

Exotic solutions in General Relativity: Traversable wormholes and “warp drive” spacetimes

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(Dated: February 2, 2008)

The General Theory of Relativity has been an extremely successful theory, with a well established experimental footing, at least for weak gravitational fields. Its predictions range from the existence of black holes, gravitational radiation to the cosmological models, predicting a primordial beginning, namely the big-bang. All these solutions have been obtained by first considering a plausible distribution of matter, i.e., a plausible stress-energy tensor, and through the Einstein field equation, the spacetime metric of the geometry is determined. However, one may solve the Einstein field equation in the reverse direction, namely, one first considers an interesting and exotic spacetime metric, then finds the matter source responsible for the respective geometry. In this manner, it was found that some of these solutions possess a peculiar property, namely “exotic matter,” involving a stress-energy tensor that violates the null energy condition. These geometries also allow closed timelike curves, with the respective causality violations. Another interesting feature of these spacetimes is that they allow “effective” superluminal travel, although, locally, the speed of light is not surpassed. These solutions are primarily useful as “gedanken-experiments” and as a theoretician’s probe of the foundations of general relativity, and include traversable wormholes and superluminal “warp drive” spacetimes. Thus, one may be tempted to denote these geometries as “exotic” solutions of the Einstein field equation, as they violate the energy conditions and generate closed timelike curves. In this article, in addition to extensively exploring interesting features, in particular, the physical properties and characteristics of these “exotic spacetimes,” we also analyze other non-trivial general relativistic geometries which generate closed timelike curves.

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I. INTRODUCTION

A. Review of wormhole physics

Wormholes act as tunnels from one region of spacetime to another, possibly through which observers may freely traverse. Interest in traversable wormholes, as hypothetical shortcuts in spacetime, has been rekindled by the classical paper by Morris and Thorne [1]. It was first introduced as a tool for teaching general relativity, as well as an allurements to attract young students into the field, but it also served to stimulate research in several branches. These developments culminated with the publication of the book *Lorentzian Wormholes: From Einstein to Hawking* by Visser [2], where a review on the subject up to 1995, as well as new ideas are developed and hinted at. However, it seems that wormhole physics can originally be traced back to Flamm in 1916 [3], when he analyzed the then recently discovered Schwarzschild solution. Paging through history one finds next that wormhole-type solutions were considered, in 1935, by Einstein and Rosen [4], where they constructed an elementary particle model represented by a “bridge” connecting two identical sheets. This mathematical representation of physical space being connected by a wormhole-type solution was denoted an “Einstein-Rosen bridge”. The field laid dormant, until Wheeler revived the subject in the 1950s. Wheeler considered wormholes, such as Reissner-Nordström or Kerr wormholes, as objects of the quantum foam connecting different regions of spacetime and operating at the Planck scale [5, 6], which were transformed later into Euclidean wormholes by Hawking [7] and others. However, these Wheeler wormholes were not traversable, and furthermore would, in principle, develop some type of singularity [8]. These objects were obtained from the coupled equations of electromagnetism and general relativity and were denoted “geons”, i.e., gravitational-electromagnetic entities.

Geons possess curious properties such as: firstly, the gravitational mass originates solely from the energy stored in the electromagnetic field, and in particular, there are no material masses present (this gave rise to the term “mass without mass”); and secondly, no charges are present (“charge without charge”). These entities were further explored by several authors in different contexts [9], but due to the extremely ambitious program and the lack of experimental evidence soon died out. Nevertheless, it is interesting to note that Misner inspired in Wheeler’s geon representation, found wormhole solutions to the source-free Einstein equations in 1960 [10]. With the introduction of multi-connected topologies in physics, the question of causality inevitably arose, as a light signal travelling through the short-cut, i.e., the wormhole, could outpace another light signal. Thus, Wheeler and Fuller examined this situation in the Schwarzschild solution and found that causality is preserved [11], as the Schwarzschild throat pinches off in a finite time, preventing the traversal of a signal from one region to another through the wormhole. However, Graves and Brill [12], considering the Reissner-Nordström metric also found wormhole-type solutions between two asymptotically flat spaces, but with an electric flux flowing through the wormhole. They found that the region of minimum radius, the “throat”, contracted, reaching a minimum and re-expanded after a finite proper time, rather than pinching off as in the Schwarzschild case. The throat, “cushioned” by the pressure of the electric field through the throat, pulsated periodically in time. The modern renaissance of wormhole physics was mainly brought about by the classic Morris-Thorne paper [1]. Thorne together with his student Morris [1], understanding that the energy conditions lay on shaky ground [13, 14], considered that wormholes, with two mouths and a throat, might be objects of nature, as stars and black holes are.

Wormhole physics is a specific example of adopting the reverse philosophy of solving the Einstein field equation, by first constructing the spacetime metric, then deducing the stress-energy tensor components. Thus, it was found that these traversable wormholes possess a stress-energy tensor that violates the null energy condition [1, 2]. In fact, they violate all the known pointwise energy conditions and averaged energy conditions, which are fundamental to the singularity theorems and theorems of classical black hole thermodynamics. The weak energy condition (WEC) assumes that the local energy density is non-negative and states that $T_{\mu\nu}U^\mu U^\nu \geq 0$, for all timelike vectors U^μ , where $T_{\mu\nu}$ is the stress energy tensor (in the frame of the matter this amounts to $\rho \geq 0$ and $\rho + p \geq 0$). By continuity, the WEC implies the null energy condition (NEC), $T_{\mu\nu}k^\mu k^\nu \geq 0$, where k^μ is a null vector. The null energy condition is the weakest of the energy conditions, and its violation signals that the other energy conditions are also violated. Although classical forms of matter are believed to obey these energy conditions [15], it is a well-known fact that they are violated by certain quantum fields, amongst which we may refer to the Casimir effect and Hawking evaporation (see [16] for a short review). It was further found that for quantum systems in classical gravitational backgrounds the weak or null energy conditions could only be violated in small amounts, and a violation at a given time through the appearance of a negative energy state, would be overcompensated by the appearance of a positive energy state soon after. Thus, violations of the pointwise energy conditions led to the averaging of the energy conditions over timelike or null geodesics [17, 18]. For instance, the averaged weak energy condition (AWEC) states that the integral of the energy density measured by a geodesic observer is non-negative, i.e., $\int T_{\mu\nu}U^\mu U^\nu d\tau \geq 0$, where τ is the observer’s proper time. Thus, the averaged energy conditions permit localized violations of the energy conditions, as long as they hold when averaged along a null or timelike geodesic [17].

Pioneering work by Ford in the late 1970’s on a new set of energy constraints [19], led to constraints on negative energy fluxes in 1991 [20]. These eventually culminated in the form of the Quantum Inequality (QI) applied to energy densities, which was introduced by Ford and Roman in 1995 [21]. The QI was proven directly from Quantum Field Theory, in four-dimensional Minkowski spacetime, for free quantized, massless scalar fields. The inequality limits the magnitude of the negative energy violations and the time for which they are allowed to exist, yielding information on the distribution of the negative energy density in a finite neighborhood [21, 22, 23, 24]. The basic applications to curved spacetimes is that these appear flat if restricted to a sufficiently small region. The application of the QI to wormhole geometries is of particular interest [22, 25]. A small spacetime volume around the throat of the wormhole was considered, so that all the dimensions of this volume are much smaller than the minimum proper radius of curvature in the region. Thus, the spacetime can be considered approximately flat in this region, so that the QI constraint may be applied. The results of the analysis is that either the wormhole possesses a throat size which is only slightly larger than the Planck length, or there are large discrepancies in the length scales which characterize the geometry of the wormhole. The analysis imply that generically the exotic matter is confined to an extremely thin band, and/or that large red-shifts are involved, which present severe difficulties for traversability, such as large tidal forces [22]. Due to these results, Ford and Roman concluded that the existence of macroscopic traversable wormholes is very improbable (see [26, 27] for a review). It was also shown that, by using the QI, enormous amounts of exotic matter are needed to support the Alcubierre warp drive and the superluminal Krasnikov tube [28, 29, 30].

Relative to the energy conditions, the situation has changed drastically, as it has been now shown that even classical systems, such as those built from scalar fields non-minimally coupled to gravity, violate all the energy conditions [31]. It is interesting to note that recent observations in cosmology strongly suggest that the cosmological fluid violates the strong energy condition (SEC), and provides tantalizing hints that the NEC *might* possibly be violated in a classical

regime [32, 33, 34]. Thus, gradually the weak and null energy conditions, and with it the other energy conditions, are losing their status of a kind of law [35]. Surely, this has had implications on the construction of wormholes. In the original paper [1], Morris and Thorne first provided a spherically symmetric spacetime metric, then deduced that it needed exotic matter to sustain the wormhole geometry. The engineering work was left to an absurdly advanced civilization, which could manufacture such matter and construct these wormholes. Then, once it was understood that quantum effects should enter in the stress-energy tensor, a self-consistent wormhole solution of semiclassical gravity was found [36], presumably obeying the quantum inequalities. Thus, it seems that these exotic spacetimes arise naturally in the quantum regime, as a large number of quantum systems have been shown to violate the energy conditions, such as the Casimir effect. Indeed, various wormhole solutions in semi-classical gravity have been considered in the literature. For instance, semi-classical wormholes were found in the framework of the Frolov-Zelnikov approximation for $\langle T_{\mu\nu} \rangle$ [37]. Analytical approximations of the stress-energy tensor of quantized fields in static and spherically symmetric wormhole spacetimes were also explored in Refs. [38]. However, the first self-consistent wormhole solution coupled to a quantum scalar field was obtained in Ref. [36]. The ground state of a massive scalar field with a non-conformal coupling on a short-throat flat-space wormhole background was computed in Ref. [39], by using a zeta renormalization approach. The latter wormhole model, which was further used in the context of the Casimir effect [40], was constructed by excising spherical regions from two identical copies of Minkowski spacetime, and finally surgically grafting the boundaries (A more realistic geometry was considered in Ref. [41]). Recently, semi-classical wormholes have also been obtained using a one-loop graviton contribution approach [42, 43]. Finally with the realization that nonminimal scalar fields violate the weak energy condition, a set of self-consistent classical wormholes was found [44].

It is fair to say that, though outside this mainstream, classical wormholes were found by Homer Ellis back in 1973 [45] and further explored in [46], and related self-consistent solutions were found by Kirill Bronnikov in 1973 [47], Takeshi Kodama in 1978 [48], and Gérard Clément in 1981 [49]. These papers written much before the wormhole boom originated from Morris and Thorne's work [1] (see [50] for a short account of these previous solutions). A self-consistent Ellis wormhole was found again by Harris [51] by solving, through an exotic scalar field, an exercise for students posed in [1]. Visser [52] motivated by the aim of minimizing the violation of the energy conditions and the possibility of a traveller not encountering regions of exotic matter in a traversal through a wormhole, constructed polyhedral wormholes and, in particular, cubic wormholes. These contained exotic matter concentrated only at the edges and the corners of the geometrical structure, and a traveller could pass through the flat faces without encountering matter, exotic or otherwise. He further generalized a suggestion of Roman for a configuration with two wormholes [1] into a Roman ring [53], and in 1999 analyzed generic dynamical traversable wormhole throats [54]. Furthermore, in 1999, together with Barcelo, he found classically consistent solutions with scalar fields [31], and in collaboration with Dadhich, Kar and Mukherjee has also found self-dual solutions [55]. Other authors have also made interesting studies and significant contributions.

Before Visser's book we can quote an interesting application to wormhole physics by Frolov and Novikov, where they relate wormhole and black hole physics [56]. An interesting application of wormholes to cosmology was treated by Hochberg and Kephart relatively to the horizon problem [57]. These authors speculated that wormholes allow the two-way passage of signals between spatially separated regions of spacetime, and could permit the thermalization of these respective regions. After the Matt Visser book, González-Díaz generalized the static spherically symmetric traversable wormhole solution to that of a torus-like topology [58]. This geometrical construction was denoted as a ringhole. González-Díaz went on to analyze the causal structure of the solution, i.e., the presence of closed timelike curves, and has recently studied the ringhole evolution due to the accelerating expansion of the universe, in the presence of dark energy [59]. Wormhole solutions inside cosmic strings were found by Clément [60], and Aros and Zamorano [61]; wormholes supported by strings by Schein and Aichelburg [62, 63]; the maintenance of a wormhole with a scalar field was considered by Vollick [64], solutions with minimal and non-minimal scalar fields were explored by Kim and Kim [65] and exact solutions of charged wormholes considering the back reaction to the traversable Lorentzian wormhole spacetime by a scalar field or electric charge were found by Kim and Lee [66]; rotating wormholes solutions were analyzed by Teo [67], and further generalized by Kuhfittig [68]; a solution was found in which the exotic matter is controlled by an external magnetic field by Parisio [69]; wormholes with stress-energy tensor of massless neutrinos and other massless fields were considered by Krasnikov [70]; wormholes made of a crossflow of dust null streams were discussed by Hayward [71] and Gergely [72]; self consistent charged solutions were found by Bronnikov and Grinyok [73]; and the possible existence of wormhole geometries in the context of nonlinear electrodynamics was also explored [74]. It is interesting to note that building on [71], Hayward and Koyama, using a model of pure phantom radiation, i.e., pure radiation with negative energy density, and the idealization of impulsive radiation, considered analytic solutions describing the theoretical construction of a traversable wormhole from a Schwarzschild black hole [75], and the respective enlargement of the wormhole [76]. More recently, exact solutions of traversable wormholes were found under the assumption of spherical symmetry and the existence of a *non-static* conformal symmetry, which presents a more systematic approach in searching for exact wormhole solutions [77].

One of the main areas in wormhole research is to try to avoid as much as possible the violation of the null energy

condition. For static wormholes the null energy condition is violated [1, 2], and thus, several attempts have been made to overcome somehow this problem. In the original article [1], Morris and Thorne had already tried to minimize the violating region by constructing specific examples of wormhole geometries. As mentioned before, Visser [52] found solutions where observers can pass the throat without interacting with the exotic matter, which was pushed to the corners, and Kuhfittig [78] has found that the region made of exotic matter can be made arbitrarily small. For dynamic wormholes, the null energy condition, more precisely the averaged null energy condition can be avoided in certain regions [54, 79, 80, 81, 82, 83]. More recently, Visser *et al* [84, 85], noting the fact that the energy conditions do not actually quantify the “total amount” of energy condition violating matter, developed a suitable measure for quantifying this notion by introducing a “volume integral quantifier”. This notion amounts to calculating the definite integrals $\int T_{\mu\nu}U^\mu U^\nu dV$ and $\int T_{\mu\nu}k^\mu k^\nu dV$, and the amount of violation is defined as the extent to which these integrals become negative. Although the null energy and averaged null energy conditions are always violated for wormhole spacetimes, Visser *et al* considered specific examples of spacetime geometries containing wormholes that are supported by arbitrarily small quantities of averaged null energy condition violating matter.

Some papers have added a cosmological constant to the wormhole construction analysis. Thin-shell wormhole solutions with Λ , in the spirit of Visser [2, 86] were analyzed in [87, 88]. Roman [89] found a wormhole solution inflating in time to test whether one could evade the violation of the energy conditions, and Delgaty and Mann [90] looked for new wormhole solutions with Λ . Construction of wormhole solutions by matching an interior wormhole spacetime to an exterior vacuum solution, at a junction surface, were also recently analyzed extensively [91, 92, 93]. In particular, a thin shell around a traversable wormhole, with a zero surface energy density was analyzed in [91], and with generic surface stresses in [92]. A similar analysis for the plane symmetric case, with a negative cosmological constant, is done in [94]. A general class of wormhole geometries with a cosmological constant and junction conditions was analyzed by DeBenedictis and Das [95], and further explored in higher dimensions [96]. To know the stability of an object against several types of perturbation is always an important issue. In particular, the stability of thin-shell wormholes, constructed using the cut-and-paste technique, by considering specific equations of state [86, 87, 90, 97, 98, 99], or by applying a linearized radial perturbation around a stable solution [88, 100, 101, 102, 103], were analyzed. For the Ellis’ drainhole [45, 46], Armendáriz-Picón [104] finds that it is stable against linear perturbations, whereas Shinkai and Hayward [105] find this same class unstable to nonlinear perturbations. Bronnikov and Grinyok [73, 106] found that the consistent wormholes of Barceló and Visser [44] are unstable.

In alternative theories to general relativity wormhole solutions have been worked out. In higher dimensions, solutions have been found by Chodos and Detweiler [107], Clément [108], and DeBenedictis and Das [96]; in the nonsymmetric gravitational theory traversable wormholes were found by Moffat and Svoboda [109]; in Brans-Dicke theory by Agnese and Camera [110], Anchordoqui *at al* [111], Nandi and collaborators [112, 113, 114, 115], and He and Kim [116]; in Kaluza-Klein theory by Shen and collaborators [117]; in Einstein-Gauss-Bonnet by Bhawal and Kar [118]; and Koyama, Hayward and Kim [119] examined wormholes in a two-dimensional dilatonic theory. Anchordoqui and Bergliaffa found a wormhole solution in a brane world scenario [120], further examined by Barceló and Visser [121], and Bronnikov and Kim considered possible traversable wormhole solutions in a brane world, by imposing the condition $R = 0$, where R is the four-dimensional scalar curvature [122]; the latter solution was generalized in Ref. [123]. La Camera using the simplest form of the Randall-Sundrum model, considered the metric generated by a static, spherically symmetric distribution of matter on the physical brane, and found that the solution to the five-dimensional Einstein equations, obtained numerically, describes a wormhole geometry [124]. Recently, wormhole throats were also analyzed in a higher derivative gravity model governed by the Einstein-Hilbert Lagrangian, supplemented with $1/R$ and R^2 curvature scalar terms [125]. Using the resulting equations of motion, it was found that the weak energy condition may be respected in the throat vicinity, with conditions compatible with those required for stability [126] and an acceptable Newtonian limit [127]. The R^2 theory was meticulously studied by Ghoroku and Soma [128], where it was concluded that, under the assumption that an asymptotically flat global solution exists, no weak energy condition respecting wormhole solution may exist in such a theory.

