^2.$$

(c) (Field equations) + (absence of horizon at throat) $\Rightarrow \tau > \rho c^2$ in throat $\Rightarrow$ traveler moving through throat at very high speed sees negative mass-energy density $\Rightarrow$ violation of weak, strong, and dominant energy conditions in throat ("exotic" matter that *might*—but it is not known for sure—be forbidden by laws of physics).

(d) One might wish to require $\rho \geqslant 0$ everywhere (static observers see nonnegative mass-energy density); this implies $b' \geqslant 0$ everywhere.

II. DESIRED PROPERTIES OF TRAVERSIBLE WORMHOLES

We have seen in Sec. I a plethora of objections to interstellar transportation systems based on black holes and Schwarzschild wormholes. These objections motivate us to initiate our study of traversible wormholes by listing all the properties that we might like them to have. This section presents such a list in the order that we shall use them in Sec. III's study of the traversible wormholes. Our list will be stated verbally; mathematical details will be omitted until Sec. III.

(1) The metric should be both spherically symmetric and static (time independent). This requirement is imposed only to simplify the calculations, and one should keep in mind that the wormhole might be unstable to spherical or nonspherical perturbations.

(2) The solution must everywhere obey the Einstein field equations. We assume the correctness of general relativity theory.

(3) To be a wormhole the solution must have a throat that connects two asymptotically flat regions of spacetime; i.e., an equatorial embedding diagram must have qualitatively the form of Fig. 1.

(4) There should be no horizon, since a horizon, if present, would prevent two-way travel through the wormhole.

(5) The tidal gravitational forces experienced by a traveler must be bearably small.

(6) A traveler must be able to cross through the wormhole in a finite and reasonably small proper time (e.g., less than a year) as measured not only by herself, but also by observers who remain behind or who await her outside the wormhole.

(7) The matter and fields that generate the wormhole's spacetime curvature must have a physically reasonable stress-energy tensor. It turns out that the form of the stress-energy tensor is strongly constrained by the preceding six properties. That constrained form in fact violates what we usually mean by "physically reasonable." In Secs. III F 2 and III F 3 we shall discuss the prospects for such a stress-energy actually to be achieved, and we shall take some care to minimize the violation of physical reasonableness.

(8) The solution should be perturbatively stable (especially as a spaceship passes through). Enforcing this re-

quirement would involve a time-dependent and nonspherical analysis, which is beyond the scope of this article. In Sec. III G this is discussed (albeit briefly).

(9) It should be possible to assemble the wormhole. For instance, the assembly should require both much less than the mass of the universe and much less than the age of the universe. Although not enough is known to permit a quantitative analysis, present knowledge of quantum gravity suggests that assembly *might* be possible; see Sec. III H.

Properties 1 through 4 we shall call the "basic wormhole criteria." Properties 5 and 6 will help us tune the wormhole's parameters for human physiological comfort, so we shall call them "usability criteria." By means of Property 7, we shall tune the parameters for our own aesthetic comfort—i.e., we shall tune them to make the wormhole's construction material as compatible as possible with our present prejudices about the forms of matter allowed by the laws of physics.

In summary, we shall build a solution of the Einstein equations using properties 1 through 4 and then shall adjust the wormhole's parameters by seeking a balance among conditions 5, 6, and 7.

III. MATHEMATICAL DETAILS OF TRAVERSIBLE WORMHOLES

In this section we present a general mathematical treatment of wormholes that possess the above properties, and in the Appendix we present specific examples of such wormholes. Since the Appendix examples illustrate many of the issues treated in this section, some readers might wish to study the Appendix in parallel with this section.

A. Form of the metric

Property 1 of Sec. II requires that the spacetime metric for the wormhole be expressible in the static, spherically symmetric form [Ref. 20, Eq. (6.4) or Ref. 1, Eq. (23.7)]:

$$ds^2 = -e^{2\Phi}c^2\,dt^2 + dr^2/(1-b/r) + r^2(d\theta^2 + \sin^2\theta\,d\phi^2). \quad (1)$$

Here, $\Phi = \Phi(r)$ and $b = b(r)$ are two arbitrary functions of radius only, to be constrained by the enumerated properties in Sec. II, and c is the speed of light. As we shall see below, $b(r)$ determines the spatial shape of the wormhole, so we shall call it the "shape function," and $\Phi(r)$ determines the gravitational redshift, so we shall call it the "redshift function." Notice that the radial coordinate r has special geometric significance: $2\pi r$ is the circumference of a circle centered on the wormhole's throat, and thus r is equal to the embedding-space radial coordinate of Fig. 1. As a result, r is nonmonotonic: It decreases from $+\infty$ to a minimum value, b_0, as one moves through the lower universe of Fig. 1 toward the wormhole and into the throat; then it increases from b_0 back to $+\infty$ as one moves out of the throat and into the upper universe. [In the specific wormhole solution of Box 2, above, the radial coordinate l is related to r by $l = \pm(r^2 - b_0^2)^{1/2}$ with $+$ in the upper universe and $-$ in the lower, and the functions $\Phi(r)$ and $b(r)$ are $\Phi = 0$, $b = b_0^2/r$.]

B. Equations of structure for the wormhole

1. *The Riemann, Ricci, and Einstein tensors*

In order to impose the Einstein field equations and in order to evaluate the tidal forces felt by travelers who cross the wormhole, we shall need the Riemann and Einstein tensors for the metric [Eq. (1)]. These are worked out in most advanced textbooks, but for the benefit of readers who have had only an elementary introduction to general relativity, e.g., Ref. 20, we shall sketch a derivation.

From our metric (1) written in the form,

$$ds^2 = g_{\alpha\beta}\,dx^\alpha\,dx^\beta, \quad x^0 = ct, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi, \quad (2)$$

the Christoffel symbols (connection coefficients) $\Gamma^\alpha_{\beta\gamma}$ and the components $R^{\alpha}{}_{\beta\gamma\delta}$ of the Riemann curvature tensor are computed using the standard formulas [Ref. 20, Eqs. (3.45) and (4.22)],

$$\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\lambda}(g_{\lambda\beta,\gamma} + g_{\lambda\gamma,\beta} - g_{\beta\gamma,\lambda}), \quad (3)$$

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\alpha_{\lambda\gamma}\Gamma^\lambda_{\beta\delta} - \Gamma^\alpha_{\lambda\delta}\Gamma^\lambda_{\beta\gamma}, \quad (4)$$

where the comma denotes a partial derivative ($g_{\alpha\beta,\gamma} = \partial g_{\alpha\beta}/\partial x^\gamma$).

By applying these equations to the metric (1) we readily find the 24 nonzero components of the Riemann tensor:

$$\begin{aligned}
R^{t}{}_{rtr} &= -R^{t}{}_{rrt} = (1-b/r)^{-1}e^{-2\Phi}R^{r}{}_{ttr} \\
&= -(1-b/r)^{-1}e^{-2\Phi}R^{r}{}_{trt} \\
&= -\Phi'' + (b'r-b)[2r(r-b)]^{-1}\Phi' - (\Phi')^2, \\
R^{t}{}_{\theta t\theta} &= -R^{t}{}_{\theta\theta t} = r^2e^{-2\Phi}R^{\theta}{}_{tt\theta} = -r^2e^{-2\Phi}R^{\theta}{}_{t\theta t} \\
&= -r\Phi'(1-b/r), \\
R^{t}{}_{\phi t\phi} &= -R^{t}{}_{\phi\phi t} = r^2e^{-2\Phi}\sin^2\theta R^{\phi}{}_{tt\phi} \\
&= -r^2e^{-2\Phi}\sin^2\theta R^{\phi}{}_{t\phi t} \\
&= -r\Phi'(1-b/r)\sin^2\theta, \\
R^{r}{}_{\theta r\theta} &= -R^{r}{}_{\theta\theta r} = -r^2(1-b/r)R^{\theta}{}_{rr\theta} \\
&= r^2(1-b/r)R^{\theta}{}_{r\theta r} \\
&= (b'r-b)/2r, \\
R^{r}{}_{\phi r\phi} &= -R^{r}{}_{\phi\phi r} = -r^2(1-b/r)\sin^2\theta R^{\phi}{}_{rr\phi} \\
&= r^2(1-b/r)\sin^2\theta R^{\phi}{}_{r\phi r} \\
&= (b'r-b)\sin^2\theta/2r, \\
R^{\theta}{}_{\phi\theta\phi} &= -R^{\theta}{}_{\phi\phi\theta} = \sin^2\theta R^{\phi}{}_{\theta\phi\theta} = -\sin^2\theta R^{\phi}{}_{\theta\theta\phi}, \\
&= (b/r)\sin^2\theta,
\end{aligned} \quad (5)$$

where the prime denotes a derivative with respect to the radial coordinate r, and the basis vectors being used are those $(\mathbf{e}_t, \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ associated with the coordinate system ct, r, θ, ϕ, i.e., such that the vector separation between two events with coordinate separation $(\Delta t, \Delta r, \Delta\theta, \Delta\phi)$ is $\Delta\mathbf{s} = c\Delta t\mathbf{e}_t + \Delta r\mathbf{e}_r + \Delta\theta\mathbf{e}_\theta + \Delta\phi\mathbf{e}_\phi$. [In the notation preferred by differential geometers, $\mathbf{e}_t = c^{-1}\partial/\partial t$, $\mathbf{e}_r = \partial/\partial r$, $\mathbf{e}_\theta = \partial/\partial\theta$, and $\mathbf{e}_\phi = \partial/\partial\phi$.]

The details of subsequent mathematics and of physical interpretations will be simplified by switching to a set of orthonormal basis vectors—the "proper reference frame" of a set of observers who remain always at rest in the coordinate system (r,θ,ϕ constant):

$$\mathbf{e}_{\hat{t}} = e^{-\Phi}\mathbf{e}_t, \quad \mathbf{e}_{\hat{r}} = (1 - b/r)^{1/2}\mathbf{e}_r, \qquad (6)$$
$$\mathbf{e}_{\hat{\theta}} = r^{-1}\mathbf{e}_\theta, \quad \mathbf{e}_{\hat{\phi}} = (r\sin\theta)^{-1}\mathbf{e}_\phi.$$

In this basis the metric coefficients take on their standard, special relativity forms,

$$g_{\hat{\alpha}\hat{\beta}} = \mathbf{e}_{\hat{\alpha}}\cdot\mathbf{e}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} \equiv \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (7)$$

and the 24 nonzero components of the Riemann tensor, Eqs. (5) above, take on the much simplified forms:

$$\begin{aligned}
R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} &= -R^{\hat{t}}{}_{\hat{r}\hat{r}\hat{t}} = R^{\hat{r}}{}_{\hat{t}\hat{t}\hat{r}} = -R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} \\
&= (1-b/r)\{-\Phi'' \\
&\quad + (b'r-b)[2r(r-b)]^{-1}\Phi' - (\Phi')^2\}, \\
R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} &= -R^{\hat{t}}{}_{\hat{\theta}\hat{\theta}\hat{t}} = R^{\hat{\theta}}{}_{\hat{t}\hat{t}\hat{\theta}} = -R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} = -(1-b/r)\Phi'/r, \\
R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{\phi}} &= -R^{\hat{t}}{}_{\hat{\phi}\hat{\phi}\hat{t}} = R^{\hat{\phi}}{}_{\hat{t}\hat{t}\hat{\phi}} = -R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} = -(1-b/r)\Phi'/r, \\
R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} &= -R^{\hat{r}}{}_{\hat{\theta}\hat{\theta}\hat{r}} = R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} = -R^{\hat{\theta}}{}_{\hat{r}\hat{r}\hat{\theta}} = (b'r-b)/2r^3, \quad (8) \\
R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} &= -R^{\hat{r}}{}_{\hat{\phi}\hat{\phi}\hat{r}} = R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} = -R^{\hat{\phi}}{}_{\hat{r}\hat{r}\hat{\phi}} = (b'r-b)/2r^3, \\
R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}} &= -R^{\hat{\theta}}{}_{\hat{\phi}\hat{\phi}\hat{\theta}} = R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} = -R^{\hat{\phi}}{}_{\hat{\theta}\hat{\theta}\hat{\phi}} = b/r^3.
\end{aligned}$$