If it is true that wormholes act as shortcuts between two regions of spacetime, then it is interesting to note that shortcuts also exist in the context of brane cosmology [129]. The latter model stipulates that our Universe is a three-brane embedded in a five-dimensional anti-de Sitter spacetime, in which matter is confined to the brane and gravity exists throughout the bulk. This implies the causal propagation of light and gravitational signals is in general different [130]. A gravitational signal travelling between two points on the brane may propagate through the bulk, taking a shortcut, and appearing quicker than a photon which propagates on the brane between the two respective points [131, 132]. It is then expected that these shortcuts would play an important role in solving the horizon problem [132, 133].

An important side effect of wormholes is that they can theoretically generate closed timelike curves with relative ease, by performing a sufficient delay to the time of one mouth in relation to the other (see [134] for a review on closed timelike curves). This can be done either by the special relativistic twin paradox method [135] or by the general relativistic redshift way [136]. The importance of wormholes in the study of time machines is that they provide a

non-eternal time machine, where closed timelike curves appear to the future of some hypersurface, the chronology horizon (a special case of a Cauchy horizon which is the onset of the nonchronal region containing closed timelike curves) which is generated in a compact region in this case [137, 138]. Since time travel to the past is in general unwelcome, it is possible to test whether classical or semiclassical effects will destroy the time machine. It is found that classically it can be easily stabilized [2, 135]. Semiclassically, there are calculations that favor the destruction [139, 140, 141], leading to chronology protection [140], others that maintain the time machine [53, 142, 143]. Other simpler systems that simulate a wormhole, such as Misner spacetime which is a species of two-dimensional wormhole, have been studied more thoroughly, with no conclusive answer. For Misner spacetime the debate still goes on, favoring chronology protection [144], disfavoring it [145], and back in favoring [146]. The upshot is that semiclassical calculations will not settle the issue of chronology protection [147], one needs a quantum gravity, as has been foreseen sometime before by Thorne [148].

In a cosmological context, it is extraordinary that recent observations have confirmed that the Universe is undergoing a phase of accelerated expansion. Evidence of this cosmological expansion, coming from measurements of supernovae of type Ia (SNe Ia) [149, 150] and independently from the cosmic microwave background radiation [151, 152], shows that the Universe additionally consists of some sort of negative pressure “dark energy”. The Wilkinson Microwave Anisotropy Probe (WMAP), designed to measure the CMB anisotropy with great precision and accuracy, has recently confirmed that the Universe is composed of approximately 70 percent of dark energy [151]. Several candidates representing dark energy have been proposed in the literature, namely, a positive cosmological constant, the quintessence fields, generalizations of the Chaplygin gas and so-called tachyon models. A simple way to parameterize the dark energy is by an equation of state of the form $\omega \equiv p/\rho$, where p is the spatially homogeneous pressure and ρ the energy density of the dark energy [153]. A value of $\omega < -1/3$ is required for cosmic expansion, and $\omega = -1$ corresponds to a cosmological constant [154]. A possibility that has been widely explored, is that of quintessence, where the parameter range is $-1 < \omega < -1/3$. However, a note on the choice of the imposition $\omega > -1$ is in order. This is considered to ensure that the null energy condition, $T_{\mu\nu} k^\mu k^\nu \geq 0$, is satisfied. If $\omega < -1$ [155, 156, 157], a case certainly not excluded by observation, then the null energy condition is violated, $\rho + p < 0$, and consequently all of the other energy conditions. Matter with the property $\omega < -1$ has been denoted “phantom energy” [158]. As the possibility of phantom energy implies the violation of the null energy condition, this leads us back to wormhole physics. This possibility has been explored with wormholes being supported by phantom energy [159, 160, 161], the generalized Chaplygin gas [162], and the van der Waals equation of state [163]. An interesting feature is that due to the fact of the accelerated expansion of the Universe, macroscopic wormholes could naturally be grown from the submicroscopic constructions that originally pervaded the quantum foam. In Ref. [59] the evolution of wormholes and ringholes embedded in a background accelerating Universe driven by dark energy, was analyzed. An interesting feature is that the wormhole’s size increases by a factor which is proportional to the scale factor of the Universe, and still increases significantly if the cosmic expansion is driven by phantom energy. The accretion of dark and phantom energy onto Morris-Thorne wormholes [164, 165], was further explored, and it was shown that this accretion gradually increases the wormhole throat which eventually overtakes the accelerated expansion of the universe, consequently engulfing the entire Universe, and becomes infinite at a time in the future before the big rip. This process was dubbed the “Big Trip” [164, 165]. It was shown that using k -essence dark energy also leads to the big rip [166], although, in an interesting article [167], considering a generalized Chaplygin gas the big rip may be avoided altogether.

Stars are common for everyone to see, black holes also inhabit the universe in billions, so one may tentatively assume that wormholes, formed or constructed from one way or another, can also appear in large amounts. If they inhabit the cosmological space, they will produce microlensing effects on point sources at non-cosmological distances [168], as well as at cosmological distances, in this case gamma-ray burts could be the objects microlensed [169, 170]. If peculiarly large, then wormholes will produce macrolensing effects [171]. There is now a growing consensus that wormholes are in the same chain of stars and black holes. For instance, González-Días [58] understood that an enormous pressure on the center ultimately meant a negative energy density to open up the tunnel; DeBenedictis and Das [95] mention that the stress-energy supporting the structure consists of an anisotropic brown dwarf ‘star’; and the wormhole joining one Friedmann-Robertson-Walker universe with Minkowski spacetime or joining two Friedmann-Robertson-Walker universes [54] could be interpreted, after further matchings, as a wormhole joining a collapsing (or expanding) star to Minkowski spacetime or a wormhole joining two dynamical stars, respectively. It has also been recognized, and emphasized by Hayward [172], that wormholes and black holes can be treated in a unified way, the black hole being described by a null outer trapped surface, and the wormhole by a timelike outer trapped surface, this surface being the throat where incoming null rays start to diverge [80, 172]. Thus, it seems there is a continuum of objects from stars to wormholes passing through black holes, where stars are made of normal matter, black holes of vacuum, and wormholes of exotic matter. Although not so appealing perhaps, wormholes could be called “exotic stars”.

B. “Warp drive” spacetimes and superluminal travel

Much interest has been revived in superluminal travel in the last few years. Despite the use of the term superluminal, it is not “really” possible to travel faster than light, in any *local* sense. The point to note is that one can make a round trip, between two points separated by a distance D , in an arbitrarily short time as measured by an observer that remained at rest at the starting point, by varying one’s speed or by changing the distance one is to cover. Providing a general *global* definition of superluminal travel is no trivial matter [173, 174], but it is clear that the spacetimes that allow “effective” superluminal travel generically suffer from the severe drawback that they also involve significant negative energy densities. More precisely, superluminal effects are associated with the presence of *exotic* matter, that is, matter that violates the null energy condition (see [28] for a review). In fact, superluminal spacetimes violate all the known energy conditions, and Ken Olum demonstrated that negative energy densities and superluminal travel are intimately related [175].

Apart from wormholes [1, 2], two spacetimes which allow superluminal travel are the Alcubierre warp drive [176] and the solution known as the Krasnikov tube [30, 177]. Alcubierre demonstrated that it is theoretically possible, within the framework of general relativity, to attain arbitrarily large velocities [176]. A warp bubble is driven by a local expansion behind the bubble, and an opposite contraction ahead of it. However, by introducing a slightly more complicated metric, José Natário [178] dispensed with the need for expansion. The Natário version of the warp drive can be thought of as a bubble sliding through space.

It is interesting to note that Krasnikov [177] discovered a fascinating aspect of the warp drive, in which an observer on a spaceship cannot create nor control on demand an Alcubierre bubble, with $v > c$, around the ship [177], as points on the outside front edge of the bubble are always spacelike separated from the centre of the bubble. However, causality considerations do not prevent the crew of a spaceship from arranging, by their own actions, to complete a *round trip* from the Earth to a distant star and back in an arbitrarily short time, as measured by clocks on the Earth, by altering the metric along the path of their outbound trip. Thus, Krasnikov introduced a two-dimensional metric with an interesting property that although the time for a one-way trip to a distant destination cannot be shortened, the time for a round trip, as measured by clocks at the starting point (e.g. Earth), can be made arbitrarily short. Soon after, Everett and Roman generalized the Krasnikov two-dimensional analysis to four dimensions, denoting the solution as the *Krasnikov tube* [30], where they analyzed the superluminal features, the energy condition violations, the appearance of closed timelike curves and applied the Quantum Inequality.

Recently, linearized gravity was applied to warp drive spacetimes, testing the energy conditions at first and second order of the non-relativistic warp-bubble velocity [179], $v \ll 1$. Thus, attention was not focussed on the “superluminal” aspects of the warp bubble [180], such as the appearance of horizons [181, 182, 183] and of closed timelike curves [184], but rather on a secondary unremarked effect: The warp drive (*if it can be realised in nature*) appears to be an example of a “reaction-less drive” wherein the warp bubble moves by interacting with the geometry of spacetime instead of expending reaction mass. A particularly interesting aspect of this construction is that one may place a finite mass spaceship at the origin and consequently analyze how the warp field compares with the mass-energy of the spaceship. This is not possible in the usual finite-strength warp field, since by definition the point in the center of the warp bubble moves along a geodesic and is “massless”. That is, in the usual formalism the spaceship is always treated as a test particle, while in the linearized theory one can treat the spaceship as a finite mass object.

For warp drive spacetimes, by using the “quantum inequality” deduced by Ford and Roman [21], it was soon verified that enormous amounts of energy are needed to sustain superluminal warp drive spacetimes [22, 29]. To reduce the enormous amounts of exotic matter needed in the superluminal warp drive, van den Broeck proposed a slight modification of the Alcubierre metric which considerably ameliorates the conditions of the solution [185]. It is also interesting to note that, by using the “quantum inequality”, enormous quantities of negative energy densities are needed to support the superluminal Krasnikov tube [30]. Gravel and Plante [186, 187] in a way similar in spirit to the van den Broeck analysis, showed that it is theoretically possible to lower significantly the mass of the Krasnikov tube. However, in the linearized analysis, no *a priori* assumptions as to the ultimate source of the energy condition violations were made, so that the quantum inequalities were not used nor needed. This means that the restrictions derived on warp drive spacetimes are more generic than those derived using the quantum inequalities – the restrictions derived in [179] hold regardless of whether the warp drive is assumed to be classical or quantum in its operation. It was not meant to suggest that such a “reaction-less drive” is achievable with current technology, as indeed extremely stringent conditions on the warp bubble were obtained, in the weak-field limit. These conditions are so stringent that it appears unlikely that the “warp drive” will ever prove technologically useful.

C. Closed timelike curves

As time is incorporated into the proper structure of the fabric of spacetime, it is interesting to note that general relativity is contaminated with non-trivial geometries which generate *closed timelike curves* [2, 26, 28, 91, 134, 188, 189]. A closed timelike curve (CTC) allows time travel, in the sense that an observer which travels on a trajectory in spacetime along this curve, returns to an event which coincides with the departure. The arrow of time leads forward, as measured locally by the observer, but globally he/she may return to an event in the past. This fact apparently violates causality, opening Pandora's box and producing time travel paradoxes [190], throwing a veil over our understanding of the fundamental nature of Time. The notion of causality is fundamental in the construction of physical theories, therefore time travel and its associated paradoxes have to be treated with great caution. The paradoxes fall into two broad groups, namely the *consistency paradoxes* and the *causal loops*.

The consistency paradoxes include the classical grandfather paradox. Imagine travelling into the past and meeting one's grandfather. Nurturing homicidal tendencies, the time traveller murders his grandfather, impeding the birth of his father, therefore making his own birth impossible. In fact, there are many versions of the grandfather paradox, limited only by one's imagination. The consistency paradoxes occur whenever possibilities of changing events in the past arise.

The paradoxes associated to causal loops are related to self-existing information or objects, trapped in spacetime. Imagine a time traveller going back to his past, handing his younger self a manual for the construction of a time machine. The younger version then constructs the time machine over the years, and eventually goes back to the past to give the manual to his younger self. The time machine exists in the future because it was constructed in the past by the younger version of the time traveller. The construction of the time machine was possible because the manual was received from the future. Both parts considered by themselves are consistent, and the paradox appears when considered as a whole. One is liable to ask, what is the origin of the manual, for it apparently surges out of nowhere. There is a manual never created, nevertheless existing in spacetime, although there are no causality violations. An interesting variety of these causal loops was explored by Gott and Li [191], where they analyzed the idea of whether there is anything in the laws of physics that would prevent the Universe from creating itself. Thus, tracing backwards in time through the original inflationary state a region of CTCs may be encountered, giving *no* first-cause.

A great variety of solutions to the Einstein Field Equations (EFEs) containing CTCs exist, but, two particularly notorious features seem to stand out. Solutions with a tipping over of the light cones due to a rotation about a cylindrically symmetric axis; and solutions that violate the Energy Conditions of general relativity, which are fundamental in the singularity theorems and theorems of classical black hole thermodynamics [15]. A great deal of attention has also been paid to the quantum aspects of closed timelike curves [192, 193, 194].

Thus, as shall be shown in Section IV, it is possible to find solutions to the EFEs, with certain ease, which generate CTCs, which This implies that if we consider general relativity valid, we need to include the *possibility* of time travel in the form of CTCs. A typical reaction is to exclude time travel due to the associated paradoxes. But the paradoxes do not prove that time travel is mathematically or physically impossible. Consistent mathematical solutions to the EFEs have been found, based on plausible physical processes. What they do seem to indicate is that local information in spacetimes containing CTCs are restricted in unfamiliar ways. The grandfather paradox, without doubt, does indicate some strange aspects of spacetimes that contain CTCs. It is logically inconsistent that the time traveller murders his grandfather. But, one can ask, what exactly impeded him from accomplishing his murderous act if he had ample opportunities and the free-will to do so. It seems that certain conditions in local events are to be fulfilled, for the solution to be globally self-consistent. These conditions are denominated *consistency constraints* [195]. To eliminate the problem of free-will, mechanical systems were developed as not to convey the associated philosophical speculations on free-will [196, 197]. Much has been written on two possible remedies to the paradoxes, namely the Principle of Self-Consistency [137, 197, 198, 199] and the Chronology Protection Conjecture [140, 147, 200].

One current of thought, led by Igor Novikov, is the Principle of Self-Consistency, which stipulates that events on a CTC are self-consistent, i.e., events influence one another along the curve in a cyclic and self-consistent way. In the presence of CTCs the distinction between past and future events are ambiguous, and the definitions considered in the causal structure of well-behaved spacetimes break down. What is important to note is that events in the future can influence, but cannot change, events in the past. The Principle of Self-Consistency permits one to construct local solutions of the laws of physics, only if these can be prolonged to a unique global solution, defined throughout non-singular regions of spacetime. Therefore, according to this principle, the only solutions of the laws of physics that are allowed locally, reinforced by the consistency constraints, are those which are globally self-consistent.

Hawking's Chronology Protection Conjecture [140] is a more conservative way of dealing with the paradoxes. Hawking notes the strong experimental evidence in favour of the conjecture from the fact that "we have not been invaded by hordes of tourists from the future". An analysis reveals that the value of the renormalized expectation quantum stress-energy tensor diverges in the imminence of the formation of CTCs. This conjecture permits the existence of traversable wormholes, but prohibits the appearance of CTCs. The transformation of a wormhole into

a time machine results in enormous effects of the vacuum polarization, which destroys it's internal structure before attaining the Planck scale. Nevertheless, Li has shown given an example of a spacetime containing a time machine that might be stable against vacuum fluctuations of matter fields [201], implying that Hawking's suggestion that the vacuum fluctuations of quantum fields acting as a chronology protection might break down. There is no convincing demonstration of the Chronology Protection Conjecture, but the hope exists that a future theory of quantum gravity may prohibit CTCs.

Visser still considers the possibility of two other conjectures [2]. The first is the radical reformulation of physics conjecture, in which one abandons the causal structure of the laws of physics and allows, without restriction, time travel, reformulating physics from the ground up. The second is the boring physics conjecture, in which one simply ceases to consider the solutions to the EFEs generating CTCs. Perhaps an eventual quantum gravity theory will provide us with the answers. But, as stated by Thorne [148], it is by extending the theory to it's extreme predictions that one can get important insights to it's limitations, and probably ways to overcome them. Therefore, time travel in the form of CTCs, is more than a justification for theoretical speculation, it is a conceptual tool and an epistemological instrument to probe the deepest levels of general relativity and extract clarifying views.

II. TRAVERSABLE LORENTZIAN WORMHOLES

One adopt the reverse philosophy in solving the Einstein field equation, namely, one first considers an interesting and exotic spacetime metric, then finds the matter source responsible for the respective geometry. In this manner, it was found that some of these solutions possess a peculiar property, namely "exotic matter", involving a stress-energy tensor that violates the null energy condition, $T_{\mu\nu}k^\mu k^\nu \geq 0$, where k^μ is a null vector. These geometries also allow closed timelike curves, with the respective causality violations. Another interesting feature of these spacetimes is that they allow "effective" superluminal travel, although, locally, the speed of light is not surpassed. These solutions are primarily useful as "gedanken-experiments" and as a theoretician's probe of the foundations of general relativity, and include traversable wormholes, which shall be extensively reviewed in this Section.

A. Spacetime metric

Consider the following spherically symmetric and static wormhole solution

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

where $\Phi(r)$ and $b(r)$ are arbitrary functions of the radial coordinate r . $\Phi(r)$ is denoted the redshift function, for it is related to the gravitational redshift, and $b(r)$ is denoted the shape function, because as can be shown by embedding diagrams, it determines the shape of the wormhole [1]. The coordinate r is non-monotonic in that it decreases from $+\infty$ to a minimum value r_0 , representing the location of the throat of the wormhole, where $b(r_0) = r_0$, and then it increases from r_0 to $+\infty$. The proper circumference of a circle of fixed r is given by $2\pi r$. Although the metric coefficient g_{rr} becomes divergent at the throat, which is signalled by the coordinate singularity, the proper radial distance

$$l(r) = \pm \int_{r_0}^r \frac{dr}{(1 - b(r)/r)^{1/2}}, \quad (2)$$

is required to be finite everywhere. Note that as $0 \leq 1 - b(r)/r \leq 1$, the proper distance is greater than or equal to the coordinate distance, i.e., $|l(r)| \geq r - r_0$. The metric (1) may be written in terms of the proper radial distance as

$$ds^2 = -e^{2\Phi(l)} dt^2 + dl^2 + r^2(l)(d\theta^2 + \sin^2 \theta d\phi^2). \quad (3)$$

The proper distance decreases from $l = +\infty$, in the upper universe, to $l = 0$ at the throat, and then from zero to $-\infty$ in the lower universe. For the wormhole to be traversable it must have no horizons, which implies that $g_{tt} = -e^{2\Phi(r)} \neq 0$, so that $\Phi(r)$ must be finite everywhere.

The four-velocity of a static observer is $U^\mu = dx^\mu/d\tau = (U^t, 0, 0, 0) = (e^{-\Phi(r)}, 0, 0, 0)$. The observer's four-acceleration is $a^\mu = U^\mu{}_{;\nu} U^\nu$, so that taking into account Eq. (1) we have

$$\begin{aligned} a^t &= 0, \\ a^r &= \Gamma_{tt}^r \left(\frac{dt}{d\tau} \right)^2 = \Phi' (1 - b/r), \end{aligned} \quad (4)$$

where the prime denotes a derivative with respect to the radial coordinate r . From the geodesic equation, a radially moving test particle which starts from rest initially has the equation of motion

$$\frac{d^2 r}{d\tau^2} = -\Gamma_{tt}^r \left(\frac{dt}{d\tau} \right)^2 = -a^r. \quad (5)$$

Therefore, a^r is the radial component of proper acceleration that an observer must maintain in order to remain at rest at constant r , θ , ϕ . Note that from Eq. (4), a static observer at the throat for generic $\Phi(r)$ is a geodesic observer. In particular, for a constant redshift function, $\Phi'(r) = 0$, static observers are also geodesic. It is interesting to note that a wormhole is “attractive” if $a^r > 0$, i.e., observers must maintain an outward-directed radial acceleration to keep from being pulled into the wormhole; and “repulsive” if $a^r < 0$, i.e., observers must maintain an inward-directed radial acceleration to avoid being pushed away from the wormhole. This distinction depends on the sign of Φ' , as is transparent from Eq. (4).

B. The mathematics of embedding and generic static throat

We can use embedding diagrams to represent a wormhole and extract some useful information for the choice of the shape function, $b(r)$. Due to the spherically symmetric nature of the problem, one may consider an equatorial slice, $\theta = \pi/2$, without loss of generality. The respective line element, considering a fixed moment of time, $t = \text{const}$, is given by

$$ds^2 = \frac{dr^2}{1 - b(r)/r} + r^2 d\phi^2. \quad (6)$$

To visualize this slice, one embeds this metric into three-dimensional Euclidean space, in which the metric can be written in cylindrical coordinates, (r, ϕ, z) , as

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2. \quad (7)$$

Now, in the three-dimensional Euclidean space the embedded surface has equation $z = z(r)$, and thus the metric of the surface can be written as,

$$ds^2 = \left[1 + \left(\frac{dz}{dr} \right)^2 \right] dr^2 + r^2 d\phi^2. \quad (8)$$

Comparing Eq. (8) with (6), we have the equation for the embedding surface, given by

$$\frac{dz}{dr} = \pm \left(\frac{r}{b(r)} - 1 \right)^{-1/2}. \quad (9)$$

To be a solution of a wormhole, the geometry has a minimum radius, $r = b(r) = r_0$, denoted as the throat, at which the embedded surface is vertical, i.e., $dz/dr \rightarrow \infty$, see Figure 1. Far from the throat consider that space is asymptotically flat, $dz/dr \rightarrow 0$ as $r \rightarrow \infty$.

To be a solution of a wormhole, one needs to impose that the throat flares out, as in Figure 1. Mathematically, this flaring-out condition entails that the inverse of the embedding function $r(z)$, must satisfy $d^2 r/dz^2 > 0$ at or near the throat r_0 . Differentiating $dr/dz = \pm(r/b(r) - 1)^{1/2}$ with respect to z , we have

$$\frac{d^2 r}{dz^2} = \frac{b - b'r}{2b^2} > 0. \quad (10)$$

At the throat we verify that the form function satisfies the condition $b'(r_0) < 1$. We will see below that this condition plays a fundamental role in the analysis of the violation of the energy conditions.

This treatment has the drawback of being highly coordinate dependent. However, for a covariant treatment we follow the analysis by Hochberg and Visser [79, 202]. Consider a generic static spacetime given by the following metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -e^{2\Phi(r)} dt^2 + g_{ij} dx^i dx^j. \quad (11)$$

Consider that Greek indices run from 0 to 3; latin indices (i, j, k, \dots) run from 1 to 3; and (a, b, c, \dots) run from 1 to 2 and refer to the wormhole throat and direction parallel to the throat.

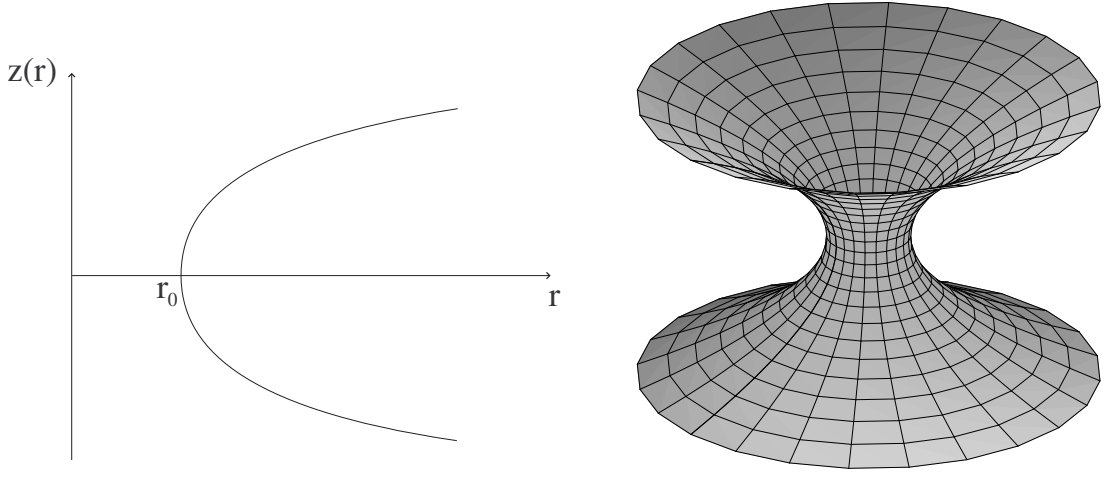


FIG. 1: The embedding diagram of a two-dimensional section along the equatorial plane ($t = \text{const}$, $\theta = \pi/2$) of a traversable wormhole. For a full visualization of the surface sweep through a 2π rotation around the z -axis, as can be seen from the graphic on the right.

A wormhole throat Σ is defined to be a two-dimensional hypersurface of minimal area taken in one of the constant-time spatial slices, and is given by

$$A(\Sigma) = \int \sqrt{{}^{(2)}g} \, d^2x. \quad (12)$$

The two-surface is embedded in a three-dimensional space, so that the definition of the extrinsic curvature is well defined. Consider Gaussian coordinates $x^i = (x^a, n)$, so that the hypersurface Σ lies at $n = 0$, the three-dimensional spatial metric is given by

$${}^{(3)}g_{ij} dx^i dx^j = {}^{(2)}g_{ab} dx^a dx^b + dn^2. \quad (13)$$

The variation in surface area is given by

$$\delta A(\Sigma) = \int \frac{\partial \sqrt{{}^{(2)}g}}{\partial n} \delta n(x) \, d^2x = \int \sqrt{{}^{(2)}g} \frac{1}{2} g^{ab} \frac{\partial g_{ab}}{\partial n} \delta n(x) \, d^2x. \quad (14)$$

Using Gaussian normal coordinates the extrinsic curvature is defined by [203]

$$K_{ab} = -\frac{1}{2} \frac{\partial g_{ab}}{\partial n}, \quad (15)$$

which substituted into Eq. (14), provides

$$\delta A(\Sigma) = - \int \sqrt{{}^{(2)}g} \, \text{tr}(K) \delta n(x) \, d^2x, \quad (16)$$

where the definition $\text{tr}(K) = g^{ab} K_{ab}$ is used. The condition that the area be extremal, for arbitrary $\delta n(x)$, is then simply $\text{tr}(K) = 0$.

For the area to be minimal, we have the additional requirement that $\delta^2 A(\Sigma) > 0$. We have then

$$\delta^2 A(\Sigma) = - \int \sqrt{{}^{(2)}g} \left(\frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K)^2 \right) \delta n(x) \delta n(x) \, d^2x. \quad (17)$$

Taking into account the extremal condition, $\text{tr}(K) = 0$, reduces the minimality constraint to

$$\delta^2 A(\Sigma) = - \int \sqrt{{}^{(2)}g} \left(\frac{\partial \text{tr}(K)}{\partial n} \right) \delta n(x) \delta n(x) \, d^2x. \quad (18)$$

Considering an arbitrary $\delta n(x)$, at the throat we have the following important condition

$$\frac{\partial \text{tr}(K)}{\partial n} < 0, \quad (19)$$

which is the covariant generalization of the Morris-Thorne flaring-out condition to arbitrary static wormhole throats.

C. Einstein field equation

The mathematical analysis and the physical interpretation will be simplified using a set of orthonormal basis vectors. These may be interpreted as the proper reference frame of a set of observers who remain at rest in the coordinate system (t, r, θ, ϕ) , with (r, θ, ϕ) fixed. Denote the basis vectors in the coordinate system as $(\mathbf{e}_t, \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$. Thus, the orthonormal basis vectors are given by

$$\begin{cases} \mathbf{e}_{\hat{t}} = e^{-\Phi} \mathbf{e}_t \\ \mathbf{e}_{\hat{r}} = (1 - b/r)^{1/2} \mathbf{e}_r \\ \mathbf{e}_{\hat{\theta}} = r^{-1} \mathbf{e}_\theta \\ \mathbf{e}_{\hat{\phi}} = (r \sin \theta)^{-1} \mathbf{e}_\phi. \end{cases} \quad (20)$$

The Einstein tensor, given in the orthonormal reference frame by $G_{\hat{\mu}\hat{\nu}} = R_{\hat{\mu}\hat{\nu}} - \frac{1}{2}R g_{\hat{\mu}\hat{\nu}}$, yields for the metric (1), the following non-zero components

$$G_{\hat{t}\hat{t}} = \frac{b'}{r^2}, \quad (21)$$

$$G_{\hat{r}\hat{r}} = -\frac{b}{r^3} + 2 \left(1 - \frac{b}{r}\right) \frac{\Phi'}{r}, \quad (22)$$

$$G_{\hat{\theta}\hat{\theta}} = \left(1 - \frac{b}{r}\right) \left[\Phi'' + (\Phi')^2 - \frac{b'r - b}{2r(r-b)} \Phi' - \frac{b'r - b}{2r^2(r-b)} + \frac{\Phi'}{r} \right], \quad (23)$$

$$G_{\hat{\phi}\hat{\phi}} = G_{\hat{\theta}\hat{\theta}}. \quad (24)$$

The Einstein field equation, $G_{\hat{\mu}\hat{\nu}} = 8\pi T_{\hat{\mu}\hat{\nu}}$, stipulates that the stress energy tensor $T_{\hat{\mu}\hat{\nu}}$ should be proportional to the Einstein tensor. Thus $T_{\hat{\mu}\hat{\nu}}$ has the same algebraic structure as $G_{\hat{\mu}\hat{\nu}}$, Eqs. (21)-(24), and the only nonzero components are precisely the diagonal terms $T_{\hat{t}\hat{t}}$, $T_{\hat{r}\hat{r}}$, $T_{\hat{\theta}\hat{\theta}}$ and $T_{\hat{\phi}\hat{\phi}}$. Using the orthonormal basis, these components carry a simple physical interpretation, i.e.,

$$T_{\hat{t}\hat{t}} = \rho(r), \quad T_{\hat{r}\hat{r}} = -\tau(r), \quad T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = p(r), \quad (25)$$

in which $\rho(r)$ is the energy density, $\tau(r)$ is the radial tension, with $\tau(r) = -p_r(r)$, i.e., it is the negative of the radial pressure, $p(r)$ is the pressure measured in the tangential directions, orthogonal to the radial direction.

Using the Einstein field equation, $G_{\hat{\mu}\hat{\nu}} = 8\pi T_{\hat{\mu}\hat{\nu}}$, we verify the following stress-energy scenario

$$\rho(r) = \frac{1}{8\pi} \frac{b'}{r^2}, \quad (26)$$

$$\tau(r) = \frac{1}{8\pi} \left[\frac{b}{r^3} - 2 \left(1 - \frac{b}{r}\right) \frac{\Phi'}{r} \right], \quad (27)$$

$$p(r) = \frac{1}{8\pi} \left(1 - \frac{b}{r}\right) \left[\Phi'' + (\Phi')^2 - \frac{b'r - b}{2r^2(1 - b/r)} \Phi' - \frac{b'r - b}{2r^3(1 - b/r)} + \frac{\Phi'}{r} \right]. \quad (28)$$

Evaluated at the throat they assume the following simplified form

$$\rho(r_0) = \frac{1}{8\pi} \frac{b'(r_0)}{r_0^2}, \quad (29)$$

$$\tau(r_0) = \frac{1}{8\pi r_0^2}, \quad (30)$$

$$p(r_0) = \frac{1}{8\pi} \frac{1 - b'(r_0)}{2r_0^2} (1 + r_0 \Phi'(r_0)). \quad (31)$$

Integrating Eq. (26), we have

$$b(r) = b(r_0) + \int_{r_0}^r 8\pi \rho(r') r'^2 dr' = 2m(r). \quad (32)$$

This can be expressed in the following manner

$$m(r) = \frac{r_0}{2} + \int_{r_0}^r 4\pi \rho(r') r'^2 dr', \quad (33)$$

which is the effective mass contained in the interior of a sphere of radius r . Therefore, the form function has an interpretation which depends on the mass distribution of the wormhole. Moving out to spatial infinity, we have

$$\lim_{r \rightarrow \infty} m(r) = \frac{r_0}{2} + \int_{r_0}^{\infty} 4\pi \rho(r') r'^2 dr' = M. \quad (34)$$

By taking the derivative with respect to the radial coordinate r , of Eq. (27), and eliminating b' and Φ'' , given in Eq. (26) and Eq. (28), respectively, we obtain the following equation

$$\tau' = (\rho - \tau)\Phi' - \frac{2}{r}(p + \tau). \quad (35)$$

Equation (35) is the relativistic Euler equation, or the hydrostatic equation for equilibrium for the material threading the wormhole, and can also be obtained using the conservation of the stress-energy tensor, $T^{\hat{\mu}\hat{\nu}}{}_{;\hat{\nu}} = 0$, inserting $\hat{\mu} = r$. The conservation of the stress-energy tensor, in turn can be deduced from the Bianchi identities,

$$R^{\hat{\alpha}}{}_{\hat{\beta}[\hat{\lambda}\hat{\mu};\hat{\nu}]} = 0, \quad (36)$$

which are equivalent to $G^{\hat{\mu}\hat{\nu}}{}_{;\hat{\nu}} = 0$, also called the contracted Bianchi identities.

D. Exotic matter

To gain some insight into the matter threading the wormhole, Morris and Thorne defined the dimensionless function $\xi = (\tau - \rho)/|\rho|$ [1]. Using equations (26)-(27) one finds

$$\xi = \frac{\tau - \rho}{|\rho|} = \frac{b/r - b' - 2r(1 - b/r)\Phi'}{|b'|}. \quad (37)$$

Combining Eq. (37) with the flaring-out condition, Eq. (10), the exoticity function takes the form

$$\xi = \frac{2b^2}{r|b'|} \frac{d^2r}{dz^2} - 2r \left(1 - \frac{b}{r}\right) \frac{\Phi'}{|b'|}. \quad (38)$$

Considering the finite character of ρ , and therefore of b' , and the fact that $(1 - b/r)\Phi' \rightarrow 0$ at the throat, we have the following relationship

$$\xi(r_0) = \frac{\tau_0 - \rho_0}{|\rho_0|} > 0. \quad (39)$$

The restriction $\tau_0 > \rho_0$ is an extremely troublesome condition, as it states that the radial tension at the throat should exceed the energy density. Thus, Morris and Thorne coined matter restricted by this condition “exotic matter” [1]. We shall verify below that this is matter that violates the null energy condition (in fact, it violates all the energy conditions) [1, 2, 54].

For instance, consider a specific class of particularly simple solutions corresponding to the choice of $b = b(r)$ and $\Phi(r) = 0$ [1]. Equations (26)-(28) reduce to

$$\rho(r) = \frac{b'(r)}{8\pi r^2}, \quad \tau(r) = \frac{b(r)}{8\pi r^3}, \quad p(r) = \frac{b(r) - b'r}{16\pi r^3}. \quad (40)$$

Note that the sign of the energy density depends on the sign of $b'(r)$. In particular, consider the form function given by $b(r) = r_0^2/r$. This corresponds to an embedding function $z(r)$ given by $z(r) = r_0 \cosh^{-1}(r/r_0)$, which has the shape of a catenary, i.e.,

$$\frac{dz}{dr} = \frac{r_0}{\sqrt{r^2 - r_0^2}}. \quad (41)$$

The wormhole material is everywhere exotic, i.e., $\xi > 0$ everywhere, extending outward from the throat, with ρ , τ , and p tending to zero as $r \rightarrow +\infty$.