From here we can contract the Riemann tensor to calculate the Ricci tensor $R_{\hat{\mu}\hat{\nu}}$ and the scalar curvature R,

$$R_{\hat{\mu}\hat{\nu}} = R^{\hat{\alpha}}{}_{\hat{\mu}\hat{\alpha}\hat{\nu}}, \qquad (9)$$
$$R = g^{\hat{\mu}\hat{\nu}}R_{\hat{\mu}\hat{\nu}}, \qquad (10)$$

and from these we can compute the Einstein tensor that enters into the Einstein field equations:

$$G_{\hat{\mu}\hat{\nu}} = R_{\hat{\mu}\hat{\nu}} - \tfrac{1}{2}Rg_{\hat{\mu}\hat{\nu}}. \qquad (11)$$

This computation yields the only nonzero components of the Einstein tensor:

$$\begin{aligned}
G_{\hat{t}\hat{t}} &= b'/r^2; \\
G_{\hat{r}\hat{r}} &= -b/r^3 + 2(1-b/r)\Phi'/r; \qquad (12) \\
G_{\hat{\theta}\hat{\theta}} &= G_{\hat{\phi}\hat{\phi}} = \left(1 - \frac{b}{r}\right)\left(\Phi'' - \frac{b'r-b}{2r(r-b)}\Phi' + (\Phi')^2 \right. \\
&\quad \left. + \frac{\Phi'}{r} - \frac{b'r-b}{2r^2(r-b)}\right).
\end{aligned}$$

2. The stress-energy tensor

Birkhoff's theorem (Ref. 1, Sec. 32.2) tells us that only one kind of *vacuum*, spherical wormhole is allowed by the Einstein field equations: a (nontraversible) Schwarzschild wormhole. Thus a traversible wormhole must be threaded by matter or fields with a nonzero (nonvacuum) stress-energy tensor.

Since the Einstein field equations require that the stress-energy tensor be proportional to the Einstein tensor, in our orthonormal basis the stress-energy tensor $T_{\hat{\mu}\hat{\nu}}$ must have the same algebraic structure as the $G_{\hat{\mu}\hat{\nu}}$ of Eq. (12): The only nonzero components must be $T_{\hat{t}\hat{t}}$, $T_{\hat{r}\hat{r}}$, and $T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}}$. Since the basis vectors are those used by static observers, each of these components has a simple physical interpretation in terms of measurements that static observers might make:

$$T_{\hat{t}\hat{t}} = \rho(r)c^2, \quad T_{\hat{r}\hat{r}} = -\tau(r), \quad \text{and} \quad T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = p(r), \qquad (13)$$

where $\rho(r)$ is the total density of mass-energy that they measure (in units of g/cm^3); $\tau(r)$ is the tension per unit area that they measure in the radial direction (i.e., it is the negative of the radial pressure and has units dyn/cm^2); and $p(r)$ is the pressure (in dyn/cm^2) that they measure in lateral directions (directions orthogonal to radial).

The stress-energy tensor of an ordinary "perfect fluid" is a special case of Eq. (13); it has $-\tau = p$. Another special case is a radially pointing electric field of strength $E(r)$, for which $\tau = p = \rho c^2 = E^2/8\pi$.

3. The Einstein field equations

The Einstein field equations,

$$G_{\hat{\alpha}\hat{\beta}} = 8\pi G c^{-4} T_{\hat{\alpha}\hat{\beta}},$$

as evaluated from the Einstein tensor (12) and the stress-energy tensor (13) become, after a bit of manipulation,

$$b' = 8\pi G c^{-2} r^2 \rho, \qquad (14)$$
$$\Phi' = (-8\pi G c^{-4}\tau r^3 + b)/[2r(r-b)], \qquad (15)$$
$$\tau' = (\rho c^2 - \tau)\Phi' - 2(p+\tau)/r. \qquad (16)$$

Equations (14) and (15) are the temporal and radial parts of the field equations, respectively. Equation (16) is the lateral $(\hat{\theta},\hat{\phi})$ part of the field equations with Φ'' eliminated using the radial derivative of Eq. (15).

We may also interpret Eq. (16) in a straightforward physical manner. It is simply the equation of hydrostatic equilibrium for the material threading the wormhole. Indeed, we encourage readers to derive this equation for themselves by a physical argument that balances the forces on a small chunk of the wormhole's material—forces due to the radial tension gradient, the lateral "Roman arch" pressure, and the gravitational pull. In evaluating the gravitational pull on the chunk, readers will need to know the tensorial nature of inertial mass (see, e.g., Ref. 1, Exercise 5.4) and will need to recognize that the gravitational acceleration felt by the chunk is the negative of the chunk's four-acceleration ("Einstein elevator experiment"). The redundancy between the Einstein field equations and the law of force balance (or, better, the law of four-momentum conservation) is a deep and important aspect of general relativity (see, e.g., Ref. 1, Sec. 17.2).

The field equations (14)–(16) are three differential equations relating five unknown functions of r: b, Φ, ρ, τ, and p. The normal approach to solving these equations would be to assume some specific type of matter or fields for the source of the stress-energy tensor, and from the physics of that source to derive "equations of state" for the radial tension as a function of mass-energy density $\tau(\rho)$ and for the lateral pressure as a function of mass-energy density $p(\rho)$. These equations of state, plus the three field equations, would be five equations for five unknown functions (b, Φ, ρ, τ, and p) of r. For example, if we were studying the structures of neutron stars, we would set $-\tau(\rho) = p(\rho) =$ (equation of state derived from nuclear theory); see, e.g., Ref. 1, Chap. 23. As another example, if we were studying an electrically charged black hole we would set $\tau = p = \rho c^2$ in accord with the stress-energy tensor of a radial electric field, and we then would obtain from Eqs. (14)–(16) the Reissner–Nordstrom solution of the

Einstein field equations; see, e.g., Ref. 1, Exercise 31.8.

In our study of traversible wormholes the philosophy of solving the field equations (14)–(16) must be altered somewhat from the usual one. We desire solutions with certain properties (enumerated in Sec. II), and to achieve them we must be willing to let the builders of a wormhole synthesize, or search throughout the universe for, materials or fields with whatever stress-energy tensor might be required. Stated mathematically, we wish to control the functions $b(r)$ and $\Phi(r)$ so as to shape the wormhole to our specifications; and, accordingly, we must let the relationships betwen ρ, τ, and p dangle, only to be fixed by the field equations and our restricted choices for b and Φ.

In accord with this philosophy, it is convenient to rewrite Eqs. (14)–(16) in the slightly different form:

$$\rho = b'/[8\pi G c^{-2} r^2], \tag{17}$$

$$\tau = [b/r - 2(r-b)\Phi']/[8\pi G c^{-4} r^2], \tag{18}$$

$$p = (r/2)[(\rho c^2 - \tau)\Phi' - \tau'] - \tau. \tag{19}$$

The forms of these equations suggest the strategy for solution: Tailor $b(r)$ and $\Phi(r)$ to make a nice wormhole; Eq. (17) and our choice for $b(r)$ will then give us $\rho(r)$; Eq. (18) and our choices for both $b(r)$ and $\Phi(r)$ will then yield $\tau(r)$; and, finally, Eq. (19) together with the above will determine $p(r)$.

4. Boundary conditions

In some cases, we may wish to let the stress-energy that generates the wormhole's curvature extend out to arbitrarily large radii. In others, we may confine it to the interior of a sphere of some surface radius $r = R_S$, i.e., we may require that ρ, τ, and p vanish at all radii $r > R_S$. In this latter case, Eqs. (17)–(19) require that the radial tension τ go smoothly to zero as one approaches $r = R_S$ from below, but they permit ρ and p to be cut off discontinuously:

$$\tau \to 0, \quad \text{but } \rho \text{ and } p \text{ may remain finite}$$
$$\text{in} \quad \lim r \to R_S \text{ from below.} \tag{20}$$

(These are special cases of "junction conditions" discussed more generally in Ref. 1, Sec. 21.13.) Equations (14)–(16) evaluated in the vacuum region outside $r = R_S$ constrain the external spacetime geometry to have the standard Schwarzschild form,

$$b(r) = b(R_S) = \text{const} \equiv B \quad \text{at} \quad r > R_S, \tag{21}$$

$$\Phi(r) = \tfrac{1}{2}\ln(1 - B/r) \quad \text{at} \quad r > R_S. \tag{22}$$

If we had chosen not to cut the matter field off but to join discontinuously in radius various kinds of matter stress-energy, the field equations would enforce continuity of τ, b, and Φ but permit discontinuities of ρ and p; cf. Ref. 1, Sec. 21.13.

If there is no cutoff in the stress-energy, we shall still require that the field die out fast enough radially that spacetime is asymptotically flat:

$$b/r \to 0 \quad \text{and} \quad \Phi \to 0 \quad \text{as } r \to \infty. \tag{23}$$

C. Spatial geometry of the wormhole

1. The mathematics of embedding

Below we shall use embedding diagrams to help us impose the demand that the spacetime metric (1) describe a wormhole. Of particular interest is the geometry of three-dimensional space at a fixed moment of time t. That geometry is spherically symmetric, so without significant loss of information we can confine attention to an equatorial slice, $\theta = \pi/2$, through it. The line element for such a slice is obtained by setting $t = \text{const}$, $\theta = \pi/2$ in Eq. (1):

$$ds^2 = (1 - b/r)^{-1} dr^2 + r^2 d\phi^2. \tag{24}$$

We wish to construct, in three-dimensional Euclidean space, a two-dimensional surface with the same geometry as this slice, i.e., we wish to visualize this slice as removed from spacetime and embedded in Euclidean space (cf. Ref. 1, Sec. 23.8). In the embedding Euclidean space we introduce cylindrical coordinates z, r, and ϕ. Then the Euclidean metric of the embedding space has the form

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2. \tag{25}$$

The embedded surface will be axially symmetric and thus can be described by the single function $z = z(r)$. On that surface the line element will be

$$ds^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\phi^2. \tag{26}$$

This line element will be the same as that of our equatorial slice through the wormhole [Eq. (24)] if we identify the coordinates (r,ϕ) of the embedding space with the (r,ϕ) of the wormhole's spacetime, and if we require the function $z(r)$, which describes the embedded surface, to satisfy

$$\frac{dz}{dr} = \pm\left(\frac{r}{b(r)} - 1\right)^{-1/2}. \tag{27}$$

This surface $z = z(r)$ is what is pictured in Fig. 1(a); and Eq. (27) displays the manner in which the function $b = b(r)$ shapes the wormhole's spatial geometry.