Exotic matter is particularly troublesome for measurements made by observers traversing through the throat with a radial velocity close to the speed of light. Consider a Lorentz transformation, $x^{\hat{\mu}} = \Lambda^{\hat{\mu}'}_{\hat{\nu}} x^{\hat{\nu}}$, with $\Lambda^{\hat{\mu}}_{\hat{\alpha}'} = \Lambda^{\hat{\alpha}'}_{\hat{\nu}} = \delta^{\hat{\mu}}_{\hat{\nu}}$ and $\Lambda^{\hat{\mu}}_{\hat{\nu}'}$ defined as

$$(\Lambda^{\hat{\mu}}_{\hat{\nu}'}) = \begin{bmatrix} \gamma & 0 & 0 & \gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{bmatrix}. \quad (42)$$

The energy density measured by these observers is given by $T_{\hat{0}'\hat{0}'} = \Lambda^{\hat{\mu}}_{\hat{0}'} \Lambda^{\hat{\nu}}_{\hat{0}'} T_{\hat{\mu}\hat{\nu}}$, i.e.,

$$T_{\hat{0}'\hat{0}'} = \gamma^2 (\rho_0 - v^2 \tau_0), \quad (43)$$

with $\gamma = (1 - v^2)^{-1/2}$. For sufficiently high velocities, $v \rightarrow 1$, the observer will measure a negative energy density, $T_{\hat{0}'\hat{0}'} < 0$.

This feature also holds for any traversable, nonspherical and nonstatic wormhole. To see this, one verifies that a bundle of null geodesics that enters the wormhole at one mouth and emerges from the other must have a cross-sectional area that initially increases, and then decreases. This conversion of decreasing to increasing is due to the gravitational repulsion of matter, requiring a negative energy density, through which the bundle of null geodesics traverses.

E. Traversability conditions

We will be interested in specific solutions for traversable wormholes and assume that a traveller of an absurdly advanced civilization, with human traits, begins the trip in a space station in the lower universe, at proper distance $l = -l_1$, and ends up in the upper universe, at $l = l_2$. Assume that the traveller has a radial velocity $v(r)$, as measured by a static observer positioned at r . One may relate the proper distance travelled dl , radius travelled dr , coordinate time lapse dt , and proper time lapse as measured by the observer $d\tau$, by the following relationships

$$v = e^{-\Phi} \frac{dl}{dt} = \mp e^{-\Phi} \left(1 - \frac{b}{r}\right)^{-1/2} \frac{dr}{dt}, \quad (44)$$

$$v\gamma = \frac{dl}{d\tau} = \mp \left(1 - \frac{b}{r}\right)^{-1/2} \frac{dr}{d\tau}. \quad (45)$$

It is also important to impose certain conditions at the space stations [1]. Firstly, consider that space is asymptotically flat at the stations, i.e., $b/r \ll 1$. Secondly, the gravitational redshift of signals sent from the stations to infinity should be small, i.e., $\Delta\lambda/\lambda = e^{-\Phi} - 1 \approx -\Phi$, so that $|\Phi| \ll 1$. The condition $|\Phi| \ll 1$ imposes that the proper time at the station equals the coordinate time. Thirdly, the gravitational acceleration measured at the stations, given by $g = -(1 - b/r)^{-1/2} \Phi' \simeq -\Phi'$, should be less than or equal to the Earth's gravitational acceleration, $g \leq g_{\oplus}$, so that the condition $|\Phi'| \leq g_{\oplus}$ is met.

For a convenient trip through the wormhole, certain conditions should also be imposed [1]. Firstly, the entire journey should be done in a relatively short time as measured both by the traveller and by observers who remain at rest at the stations. Secondly, the acceleration felt by the traveller should not exceed the Earth's gravitational acceleration, g_{\oplus} . Finally, the tidal accelerations between different parts of the traveller's body, should not exceed, once again, Earth's gravity.

Total time in a traversal. The trip should take a relatively short time, for instance Morris and Thorne considered one year, as measured by the traveler and for observers that stay at rest at the space stations, $l = -l_1$ and $l = l_2$, i.e.,

$$\Delta\tau_{\text{traveler}} = \int_{-l_1}^{+l_2} \frac{dl}{v\gamma} \leq 1 \text{ year}, \quad (46)$$

$$\Delta t_{\text{spacestation}} = \int_{-l_1}^{+l_2} \frac{dl}{ve^{\Phi}} \leq 1 \text{ year}, \quad (47)$$

respectively.

Acceleration felt by a traveler. An important traversability condition required is that the acceleration felt by the traveller should not exceed Earth's gravity [1]. Consider an orthonormal basis of the traveller's proper reference frame, $(\mathbf{e}_{\hat{0}'}, \mathbf{e}_{\hat{1}'}, \mathbf{e}_{\hat{2}'}, \mathbf{e}_{\hat{3}'})$, given in terms of the orthonormal basis vectors of Eqs. (20) of the static observers, by a Lorentz transformation, i.e.,

$$\mathbf{e}_{\hat{0}'} = \gamma \mathbf{e}_{\hat{t}} \mp \gamma v \mathbf{e}_{\hat{r}}, \quad \mathbf{e}_{\hat{1}'} = \mp \gamma \mathbf{e}_{\hat{r}} + \gamma v \mathbf{e}_{\hat{t}}, \quad \mathbf{e}_{\hat{2}'} = \mathbf{e}_{\hat{\theta}}, \quad \mathbf{e}_{\hat{3}'} = \mathbf{e}_{\hat{\phi}}, \quad (48)$$

where $\gamma = (1-v^2)^{-1/2}$, and $v(r)$ being the velocity of the traveller as he passes r , as measured by a static observer positioned there. Thus, the traveller's four-acceleration expressed in his proper reference frame, $a^{\hat{\mu}'} = U^{\hat{\nu}'} U^{\hat{\mu}'}_{;\hat{\nu}'}$, yields the following restriction

$$|\vec{a}| = \left| \left(1 - \frac{b}{r}\right)^{1/2} e^{-\Phi} (\gamma e^{\Phi})' \right| \leq g_{\oplus}. \quad (49)$$

For the particular case of $\Phi' = 0$, this restriction reduces to

$$|\vec{a}| = \left| \left(1 - \frac{b}{r}\right)^{1/2} \gamma' c^2 \right| \leq g_{\oplus}. \quad (50)$$

For observers traversing the wormhole with a constant velocity, $v = \text{const}$, one has $|\vec{a}| = 0$, of course!

Tidal acceleration felt by a traveler. Its important that an observer traversing through the wormhole should not be ripped apart by enormous tidal forces. Thus, another of the traversability conditions required is that the tidal accelerations felt by the traveller should not exceed, for instance, the Earth's gravitational acceleration [1]. The tidal acceleration felt by the traveller is given by $\Delta a^{\hat{\mu}'} = -R^{\hat{\mu}'}_{\hat{\nu}'\hat{\alpha}'\hat{\beta}'} U^{\hat{\nu}'} \eta^{\hat{\alpha}'} U^{\hat{\beta}'}$, where $U^{\hat{\mu}'} = \delta^{\hat{\mu}'}_{\hat{0}'}$ is the traveller's four velocity and $\eta^{\hat{\alpha}'}$ is the separation between two arbitrary parts of his body. Note that $\eta^{\hat{\alpha}'}$ is purely spatial in the traveller's reference frame, as $U^{\hat{\mu}'} \eta_{\hat{\mu}'} = 0$, so that $\eta^{\hat{0}'} = 0$. For simplicity, assume that $|\eta^{\hat{i}'}| \approx 2$ m along any spatial direction in the traveller's reference frame. Taking into account the antisymmetric nature of $R^{\hat{\mu}'}_{\hat{\nu}'\hat{\alpha}'\hat{\beta}'}$ in its first two indices, we verify that $\Delta a^{\hat{\mu}'}$ is purely spatial with the components

$$\Delta a^{\hat{i}'} = -R^{\hat{i}'}_{\hat{0}'\hat{j}'\hat{0}'} \eta^{\hat{j}'} = -R_{\hat{i}'\hat{0}'\hat{j}'\hat{0}'} \eta^{\hat{j}'}. \quad (51)$$

By using a Lorentz transformation of the Riemann tensor components in the static observer's frame, $(\mathbf{e}_{\hat{t}}, \mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\theta}}, \mathbf{e}_{\hat{\phi}})$, to the traveller's frame, $(\mathbf{e}_{\hat{0}'}, \mathbf{e}_{\hat{1}'}, \mathbf{e}_{\hat{2}'}, \mathbf{e}_{\hat{3}'})$, the nonzero components of $R_{\hat{i}'\hat{0}'\hat{j}'\hat{0}'}$ are given by

$$\begin{aligned} R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'} &= R_{\hat{r}\hat{t}\hat{r}\hat{t}} \\ &= -\left(1 - \frac{b}{r}\right) \left[-\Phi'' - (\Phi')^2 + \frac{b'r - b}{2r(r-b)} \Phi' \right], \end{aligned} \quad (52)$$

$$\begin{aligned} R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'} &= R_{\hat{3}'\hat{0}'\hat{3}'\hat{0}'} = \gamma^2 R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} + \gamma^2 v^2 R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}} \\ &= \frac{\gamma^2}{2r^2} \left[v^2 \left(b' - \frac{b}{r} \right) + 2(r-b)\Phi' \right]. \end{aligned} \quad (53)$$

Thus, Eq. (51) takes the form

$$\Delta a^{\hat{1}'} = -R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'} \eta^{\hat{1}'}, \quad \Delta a^{\hat{2}'} = -R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'} \eta^{\hat{2}'}, \quad \Delta a^{\hat{3}'} = -R_{\hat{3}'\hat{0}'\hat{3}'\hat{0}'} \eta^{\hat{3}'}. \quad (54)$$

The constraint $|\Delta a^{\hat{\mu}'}| \leq g_{\oplus}$ provides the tidal acceleration restrictions as measured by a traveller moving radially through the wormhole, given by the following inequalities

$$\left| \left(1 - \frac{b}{r}\right) \left[\Phi'' + (\Phi')^2 - \frac{b'r - b}{2r(r-b)} \Phi' \right] \right| |\eta^{\hat{1}'}| \leq g_{\oplus}, \quad (55)$$

$$\left| \frac{\gamma^2}{2r^2} \left[v^2 \left(b' - \frac{b}{r} \right) + 2(r-b)\Phi' \right] \right| |\eta^{\hat{2}'}| \leq g_{\oplus}. \quad (56)$$

The radial tidal constraint, Eq. (55), constrains the redshift function, and the lateral tidal constraint, Eq. (56), constrains the velocity with which observers traverse the wormhole. These inequalities are particularly simple at the throat, r_0 ,

$$|\Phi'(r_0)| \leq \frac{2g_{\oplus} r_0}{(1-b') |\eta^{\hat{1}'}|}, \quad (57)$$

$$\gamma^2 v^2 \leq \frac{2g_{\oplus} r_0^2}{(1-b') |\eta^{\hat{2}'}|}, \quad (58)$$

For the particular case of a constant redshift function, $\Phi' = 0$, the radial tidal acceleration is zero, and Eq. (56) reduces to

$$\frac{\gamma^2 v^2}{2r^2} \left| \left(b' - \frac{b}{r} \right) \right| |\eta^{\hat{2}'}| \leq g_{\oplus}, \quad (59)$$

For this specific case one verifies that stationary observers with $v = 0$ measure null tidal forces.

It is interesting to note that if the tidal forces are velocity independent, then the wormhole is not traversable. For instance, consider the lateral tidal constraint, Eq. (56), which is the only component of the tidal acceleration that is velocity dependent. This velocity dependence cancels out if and only if $R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = -R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}}$ (see Eq. (53)), or

$$b' - \frac{b}{r} = -2r \left(1 - \frac{b}{r} \right) \Phi'. \quad (60)$$

Integrating this restriction yields

$$e^{2\Phi(r)} = e^{2\Phi(\infty)} \left(1 - \frac{b}{r} \right), \quad (61)$$

which indicates that a horizon is present at $r = r_0$, and that the wormhole is not traversable.

F. Energy conditions

1. Pointwise energy conditions

Given the fact that wormhole spacetimes are supported by exotic matter, we shall specify the energy conditions for the specific case in which the stress-energy tensor is diagonal [15], i.e.,

$$T^{\mu\nu} = \text{diag}(\rho, p_1, p_2, p_3), \quad (62)$$

where ρ is the mass density and the p_j are the three principal pressures. In the case that $p_1 = p_2 = p_3$ this reduces to the perfect fluid stress-energy tensor. Although classical forms of matter are believed to obey these energy conditions, it is a well-known fact that they are violated by certain quantum fields, amongst which we may refer to the Casimir effect.

Null energy condition (NEC). The NEC asserts that for any null vector k^μ

$$T_{\mu\nu} k^\mu k^\nu \geq 0. \quad (63)$$

In the case of a stress-energy tensor of the form (62), we have

$$\forall i, \quad \rho + p_i \geq 0. \quad (64)$$

Weak energy condition (WEC). The WEC states that for any timelike vector U^μ

$$T_{\mu\nu} U^\mu U^\nu \geq 0. \quad (65)$$

One can physically interpret $T_{\mu\nu} U^\mu U^\nu$ as the energy density measured by any timelike observer with four-velocity U^μ . Thus, the WEC requires that this quantity to be positive. In terms of the principal pressures this gives

$$\rho \geq 0 \quad \text{and} \quad \forall i, \quad \rho + p_i \geq 0. \quad (66)$$

By continuity, the WEC implies the NEC.

Strong energy condition (SEC). The SEC asserts that for any timelike vector U^μ the following inequality holds

$$\left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}\right) U^\mu U^\nu \geq 0, \quad (67)$$

where T is the trace of the stress energy tensor.

In terms of the diagonal stress energy tensor (62) the SEC reads

$$\forall i, \quad \rho + p_i \geq 0 \quad \text{and} \quad \rho + \sum_i p_i \geq 0. \quad (68)$$

The SEC implies the NEC but not necessarily the WEC.

Dominant energy condition (DEC). The DEC states that for any timelike vector U^μ

$$T_{\mu\nu} U^\mu U^\nu \geq 0 \quad \text{and} \quad T_{\mu\nu} U^\nu \text{ is not spacelike} \quad (69)$$

These conditions imply that the locally observed energy density be positive and that the energy flux should be timelike or null. The DEC implies the WEC, and therefore the NEC, but not necessarily the SEC. In the case of a stress-energy tensor of the form (62), we have

$$\rho \geq 0 \quad \text{and} \quad \forall i, \quad p_i \in [-\rho, +\rho]. \quad (70)$$

One may readily verify that wormhole spacetimes violate all the pointwise energy conditions. Taking into account Eqs. (26)-(27), we have

$$\rho(r) - \tau(r) = \frac{1}{8\pi} \left[\frac{b'r - b}{r^3} + 2 \left(1 - \frac{b}{r} \right) \frac{\Phi'}{r} \right]. \quad (71)$$

Due to the flaring out condition of the throat deduced from the mathematics of embedding, Eq. (10), i.e., $(b-b'r)/b^2 > 0$ [1, 2, 91], we verify that at the throat $b(r_0) = r = r_0$, and due to the finiteness of $\Phi(r)$, from Eq. (71) we have $\rho(r) - \tau(r) < 0$. From this we verify that all the energy conditions are violated. However, Eq. (71) is precisely the definition of the NEC, i.e., $T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}} k^{\hat{\nu}} = \rho(r) - \tau(r)$, with $k^{\hat{\mu}} = (1, 1, 0, 0)$. Matter that violates the NEC is denoted as “exotic matter”.

2. Averaged energy conditions

Violations of the pointwise energy conditions led to the averaging of the energy conditions over timelike or null geodesics [17]. The averaged energy conditions are somewhat weaker than the pointwise energy conditions, as they permit localized violations of the energy conditions, as long on average the energy conditions hold when integrated along timelike or null geodesics.

Averaged null energy condition (ANEC). The ANEC is satisfied along a null curve, Γ , if the following holds

$$\int_{\Gamma} T_{\mu\nu} k^\mu k^\nu d\lambda \geq 0, \quad (72)$$

where λ is the generalized affine parameter, and k^μ is a null vector. If the curve Γ is a null geodesic, then λ is reduced to the ordinary affine parameter.

Averaged weak energy condition (AWEC). The AWEC is satisfied along a timelike curve, Γ , if

$$\int_{\Gamma} T_{\mu\nu} U^\mu U^\nu ds \geq 0, \quad (73)$$

where s is a parameterization, the proper time of the timelike curve, and U^μ is the respective tangent vector.

It can be shown, under general conditions, that traversable wormholes violate the ANEC in the region of the throat using the Raychaudhuri equation for null geodesics [2, 15, 204].

3. Volume Integral Quantifier

Unfortunately the ANEC involves a line integral, with dimensions (mass)/(area), not a volume integral, and therefore gives no useful information regarding the “total amount” of energy-condition violating matter. Therefore, this prompted Visser *et al* [84, 85] to propose a “volume integral quantifier” which amounts to calculating the following definite integrals

$$\int T_{\mu\nu} U^\mu U^\nu dV \quad \text{and} \quad \int T_{\mu\nu} k^\mu k^\nu dV. \quad (74)$$

The amount of energy condition violations is then the extent that these integrals become negative. A more precise measure was analyzed in Ref. [205] by considering the proper volume in the integral.

To develop the key volume-integral result note that $T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}} k^{\hat{\nu}}$, with the null vector given by $k^{\hat{\mu}} = (1, 1, 00)$, can be written in the following manner

$$\rho - \tau = \frac{1}{8\pi r} \left(1 - \frac{b}{r}\right) \left[\ln \left(\frac{e^{2\Phi}}{1 - b/r} \right) \right]'. \quad (75)$$

Thus, integrating by parts, we have

$$I_V = \int (\rho - \tau) dV = \left[(r - b) \ln \left(\frac{e^{2\Phi}}{1 - b/r} \right) \right]_{r_0}^{\infty} - \int_{r_0}^{\infty} (1 - b') \left[\ln \left(\frac{e^{2\Phi}}{1 - b/r} \right) \right] dr. \quad (76)$$

The boundary term at r_0 vanishes by the construction of the wormhole, and the boundary term at infinity also vanishes because of the assumed asymptotic behaviour. Thus, Eq. (76) reduces to

$$I_V = \int (\rho - \tau) dV = - \int_{r_0}^{\infty} (1 - b') \left[\ln \left(\frac{e^{2\Phi}}{1 - b/r} \right) \right] dr. \quad (77)$$

This volume-integral theorem provides information about the “total amount” of ANEC violating matter in the space-time, and one may now consider specific cases, by choosing the form function and the redshift function.