2. Schwarzschild wormhole

As a specific example, consider a Schwarzschild wormhole for which $b(r) = \text{const} = B$. The embedded surface [solution of Eq. (27)] in this case is

$$z(r) = \pm 2B(r/B - 1)^{1/2}; \tag{28}$$

cf. Ref. 1, Eq. (23.34). The wormhole's throat in this case is located at $r = B$ ("Schwarzschild radius"). Notice that dz/dr is infinite at the throat [Eq. (27)]. Of course, this is true for any wormhole, not just for Schwarzschild, because $dz/dr = \infty$ corresponds to a vertical slope of the embedding surface in Fig. 1, which is precisely what we mean by "throat."

Because of the divergence of dz/dr at the throat, r is not a good coordinate to use in the throat's vicinity. Much better is proper radial distance as measured by static observers:

$$dl = \pm[1 - B/r]^{-1/2} dr; \tag{29a}$$

i.e.,

$$l = \pm[\sqrt{r(r-B)} + B\ln(\sqrt{r/B} + \sqrt{r/B - 1})]. \tag{29b}$$

This radial distance is positive [+ sign in Eqs. (29a,b)] above the throat ("upper universe") and negative [$-$ sign in Eqs. (29a,b)] below the throat ("lower universe"); cf. Fig. 1.

Very far from the Schwarzschild throat the embedding surface becomes flat

$$\frac{dz}{dr}(l \to \pm\infty) = 0, \tag{30}$$

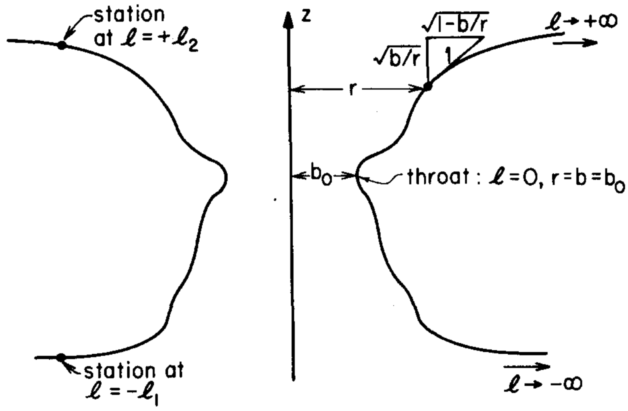

Fig. 2. Embedding diagram for a general wormhole, as seen in profile. (The diagram must be rotated about the vertical z axis to make it complete; cf. Fig. 1.).

corresponding to the two asymptotically flat regions ($l \to +\infty$ and $l \to -\infty$), which the wormhole connects.

3. General wormhole

Every wormhole, by definition of "wormhole," must have a minimum radius $r = b_0$ (the wormhole throat) at which its embedded surface is vertical, i.e., at which expression (27) is divergent, i.e., at which $b(r) = r$. We shall denote the common value of r and b at this throat by b_0:

there exists a minimum radius $r = b_0$ in the wormhole;

$$\text{and at } r = b_0,\ b = b_0. \tag{31}$$

As for a Schwarzschild wormhole, so also in the general case, the radial coordinate r is ill behaved near the throat; but proper radial distance

$$l(r) = \pm \int_{b_0}^{r} \frac{dr}{[1 - b(r)/r]^{1/2}} \tag{32}$$

must be well behaved everywhere; i.e., we must require that

$$l(r) \text{ is finite throughout spacetime,} \tag{33}$$

which also implies that

$$1 - b/r \geqslant 0 \text{ throughout spacetime.} \tag{34}$$

Far from the throat in both radial directions space must become asymptotically flat; i.e., $dz/dr = \pm (r/b - 1)^{-1/2}$ must approach zero as $l \to \pm\infty$; i.e.,

$$b/r \to 0, \quad \text{as} \quad l \to \pm\infty. \tag{35}$$

Note that Eqs. (27) and (32) imply that for the embedded wormhole

$$\frac{dz}{dl} = \pm \sqrt{\frac{b}{r}} \text{ and } \frac{dr}{dl} = \pm \sqrt{1 - \frac{b}{r}}. \tag{36}$$

Figure 2 depicts a somewhat general wormhole shape and the geometrical meanings of Eqs. (36).

D. The absence of a horizon

In any static, asymptotically flat spacetime, including that of a wormhole, it is easy to identify horizons: They are the physically nonsingular surfaces at which $g_{00} \equiv -e^{2\Phi} \to 0$ (vanishing proper time lapse during any finite coordinate time).[21] For example, a Schwarzschild wormhole possesses a horizon precisely at its throat, $r = B$. The demand that our traversible wormholes not possess any horizons corresponds, then, to

$$\Phi(r) \text{ is everywhere finite.} \tag{37}$$

E. Tidal gravitational forces and time to traverse the wormhole

Turn attention now to a thought experiment in which a traveler journeys radially through a wormhole, beginning at rest in a space station in the lower universe, at $l = -l_1$, and ending at rest in a space station in the upper universe, at $l = +l_2$. (See left side of Fig. 2.) Denote by $v(r)$ the radial velocity of the traveler as she passes radius r, as measured by a static observer there; and define $\gamma \equiv [1 - (v/c)^2]^{-1/2}$, as in special relativity. Then in terms of distance traveled dl, radius traveled dr, coordinate time lapse dt, and proper time lapse as seen by the traveler $d\tau_T$,

$$v = \frac{dl}{e^{\Phi} dt} = \mp \frac{dr}{(1 - b/r)^{1/2} e^{\Phi} dt}, \tag{38a}$$

$$v\gamma \equiv \frac{v}{[1 - (v/c)^2]^{1/2}} = \frac{dl}{d\tau_T} = \mp \frac{dr}{(1 - b/r)^{1/2}\, d\tau_T}. \tag{38b}$$

Here, the ($-$) sign refers to the first half of the trip (lower universe); the ($+$) sign to the second half (upper universe). Because the trip begins and ends at stations that are at rest, we have

$$v = 0 \quad \text{at} \quad l = -l_1 \quad \text{and} \quad l = +l_2;$$
$$v > 0 \quad \text{at} \quad -l_1 < l < +l_2. \tag{39}$$

The stations at $l = -l_1$ and $l = \pm l_2$ must be far enough from the throat for the gravitational effects of the wormhole to be small. In particular, (i) the geometry of space there must be nearly flat, $b/r \ll 1$; (ii) the gravitational redshift of signals sent from the stations to infinity must be small [$\Delta(\text{wavelength})/(\text{wavelength}) = e^{-\Phi} - 1 \cong -\Phi \ll 1$; i.e., $|\Phi| \ll 1$; cf. Ref. 1, pp. 657–659]; (iii) the "acceleration of gravity" as measured at the stations, $g = -(1 - b/r)^{1/2}\Phi' c^2 \cong -\Phi' c^2$, must be less than or of order 1 Earth gravity, $g_{\oplus} = 980\ \text{cm/s}^2$:

$$b/r \ll 1, \quad |\Phi| \ll 1,$$
$$|\Phi' c^2| \lesssim g_{\oplus} \quad \text{at} \quad l = -l_1 \quad \text{and} \quad l = +l_2. \tag{40}$$

Because $|\Phi| \ll 1$ at the stations, the proper time ticked by clocks there is equal to coordinate time t; cf. the spacetime metric (1).

If wormhole travel is to be at all convenient for human beings, the traveler's journey must satisfy three constraints: (i) the entire trip should require less than or of order 1 year as measured both by the traveler and by people who live in the stations,

$$\Delta\tau_1 = \int_{-l_1}^{l_2} \frac{dl}{v\gamma} \lesssim 1 \text{ yr.,} \tag{41a}$$

$$\Delta t = \int_{-l_1}^{l_2} \frac{dl}{v e^{\Phi}} \lesssim 1 \text{ yr.;} \tag{41b}$$

(ii) the acceleration **a** felt by the traveler must not exceed by much 1 Earth gravity; (iii) the tidal accelerations $\mathbf{\Delta a}$ between various parts of the traveler's body must not exceed 1 Earth gravity.

As an aid to discussing the accelerations felt by the traveler, we introduce the orthonormal basis of her own reference frame, $\mathbf{e}_{\hat{0}'}, \mathbf{e}_{\hat{1}'}, \mathbf{e}_{\hat{2}'}, \mathbf{e}_{\hat{3}'}$. Expressed in terms of the orthonormal basis of the static observers, $\mathbf{e}_{\hat{t}},\mathbf{e}_{\hat{r}},\mathbf{e}_{\hat{\theta}},\mathbf{e}_{\hat{\phi}}$, the traveler's basis is given by the standard special relativity Lorentz transformation,

$$\mathbf{e}_{\hat{0}'} = \mathbf{u} = \gamma \mathbf{e}_{\hat{t}} \mp \gamma(v/c)\mathbf{e}_{\hat{r}}, \quad \mathbf{e}_{\hat{1}'} = \mp \gamma \mathbf{e}_{\hat{r}} + \gamma(v/c)\mathbf{e}_{\hat{t}},$$
$$\mathbf{e}_{\hat{2}'} = \mathbf{e}_{\hat{\theta}}, \quad \mathbf{e}_{\hat{3}'} = \mathbf{e}_{\hat{\phi}}. \tag{42}$$

Here, $\mathbf{u}$ is the traveler's four-velocity. Note that $\mathbf{e}_{\hat{1}'}$ points along the direction of travel (toward increasing l).

The traveler's four-acceleration $a^{\hat{\alpha}'} = u^{\hat{\alpha}'}{}_{;\hat{\beta}'} u^{\hat{\beta}'} c^2$ is the acceleration that her body feels. Since four-acceleration is always orthogonal to four-velocity, $\mathbf{a}\cdot\mathbf{u} = \mathbf{a}\cdot\mathbf{e}_{\hat{0}'}$, $= a_{\hat{0}'} = -a^{\hat{0}'}$ vanishes. Because the traveler moves radially, her acceleration must be radial, so $a_{\hat{2}'} = a_{\hat{3}'} = 0$ and $\mathbf{a} = a\mathbf{e}_{\hat{1}'}$, with a the magnitude of the acceleration. The easiest way to compute a is to regard u_α as a function of the traveler's radial location r, to evaluate $a_t/c^2 = u_{t;\alpha}u^\alpha = u_{t,r}u^r - \Gamma_{\alpha t\beta}u^\alpha u^\beta$ in the (ct,r,θ,ϕ) coordinate frame, and then to note that $a_t = \mathbf{a}\cdot\mathbf{e}_t = (a\mathbf{e}_{\hat{1}'})\cdot(\mathbf{e}_t) = -\gamma(v/c)e^{\Phi}a$ [cf. Eqs. (42), (6), and (7)]. The result is

$$a = \mp\left(1 - \frac{b}{r}\right)^{1/2} e^{-\Phi}(\gamma e^{\Phi})'c^2 = e^{-\Phi}\frac{d}{dl}(\gamma e^{\Phi})c^2. \tag{43}$$

Our demand that the traveler not feel an acceleration larger than about 1 Earth gravity corresponds, then, to

$$\left| e^{-\Phi}\frac{d(\gamma e^{\Phi})}{dl} \right| \lesssim \frac{g_\oplus}{c^2} \simeq \frac{1}{0.97\ \text{l.yr.}}. \tag{44}$$

Turn, next, to the tidal gravitational forces that the traveler feels. Denote by $\boldsymbol{\xi}$ the vector separation between two parts of her body (e.g., head to feet); $\boldsymbol{\xi}$ is purely spatial in the traveler's reference frame, i.e., $\boldsymbol{\xi}\cdot\mathbf{u} = -\xi^{\hat{0}'} = 0$, where $\mathbf{u}$ is her four-velocity. Then the tidal acceleration between the two parts of her body is given by (cf. Ref. 1, Box 37.1)

$$\Delta a^{\hat{\alpha}'} = -c^2 R^{\hat{\alpha}'}{}_{\hat{\beta}'\hat{\gamma}'\hat{\delta}'} u^{\hat{\beta}'}\xi^{\hat{\gamma}'}u^{\hat{\delta}'}. \tag{45}$$

Here, $R^{\hat{\alpha}'}{}_{\hat{\beta}'\hat{\gamma}'\hat{\delta}'}$ are the components of the Riemann curvature tensor. [For readers who have not previously met Eq. (45) but who are familiar with the equation of geodesic deviation, we point out that the right-hand side of Eq. (45) is precisely the relative acceleration (tidal acceleration) of two freely falling test particles that have separation $\boldsymbol{\xi}$ and four-velocity $\mathbf{u}$. The fact that the observer is accelerated (does not fall freely) has no influence on the relative accelerations that she feels.] Since $u^{\hat{\alpha}'} = \delta^{\hat{\alpha}'}_{\hat{0}'}$ and $\xi^{\hat{0}'} = 0$ in the traveler's frame, and since $R_{\hat{\alpha}'\hat{\beta}'\hat{\gamma}'\hat{\delta}'}$ is antisymmetric in its first two indices, $\Delta a^{\hat{\alpha}'}$ is purely spatial with components,

$$\Delta a^{\hat{j}'} = -c^2 R^{\hat{j}'}{}_{\hat{0}'\hat{k}'\hat{0}'}\xi^{\hat{k}'} = -c^2 R_{\hat{j}'\hat{0}'\hat{k}'\hat{0}'}\xi^{\hat{k}'}. \tag{46}$$

By transforming the components (8) of the Riemann tensor from the static observers' frame $\mathbf{e}_{\hat{t}},\mathbf{e}_{\hat{r}},\mathbf{e}_{\hat{\theta}},\mathbf{e}_{\hat{\phi}}$ to the traveler's frame $\mathbf{e}_{\hat{0}'},\mathbf{e}_{\hat{1}'},\mathbf{e}_{\hat{2}'},\mathbf{e}_{\hat{3}'}$ [a special relativity type transformation (Lorentz transformation) since both sets of basis vectors are orthonormal], we obtain

$$R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'} = R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\left(1 - \frac{b}{r}\right) \times\left(-\Phi'' + \frac{b'r - b}{2r(r-b)}\Phi' - (\Phi')^2\right), \tag{47a}$$

$$R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'} = R_{\hat{3}'\hat{0}'\hat{3}'\hat{0}'} = \gamma^2 R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} + \gamma^2\left(\frac{v}{c}\right)^2 R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}} = \frac{\gamma^2}{2r^2}\left[\left(\frac{v}{c}\right)^2\left(b' - \frac{b}{r}\right) + 2(r-b)\Phi'\right]. \tag{47b}$$

Since these are the only nonvanishing parts of $R_{\hat{j}'\hat{0}'\hat{k}'\hat{0}'}$ in the traveler's frame, the tidal acceleration (46) takes on the simple form

$$\Delta a^{\hat{1}'} = -c^2 R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'}\xi^{\hat{1}'}, \quad \Delta a^{\hat{2}'} = -c^2 R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'}\xi^{\hat{2}'},$$
$$\Delta a^{\hat{3}'} = -c^2 R_{\hat{3}'\hat{0}'\hat{3}'\hat{0}'}\xi^{\hat{3}'}. \tag{48}$$

We must insist that, for $|\boldsymbol{\xi}| \sim 2$ m (the size of the traveler's body) and for $\boldsymbol{\xi}$ oriented along any spatial direction in the traveler's frame, $|\Delta\mathbf{a}| \lesssim$ (1 Earth gravity) $\equiv g_\oplus$. By combining Eqs. (47) and (48) we can write this constraint as

$$|R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'}| = \left|\left(1 - \frac{b}{r}\right)\left(-\Phi'' + \frac{b'r - b}{2r(r-b)}\Phi' - (\Phi')^2\right)\right| \lesssim \frac{g_\oplus}{c^2 \times 2\ \text{m}} \simeq \frac{1}{(10^{10}\ \text{cm})^2}, \tag{49}$$

$$|R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'}| = \left|\frac{\gamma^2}{2r^2}\left[\left(\frac{v}{c}\right)^2\left(b' - \frac{b}{r}\right) + 2(r-b)\Phi'\right]\right| \lesssim \frac{g_\oplus}{c^2 \times 2\ \text{m}} \simeq \frac{1}{(10^{10}\ \text{cm})^2}. \tag{50}$$

The radial tidal constraint (49) can be regarded as constraining the metric coefficient Φ. That constraint is most easily satisfied by setting $\Phi' = 0$ everywhere—a class of wormhole solutions which we shall discuss in Sec. 1 of the Appendix. The lateral tidal constraint (50) can be regarded as constraining the speed v with which the traveler crosses the wormhole. In the Appendix, we shall study the implications of both of these constraints for specific wormhole solutions.

F. The stress-energy that generates the wormhole's spacetime curvature

1. Constraints on the tension and mass density at the throat

The constraints that we have placed on the wormhole's shape function $b(r)$ give rise, via the Einstein field equations (17)–(19), to constraints on the mass density ρ, radial tension τ, and lateral pressure p, which generate the spacetime curvature.

The most severe constraints occur in the wormhole's throat. The fact that $r = b = b_0$ at the throat, together with the fact the $(r-b)\Phi' \to 0$ there [which follows from finiteness of ρ and hence of b', Eq. (17), and from the absence of a horizon and hence the finiteness of Φ] implies, via the field equation (18), that

$$\tau_0 \equiv (\text{tension in throat}) = \frac{1}{8\pi G c^{-4} b_0^2} \sim 5\times 10^{41}\ \frac{\text{dyn}}{\text{cm}^2}\left(\frac{10\ \text{m}}{b_0}\right)^2 \sim 5\times 10^{11}\ \frac{\text{dyn}}{\text{cm}^2}\left(\frac{1\ \text{l. yr.}}{b_0}\right)^2. \tag{51}$$

This is an enormous tension. When $b_0 \sim 3$ km, τ_0 has the same magnitude, $\sim 10^{37}$ dyn/cm^2, as the pressure at the center of the most massive of neutron stars. Even for the extremely large throat size $b_0 = 1$ l. yr., τ_0 could be produced by a magnetic field only if the field strength were $B \sim 10^6$ Gauss.

In the neighborhood of the throat we can investigate another key aspect of this tension by defining the dimensionless function,

$$\zeta \equiv \frac{\tau - \rho c^2}{|\rho c^2|} = \frac{b/r - b' - 2(r-b)\Phi'}{|b'|}, \qquad (52)$$

where we have used the Einstein field equations (17) and (18) to replace the stress-energy functions τ and ρ by their geometric counterparts b and Φ. This dimensionless function $\zeta(l)$ enters into the following discussion.

The requirement that the wormhole be connectible to asymptotically flat spacetime entails at the throat that the embedding surface flare outward as shown in Figs. 1(a) and 2. The outward flaring of the throat means mathematically that the inverse of the embedding function $r(z)$ must satisfy $d^2r/dz^2 > 0$ at or near the throat, $r = b$. In exploring the consequences of this constraint, we start with Eq. (27) in the inverted form,

$$\frac{dr}{dz} = \pm\left(\frac{r}{b(r)} - 1\right)^{1/2}. \qquad (53)$$

Differentiating this with respect to z we obtain one version of the flaring-out condition:

$$\frac{d^2r}{dz^2} = \frac{b - b'r}{2b^2} > 0 \text{ at or near the throat, } \quad r = b. \qquad (54)$$

A second version can be obtained as follows. By combining Eqs. (52) and (54) we may rewrite ζ at any radius r as

$$\zeta = \frac{2b^2}{r|b'|}\left(\frac{d^2r}{dz^2}\right) - 2(r-b)\frac{\Phi'}{|b'|}. \qquad (55)$$

This relation, together with finiteness of ρ and hence of b' [Eq. (17)] and the fact that $(r-b)\Phi' \to 0$ at the throat (see above), enables us to rewrite the flaring-out condition (54) as

$$\zeta_0 = \frac{\tau_0 - \rho_0 c^2}{|\rho_0 c^2|} > 0 \text{ at or near the throat }, \quad r = b = b_0. \qquad (56)$$

2. *Troublesome aspects of* $\tau_0 > \rho_0 c^2$

The constraint $\tau_0 > \rho_0 c^2$ is deeply troublesome; it says that in the throat the tension must be so large as to exceed the total density of mass-energy $\rho_0 c^2$. We shall call material with this property, $\tau > \rho c^2 > 0$, "exotic."

The exotic nature of the wormhole's throat material, $\tau_0 > \rho_0 c^2$ is especially troublesome because of its implications for measurements made by an observer who moves through the throat with a radial velocity close to the speed of light, $\gamma \gg 1$. Such an observer sees an energy density [projection of the stress-energy tensor (13) on her time basis vector $\mathbf{e}_{\hat{0}'} = \gamma \mathbf{e}_{\hat{t}} \mp \gamma(v/c)\mathbf{e}_{\hat{r}}$] given by

$$T_{\hat{0}'\hat{0}'} = \gamma^2 T_{\hat{t}\hat{t}} \mp 2\gamma^2(v/c)^2 T_{\hat{t}\hat{r}} + \gamma^2(v/c)^2 T_{\hat{r}\hat{r}}$$
$$= \gamma^2[\rho_0 c^2 - (v/c)^2\tau_0] = \gamma^2(\rho_0 c^2 - \tau_0) + \tau_0. \qquad (57)$$

If such an observer moves sufficiently fast (sufficiently large γ), the observer will see a negative density of mass-energy! It, perhaps, is only a small step from this unavoidable property of the wormhole's throat material to the use of material in which static observers also see a negative energy density, $\rho_0 c^2 < 0$. However, some readers may wish to minimize the "exoticity" of the material by demanding

$$\rho c^2 \geqslant 0 \text{ everywhere: a possible constraint.} \qquad (58)$$

[*Note added in proof:* Don Page has pointed out to us that not only must a static, spherical wormhole throat be threaded by matter whose mass-energy density as seen by some observers is negative ("exotic matter"), but this is also true for any traversible, nonspherical, and nonstatic wormhole. Roughly speaking, the reason is that bundles of light rays (null geodesics) that enter the wormhole at one mouth and emerge from the other must have cross-sectional areas that initially decrease and then increase. The conversion from decreasing to increasing can only be produced by gravitational repulsion of matter through which the light rays pass, a repulsion that requires negative energy density. (A more rigorous statement is this: A roughly spherical surface on one side of the wormhole throat, from the viewpoint of the other side, is an "outer trapped surface"—which, by Proposition 9.2.8 of Ref. 22, is possible only if the "weak energy condition" is violated, i.e., only if some observers near the throat see a negative mass-energy density.)]