Consider the solution obtained by setting the following choices for the redshift function and form function: $\Phi = 0$ and $b = r_0^2/r$, respectively. In fact, this is a solution obtained by Homer Ellis [45] in 1973. The properties are commented in [1] and [50]. Harris showed that it is a solution of the EFE with a stress-energy tensor of a peculiar massless scalar field [51]. In terms of the proper radial distance $l(r)$, the metric takes the form

$$ds^2 = -dt^2 + dl^2 + (r_0^2 + l^2) (d\theta^2 + \sin^2\theta d\phi^2), \quad (78)$$

where $l = \pm(r^2 - r_0^2)^{1/2}$. The stress-tensor components are given by

$$\rho = -\tau = -p = -\frac{r_0^2}{8\pi r^4} = -\frac{r_0^2}{8\pi(r_0^2 + l^2)^2}. \quad (79)$$

Suppose now that the wormhole extends from the throat, r_0 , to a radius situated at a . Evaluating the volume integral, Eq. (77), one deduces

$$I_V = \int (\rho - \tau) dV = \frac{1}{a} \left[(a^2 - r_0^2) \ln \left(1 - \frac{r_0^2}{a^2} \right) + 2r_0(r_0 - a) \right]. \quad (80)$$

Taking the limit as $a \rightarrow r_0^+$, one verifies that $\int (\rho - \tau) dV \rightarrow 0$. Thus, as in the examples presented in [84, 85], one may construct a traversable wormhole with arbitrarily small quantities of ANEC violating matter. The exotic matter threading the wormhole extends from the throat at r_0 to the junction boundary situated at a , where the interior solution is matched to an exterior vacuum spacetime.

4. Quantum Inequality

Pioneering work by Ford in the late 1970's on a new set of energy constraints [19], led to constraints on negative energy fluxes in 1991 [20]. These eventually culminated in the form of the Quantum Inequality (QI) applied to energy densities, which was introduced by Ford and Roman in 1995 [21]. The QI was proven directly from Quantum Field

Theory, in four-dimensional Minkowski spacetime, for free quantized, massless scalar fields and takes the following form

$$\frac{\tau_0}{\pi} \int_{-\infty}^{+\infty} \frac{\langle T_{\mu\nu} U^\mu U^\nu \rangle}{\tau^2 + \tau_0^2} d\tau \geq -\frac{3}{32\pi^2 \tau_0^4}, \quad (81)$$

in which, U^μ is the tangent to a geodesic observer's worldline; τ is the observer's proper time and τ_0 is a sampling time. The expectation value $\langle \rangle$ is taken with respect to an arbitrary state $|\Psi\rangle$. Contrary to the averaged energy conditions, one does not average over the entire worldline of the observer, but weights the integral with a sampling function of characteristic width, τ_0 . The inequality limits the magnitude of the negative energy violations and the time for which they are allowed to exist. The physical interpretation of Eq. (81) is that the more negative the energy density is in an interval, the shorter must be the duration of the interval.

The basic applications to curved spacetimes is that these appear flat if restricted to a sufficiently small region. The application of the QI to wormhole geometries is of particular interest [22]. A small spacetime volume around the throat of the wormhole is considered, so that all the dimensions of this volume are much smaller than the minimum proper radius of curvature in the region. Thus, the spacetime can be approximately flat in this region, so that the QI constraints may be applied. The sampling time τ_0 is restricted to be small compared to the local proper radii of curvature and the proper distance to any boundaries in the spacetime.

The Riemann tensor components will play a fundamental role in the analysis that follows. At the throat, the components of the Riemann tensor reduce to

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}}|_{r_0} = \frac{\Phi'_0}{2r_0} (1 - b'_0), \quad (82)$$

$$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}}|_{r_0} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}}|_{r_0} = 0, \quad (83)$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}|_{r_0} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}}|_{r_0} = -\frac{1}{2r_0^2} (1 - b'_0), \quad (84)$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}|_{r_0} = \frac{1}{r_0^2}. \quad (85)$$

All the other components vanish, except for those related to the above by symmetry.

Let the magnitude of the maximum curvature component be R_{\max} , and the smallest proper radius of curvature be given by $r_c \approx 1/\sqrt{R_{\max}}$. The QI-bound is applied to a small spacetime volume around the throat of the wormhole such that all dimensions of this volume are much smaller than r_c , the smallest proper radius of curvature anywhere in the region, so that in the absence of boundaries, spacetime can be considered to be approximately Minkowskian in the respective region [22].

As specific example, consider QI-bound applied to the Ellis drainhole geometry. Consider the Ellis drainhole, given by $\Phi = 0$ and $b = r_0^2/r$. The metric is given by Eq. (78), and the Riemann curvature components are

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = -R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = -R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = \frac{r_0^2}{(r_0^2 + l^2)^2}. \quad (86)$$

Note that all the curvature components are equal in magnitude, and have their maximum magnitude at the throat, i.e., $1/r_0^2$. The same holds true for the stress-tensor components given by Eq. (79).

Applying the QI-bound to a static observer at $r = r_0$, and as the energy density seen by this static observer is constant, we have

$$\frac{\tau_0}{\pi} \int_{-\infty}^{\infty} \frac{\langle T_{\mu\nu} u^\mu u^\nu \rangle d\tau}{\tau^2 + \tau_0^2} = \rho_0 \geq -\frac{3}{32\pi^2 \tau_0^4}, \quad (87)$$

where τ is the observer's proper time, and τ_0 is the sampling time. If the sampling time is chosen to be $\tau_0 = fr_m = fr_0 \ll r_c$, with $f \ll 1$ (recall that the QI is applicable if τ_0 is smaller than the local proper radius of curvature), using $\rho_0 = -1/(8\pi r_0^2)$ and from Eq. (87), one finally deduces

$$r_0 \leq \frac{l_p}{2f^2}, \quad (88)$$

where l_p is the Planck length. For any reasonable choice of f gives a value of r_0 which is not much larger than l_p . For example, for $f \approx 0.01$ one has $r_0 \leq 10^4 l_p = 10^{-31}$ m. Note from Eqs. (79) and (86) that if the spacetime region is such that $l \ll r_0$, then the curvature and stress-tensor components do not change very much [22].

Ford and Roman considered more specific examples by choosing appropriate definitions of length scales. They also found general bounds on the relative size scales of arbitrary static and spherically symmetric Morris-Thorne wormholes, i.e., for generic $\Phi(r)$ and $b(r)$. The results of the analysis is that either the wormhole possesses a throat size which is only slightly larger than the Planck length, or there are large discrepancies in the length scales which characterize the geometry of the wormhole. The analysis imply that generically the exotic matter is confined to an extremely thin band, and/or that large red-shifts are involved, which present severe difficulties for traversability, such as large tidal forces [22].

Due to these results, Ford and Roman argued that the existence of macroscopic traversable wormholes is very improbable. But, there are a series of considerations that can be applied to the QI [26]. Firstly, the QI is only of interest if one is relying on quantum field theory to provide the exotic matter to support the wormhole throat. But there are classical systems (non-minimally coupled scalar fields) that violate the null and the weak energy conditions [31], whilst presenting plausible results when applying the QI. Secondly, even if one relies on quantum field theory to provide exotic matter, the QI does not rule out the existence of wormholes, although they do place serious constraints on the geometry. Thirdly, it may be possible to reformulate the QI in a more transparent covariant notation, and to prove it for arbitrary background geometries.

G. Rotating wormholes

Now, consider the stationary and axially symmetric $(3+1)$ -dimensional spacetime, it possesses a time-like Killing vector field, which generates invariant time translations, and a spacelike Killing vector field, which generates invariant rotations with respect to the angular coordinate ϕ . We have the following metric

$$ds^2 = -N^2 dt^2 + e^\mu dr^2 + r^2 K^2 [d\theta^2 + \sin^2 \theta (d\phi - \omega dt)^2] \quad (89)$$

where N , K , ω and μ are functions of r and θ [67]. $\omega(r, \theta)$ may be interpreted as the angular velocity $d\phi/dt$ of a particle that falls freely from infinity to the point (r, θ) . For simplicity, we shall consider the definition [67]

$$e^{-\mu(r, \theta)} = 1 - \frac{b(r, \theta)}{r}, \quad (90)$$

which is well suited to describe a traversable wormhole. Assume that $K(r, \theta)$ is a positive, nondecreasing function of r that determines the proper radial distance R , i.e., $R \equiv rK$ and $R_r > 0$ [67], as for the $(2+1)$ -dimensional case. We shall adopt the notation that the subscripts $_r$ and $_\theta$ denote the derivatives in order of r and θ , respectively [67].

Note that an event horizon appears whenever $N = 0$ [67]. The regularity of the functions N , b and K are imposed, which implies that their θ derivatives vanish on the rotation axis, $\theta = 0, \pi$, to ensure a non-singular behavior of the metric on the rotation axis. The metric (89) reduces to the Morris-Thorne spacetime metric (1) in the limit of zero rotation and spherical symmetry

$$N(r, \theta) \rightarrow e^{\Phi(r)}, \quad b(r, \theta) \rightarrow b(r), \quad K(r, \theta) \rightarrow 1, \quad \omega(r, \theta) \rightarrow 0. \quad (91)$$

In analogy with the Morris-Thorne case, $b(r_0) = r_0$ is identified as the wormhole throat, and the factors N , K and ω are assumed to be well-behaved at the throat.

The scalar curvature of the space-time (89) is extremely messy, but at the throat $r = r_0$ simplifies to

$$R = -\frac{1}{r^2 K^2} \left(\mu_{\theta\theta} + \frac{1}{2} \mu_\theta^2 \right) - \frac{\mu_\theta}{N r^2 K^2} \frac{(N \sin \theta)_\theta}{\sin \theta} - \frac{2}{N r^2 K^2} \frac{(N_\theta \sin \theta)_\theta}{\sin \theta} - \frac{2}{r^2 K^3} \frac{(K_\theta \sin \theta)_\theta}{\sin \theta} + e^{-\mu} \mu_r [\ln(N r^2 K^2)]_r + \frac{\sin^2 \theta \omega_\theta^2}{2N^2} + \frac{2}{r^2 K^4} (K^2 + K_\theta^2). \quad (92)$$

The only troublesome terms are the ones involving the terms with μ_θ and $\mu_{\theta\theta}$, i.e.,

$$\mu_\theta = \frac{b_\theta}{(r-b)}, \quad \mu_{\theta\theta} + \frac{1}{2} \mu_\theta^2 = \frac{b_{\theta\theta}}{r-b} + \frac{3}{2} \frac{b_\theta^2}{(r-b)^2}. \quad (93)$$

Note that one needs to impose that $b_\theta = 0$ and $b_{\theta\theta} = 0$ at the throat to avoid curvature singularities. This condition shows that the throat is located at a constant value of r .

Thus, one may conclude that the metric (89) describes a rotating wormhole geometry, with an angular velocity ω . The factor K determines the proper radial distance. N is the analog of the redshift function in the Morris-Thorne wormhole and is finite and nonzero to ensure that there are no event horizons or curvature singularities. b is the shape

function which satisfies $b \leq r$; it is independent of θ at the throat, i.e., $b_\theta = 0$; and obeys the flaring out condition $b_r < 1$.

The analysis is simplified using an orthonormal reference frame, with the following orthonormal basis vectors

$$\mathbf{e}_{\hat{t}} = \frac{1}{N} \mathbf{e}_t + \frac{\omega}{N} \mathbf{e}_\phi, \quad \mathbf{e}_{\hat{r}} = \left(1 - \frac{b}{r}\right)^{1/2} \mathbf{e}_r, \quad \mathbf{e}_{\hat{\theta}} = \frac{1}{rK} \mathbf{e}_\theta, \quad \mathbf{e}_{\hat{\phi}} = \frac{1}{rK \sin \theta} \mathbf{e}_\phi. \quad (94)$$

Now the stress-energy tensor components are extremely messy, but assume a more simplified form using the orthonormal reference frame and evaluated at the throat. They have the following non-zero components

$$8\pi T_{\hat{t}\hat{t}} = -\frac{(K_\theta \sin \theta)_\theta}{r^2 K^3 \sin \theta} - \frac{\omega_\theta^2 \sin^2 \theta}{4N^2} + e^{-\mu} \mu_r \frac{(rK)_r}{rK} + \frac{K^2 + K_\theta^2}{r^2 K^4}, \quad (95)$$

$$8\pi T_{\hat{r}\hat{r}} = \frac{(K_\theta \sin \theta)_\theta}{r^2 K^3 \sin \theta} - \frac{\omega_\theta^2 \sin^2 \theta}{4N^2} + \frac{(N_\theta \sin \theta)_\theta}{Nr^2 K^2 \sin \theta} - \frac{K^2 + K_\theta^2}{r^2 K^4}, \quad (96)$$

$$8\pi T_{\hat{\theta}\hat{\theta}} = \frac{N_\theta (K \sin \theta)_\theta}{Nr^2 K^3 \sin \theta} + \frac{\omega_\theta^2 \sin^2 \theta}{4N^2} - \frac{\mu_r e^{-\mu} (NrK)_r}{2NrK}, \quad (97)$$

$$8\pi T_{\hat{\phi}\hat{\phi}} = -\frac{\mu_r e^{-\mu} (NKr)_r}{2NKr} - \frac{3 \sin^2 \theta \omega_\theta^2}{4N^2} + \frac{N_{\theta\theta}}{Nr^2 K^2} - \frac{N_\theta K_\theta}{Nr^2 K^3}, \quad (98)$$

$$8\pi T_{\hat{t}\hat{\phi}} = \frac{1}{4N^2 K^2 r} \left(6NK \omega_\theta \cos \theta + 2NK \sin \theta \omega_{\theta\theta} - \mu_r e^{-\mu} r^2 NK^3 \sin \theta \omega_r + 4N \omega_\theta \sin \theta K_\theta - 2K \sin \theta N_\theta \omega_\theta \right). \quad (99)$$

The components $T_{\hat{t}\hat{t}}$ and $T_{\hat{r}\hat{r}}$ have the usual physical interpretations, and in particular, $T_{\hat{t}\hat{\phi}}$ characterizes the rotation of the matter distribution. Taking into account the Einstein tensor components above, the NEC at the throat is given by

$$8\pi T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}} k^{\hat{\nu}} = e^{-\mu} \mu_r \frac{(rK)_r}{rK} - \frac{\omega_\theta^2 \sin^2 \theta}{2N^2} + \frac{(N_\theta \sin \theta)_\theta}{(rK)^2 N \sin \theta}. \quad (100)$$

Rather than reproduce the analysis here, we refer the reader to Ref. [67], where it was shown that the NEC is violated in certain regions, and is satisfied in others. Thus, it is possible for an infalling observer to move around the throat, and avoid the exotic matter supporting the wormhole. However, it is important to emphasize that one cannot avoid the use of exotic matter altogether.

H. Evolving wormholes in a cosmological background

Consider the metric element of a wormhole in a cosmological background given by

$$ds^2 = \Omega^2(t) \left[-e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - kr^2 - \frac{b(r)}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (101)$$

where $\Omega^2(t)$ is the conformal factor, which is finite and positive definite throughout the domain of t . It is also possible to write the metric (101) using ‘‘physical time’’ instead of ‘‘conformal time’’, by replacing t by $\tau = \int \Omega(t) dt$ and therefore $\Omega(t)$ by $R(\tau)$, where the latter is the functional form of the metric in the τ coordinate [81, 82]. When the form function and the redshift function vanish, $b(r) \rightarrow 0$ and $\Phi(r) \rightarrow 0$, respectively, the metric (101) becomes the FRW metric. As $\Omega(t) \rightarrow \text{const}$ and $k \rightarrow 0$, it approaches the static wormhole metric, Eq. (1).

The Einstein field equation will be written $G_{\hat{\mu}\hat{\nu}} = R_{\hat{\mu}\hat{\nu}} - \frac{1}{2} g_{\hat{\mu}\hat{\nu}} R = 8\pi T_{\hat{\mu}\hat{\nu}}$, in an orthonormal reference frame, so that any cosmological constant terms will be incorporated as part of the stress-energy tensor $T_{\hat{\mu}\hat{\nu}}$. The components of the stress-energy tensor $T_{\hat{\mu}\hat{\nu}}$ are given by

$$T_{\hat{t}\hat{t}} = \rho(r, t), \quad T_{\hat{r}\hat{r}} = -\tau(r, t), \quad T_{\hat{t}\hat{r}} = -f(r, t), \quad T_{\hat{\phi}\hat{\phi}} = T_{\hat{\theta}\hat{\theta}} = p(r, t), \quad (102)$$

with

$$\rho(r, t) = \frac{1}{8\pi} \frac{1}{\Omega^2} \left[3e^{-2\Phi} \left(\frac{\dot{\Omega}}{\Omega} \right)^2 + \left(3k + \frac{b'}{r^2} \right) \right], \quad (103)$$

$$\tau(r, t) = -\frac{1}{8\pi} \frac{1}{\Omega^2} \left\{ e^{-2\Phi(r)} \left[\left(\frac{\dot{\Omega}}{\Omega} \right)^2 - 2 \frac{\ddot{\Omega}}{\Omega} \right] - \left[k + \frac{b}{r^3} - 2 \frac{\Phi'}{r} \left(1 - kr^2 - \frac{b}{r} \right) \right] \right\}, \quad (104)$$

$$f(r, t) = -\frac{1}{8\pi} \left[2 \frac{\dot{\Omega}}{\Omega^3} e^{-\Phi} \Phi' \left(1 - kr^2 - \frac{b}{r} \right)^{1/2} \right], \quad (105)$$

$$p(r, t) = \frac{1}{8\pi} \frac{1}{\Omega^2} \left\{ e^{-2\Phi(r)} \left[\left(\frac{\dot{\Omega}}{\Omega} \right)^2 - 2 \frac{\ddot{\Omega}}{\Omega} \right] + \left(1 - kr^2 - \frac{b}{r} \right) \times \right. \\ \left. \times \left[\Phi'' + (\Phi')^2 - \frac{2kr^3 + b'r - b}{2r(r - kr^3 - b)} \Phi' - \frac{2kr^3 + b'r - b}{2r^2(r - kr^3 - b)} + \frac{\Phi'}{r} \right] \right\}. \quad (106)$$

The overdot denotes a derivative with respect to t , and the prime a derivative with respect to r . The physical interpretation of $\rho(r, t)$, $\tau(r, t)$, $f(r, t)$, and $p(r, t)$ are the following: the energy density, the radial tension per unit area, energy flux in the (outward) radial direction, and lateral pressures as measured by observers stationed at constant r , θ , ϕ , respectively. The stress-energy tensor has a non-diagonal component due to the time dependence of $\Omega(t)$ and/or the dependence of the redshift function on the radial coordinate. The stress-energy tensor of an imperfect fluid was analyzed in [206].

A particularly interesting case of the metric (101) is that of a wormhole in a time-dependent inflationary background, considered by Thomas Roman [89]. The primary goal in the Roman analysis was to use inflation to enlarge an initially small [89], possibly submicroscopic, wormhole. $\Phi(r)$ and $b(r)$ are chosen to give a reasonable wormhole at $t = 0$, which is assumed to be the onset of inflation. Roman [89] went on to explore interesting properties of the inflating wormholes, in particular, by analyzing constraints placed on the initial size of the wormhole, if the mouths were to remain in causal contact throughout the inflationary period; and the maintenance of the wormhole during and after the decay of the false vacuum. It is also possible that the wormhole will continue to be enlarged by the subsequent FRW phase of expansion. One could perform a similar analysis to ours by replacing the deSitter scale factor by an FRW scale factor $a(t)$ [81, 82, 83]. In particular, in Refs.[81, 82] specific examples for evolving wormholes that exist only for a finite time were considered, and a special class of scale factors which exhibit ‘flashes’ of the WEC violation were also analyzed.

I. Thin shells

Consider two distinct spacetime manifolds, \mathcal{M}_+ and \mathcal{M}_- , with metrics given by $g_{\mu\nu}^+(x_+^\mu)$ and $g_{\mu\nu}^-(x_-^\mu)$, in terms of independently defined coordinate systems x_+^μ and x_-^μ . The manifolds are bounded by hypersurfaces Σ_+ and Σ_- , respectively, with induced metrics g_{ij}^+ and g_{ij}^- . The hypersurfaces are isometric, i.e., $g_{ij}^+(\xi) = g_{ij}^-(\xi) = g_{ij}(\xi)$, in terms of the intrinsic coordinates, invariant under the isometry. A single manifold \mathcal{M} is obtained by gluing together \mathcal{M}_+ and \mathcal{M}_- at their boundaries, i.e., $\mathcal{M} = \mathcal{M}_+ \cup \mathcal{M}_-$, with the natural identification of the boundaries $\Sigma = \Sigma_+ = \Sigma_-$. In particular, assuming the continuity of the four-dimensional coordinates x_\pm^μ across Σ , then $g_{\mu\nu}^- = g_{\mu\nu}^+$ is required, which together with the continuous derivatives of the metric components $\partial g_{\mu\nu}/\partial x^\alpha|_- = \partial g_{\mu\nu}/\partial x^\alpha|_+$, provide the Lichnerowicz conditions [207].

The three holonomic basis vectors $\mathbf{e}_{(i)} = \partial/\partial\xi^i$ tangent to Σ have the following components $e_{(i)}^\mu|_\pm = \partial x_\pm^\mu/\partial\xi^i$, which provide the induced metric on the junction surface by the following scalar product

$$g_{ij} = \mathbf{e}_{(i)} \cdot \mathbf{e}_{(j)} = g_{\mu\nu} e_{(i)}^\mu e_{(j)}^\nu|_\pm. \quad (107)$$

We shall consider a timelike junction surface Σ , defined by the parametric equation of the form $f(x^\mu(\xi^i)) = 0$. The unit normal 4–vector, n_μ , to Σ is defined as

$$n_\mu = \pm \left| g^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial f}{\partial x^\beta} \right|^{-1/2} \frac{\partial f}{\partial x^\mu}, \quad (108)$$

with $n_\mu n^\mu = +1$ and $n_\mu e_{(i)}^\mu = 0$. The Israel formalism requires that the normals point from \mathcal{M}_- to \mathcal{M}_+ [208].

The extrinsic curvature, or the second fundamental form, is defined as $K_{ij} = n_{\mu;\nu} e_{(i)}^\mu e_{(j)}^\nu$, or

$$K_{ij}^\pm = -n_\mu \left(\frac{\partial^2 x^\mu}{\partial\xi^i \partial\xi^j} + \Gamma_{\alpha\beta}^{\mu\pm} \frac{\partial x^\alpha}{\partial\xi^i} \frac{\partial x^\beta}{\partial\xi^j} \right). \quad (109)$$

Note that for the case of a thin shell K_{ij} is not continuous across Σ , so that for notational convenience, the discontinuity in the second fundamental form is defined as $\kappa_{ij} = K_{ij}^+ - K_{ij}^-$. In particular, the condition that $g_{ij}^- = g_{ij}^+$, together with the continuity of the extrinsic curvatures across Σ , $K_{ij}^- = K_{ij}^+$, provide the Darmois conditions [209].

Now, the Lanczos equations follow from the Einstein equations for the hypersurface, and are given by

$$S^i_j = -\frac{1}{8\pi} (\kappa^i_j - \delta^i_j \kappa^k_k), \quad (110)$$

where S^i_j is the surface stress-energy tensor on Σ .

The first contracted Gauss-Kodazzi equation or the ‘‘Hamiltonian’’ constraint

$$G_{\mu\nu} n^\mu n^\nu = \frac{1}{2} (K^2 - K_{ij} K^{ij} - {}^3R), \quad (111)$$

with the Einstein equations provide the evolution identity

$$S^{ij} \bar{K}_{ij} = -[T_{\mu\nu} n^\mu n^\nu - \Lambda/8\pi]_-^+. \quad (112)$$

The convention $[X]_-^+ \equiv X^+|_\Sigma - X^-|_\Sigma$ and $\bar{X} \equiv (X^+|_\Sigma + X^-|_\Sigma)/2$ is used.

The second contracted Gauss-Kodazzi equation or the ‘‘ADM’’ constraint

$$G_{\mu\nu} e^\mu_{(i)} n^\nu = K^j_{|j} - K_{,i}, \quad (113)$$

with the Lanczos equations gives the conservation identity

$$S^i_{j|i} = [T_{\mu\nu} e^\mu_{(j)} n^\nu]_-^+. \quad (114)$$

The momentum flux term in the right hand side corresponds to the net discontinuity in the momentum which impinges on the shell.

In particular, considering spherical symmetry considerable simplifications occur, namely $\kappa^i_j = \text{diag}(\kappa^\tau_\tau, \kappa^\theta_\theta, \kappa^\theta_\theta)$. The surface stress-energy tensor may be written in terms of the surface energy density, σ , and the surface pressure, \mathcal{P} , as $S^i_j = \text{diag}(-\sigma, \mathcal{P}, \mathcal{P})$. The Lanczos equations then reduce to

$$\sigma = -\frac{1}{4\pi} \kappa^\theta_\theta, \quad (115)$$

$$\mathcal{P} = \frac{1}{8\pi} (\kappa^\tau_\tau + \kappa^\theta_\theta). \quad (116)$$

Taking into account the wormhole spacetime metric (1) and the Schwarzschild solution, the non-trivial components of the extrinsic curvature are given by

$$K^\tau_\tau{}^+ = \frac{\frac{M}{a^2} + \ddot{a}}{\sqrt{1 - \frac{2M}{a} + \dot{a}^2}}, \quad (117)$$

$$K^\tau_\tau{}^- = \frac{\Phi' (1 - \frac{b}{a} + \dot{a}^2) + \ddot{a} - \frac{\dot{a}^2(b-b'a)}{2a(a-b)}}{\sqrt{1 - \frac{b(a)}{a} + \dot{a}^2}}, \quad (118)$$

and

$$K^\theta_\theta{}^+ = \frac{1}{a} \sqrt{1 - \frac{2M}{a} + \dot{a}^2}, \quad (119)$$

$$K^\theta_\theta{}^- = \frac{1}{a} \sqrt{1 - \frac{b(a)}{a} + \dot{a}^2}. \quad (120)$$

The Lanczos equation, Eq. (110), then provide us with the following expressions for the surface stresses

$$\sigma = -\frac{1}{4\pi a} \left(\sqrt{1 - \frac{2M}{a} + \dot{a}^2} - \sqrt{1 - \frac{b(a)}{a} + \dot{a}^2} \right), \quad (121)$$

$$\mathcal{P} = \frac{1}{8\pi a} \left[\frac{1 - \frac{M}{a} + \dot{a}^2 + a\ddot{a}}{\sqrt{1 - \frac{2M}{a} + \dot{a}^2}} - \frac{(1 + a\Phi') (1 - \frac{b}{a} + \dot{a}^2) + a\ddot{a} - \frac{\dot{a}^2(b-b'a)}{2(a-b)}}{\sqrt{1 - \frac{b(a)}{a} + \dot{a}^2}} \right], \quad (122)$$

where σ and \mathcal{P} are the surface energy density and the tangential surface pressure, respectively.

Using $S_{\tau|z}^i = -[\dot{\sigma} + 2\dot{a}(\sigma + \mathcal{P})/a]$, Eq. (114) provides us with

$$\sigma' = -\frac{2}{a}(\sigma + \mathcal{P}) + \Xi, \quad (123)$$

where Ξ , defined for notational convenience, is given by

$$\Xi = -\frac{1}{4\pi a^2} \left[\frac{b'a - b}{2a(1 - \frac{b}{a})} + a\Phi' \right] \sqrt{1 - \frac{b}{a} + \dot{a}^2}. \quad (124)$$

For self-completeness, we shall also include the $\sigma + \mathcal{P}$ term, which is given by

$$\sigma + \mathcal{P} = \frac{1}{8\pi a} \left[\frac{(1 - a\Phi')(1 - \frac{b}{a} + \dot{a}^2) - a\ddot{a} + \frac{\dot{a}^2(b-b'a)}{2(a-b)}}{\sqrt{1 - \frac{b(a)}{a} + \dot{a}^2}} - \frac{1 - \frac{3M}{a} + \dot{a}^2 - a\ddot{a}}{\sqrt{1 - \frac{2M}{a} + \dot{a}^2}} \right]. \quad (125)$$

Thus, taking into account Eq. (125), and the definition of Ξ , we verify that Eq. (123) finally takes the form

$$\sigma' = \frac{1}{4\pi a^2} \left(\frac{1 - \frac{3M}{a} + \dot{a}^2 - a\ddot{a}}{\sqrt{1 - \frac{2M}{a} + \dot{a}^2}} - \frac{1 - \frac{3b}{2a} + \frac{b'}{2} + \dot{a}^2 - a\ddot{a}}{\sqrt{1 - \frac{b}{a} + \dot{a}^2}} \right), \quad (126)$$

which, evaluated at a static solution a_0 , shall play a fundamental role in determining the stability regions. Note that Eq. (126) can also be deduced by taking the radial derivative of the surface energy density, Eq. (121).

The construction of dynamic shells in wormholes have been extensively analyzed in Ref. [210], where the stability of generic spherically symmetric thin shells to linearized perturbations around static solutions were considered, and applying the analysis to traversable wormhole geometries, by considering specific choices for the form function, the stability regions were deduced. It was found that the latter may be significantly increased by considering appropriate choices for the redshift function (The linearized stability analysis was also applied to dark energy stars [211]).

J. Late-time cosmic accelerated expansion and traversable wormholes

In this section, we shall explore the possibility that traversable wormholes be supported by specific equations of state responsible for the late time accelerated expansion of the Universe, namely, phantom energy, the generalized Chaplygin gas, and the van der Waals quintessence equation of state. Firstly, phantom energy possesses an equation of state of the form $\omega \equiv p/\rho < -1$, consequently violating the null energy condition (NEC), which is a fundamental ingredient necessary to sustain traversable wormholes. Thus, this cosmic fluid presents us with a natural scenario for the existence of wormhole geometries [159, 160, 161]. Secondly, the generalized Chaplygin gas (GCG) is a candidate for the unification of dark energy and dark matter, and is parametrized by an exotic equation of state given by $p_{ch} = -A/\rho_{ch}^\alpha$, where A is a positive constant and $0 < \alpha \leq 1$. Within the framework of a flat Friedmann-Robertson-Walker cosmology the energy conservation equation yields the following evolution of the energy density $\rho_{ch} = [A + Ba^{-3(1+\alpha)}]^{1/(1+\alpha)}$, where a is the scale factor, and B is normally considered to be a positive integration constant to ensure the dominant energy condition (DEC). However, it is also possible to consider $B < 0$, consequently violating the DEC, and the energy density is an increasing function of the scale function [212]. It is in the latter context that we shall explore exact solutions of traversable wormholes supported by the GCG [162]. Thirdly, the van der Waals quintessence equation of state, $p = \gamma\rho/(1 - \beta\rho) - \alpha\rho^2$, is an interesting scenario for describing the late universe, and seems to provide a solution to the puzzle of dark energy, without the presence of exotic fluids or modifications of the Friedmann equations. Note that $\alpha, \beta \rightarrow 0$ and $\gamma < -1/3$ reduces to the dark energy equation of state. The existence of traversable wormholes supported by the VDW equation of state shall also be explored [163]. Despite of the fact that, in a cosmological context, these cosmic fluids are considered homogeneous, inhomogeneities may arise through gravitational instabilities, resulting in a nucleation of the cosmic fluid due to the respective density perturbations. Thus, the wormhole solutions considered in this work may possibly originate from density fluctuations in the cosmological background.

The strategy we shall adopt is to impose an equation of state, $p_r = p_r(\rho)$, which provides four equations, together with the Einstein field equations. However, we have five unknown functions of r , i.e., $\rho(r)$, $p_r(r)$, $p_t(r)$, $b(r)$ and $\Phi(r)$. Therefore, to fully determine the system we impose restricted choices for $b(r)$ or $\Phi(r)$ [159, 162, 163]. It is also possible to consider plausible stress-energy components, and through the field equations determine the metric fields[161].

Now, using the equation of state representing phantom energy, $p_r = \omega\rho$ with $\omega < -1$, and taking into account Eqs. (26), we have the following condition

$$\Phi'(r) = \frac{b + \omega r b'}{2r^2(1 - b/r)}. \quad (127)$$

For instance, consider a constant $\Phi(r)$, so that Eq. (127) provides $b(r) = r_0(r/r_0)^{-1/\omega}$, which corresponds to an asymptotically flat wormhole geometry. It was shown that this solution can be constructed, in principle, with arbitrarily small quantities of averaged null energy condition violating phantom energy, and the traversability conditions were explored[159]. The dynamic stability of these phantom wormholes were also analyzed[160], and we refer the reader to [159, 161] for further examples.

Relative to the GCG gas equation of state, $p_r = -A/\rho^\alpha$, using the field equations, we have the following condition

$$2r \left(1 - \frac{b}{r}\right) \Phi'(r) = -Ab' \left(\frac{8\pi r^2}{b'}\right)^{1+\alpha} + \frac{b}{r}. \quad (128)$$

Solutions of the metric (1), satisfying Eq. (128) are denoted ‘‘Chaplygin wormholes’’. To be a generic solution of a wormhole, the GCG equation of state imposes the following restriction $A < (8\pi r_0^2)^{-(1+\alpha)}$, consequently violating the NEC. However, for the GCG cosmological models it is generally assumed that the NEC is satisfied, which implies $\rho \geq A^{1/(1+\alpha)}$. The NEC violation is a fundamental ingredient in wormhole physics, and it is in this context that the construction of traversable wormholes, i.e., for $\rho < A^{1/(1+\alpha)}$, are explored. Note that as emphasized in [212], considering $B < 0$ in the evolution of the energy density, one also deduces that $\rho_{ch} < A^{1/(1+\alpha)}$, which violates the DEC. We refer the reader to [162] for specific examples of Chaplygin wormholes, where the physical properties and characteristics of these geometries were analyzed in detail. The solutions found are not asymptotically flat, and the spatial distribution of the exotic GCG is restricted to the throat vicinity, so that the dimensions of these Chaplygin wormholes are not arbitrarily large.

Finally, consider the VDW equation of state for an inhomogeneous spherically symmetric spacetime, given by $p_r = \gamma\rho/(1 - \beta\rho) - \alpha\rho^2$. The Einstein field equations provide the following relationship

$$2r \left(1 - \frac{b}{r}\right) \Phi' = \frac{b}{r} + \frac{\gamma b'}{1 - \frac{\beta b'}{8\pi r^2}} - \frac{\alpha b'^2}{8\pi r^2}. \quad (129)$$

It was shown that traversable wormhole solutions may be constructed using the VDW equation of state, which are either asymptotically flat or possess finite dimensions, where the exotic matter is confined to the throat neighborhood [163]. The latter solutions are constructed by matching an interior wormhole geometry to an exterior vacuum Schwarzschild vacuum, and we refer the reader to [163] for further details.

In concluding, it is noteworthy the relative ease with which one may theoretically construct traversable wormholes with the exotic fluid equations of state used in cosmology to explain the present accelerated expansion of the Universe. These traversable wormhole variations have far-reaching physical and cosmological implications, namely, apart from being used for interstellar shortcuts, an absurdly advanced civilization may convert them into time-machines, probably implying the violation of causality.

III. ‘‘WARP DRIVE’’ SPACETIMES AND SUPERLUMINAL TRAVEL

Much interest has been revived in superluminal travel in the last few years. Despite the use of the term superluminal, it is not ‘‘really’’ possible to travel faster than light, in any *local* sense. The point to note is that one can make a round trip, between two points separated by a distance D , in an arbitrarily short time as measured by an observer that remained at rest at the starting point, by varying one’s speed or by changing the distance one is to cover. Providing a general *global* definition of superluminal travel is no trivial matter [173, 174], but it is clear that the spacetimes that allow ‘‘effective’’ superluminal travel generically suffer from the severe drawback that they also involve significant negative energy densities. More precisely, superluminal effects are associated with the presence of *exotic* matter, that is, matter that violates the null energy condition [NEC].