In the remainder of this section we shall ponder the possible existence of the exotic material necessary for wormhole construction. Our pondering will lead us to the forefront of current research—and in doing so will necessitate a change of style: Rather than present the full details as above, we shall only mention briefly the principal issues, and for each issue give references to the literature.

In the 1960s and early 1970s most physicists regarded as almost sacred the assertion that no observer should ever be able to measure a negative energy density. This assertion carries the name "weak energy condition"[22]; and when augmented by additional constraints it is called the "dominant energy condition" or the "strong energy condition."[22] These energy conditions, all of which will be violated by matter with $\tau > \rho c^2$, are key foundations for a number of important theorems—e.g., the "positive mass theorem," which says that objects made of matter that satisfies the dominant energy condition can never antigravitate (can never repel other bodies gravitationally)[23]; a variety of theorems that predict that if one or another of the energy conditions is satisfied, then spacetime singularities will be created in cosmological situations and in gravitational collapse[24]; and the "second law of black hole mechanics," which says that if all stress-energy near a black hole horizon satisfies the strong energy condition, then the horizon's surface area can never decrease.[25]

The discovery, by Hawking,[26] that nonrotating black holes can evaporate and, correspondingly, that their surface areas can shrink, in violation of the second law of black hole mechanics, forced physicists to face up to the fact that quantum fields can violate the energy conditions. Stated more precisely, there are quantum states in which the (renormalized) expectation value of the stress-energy tensor violates all of the energy conditions.[27] One very general situation for such violations is the quantum mechanical creation of particles. In fact, particle creation always entails a violation of the energy conditions.[28] As an example, any static observer just above the horizon of an isolated (surrounded-by-vacuum) Schwarzschild black hole will see a time-independent, negative expectation value for the energy density.[29] This negative energy density is associated with the creation of particles near the horizon, particles that subsequently will evaporate, and correspondingly with a flow of negative energy into the horizon, negative energy that causes the horizon to shrink in response to the evaporation.

Another situation where quantum fields can have negative energy density, violating the energy conditions, is a squeezed state of the electromagnetic field.[30] Such a state has recently become a practical reality in the laboratory[31] as a result of the nonlinear-optics technique of *squeezing*, i.e., of moving some of the quantum fluctuations of laser light out of the $\cos \omega(t - z/c)$ part of the beam and into the $\sin \omega(t - z/c)$ part. If one squeezes the vacuum[32] (i.e., if one puts vacuum rather than laser light into the input port of a squeezing device), then one gets at the output an electromagnetic field with weaker fluctuations and thus less energy density than the vacuum at locations where $\cos^2 \omega(t - z/c) \simeq 1$ and $\sin^2 \omega(z - t/c) \ll 1$; but with greater fluctuations and thus greater energy density than the vacuum at locations where $\cos^2 \omega(t - z/c) \ll 1$ and $\sin^2 \omega(t - z/c) \simeq 1$. Since the vacuum is defined to have vanishing energy density, any region with less energy density than the vacuum actually has a negative (renormalized) expectation value for the energy density. Thus a "squeezed vacuum state" consists of a traveling electromagnetic wave that oscillates back and forth between negative energy density and positive energy density, but has positive time-averaged energy density (by contrast with the near-horizon region of an evaporating black hole where the negative energy density is time independent).

These examples of violations of the energy conditions give warning that one should not blithely assume the impossibility of the exotic material that is required for the throat of a traversible wormhole. Another warning comes from the fact that a time-independent, radial electric field or magnetic field threading the wormhole is right on the borderline of being exotic; if its tension were infinitesimally larger, for a given energy density, it would satisfy our wormhole-building needs.

It may well be that the fundamental laws of physics forbid exotic material on the macroscopic scales required for wormhole building; but the authors know of no way to prove so and, in fact, would not be extremely surprised if a quantum-field-theoretic example of such material were found in the near future. The search for such an example, or an impossibility proof, is an interesting challenge.

Sometimes one sees assertions that the speed with which signals should propagate in exotic material is $|\tau/\rho|^{1/2}$ which exceeds the speed of light, and that, therefore, macroscopic exotic material is forbidden. However, this is not necessarily so. To prove that $|\tau/\rho|^{1/2}$ is the signal speed, one needs a detailed theory of the material not just in static situations like that of our wormholes but also in dynamical situations, and one must study, using such a detailed theory, the speed of propagation of signals (group velocity), not just that of monochromatic waves (phase velocity). Situations in which the group velocity is much more complicated than simply $|\tau/\rho|^{1/2}$ are very common, e.g., in plasma physics.[33] For further discussion see, e.g., Ref. 34.

As a warning that one should not assert too strongly the impossibility of exotic matter on macroscopic scales, one can look back at the history of physicists' beliefs about extreme equations of state. Prior to 1961, it seems to have been believed universally that the trace of the stress-energy tensor must always be positive, and thus that in a medium with isotropic pressure p and mass density ρ the pressure can never exceed one-third the mass-energy density: $p \leqslant \rho c^2/3$. One finds this asserted without proof, e.g., in the classic papers of Oppenheimer and colleagues[35] (1939) and of Wheeler and colleagues[36] (1957) on neutron-star equations of state. However, in 1961, Zel'dovich[37] gave an explicit example, in quantum field theory, of a field that leads macroscopically to an isotropic equation of state with $p = \rho c^2$, and many experts today believe that matter actually behaves in this manner at densities above about 10 times nuclear.[38] Similarly, the beliefs[22] of the 1960s and early 1970s that matter must always possess positive energy density and satisfy $|\tau| \leqslant \rho c^2$, even on microscopic length scales, have been supplanted more recently by the realization that this is not so.[27] It may well be that today's widely held prejudices for $|\tau| \leqslant \rho c^2$ when averaged over macroscopic scales and over time (exotic material) will also fall when we better understand the laws of physics. Such better understanding (the discovery by Eda of a field that produces an anisotropic stress with $\tau > \rho c^2 > 0$ along one direction), is the key in Sagan's novel (Box 1) to an understanding of the characters' wormhole travel experiences.

3. *Ways to minimize the use of exotic material*

Since exotic material is so problematic, it behooves us to use as little of it as possible in our wormhole solutions. We shall quantify the amount of exotic material used by the function $\zeta(r) = (\tau - \rho c^2)/\rho c^2$, and in constructing specific wormhole solutions (in the Appendix), we shall rely on three different methods to limit the amount used.

(a) Use exotic material ($\zeta > 0$) throughout the wormhole, but insist that the density of exotic material fall off rapidly with radius as one moves away from the throat. This is the least pleasing of the three methods. An example is $b = \text{const}$, $\Phi = 0$, which has the following stress-energy profile:

$$\rho(r) = 0, \quad \tau(r) = b_0/(8\pi G c^{-4} r^3),$$
$$p(r) = b_0/(16\pi G c^{-4} r^3), \quad \zeta = \infty. \tag{59}$$

This solution has the unattractive feature that ζ is positive and huge everywhere, but the exotic material does fall off rapidly with radius.

(b) Use exotic material as the only source of curvature, but cut it off completely at some radius R_S: $\zeta > 0$ for all $r < R_S$, and $\rho c^2 = \tau = p = 0$ for $r > R_S$. This method may be better than the first, but not so good as the following.

(c) Relegate the exotic material to a tiny central region $-l_c < l < +l_c$ around the throat, and surround that tiny region with normal matter: $\zeta > 0$ for $|l| < l_c$; and $\zeta \leqslant 0$ for $|l| \geqslant l_c$.

In the Appendix we exhibit wormhole solutions that make use of each of these three methods.

4. *Physical coupling of the wormhole material to a space traveler*

It could be extremely uncomfortable for a human traveler to interact with a material that has tensions as large as $\tau_0 \sim 5 \times 10^{41}$ dyn/cm^2$(10\ \text{m}/b_0)^2$ [Eq. (51)]. There are two ways to protect the traveler from such interaction: (a) The spherical symmetry of the wormhole might be broken by passing a vacuum tube with diameter $\ll b_0$ down the wormhole, and by using stresses in the tube's walls to hold the exotic matter out. This seems to have been the method used in Sagan's novel[19]; but the only way to check its viability is by a study of nonspherical wormhole solutions of the Einstein equations—a study well beyond the scope of this article. (b) The wormhole material, like neutrinos and gravitational waves, might couple only very weakly to the human body. Then, despite its huge stress and mass density, the material could penetrate the traveler's body and not exert noticeable forces on her.

G. Stability of the wormhole

In the absence of a detailed understanding of the exotic matter that threads the wormhole's throat, it is impossible to say anything concrete about the stability of the wormhole against small or large perturbations—such as those produced by a traversing spacecraft. However, one should keep in mind that, even if the wormhole might naturally be unstable, an advanced civilization might be able to monitor its structure and use feedback forces to prevent instabilities from growing. [Note that a wormhole, being something that—if stable or stabilized—has a spatial structure that persists over time, is quite different from a white hole antihorizon or Kerr tunnel Cauchy horizon (see Sec. I A 4), which as seen by any physical observer at its location has only a transient existence. The difference is that of a timelike entity (wormhole) versus a lightlike entity (antihorizon or Cauchy horizon). This difference may well make it far easier for an advanced civilization to stabilize a wormhole than an antihorizon or Cauchy horizon.]

H. Assembly of the wormhole

Even Sagan[19] declines to face the problem of how to assemble a traversible wormhole; he leaves that in the hands of an ancient, extinct civilization. The assembly might seem especially daunting because it entails a change in the topology of space; and in classical general relativity such changes probably entail spacetime singularities,[39] which will only be properly understood after gravity has successfully been quantized. On the other hand, there is strong reason to believe[40] that on length scales of order of the Planck–Wheeler length, $l_{P-W} \equiv (G\hbar/c^3)^{1/2} = 1.6 \times 10^{-33}$ cm, quantum-gravity effects dominate and produce a foamlike, multiply connected spacetime structure. One could *imagine* an exceedingly advanced civilization pulling a wormhole out of this submicroscopic, quantum mechanical, spacetime foam and enlarging it and moving its openings around the universe until it has assumed the size, shape, and location required for some specific interstellar travel project. Note the emphasis on the word imagine; we today are far from being able to analyze such a process theoretically. Any such analysis will require a reliable understanding of quantum gravity.

I. Backward time travel using two wormholes

We may picture our wormhole as connecting two widely separated regions of flat spacetime [cf. Fig. 1(b); the distance is large between Earth and Vega the "long way around"]. If the (exterior) distance between the "mouths" of the wormhole is long enough and the (interior) wormhole distance is short enough, then the two events representing (i) a given traveler entering one mouth and (ii) the same traveler leaving the wormhole by the other mouth will be seen by observers outside the wormhole as having a spacelike separation. In an external inertial reference frame that moves at high speed from the first mouth toward the second, the exit event (ii) in fact will precede the entry event (i). If there were a second wormhole with mouths at rest in this high-speed frame, the traveler upon exiting from the first wormhole could accelerate up to the speed of the second wormhole's mouths, then plunge down the second wormhole, and return through it to her starting point before she started [before event (i)].