In fact, superluminal spacetimes violate all the known energy conditions, and Ken Olum demonstrated that negative energy densities and superluminal travel are intimately related [175]. Although most classical forms of matter are thought to obey the energy conditions, they are certainly violated by certain quantum fields [35]. Additionally, certain classical systems (such as non-minimally coupled scalar fields) have been found that violate the null and the weak energy conditions [31, 44]. It is also interesting to note that recent observations in cosmology strongly suggest that

the cosmological fluid violates the strong energy condition [SEC], and provides tantalizing hints that the NEC *might* possibly be violated in a classical regime [32, 33, 34].

Apart from wormholes [1, 2], two spacetimes which allow superluminal travel are the Alcubierre warp drive [176] and the solution known as the Krasnikov tube [30, 177]. Alcubierre demonstrated that it is theoretically possible, within the framework of general relativity, to attain arbitrarily large velocities [176]. A warp bubble is driven by a local expansion behind the bubble, and an opposite contraction ahead of it. However, by introducing a slightly more complicated metric, José Natário [178] dispensed with the need for expansion. Thus, the Natário version of the warp drive can be thought of as a bubble sliding through space.

It is interesting to note that Krasnikov [177] discovered a fascinating aspect of the warp drive, in which an observer on a spaceship cannot create nor control on demand an Alcubierre bubble, with $v > c$, around the ship [177], as points on the outside front edge of the bubble are always spacelike separated from the centre of the bubble. However, causality considerations do not prevent the crew of a spaceship from arranging, by their own actions, to complete a *round trip* from the Earth to a distant star and back in an arbitrarily short time, as measured by clocks on the Earth, by altering the metric along the path of their outbound trip. Thus, Krasnikov introduced a two-dimensional metric with an interesting property that although the time for a one-way trip to a distant destination cannot be shortened, the time for a round trip, as measured by clocks at the starting point (e.g. Earth), can be made arbitrarily short. Soon after, Everett and Roman generalized the Krasnikov two-dimensional analysis to four dimensions, denoting the solution as the *Krasnikov tube* [30], where they analyzed the superluminal features, the energy condition violations, the appearance of closed timelike curves and applied the Quantum Inequality.

A. “Warp drive” spacetime metric

Within the framework of general relativity, Alcubierre demonstrated that it is in principle possible to warp spacetime in a small *bubble-like* region, in such a way that the bubble may attain arbitrarily large velocities. Inspired by the inflationary phase of the early universe, the enormous speed of separation arises from the expansion of spacetime itself. The simplest model for hyper-fast travel is to create a local distortion of spacetime, producing an expansion behind the bubble, and an opposite contraction ahead of it. Natário’s version of the warp drive dispensed with the need for expansion at the cost of introducing a slightly more complicated metric.

The warp drive spacetime metric, in cartesian coordinates, is given by (with $G = c = 1$)

$$ds^2 = -dt^2 + [d\vec{x} - \vec{\beta}(x, y, z - z_0(t)) dt] \cdot [d\vec{x} - \vec{\beta}(x, y, z - z_0(t)) dt]. \quad (130)$$

In terms of the well-known ADM formalism this corresponds to a spacetime wherein *space* is flat, while the “lapse function” is identically unity, and the only non-trivial structure lies in the “shift vector” $\beta(t, \vec{x})$. Thus warp drive spacetimes can also be viewed as specific examples of “shift-only” spacetimes [179]. The Alcubierre warp drive corresponds to taking the shift vector to always lie in the direction of motion

$$\vec{\beta}(x, y, z - z_0(t)) = v(t) \hat{z} f(x, y, z - z_0(t)), \quad (131)$$

in which $v(t) = dz_0(t)/dt$ is the velocity of the warp bubble, moving along the positive z -axis, whereas in the Natário warp drive the shift vector is constrained by being divergence-free

$$\nabla \cdot \vec{\beta}(x, y, z) = 0. \quad (132)$$

B. Alcubierre warp drive

In the Alcubierre warp drive the spacetime metric is

$$ds^2 = -dt^2 + dx^2 + dy^2 + [dz - v(t) f(x, y, z - z_0(t)) dt]^2. \quad (133)$$

The form function $f(x, y, z)$ possesses the general features of having the value $f = 0$ in the exterior and $f = 1$ in the interior of the bubble. The general class of form functions, $f(x, y, z)$, chosen by Alcubierre was spherically symmetric: $f(r)$ with $r = \sqrt{x^2 + y^2 + z^2}$. Then

$$f(x, y, z - z_0(t)) = f(r(t)) \quad \text{with} \quad r(t) = \{[(z - z_0(t))^2 + x^2 + y^2]^{1/2}. \quad (134)$$

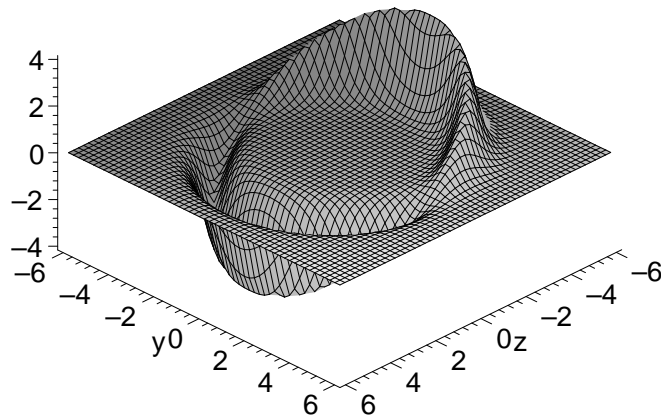


FIG. 2: The expansion of the volume elements. These are expanding behind the spaceship, and contracting in front of it.

Whenever a more specific example is required we adopt

$$f(r) = \frac{\tanh[\sigma(r+R)] - \tanh[\sigma(r-R)]}{2 \tanh(\sigma R)}, \quad (135)$$

in which $R > 0$ and $\sigma > 0$ are two arbitrary parameters. R is the “radius” of the warp-bubble, and σ can be interpreted as being inversely proportional to the bubble wall thickness. If σ is large, the form function rapidly approaches a *top hat* function, i.e.,

$$\lim_{\sigma \rightarrow \infty} f(r) = \begin{cases} 1, & \text{if } r \in [0, R], \\ 0, & \text{if } r \in (R, \infty). \end{cases} \quad (136)$$

It can be shown that observers with the four velocity

$$U^\mu = (1, 0, 0, vf), \quad U_\mu = (-1, 0, 0, 0). \quad (137)$$

move along geodesics, as their 4-acceleration is zero, *i.e.*, $a^\mu = U^\nu U^\mu{}_{;\nu} = 0$. They were denoted Eulerian observers by Alcubierre. The spaceship, which in the original formulation is treated as a test particle which moves along the curve $z = z_0(t)$, can easily be seen to always move along a timelike curve, regardless of the value of $v(t)$. One can also verify that the proper time along this curve equals the coordinate time, by simply substituting $z = z_0(t)$ in Eq. (133). This reduces to $d\tau = dt$, taking into account $dx = dy = 0$ and $f(0) = 1$.

Consider a spaceship placed within the Alcubierre warp bubble. The expansion of the volume elements, $\theta = U^\mu{}_{;\mu}$, is given by $\theta = v (\partial f / \partial z)$. Taking into account Eq. (135), we have (for Alcubierre’s version of the warp bubble)

$$\theta = v \frac{z - z_0}{r} \frac{df(r)}{dr}. \quad (138)$$

The center of the perturbation corresponds to the spaceship’s position $z_0(t)$. The volume elements are expanding behind the spaceship, and contracting in front of it, as shown in Figure 2.

C. The violation of the energy conditions

If we attempt to treat the spaceship as more than a test particle, we must confront the fact that by construction we have forced $f = 0$ outside the warp bubble. [Consider, for instance, the explicit form function of Eq. (135) in the limit $r \rightarrow \infty$.] This implies that the spacetime geometry is asymptotically Minkowski space, and in particular the ADM mass (defined by taking the limit as one moves to spacelike infinity i^0) is zero. That is, the ADM mass of the spaceship and the warp field generators must be exactly compensated by the ADM mass due to the stress-energy of the warp-field itself. Viewed in this light it is now patently obvious that there must be massive violations of the classical energy conditions (at least in the original version of the warp-drive spacetime), and the interesting question becomes “Where are these energy condition violations localized?”

One of our tasks in the current Section will be to see if we can first avoid this exact cancellation of the ADM mass, and second, to see if we can make qualitative and quantitative statements concerning the localization and

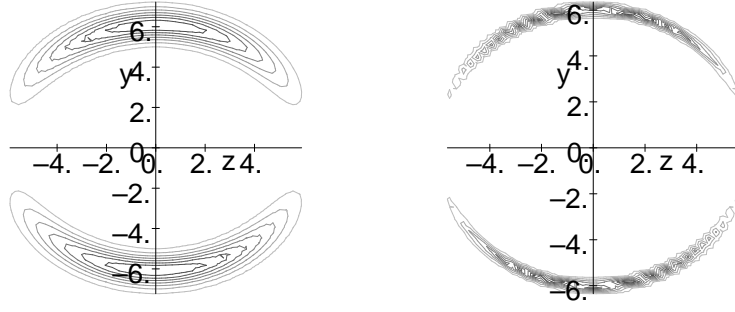


FIG. 3: The energy density is distributed in a toroidal region perpendicular to the direction of travel of the spaceship, which is situated at $z_0(t)$. We have considered the following values, $v = 2$ and $R = 6$, with $\sigma = 1$ and $\sigma = 4$ in (a) and (b), respectively.

“total amount” of energy condition violations. (A similar attempt at quantification of the “total amount” of energy condition violation in traversable wormholes was recently presented in [84, 85]). By using the Einstein field equation, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, we can make rather general statements regarding the nature of the stress energy required to support a warp bubble.

1. The violation of the WEC

The WEC states $T_{\mu\nu} U^\mu U^\nu \geq 0$, in which U^μ is a timelike vector and $T_{\mu\nu}$ is the stress-energy tensor. Its physical interpretation is that the local energy density is positive. By continuity it implies the NEC. We verify that for the warp drive metric, the WEC is violated, *i.e.*,

$$T_{\mu\nu} U^\mu U^\nu = -\frac{v^2}{32\pi} \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] < 0, \quad (139)$$

or by taking into account the Alcubierre form function (135), we have

$$T_{\mu\nu} U^\mu U^\nu = -\frac{1}{32\pi} \frac{v^2(x^2 + y^2)}{r^2} \left(\frac{df}{dr} \right)^2 < 0. \quad (140)$$

By considering an orthonormal basis, we verify that the energy density of the warp drive spacetime is given by $T_{\hat{t}\hat{t}} = T_{\hat{\mu}\hat{\nu}} U^{\hat{\mu}} U^{\hat{\nu}}$, that is, Eq. (140). It is easy to verify that the energy density is distributed in a toroidal region around the z -axis, in the direction of travel of the warp bubble [29], as may be verified from Figure 3. It is perhaps instructive to point out that the energy density for this class of spacetimes is nowhere positive. That the total ADM mass can nevertheless be zero is due to the intrinsic nonlinearity of the Einstein equations.

It is interesting to note that the inclusion of a generic lapse function $\alpha(x, y, z, t)$, decreases the negative energy density, which is given by

$$T_{\hat{t}\hat{t}} = -\frac{v^2}{32\pi \alpha^2} \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right]. \quad (141)$$

Now, α may be taken as unity in the exterior and interior of the warp bubble, so proper time equals coordinate time. In order to significantly decrease the negative energy density in the bubble walls, one may impose an extremely large value for the lapse function. However, the inclusion of the lapse function suffers from an extremely severe drawback, as proper time as measured in the bubble walls becomes absurdly large, $d\tau = \alpha dt$, for $\alpha \gg 1$.

We can (in analogy with the definitions in [84, 85]) quantify the “total amount” of energy condition violating matter in the warp bubble by defining

$$\begin{aligned} M_{\text{warp}} &= \int \rho_{\text{warp}} d^3x = \int T_{\mu\nu} U^\mu U^\nu d^3x \\ &= -\frac{v^2}{32\pi} \int \frac{x^2 + y^2}{r^2} \left(\frac{df}{dr} \right)^2 r^2 dr d^2\Omega = -\frac{v^2}{12} \int \left(\frac{df}{dr} \right)^2 r^2 dr. \end{aligned} \quad (142)$$

This is emphatically not the total mass of the spacetime, but it characterizes how much (negative) energy one needs to localize in the walls of the warp bubble. For the specific shape function (135) we can estimate

$$M_{\text{warp}} \approx -v^2 R^2 \sigma. \quad (143)$$

(The integral can be done exactly, but the exact result in terms of polylog functions is unhelpful.) Note that the energy requirements for the warp bubble scale quadratically with bubble velocity, quadratically with bubble size, and inversely as the thickness of the bubble wall [179].

2. The violation of the NEC

The NEC states that $T_{\mu\nu} k^\mu k^\nu \geq 0$, where k^μ is *any* arbitrary null vector and $T_{\mu\nu}$ is the stress-energy tensor. The NEC for a null vector oriented along the $\pm\hat{z}$ directions takes the following form

$$T_{\mu\nu} k^\mu k^\nu = -\frac{v^2}{8\pi} \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] \pm \frac{v}{8\pi} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right). \quad (144)$$

In particular if we average over the $\pm\hat{z}$ directions we have

$$\frac{1}{2} \{ T_{\mu\nu} k_{+\hat{z}}^\mu k_{+\hat{z}}^\nu + T_{\mu\nu} k_{-\hat{z}}^\mu k_{-\hat{z}}^\nu \} = -\frac{v^2}{8\pi} \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right], \quad (145)$$

which is manifestly negative, and so the NEC is violated for all v . Furthermore, note that even if we do not average, the coefficient of the term linear in v must be nonzero *somewhere* in the spacetime. Then at low velocities this term will dominate and at low velocities the un-averaged NEC will be violated in either the $+\hat{z}$ or $-\hat{z}$ directions.

To be a little more specific about how and where the NEC is violated consider the Alcubierre form function. We have

$$T_{\mu\nu} k_{\pm\hat{z}}^\mu k_{\pm\hat{z}}^\nu = -\frac{1}{8\pi} \frac{v^2(x^2 + y^2)}{r^2} \left(\frac{df}{dr} \right)^2 \pm \frac{v}{8\pi} \left[\frac{x^2 + y^2 + 2(z - z_0(t))^2}{r^3} \frac{df}{dr} + \frac{x^2 + y^2}{r^2} \frac{d^2 f}{dr^2} \right]. \quad (146)$$

The first term is manifestly negative everywhere throughout the space. As f decreases monotonically from the center of the warp bubble, where it takes the value of $f = 1$, to the exterior of the bubble, with $f \approx 0$, we verify that df/dr is negative in this domain. The term $d^2 f/dr^2$ is also negative in this region, as f attains its maximum in the interior of the bubble wall. Thus, the term in square brackets unavoidably assumes a negative value in this range, resulting in the violation of the NEC.

Equation (146) is plotted in Figures 4 and 5, for various values of the parameters. Figure 4 represents the null vector oriented along the $+\hat{z}$ direction, and Figure 5 along the $-\hat{z}$ direction. We have considered the following values of $v = 2$, $\sigma = 2$ and $R = 6$, for the parameters.

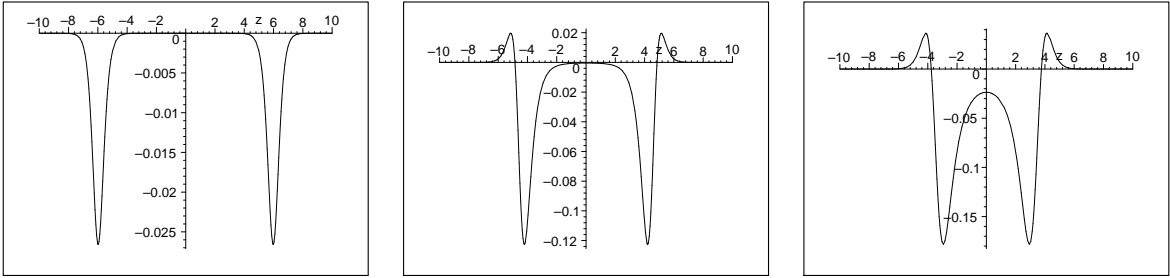


FIG. 4: The NEC for a null vector oriented along the $+\hat{z}$ direction. Taking into account the Alcubierre form function, we have considered the parameters $v = 2$, $\sigma = 2$ and $R = 6$. Considering the definition $\rho = \sqrt{x^2 + y^2}$, the plots have the respective values of $\rho = 0$, $\rho = 4$ and $\rho = 5$.

For a null vector oriented perpendicular to the direction of motion (for definiteness take $\hat{k} = \pm\hat{x}$) the NEC takes the following form

$$T_{\mu\nu} k_{\pm\hat{x}}^\mu k_{\pm\hat{x}}^\nu = -\frac{v^2}{8\pi} \left[\frac{1}{2} \left(\frac{\partial f}{\partial y} \right)^2 + \left(\frac{\partial f}{\partial z} \right)^2 - (1 - f) \frac{\partial^2 f}{\partial z^2} \right] \mp \frac{v}{8\pi} \left(\frac{\partial^2 f}{\partial x \partial z} \right). \quad (147)$$

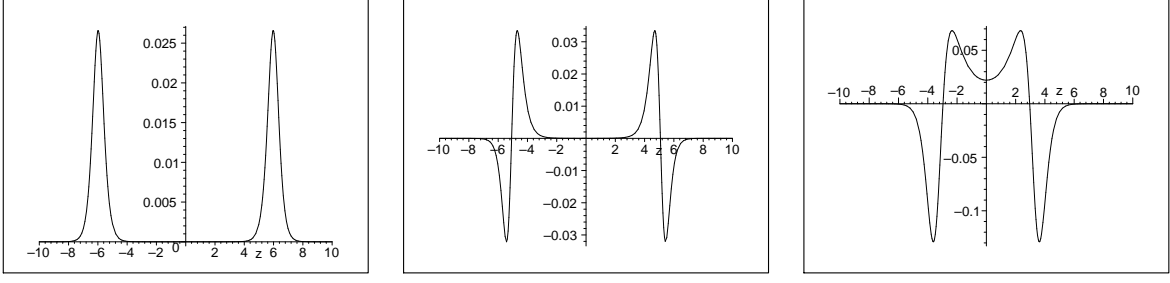


FIG. 5: The NEC for a null vector oriented along the $-\hat{z}$ direction. Taking into account the Alcubierre form function, we have considered the parameters $v = 2$, $\sigma = 2$ and $R = 6$. Considering the definition $\rho = \sqrt{x^2 + y^2}$, the plots have the respective values of $\rho = 0$, $\rho = 3$ and $\rho = 5$.