Thus it would seem that if advanced civilizations can build multiple wormhole spacetimes with adjustable relative velocities, then such civilizations can use them for backward time travel and causality violation. There seems to be no *a priori* reason that would preclude such multiple wormholes if single wormholes were constructable. Some readers may regard this as indicating that the laws of physics will prevent the assembly of even single wormholes. Other readers will await a definitive answer from future research as to what the laws of physics prevent and what they permit. [*Note added in proof:* Since writing this, we have discovered that from a single wormhole an arbitrarily advanced civilization can construct a machine for backward time travel.[41]]

IV. CONCLUSION

The wormhole solutions to Einstein's equations presented in this article are not only a pedagogical tool for teaching general relativity. Today they are also an intriguing possibility for actual construction by advanced civilizations. However, any hope that they might be constructable must rely on the future discovery of an exotic field or quantum state of known fields with tension that exceeds energy density on macroscopic length scales. We must keep in mind that such exotic fields or states *might* eventually be ruled out on fundamental microphysical grounds, and that such an exclusion would prevent our wormholes by fiat. Moreover, even if an exotic field or state were available, several other difficulties might prevent construction of real traversible wormholes: The topology change required for wormhole formation may not be classically allowed, is not quantum mechanically understood, and might be quantum mechanically forbidden. The wormholes might be unstable and even unstabilizable. The existing exotic field might interact only very strongly with ordinary matter—preventing human travel because of overbearing field stresses. And the backward time travel that appears to be permitted by such wormholes might, in some as yet unimagined way, prevent their construction. Nevertheless, we do not know today enough to either affirm or refute these difficulties, and we correspondingly cannot now rule out traversible spacetime wormholes.

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APPENDIX: SPECIFIC WORMHOLE SOLUTIONS

In this Appendix we present three specific solutions to the Einstein equations (17)–(19) for traversible wormholes. Each of these solutions is designed to satisfy all the constraints discussed in Sec. III and summarized in Box 3; and each uses a different method [from Sec. III F 3] of trying to limit the amount of exotic matter.

1. The zero-tidal-force solutions

A simple class of solutions results when we set $\Phi = 0$ everywhere—corresponding to precisely zero tidal force as seen by stationary observers [cf. Eqs. (49) and (50)]:

$$b = b(r), \quad \Phi = 0, \tag{A1a}$$

$$\rho(r) = b'(r)/(8\pi G c^{-2} r^2), \tag{A1b}$$

$$\tau(r) = b(r)/(8\pi G c^{-4} r^3), \tag{A1c}$$

$$p(r) = (b - b'r)/(16\pi G c^{-4} r^3), \tag{A1d}$$

$$z(r) = \pm \int_{b_0}^{r} \frac{dr}{[r/b(r) - 1]^{1/2}}, \tag{A1e}$$

$$\zeta(r) = (b - b'r)/|b'r|. \tag{A1f}$$

The shape function $b(r)$ that generates these solutions must satisfy the wormhole-shaping conditions of Sec. III C. Two particularly simple examples are $b(r) = (b_0 r)^{1/2}$ and $b(r) = \frac{4}{3} b_0 - \frac{1}{3} b_0 (b_0/r)^2$. For both of these shape functions the wormhole's material extends all the way from the throat out to $l = \pm \infty$, and it is everywhere exotic ($\tau > \rho c^2 > 0$ everywhere). Of course, the density, tension, and pressure ρ, τ, and p all asymptote toward zero as $l \to \pm \infty$, so that (in the neighborhood of Earth, for example) the exotic material will be unmeasurably weak.

Consider a spaceship traveling radially through the wormhole with its propulsion power shut off. Equation (43) tells us that, since travelers in the spaceship feel no acceleration (since $a = 0$), the spaceship must travel with constant γe^{Φ}. For the zero-tidal-force solutions ($\Phi = 0$), this corresponds to constant $\gamma = (1 - v^2/c^2)^{-1/2}$ and hence to constant speed $v = dl/dt$ as measured by static observers,

$$v = \frac{dl}{dt} = \text{const for unpowered spaceship.} \tag{A2}$$

Consider specifically now our first choice,

$$b(r) = (b_0 r)^{1/2} \tag{A3}$$

for the wormhole shape function. Plugging into Eqs. (A1) we find that this choice requires the exotic material to have the following equations of state:

$$\tau/2 = 2p = \rho c^2, \quad \rho > 0 \text{ everywhere.} \tag{A4}$$

We can integrate Eq. (A1e) for this simple choice of $b(r)$ to find the embedding function, $z(r)$:

$$z(r) = \pm 4 b_0 [(\sqrt{r/b_0} - 1)^{3/2}/3 + (\sqrt{r/b_0} - 1)^{1/2}], \tag{A5}$$

and we can similarly determine proper distance through the wormhole [Eq. (32)]:

$$l(r) = \pm 4 b_0 \left[\left(\frac{r}{b_0}\right)^{3/4} \frac{(\sqrt{r/b_0} - 1)^{1/2}}{4} \right] \pm 4 b_0 \left\{ \frac{3}{8} \left(\frac{r}{b_0}\right)^{1/4} \left(\sqrt{\frac{r}{b_0}} - 1\right)^{1/2} + \frac{3}{8} \ln \left[\left(\frac{r}{b_0}\right)^{1/4} + \left(\sqrt{\frac{r}{b_0}} - 1\right)^{1/2} \right] \right\}. \tag{A6}$$

We shall locate our two space stations (cf. Sec. III E) at large enough radii that the factor $(1 - b(r)/r)$ differs from unity by only 1% [in a region that is very nearly flat, cf. constraint (40)]. That is, we take the radial location of the two stations to be $r_1 = r_2 = 10^4 b_0$, corresponding to $l_1 = l_2 \simeq 10^4 b_0$. Now we shall proceed to calculate how fast a traveler can traverse the entire wormhole from station 1 at $-l_1$ in the lower universe to station 2 at $\pm l_2$ in the upper universe (cf. Fig. 2). Consider the acceleration and tidal-force constraints (44), (49), and (50). We shall at first ignore the acceleration at leaving station 1 and the deceleration upon arriving at station 2 and instead assume that our traveler maintains constant speed v throughout her trip. Then the acceleration constraint (44) is trivially satisfied since $\Phi = 0$ and γ remains constant for the trip. The radial tidal acceleration (49) is also identically zero since $\Phi = 0$ everywhere. We are thus left with constraint (50) limiting the tidal forces associated with motion through the tunnel:

$$\frac{\gamma^2}{2r^2}\left(\frac{v}{c}\right)^2 \left| b' - \frac{b}{r} \right| \lesssim \frac{1}{(10^{10}\ \text{cm})^2}. \tag{A7}$$

We substitute our particular solution to obtain

$$\frac{\gamma^2}{4r^3}\left(\frac{v}{c}\right)^2 (b_0 r)^{1/2} \lesssim \frac{1}{(10^{10}\ \text{cm})^2}. \tag{A8}$$

This constraint is most severe for the smallest radius $r = b_0$ (at the throat)

$$\gamma v/c \lesssim 2 \times 10^{-7} (b_0/10\ \text{m}). \tag{A9}$$

In the limit that the motion is nonrelativistic ($v/c \ll 1, \gamma \simeq 1$) we obtain:

$$v \lesssim 60\ \text{m/s} (b_0/10\ \text{m}). \tag{A10}$$

Correspondingly, the total time lapse for travel from station 1 to station 2 [Eqs. (41)] is the same (since $\gamma \simeq 1$, $\Phi = 0$) for clocks ticking in the stations and on board the spaceship:

$$\Delta\tau_T \simeq \Delta t \simeq \int_{-l_1}^{l_2} \frac{dl}{v} \simeq 2 \times 10^4 \frac{b_0}{v} \simeq 3 \times 10^3\ \text{s} (v/60\ \text{m/s})^{-1}. \tag{A11}$$

Thus the total trip time through such a tunnel can be made a comfortable hour with maximum tidal force of 1 Earth gravity at the midpoint of the journey.

Because the velocity of travel is so small [Eq. (A10)], acceleration at the beginning of the trip and deceleration at the end have no significant effect on the above conditions.

2. A solution with a finite radial cutoff of the stress-energy

We turn next to solutions that confine the exotic material to a finite region around the wormhole. To accomplish this we shall use a simple zero-tidal-force solution interior to a surface radius R_S and join it smoothly near there to an exterior Schwarzschild solution. We choose

$$b = b_0 (r/b_0)^{1-\eta}, \quad \text{with } 0 < \eta = \text{const} < 1,$$

$$\Phi = \Phi_0 = \text{const}, \quad \text{for } b_0 \leqslant r < R_S. \tag{A12}$$

Given this form for the interior wormhole functions, $b(r)$ and $\Phi(r)$, we obtain from Eqs. (17)–(19) the stress-energy profile,

$$\rho(r) = (1-\eta) b(r)/(8\pi G c^{-2} r^3), \quad \tau(r) = \rho c^2/(1-\eta), \quad p(r) = \eta \rho c^2 / 2(1-\eta). \tag{A13}$$

Correspondingly, the exoticity, $\zeta(r)$ [Eq. (52)], is a constant,

$$\zeta(r) = \eta/(1-\eta). \tag{A14}$$

We note that this "interior" solution satisfies all of the wormhole constraints: There is a throat at $r = b_0$; there are no horizons (since $\Phi = \text{const}$ everywhere); and the embedding diagram is outward flaring for $\eta > 0$. Indeed, this somewhat more general solution reduces to our first specific choice in Sec. 1 of the Appendix above when we set $\eta = \frac{1}{2}$ and $\Phi_0 = 0$. For simplicity, we shall henceforth restrict our analysis to the case $\eta = \frac{1}{2}$.

Recall that the Einstein field equations and the law of radial force balance permit discontinuities in ρ and p, but require τ to be continuous [paragraph following Eq. (22)]. Thus, in order to bring τ to zero near the surface radius R_S, we must join the above interior solution onto a transition layer at R_S, which, in turn, we join to external vacuum at $R_S + \Delta R$. A simple choice for this transition layer is

$$\rho(r) = [\tau(R_S)/c^2](R_S/\Delta R), \tag{A15a}$$

$$\tau(r) = \tau(R_S) - [\tau(R_S)/\Delta R](r - R_S), \quad \text{for } R_S < r < R_S + \Delta R \tag{A15b}$$

(constant density and linear decrease of τ to zero). The Einstein equations (14), (15), and (19) then enforce,

$$b(r) = \tfrac{8}{3}\pi G c^{-4}(r^3 - R_S^3)[R_S \tau(R_S)/\Delta R] + b(R_S), \tag{A15c}$$

$$\Phi'(r) = [-8\pi G c^{-4}\tau r^3 + b(r)]/[2r(r - b(r))], \tag{A15d}$$

$$p(r) = (r/2)[(\rho c^2 - \tau)\Phi' - \tau'] - \tau \tag{A15e}$$

We shall choose the thickness of the transition layer to be $\Delta R = b(R_S)$ for simplicity, and shall assume that it lies far from the throat, $R_S \gg b_0$, so that $\Delta R = b(R_S) \ll R_S$ [cf. Eq. (A12) with $\eta = \frac{1}{2}$]. Then Eqs. (A15) imply that, aside from fractional errors of order $\Delta R/R_S \ll 1$, b, Φ', and τ change linearly through the layer, while p and ρ are constant:

$$b(r) = b(R_S) + [(r - R_S)/\Delta R]b(R_S), \quad \text{so } B = b(R_S + \Delta R) = 2b(R_S), \tag{A16a}$$

$$\Phi'(r) = [(r - R_S)/\Delta R](B/2R_S^2), \quad \text{so } \Phi'(R_S + \Delta R) = B/2R_S^2, \tag{A16b}$$

$$\tau(r) = \tau(R_S) - [(r - R_S)/\Delta R]\tau(R_S), \quad \text{so } \tau(R_S + \Delta R) = 0, \tag{A16c}$$

$$p(r) = (R_S/2\Delta R)\tau(R_S), \tag{A16d}$$

$$\rho(r) = (R_S/\Delta R)[\tau(R_S)/c^2]. \tag{A16e}$$

Equations (A16a,b) permit a match at $r = R_S + \Delta R$ onto a vacuum Schwarzschild solution [Eqs. (21), (22)]. Equations (A16c,d) show that the transition layer is using an enormous "Roman arch" pressure to counterbalance the radial gradient of τ, as it brings τ to zero. By comparison with these two huge, counterbalancing internal forces, the gravitational force on the layer is negligible. Equations (A16c,d,e) show that the equations of state of the layer's material are $p = \rho/2$, and {ρ independent of τ, as τ varies}. Our choice for ρ in the layer [Eq. (A15a)] was determined by the desire that the layer's material be nonexotic.

We shall locate the terminal space stations at the edge of the Schwarzschild region ($r_1 = r_2 = R_S + \Delta R$) and shall require that the traveler be able to stop there without being gravitationally crushed. The accelerative constraint of Eq. (44) is then the most severe [the tidal constraints of Eqs. (49) and (50) are smaller]:

$$|\Phi'(R_S + \Delta R)| = B/2R_S^2 \leqslant (9.2\times 10^{15}\ \text{m})^{-1}. \tag{A17}$$

Here, we have used $B/2 = \Delta R \ll R_S$ and Eq. (A16b) for Φ'. By virtue of Eq. (A12) with $\eta = \frac{1}{2}$, this corresponds to

$$R_S > 1\times 10^{11}\ \text{m}(b_0/10\ \text{m})^{1/3} \simeq 0.6\ \ \text{a.u.}(b_0/10\ \text{m})^{1/3}, \tag{A18}$$

i.e., we must make the wormhole's surface radius R_S very large in order to keep small the acceleration of gravity on the terminal space stations. This large value of R_S implies, through Eqs. (A12) and (A16a), that

$$B \equiv b(R_S + \Delta R) = 2b(R_S) \simeq 1.9\times 10^6\ \text{m}(b_0/10\ \text{m})^{2/3}. \tag{A19}$$

When the exterior Schwarzschild solution,

$$b(r) = B \text{ and } \Phi(r) = \tfrac{1}{2}\ln(1 - B/r) \quad \text{for } r > R_S + \Delta R, \tag{A20}$$

is matched onto the layer (A16), we find, aside from fractional corrections of order $\Delta R/R_S \ll 1$, that Φ in the wormhole's interior has the value

$$\Phi_0 = \tfrac{1}{2}\ln(1 - B/R_S) \simeq B/2R_S. \tag{A21}$$

This shows that $e^{2\Phi_0} = (1 - B/R_S)$ differs from unity by only a very small amount so that the proper time measured by static observers is nearly the same as coordinate time t throughout the wormhole. Correspondingly, the analysis of travel through the wormhole as given in Sec. IV A remains valid here; and, in particular, travel with comfort requires $v \lesssim (60\ \text{m/s})(b_0/10\ \text{m})$ in the throat.

In Sec. 1 of the Appendix we permitted v to remain this small throughout the journey. Now, however, the stations are $\sim 10^6$ times farther from the throat (the great distance being forced by the demand that the acceleration of gravity be bearable on the stations); and, correspondingly, we must ask our traveler to hasten her journey by using a variable velocity v. Her varying velocity is constrained by tidal gravity [the "motional constraint" of Eq. (50)],

$$\left|\frac{\gamma^2}{2r^2}\left(\frac{v}{c}\right)^2\left(\frac{rb' - b}{r}\right)\right| \leqslant \frac{1}{(10^8\ \text{m})^2}, \tag{A22}$$

which reduces [by virtue of Eq. (A12) with $\eta = \frac{1}{2}$, and assuming $\gamma \simeq 1$] to

$$\frac{v}{c} \leqslant \left(\frac{2r}{10^8\ \text{m}}\right)\left(\frac{r}{b_0}\right)^{1/4}, \quad \text{i.e., } v \lesssim (60\ \text{m/s})\left(\frac{r}{b_0}\right)^{5/4}\left(\frac{b_0}{10\ \text{m}}\right). \tag{A23}$$

Her velocity is also constrained by the demand that she not feel too large an acceleration [Eq. (44) with $\gamma \simeq 1$, Φ constant, and $v \simeq dl/dt$]:

$$\left|\frac{dv}{dt}\right| = \left|\frac{d^2 l}{dt^2}\right| \lesssim g_\oplus. \tag{A24}$$

For concreteness, ask her to accelerate away from the lower station at $d^2l/dt^2 = +g_\oplus$ until she is halfway to the throat, then decelerate at $d^2l/dt^2 = -g_\oplus$ until she comes to rest at the throat, then accelerate away from the throat at $d^2l/dt^2 = +g_\oplus$ until she is halfway to the upper station, then finally decelerate at $d^2l/dt^2 = -g_\oplus$ until she comes to rest at the upper station. With this travel scheme her maximum velocity will be

$$v_{\max} = (\tfrac{1}{2} g_{\oplus} R_S)^{1/2} = 7\times 10^5 \text{ m/s}(R_S/10^{11} \text{ m})^{1/2}, \tag{A25}$$

which gives $\gamma \simeq 1$ (as assumed in our discussion) so long as

$$R_S \ll 10^{16} \text{ m}. \tag{A26}$$

The velocity profile, $v(r)$ associated with this scheme easily satisfies the tidal constraint (A23) at all radii; and it gives a total travel time from station to station,

$$\Delta\tau_T = \Delta t = (32 R_S/g_{\oplus})^{1/2} \simeq (7 \text{ days})(R_S/10^{11} \text{ m})^{1/2}. \tag{A27}$$

Thus this wormhole is very nicely suited to interstellar travel.

3. Solutions with exotic matter limited to the throat vicinity

If we allow ourselves to use matter with negative energy density as measured by static observers, $\rho c^2 < 0$, we can confine the exotic matter to an arbitrarily small throat region and thereby obtain an absurdly benign wormhole. An example is

$$b(r) = b_0[1-(r-b_0)/a_0]^2, \quad \Phi(r) = 0$$

$$\text{for } b_0 \leqslant r \leqslant b_0 + a_0, \tag{A28a}$$

$$b = \Phi = 0 \text{ for } r \geqslant b_0 + a_0. \tag{A28b}$$

We may use the Einstein equations (17)–(19) to tell us what kind of material would be necessary to produce this wormhole: At $b_0 < r < b_0 + a_0$ the material must have

$$\rho(r) = [(-b_0/a_0)/(4\pi G c^{-2} r^2)][1-(r-b_0)/a_0] < 0, \tag{A28c}$$

$$\tau(r) = b_0[1-(r-b_0)/a_0]^2/(8\pi G c^{-4} r^3), \tag{A28d}$$

$$p(r) = \tfrac{1}{2}[\tau(r) - \rho(r)c^2], \tag{A28e}$$

while at $r \geqslant b_0 + a_0$ spacetime is flat [Eq. (A28b)] and empty, $\rho = \tau = p = 0$. Because $\Phi = 0$ everywhere, if a traveler moves through the wormhole at constant speed v, accelerative forces are nonexistent and tidal forces are bearable so long as the motional constraint of Eq. (50) is satisfied:

$$\left|\frac{\gamma^2}{2r^2}\left(\frac{v}{c}\right)^2\left(\frac{b'r-b}{r}\right)\right| \lesssim \frac{1}{(10^8 \text{ m})^2}. \tag{A29}$$

This reduces, by virtue of Eqs. (A28), to

$$(v/c)^2 \lesssim a_0 b_0/(10^8 \text{ m})^2 \text{ at } b_0 \leqslant r \leqslant b_0 + a_0. \tag{A30}$$

The total traversal time, so long as $v/c \ll 1$, is

$$\Delta\tau_T \simeq \Delta t \simeq \pi a/v \gtrsim 1 \text{ sec}\sqrt{a_0/b_0}. \tag{A31}$$

Whatever may be the wormhole's circumference $2\pi b_0$, by choosing a_0 arbitrarily small we confine the exotic matter to a region of arbitrarily small thickness $\Delta l = \pi a_0$ and volume $4\pi^2 b_0^2 a_0$, and we ensure that it can be traversed with comfort arbitrarily quickly.

Unfortunately, when ρc^2 is constrained to be positive, the exotic matter cannot be confined to an arbitrarily small region and still yield a significant flaring outward of the embedding: A strategy to obtain a wormhole with maximal confinement of the exotic matter would be as follows. First we should allow the exotic matter to dominate the central region of the wormhole around the throat, and we should use a sufficient amount of this material to achieve a sizable flaring out in the geometry's embedding function $z(r)$, as quickly as possible in radius. This exotic region we should then join onto a region of near-exotic matter (matter with a stress-energy tensor that obeys all of the energy conditions but comes close to breaking them, so that we can go far out in radius with little or no flaring back in). Finally, at a large enough radius R_S for Schwarzschild gravitational forces to be comfortable, we should use a surface layer of thickness ΔR to match onto the vacuum Schwarzschild exterior solution. Unfortunately, it seems impossible to achieve sizable flaring outward on small scales (less than the throat size b_0) without resorting to negative ρ. To see this, recall the embedding slope from Eq. (27):

$$\frac{dz}{dr} = \pm\left(\frac{r}{b(r)} - 1\right)^{-1/2}.$$

If at some radius, $r_c = b_0 + \Delta r$, this slope is to reach 1 (which corresponds to 45° from flat in the embedding surface, a significant flaring from the throat), then we must have

$$(b_0 + \Delta r)/[b(b_0 + \Delta r)] = 2. \tag{A32}$$

If it were possible to choose $\Delta r/b_0 \ll 1$, then we could Taylor expand in $\Delta r/b_0$ to find:

$$\frac{b_0 + \Delta r}{b_0 + \Delta r b'(b_0)} = \frac{1 + \Delta r/b_0}{1 + (\Delta r/b_0)b'}$$

$$= 1 + \frac{\Delta r}{b_0}(1 - b') \simeq 2. \tag{A33}$$

This, however, is clearly impossible without choosing b' negative and hence ρ negative [Eq. (14)]. If we hold to solutions that keep ρ positive, we must allow the exotic matter to occupy a macroscopically large region of space ($\Delta r \gtrsim b_0$).

As an example of such a wormhole we choose our interior solution in and around the throat to have the same exotic form as in Secs. A and B of the Appendix above: $b(r) = (b_0 r)^{1/2}$, $\Phi = \Phi_0$ for $b_0 \lesssim r \lesssim r_c$. We wish to join this onto nonexotic matter at a radius r_c. It turns out that both the size of the wormhole and the resultant traversal time are dominated by the embedding slope at r_c; and, correspondingly, usability dictates that the slope be chosen rather small. With the prescience of hindsight, we pick $dz/dr(r_c) = \frac{1}{10}$, which gives $r_c = 10^4 b_0$. For $r_c \lesssim r \lesssim R_S$ we choose $b(r) = r/100$, $\Phi = \Phi_0$ with accompanying stress-energy $\tau = \rho c^2, p = 0$. Finally, at R_S we drop τ to zero, as in Sec. B of the Appendix above, in a surface layer of thickness $\Delta R = b(R_S)$ and mass density $\rho = R_S \tau(R_S)/\Delta R c^2$. This creates a gradient in Φ at the outer edge of the layer, of size

$$\Phi' = B/2R_S^2 = 1/100 R_S, \tag{A34}$$

so that $B = R_S/50$ in the external Schwarzschild region. If, as in Sec. B of the Appendix, we locate the terminal space stations just outside the transition layer, then the demand that the gravitational acceleration on the stations be bearable [Eq. (40)] implies, independently of b_0,

$$R_S \gtrsim 9.2\times 10^{13} \text{ m} \sim 600 \text{ a.u.} \tag{A35}$$

Thus the surface radius must be quite large to keep life in the stations comfortable. It is this hindsight that has prompted us to require r_c so large. We may now write down the complete wormhole solution:

$$b(r) = \begin{cases} (b_0 r)^{1/2}, \quad \text{at } b_0 \leqslant r \leqslant r_c = 10^4 b_0, \\ \frac{1}{100} r, \quad \text{at } r_c \leqslant r \leqslant R_S \cong 9.2 \times 10^{13} \text{ m}, \\ \frac{1}{3}[(r^3 - R_S{}^3)/R_S{}^2] + R_S/100, \quad \text{at } R_S \leqslant r \leqslant R_S + \Delta R, \quad \Delta R = R_S/100, \\ B \equiv R_S/50, \quad \text{at } R_S + \Delta R \leqslant r; \end{cases} \tag{A36a}$$

$$\Phi(r) = \begin{cases} \Phi_0 \cong -0.01, \quad \text{at } b_0 \leqslant r \leqslant R_S + \Delta R, \\ \frac{1}{2} \ln(1 - B/r), \quad \text{at } R_S + \Delta R \leqslant r; \end{cases} \tag{A36b}$$

$$\begin{cases} \zeta = 1 \text{ and } \rho c^2 = \tau/2 = 2p \text{ for } b_0 \leqslant r \leqslant r_c, \\ \zeta = 0 \text{ and } \rho c^2 = \tau, \ p = 0 \text{ for } r_c \leqslant r \leqslant R_S, \\ \text{transit. surf. layer with } \rho = R_S \tau(R_S)/\Delta R c^2 \text{ for } R_S \leqslant r \leqslant R_S + \Delta R, \\ \text{vacuum where } \rho c^2 = \tau = p = 0 \text{ for } R_S + \Delta R \leqslant r. \end{cases} \tag{A36c}$$

It is straightforward to verify that, if this wormhole is traversed in the same manner as that of Sec. B of the Appendix (accelerate at $g_\oplus$ from the lower station, decelerate at $g_\oplus$ into the throat, accelerate at $g_\oplus$ from the throat, and decelerate at $g_\oplus$ to the upper station), then the trip will be fully comfortable and will require a total time of about 200 days.

[*Note added in proof:* For yet another solution with exotic matter confined to the immediate vicinity of the throat—one whose exoticity $\zeta < 0$ is produced by the Casimir effect, see Ref. 41.]

[1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), cited in text as MTW.

[2] See, e.g., R. Penrose, Rivi. Nuovo Cimento **1**, special number, 252 (1969).

[3] White holes were first introduced into physics—under the alternative name "lagging core"—by I. D. Novikov, Astron. Zh. **41**, 1075 (1964) [Sov. Astron. A. J. **8**, 857 (1965)], and independently by Y. Ne'eman, Astrophys. J. **141**, 1303 (1965). For a recent bibliography on white holes see A. P. Trofimenko and V. S. Gurin, Gen. Relativ. Gravit. **18**, 53 (1986).

[4] For a brief introduction to black holes, white holes, and wormholes with emphasis on their roles as exemplifying nonlinear features of Einsteinian gravity, see K. S. Thorne, in *Nonlinear Phenomena in Physics*, edited by F. Claro (Springer-Verlag, Berlin, 1985), p. 280.

[5] D. M. Eardley, Phys. Rev. Lett. **33**, 442 (1974); the quantum-field-theory extension of this classical result is found in R. M. Wald and S. Ramaswamy, Phys. Rev. D **21**, 2736 (1980).

[6] For a popular discussion see, e.g., W. J. Kaufmann, III, *Black Holes and Warped Spacetime* (Freeman, San Francisco, 1979), Chap. 6; for the original technical analyses see R. H. Boyer and R. W. Lindquist, J. Math. Phys. **8**, 265 (1967), and B. Carter, Phys. Rev. **141**, 1242 (1966); **174**, 1559 (1968). Such tunnels also occur, and indeed were first discovered, in the Reissner–Nordstrom solution of the Einstein field equations—a solution that describes the external gravitational field of a nonrotating, electrically charged black hole; for the original analysis see J. C. Graves and D. R. Brill, Phys. Rev. **120**, 1507 (1960).

[7] This proof consists of a demonstration that the Kerr metric is the most general external gravitational field of a rotating, time-independent black hole [theorem due to W. Israel, B. Carter, S. W. Hawking, D. C. Robinson, G. Bunting, and P. O. Mazur; see references and discussion in Sec. 6.7 of the chapter by B. Carter in *General Relativity, an Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge U. P., Cambridge, 1979), p. 294, and in the chapter by P. O. Mazur in *Proceedings of the 11th International Conference on General Relativity and Gravitation*, edited by M. MacCallum (Cambridge U. P., Cambridge, 1987)]; plus theorems and examples showing how a black hole settles down into its final Kerr (or, if nonrotating, Schwarzschild) state by emitting gravitational radiation [due to I. D. Novikov, R. H. Price, and others; see Box 32.2 of MTW (Ref. 1) for discussion and references].

[8] The instability of Cauchy horizons was first speculated by R. Penrose, in *Battelles Rencontres*, edited by B. S. DeWitt and J. A. Wheeler (Benjamin, New York, 1968), and was later proved in a variety of different ways by M. Simpson and R. Penrose, Int. J. Theor. Phys. **7**, 183 (1973); J. M. McNamara, Proc. R. Soc. London Ser. A **358**, 499 (1978); J. M. McNamara, Proc. R. Soc. London Ser. A **364**, 121 (1978); Y. Gursel, V. D. Sandberg, I. D. Novikov, and A. A. Starobinsky, Phys. Rev. D **19**, 413 (1979); Y. Gursel, I. D. Novikov, V. D. Sandberg, and A. A. Starobinsky, Phys. Rev. D **20**, 1260 (1979); R. A. Matzner, N. Zamorano, and V. D. Sandberg, Phys. Rev. D **19**, 2821 (1979); S. Chandrasekhar and J. B. Hartle, Proc. R. Soc. London Ser. A **384**, 301 (1982). A pedagogical review will be found in I. D. Novikov and V. P. Frolov, *The Physics of Black Holes* (Nauka, Moscow, 1986). All of the above references except for the first paper cited by J. M. McNamara explictly treat instability of the inner Cauchy horizon of the Reissner–Nordstrom geometry. The first McNamara paper treats both the Reissner–Nordstrom and Kerr cases. Also see N. D. Birrell and P. C. W. Davies, Nature **272**, 35 (1978); I. D. Novikov and A. A. Starobinsky, Zh. Eksp. Teor. Fiz. **78**, 3 (1980) [Sov. Phys. JETP **51**, 1 (1980)].

[9] Ever since the 1950s, Wheeler has argued cogently that quantum gravity must govern the singularities at the endpoint of gravitational collapse (inside black holes); see the compilation of Wheeler quotations and references in K. S. Thorne and W. H. Zurek, eds., Found. Phys. **16**, 79 (1986).

[10] This was first pointed out in 1968, for the case of a Reissner–Nordstrom singularity, by J. M. Bardeen (unpublished); for a detailed analysis in the Reissner–Nordstrom case see L. H. Ford and L. Parker, Phys. Rev. D **17**, 1485 (1978).

[11] J. M. Bardeen, in *Abstracts of the 5th International Conference on Gravitation and the Theory of Relativity* (Publishing House of Tbilisi University, Tbilisi, USSR, 1968), p. 174. Bardeen's metric is actually more nearly like that of Reissner and Nordstrom (Ref. 6) than that of Kerr. In Schwarzschildlike coordinates it has the form $ds^2 = -\phi(r)c^2\,dt^2 + [\phi(r)]^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$ with $\phi(r) = [1 - 2mr^2/(r_0^2 + r^2)^{-3/2}]$ and $r_0^2 \geqslant 16m^2/27$, and is endowed with the stress-energy profile [our notation is that of Eq. (13)]: $\rho(r)c^2 = \tau(r) = 3mr_0^2/[4\pi Gc^{-4}(r^2 + r_0^2)^{5/2}]$, $p(r) = 3mr_0^2(3r^2 - 2r_0^2)/[8\pi Gc^{-4}(r^2 + r_0^2)^{7/2}]$. The causality structure of this spacetime is described by a Penrose diagram that is just like that for the Reissner–Nordstrom tunnels [Figure 34.4 of MTW for the case $r_0^2 > 16m^2/27$; Figure 1(a) of B. Carter, Phys. Lett. **21**, 423 (1966) for the case $r_0^2 = 16m^2/27$], except that $r = 0$ is here a smooth region rather than a singularity as in Reissner–Nordstrom.

[12] See, e.g., W. J. Kauffman, III, *The Cosmic Frontiers of General Relativity* (Little, Brown, Boston, 1977), Chap. 14.

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[15] A. Einstein and N. Rosen, Phys. Rev. **48**, 73 (1935).

[16] J. A. Wheeler, *Geometrodynamics* (Academic, New York, 1962); much of this book is a compilation of Wheeler's writings from the 1950s.

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PROBLEM: METHOD OF ELECTROSTATIC IMAGES—TREATING PLANE GEOMETRY AS A SPECIAL CASE OF SPHERICAL GEOMETRY

A conducting sphere of radius a, assumed to be at zero potential, is in a medium of infinite extent. A point charge q, placed at a point P inside the medium, produces an image q' at a point P' inside the sphere. Let $d = CP$ and $b = CP'$, where C is the center of the sphere. It is known that (i) P' lies on the line joining C and P, (ii) $q' = -(a/d)q$, and (iii) $b = a^2/d$. Use *these* results to find the magnitude and location of the image charge produced by q when it is placed at a distance l from a grounded, semi-infinite conducting region, $x < 0$. (Solution is on p. 471).