Again, note that the coefficient of the term linear in v must be nonzero *somewhere* in the spacetime. Then at low velocities this term will dominate, and at low velocities the NEC will be violated in one or other of the transverse directions. Upon considering the specific form of the spherically symmetric Alcubierre form function, we have

$$T_{\mu\nu} k_{\pm\hat{x}}^\mu k_{\pm\hat{x}}^\nu = -\frac{v^2}{8\pi} \left[\frac{y^2 + 2(z - z_0(t))^2}{2r^2} \left(\frac{df}{dr} \right)^2 - (1-f) \left(\frac{x^2 + y^2}{r^3} \frac{df}{dr} + \frac{(z - z_0(t))^2}{r^2} \frac{d^2f}{dr^2} \right) \right] \mp \frac{v}{8\pi} \frac{x(z - z_0(t))}{r^2} \left(\frac{d^2f}{dr^2} - \frac{1}{r} \frac{df}{dr} \right). \quad (148)$$

Again, the message to take from this is that localized NEC violations are ubiquitous and persist to arbitrarily low warp bubble velocities.

Using the “volume integral quantifier” (as defined in [84, 85]), we may estimate the “total amount” of averaged null energy condition violating matter in this spacetime, given by

$$\int T_{\mu\nu} k_{\pm\hat{z}}^\mu k_{\pm\hat{z}}^\nu d^3x \approx \int T_{\mu\nu} k_{\pm\hat{x}}^\mu k_{\pm\hat{x}}^\nu d^3x \approx -v^2 R^2 \sigma \approx M_{\text{warp}}. \quad (149)$$

The key things to note here are that the net volume integral of the $O(v)$ term is zero, and that the net volume average of the NEC violations is approximately the same as the net volume average of the WEC violations, which are $O(v^2)$.

D. The Quantum Inequality applied to the “warp drive”

It is of a particular interest to apply the Quantum Inequality (QI), outlined in Section II F 4, to “warp drive” spacetimes [29]. Inserting the energy density, Eq. (140), into the QI, Eq. (81), one gets

$$t_0 \int_{-\infty}^{+\infty} \frac{v(t)^2}{r^2} \left(\frac{df}{dr} \right)^2 \frac{dt}{t^2 + t_0^2} \leq \frac{3}{\rho^2 t_0^4}, \quad (150)$$

where $\rho = (x^2 + y^2)^{1/2}$ is defined for notational convenience.

The warp bubble’s velocity can be considered roughly constant, $v_s(t) \approx v_b$, if the time scale of the sampling is sufficiently small compared to the time scale over which the bubble’s velocity is changing. Taking into account the small sampling time, the $(t^2 + t_0^2)^{-1}$ term becomes strongly peaked, so that only a small portion of the geodesic is sampled by the QI integral. Consider that the observer is at the equator of the warp bubble at $t = 0$ [29], so that the geodesic is approximated by

$$x(t) \approx f(\rho)v_b t, \quad (151)$$

so that we have $r(t) = [(v_b t)^2 (f(\rho) - 1)^2 + \rho^2]^{1/2}$.

Without a significant loss of generality, one may consider a piece-wise continuous form of the shape function given by

$$f_{p.c.}(r) = \begin{cases} 1 & r < R - \frac{\Delta}{2} \\ -\frac{1}{\Delta}(r - R - \frac{\Delta}{2}) & R - \frac{\Delta}{2} < r < R + \frac{\Delta}{2} \\ 0 & r > R + \frac{\Delta}{2} \end{cases} \quad (152)$$

where R is the radius of the bubble, and Δ the bubble wall thickness [29]. Δ is related to the Alcubierre parameter σ by setting the slopes of the functions $f(r)$ and $f_{p.c.}(r)$ to be equal at $r = R$, which provides the following relationship

$$\Delta = \frac{[1 + \tanh^2(\sigma R)]^2}{2 \sigma \tanh(\sigma R)}, \quad (153)$$

Note that in the limit of large σR one obtains the approximation $\Delta \simeq 2/\sigma$. The QI-bound then becomes

$$t_0 \int_{-\infty}^{+\infty} \frac{dt}{(t^2 + \beta^2)(t^2 + t_0^2)} \leq \frac{3\Delta^2}{v_b^2 t_0^4 \beta^2} \quad (154)$$

where

$$\beta = \frac{\rho}{v_b [1 - f(\rho)]}, \quad (155)$$

and yields the following inequality

$$\frac{\pi}{3} \leq \frac{\Delta^2}{v_b^2 t_0^4} \left[\frac{v_b t_0}{\rho} (1 - f(\rho)) + 1 \right]. \quad (156)$$

It is important to emphasize that the above inequality is only valid for sampling times on which the spacetime may be considered approximately flat. Considering the Riemann tensor components in an orthonormal frame [29], the largest component is given by

$$|R_{\hat{t}\hat{y}\hat{t}\hat{y}}| = \frac{3v_b^2 y^2}{4 \rho^2} \left[\frac{df(\rho)}{d\rho} \right]^2, \quad (157)$$

which yields $r_{\min} \equiv 1/\sqrt{|R_{\hat{t}\hat{y}\hat{t}\hat{y}}|} \sim \frac{2\Delta}{\sqrt{3} v_b}$, when $y = \rho$ and the piece-wise continuous form of the shape function is used. The sampling time must be smaller than this length scale, so that one may define

$$t_0 = \alpha \frac{2\Delta}{\sqrt{3} v_b}, \quad 0 < \alpha \ll 1. \quad (158)$$

Considering $\Delta/\rho \sim v_b t_0/\rho \ll 1$, the term involving $1 - f(\rho)$ in Eq. (156) may be neglected, which provides

$$\Delta \leq \frac{3}{4} \sqrt{\frac{3}{\pi}} \frac{v_b}{\alpha^2}. \quad (159)$$

Now, for instance considering $\alpha = 1/10$, one obtains

$$\Delta \leq 10^2 v_b L_{\text{Planck}}, \quad (160)$$

where L_{Planck} is the Planck length. Thus, unless v_b is extremely large, the wall thickness cannot be much above the Planck scale.

It is also interesting to find an estimate of the total amount of negative energy that is necessary to maintain a warp metric. It was found that the energy required for a warp bubble is on the order of

$$E \leq -3 \times 10^{20} M_{\text{galaxy}} v_b, \quad (161)$$

which is an absurdly enormous amount of negative energy, roughly ten orders of magnitude greater than the total mass of the entire visible universe [29].

E. Linearized warp drive

To ever bring a ‘‘warp drive’’ into a strong-field regime, any highly-advanced civilization would first have to take it through the ‘‘weak-field’’ regime. The central point of this Section is to demonstrate that there are significant problems that already arise even in the weak-field regime, and long before strong field effects come into play. In the weak-field regime, applying linearized theory, the physics is much simpler than in the strong-field regime and this allows us to ask and answer questions that are difficult to even formulate in the strong field regime [179]. Our goal now is to try to build a more realistic model of a warp drive spacetime where the warp bubble is interacting with a finite mass spaceship. To do so we first consider the linearized theory applied to warp drive spacetimes, for non-relativistic velocities, $v \ll 1$.

1. *The WEC violation to first order of v*

It is interesting to consider the specific case of an observer which moves with an arbitrary velocity, $\tilde{\beta}$, along the positive z axis measure a negative energy density [at $O(v)$]. That is, $T_{\hat{0}\hat{0}} < 0$. The $\tilde{\beta}$ occurring here is completely independent of the shift vector $\beta(x, y, z - z_0(t))$, and is also completely independent of the warp bubble velocity v .

We have

$$T_{\hat{0}\hat{0}} = \frac{\gamma^2 \tilde{\beta} v}{8\pi} \left[\left(\frac{x^2 + y^2}{r^2} \right) \frac{d^2 f}{dr^2} + \left(\frac{x^2 + y^2 + 2(z - z_0(t))^2}{r^3} \right) \frac{df}{dr} \right] + O(v^2). \quad (162)$$

See Ref. [179] for details. A number of general features can be extracted from the terms in square brackets, without specifying an explicit form of f . In particular, f decreases monotonically from its value at $r = 0$, $f = 1$, to $f \approx 0$ at $r \geq R$, so that df/dr is negative in this domain. The form function attains its maximum in the interior of the bubble wall, so that $d^2 f/dr^2$ is also negative in this region. Therefore there is a range of r in the immediate interior neighbourhood of the bubble wall that necessarily provides negative energy density, as seen by the observers considered above. Again we find that WEC violations persist to arbitrarily low warp bubble velocities. The negative character of the energy density can be seen from Figure 6.

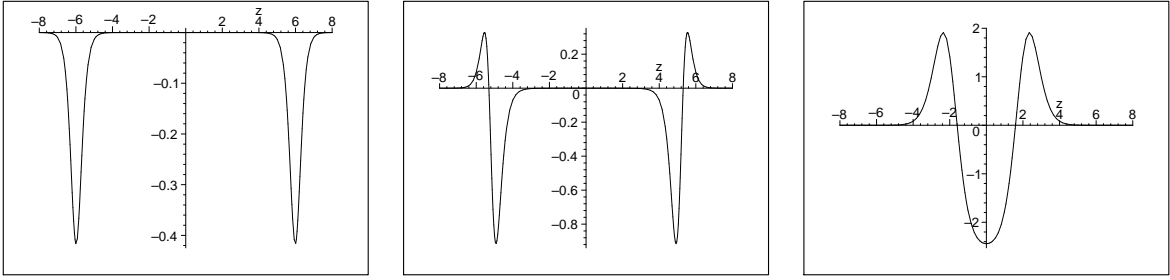


FIG. 6: The term in square brackets of Eq. (162) is plotted as a function of the z coordinate. Taking into account the Alcubierre form function, we have considered the following values for the parameters $\sigma = 2.5$ and $R = 6$. Considering the definition $\rho = \sqrt{x^2 + y^2}$, the plots have the respective values of $\rho = 0$, $\rho = 3$ and $\rho = 5.8$.

2. *Spaceship within the warp bubble*

Consider now a spaceship in the interior of an Alcubierre warp bubble, which is moving along the positive z axis with a non-relativistic constant velocity [179], i.e., $v \ll 1$. The metric is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + [dz - v f(x, y, z - vt) dt]^2 - 2\Phi(x, y, z - vt) [dt^2 + dx^2 + dy^2 + (dz - v f(x, y, z - vt) dt)^2]. \quad (163)$$

If $\Phi = 0$, the metric (163) reduces to the warp drive spacetime of Eq. (133). If $v = 0$, we have the metric representing the gravitational field of a static source.

Consider the approximation in which we keep the exact v dependence but linearize in the gravitational field of the spaceship Φ .

The WEC is given by

$$T_{\hat{\mu}\hat{\nu}} U^{\hat{\mu}} U^{\hat{\nu}} = \rho - \frac{v^2}{32\pi} \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] + O(\Phi^2), \quad (164)$$

or by taking into account the Alcubierre form function, we have

$$T_{\mu\nu} U^\mu U^\nu = \rho - \frac{1}{32\pi} \frac{v^2(x^2 + y^2)}{r^2} \left(\frac{df}{dr} \right)^2 + O(\Phi^2). \quad (165)$$

Once again, using the ‘‘volume integral quantifier’’, we find the following estimate

$$\int T_{\hat{\mu}\hat{\nu}} U^{\hat{\mu}} U^{\hat{\nu}} d^3x = M_{\text{ship}} - v^2 R^2 \sigma + \int O(\Phi^2) d^3x, \quad (166)$$

which we can recast as

$$M_{\text{ADM}} = M_{\text{ship}} + M_{\text{warp}} + \int O(\Phi^2) d^3x. \quad (167)$$

Now suppose we demand that the volume integral of the WEC at least be positive, then

$$v^2 R^2 \sigma \leq M_{\text{ship}}. \quad (168)$$

This equation is effectively the quite reasonable condition that the net total energy stored in the warp field be less than the total mass-energy of the spaceship itself, which places a powerful constraint on the velocity of the warp bubble. Re-writing this in terms of the size of the spaceship R_{ship} and the thickness of the warp bubble walls $\Delta = 1/\sigma$, we have

$$v^2 \leq \frac{M_{\text{ship}}}{R_{\text{ship}}} \frac{R_{\text{ship}} \Delta}{R^2}. \quad (169)$$

For any reasonable spaceship this gives extremely low bounds on the warp bubble velocity.

In a similar manner, the NEC, with $k^\mu = (1, 0, 0, \pm 1)$, is given by

$$T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}} k^{\hat{\nu}} = \rho \pm \frac{v}{8\pi} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) - \frac{v^2}{8\pi} \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] + O(\Phi^2). \quad (170)$$

Considering the ‘‘volume integral quantifier’’, we verify that, as before, the exact solution in terms of polylogarithmic functions is unhelpful, although we may estimate that

$$\int T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}} k^{\hat{\nu}} d^3x = M_{\text{ship}} - v^2 R^2 \sigma + \int O(\Phi^2) d^3x, \quad (171)$$

which is [to order $O(\Phi^2)$] the same integral we encountered when dealing with the WEC. This volume integrated NEC is now positive if

$$v^2 R^2 \sigma \leq M_{\text{ship}}. \quad (172)$$

Finally, considering a null vector oriented perpendicularly to the direction of motion (for definiteness take $\hat{k} = \pm \hat{x}$), the NEC takes the following form

$$T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}} k^{\hat{\nu}} = \rho - \frac{v^2}{8\pi} \left[\frac{1}{2} \left(\frac{\partial f}{\partial y} \right)^2 + \left(\frac{\partial f}{\partial z} \right)^2 - (1-f) \frac{\partial^2 f}{\partial z^2} \right] \mp \frac{v}{8\pi} \left(\frac{\partial^2 f}{\partial x \partial z} \right) + O(\Phi^2). \quad (173)$$

Once again, evaluating the ‘‘volume integral quantifier’’, we have

$$\int T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}} k^{\hat{\nu}} d^3x = M_{\text{ship}} - \frac{v^2}{4} \int \left(\frac{df}{dr} \right)^2 r^2 dr + \frac{v^2}{6} \int (1-f) \left(2r \frac{df}{dr} + r^2 \frac{d^2 f}{dr^2} \right) dr + \int O(\Phi^2) d^3x, \quad (174)$$

which, as before, may be estimated as

$$\int T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}} k^{\hat{\nu}} d^3x \approx M_{\text{ship}} - v^2 R^2 \sigma + \int O(\Phi^2) d^3x. \quad (175)$$

If we do not want the total NEC violations in the warp field to exceed the mass of the spaceship itself we must again demand

$$v^2 R^2 \sigma \leq M_{\text{ship}}. \quad (176)$$

This places an extremely stringent condition on the warp drive spacetime, namely, that for all conceivably interesting situations the bubble velocity should be absurdly low, and it therefore appears unlikely that, by using this analysis, the warp drive will ever prove to be technologically useful. Finally, we point out that any attempt at building up a ‘‘strong-field’’ warp drive starting from an approximately Minkowski spacetime will inevitably have to pass through a weak-field regime. Since the weak-field warp drives are already so tightly constrained, the analysis above implies additional difficulties for developing a ‘‘strong field’’ warp drive. See Ref. [179] for more details.

F. Interesting aspects of the Alcubierre spacetime

1. Superluminal travel in the warp drive

To demonstrate that it is possible to travel to a distant point and back in an arbitrary short time interval, let us consider two distant stars, A and B , separated by a distance D in flat spacetime. Suppose that, at the instant t_0 , a spaceship initiates its movement using the engines, moving away from A with a velocity $v < 1$. It comes to rest at a distance d from A . For simplicity, assume that $R \ll d \ll D$. It is at this instant that the perturbation of spacetime appears, centered around the spaceship's position. The perturbation pushes the spaceship away from A , rapidly attaining a constant acceleration, a . Half-way between A and B , the perturbation is modified, so that the acceleration rapidly varies from a to $-a$. The spaceship finally comes to rest at a distance, d , from B , in which the perturbation disappears. It then moves to B at a constant velocity in flat spacetime. The return trip to A is analogous.

If the variations of the acceleration are extremely rapid, the total coordinate time, T , in a one-way trip will be

$$T = 2 \left(\frac{d}{v} + \sqrt{\frac{D-2d}{a}} \right). \quad (177)$$

The proper time of the stars are equal to the coordinate time, because both are immersed in flat spacetime. The proper time measured by observers within the spaceship is given by:

$$\tau = 2 \left(\frac{d}{\gamma v} + \sqrt{\frac{D-2d}{a}} \right), \quad (178)$$

with $\gamma = (1 - v^2)^{-1/2}$. The time dilation only appears in the absence of the perturbation, in which the spaceship is moving with a velocity v , using only its engines in flat spacetime.

Using $R \ll d \ll D$, we can then obtain the following approximation

$$\tau \approx T \approx 2\sqrt{\frac{D}{a}}. \quad (179)$$

Note that T can be made arbitrarily short, by increasing the value of a . The spaceship may travel faster than the speed of light. However, it moves along a spacetime temporal trajectory, contained within its light cone, as light suffers the same distortion of spacetime [176].

2. The Krasnikov analysis

Krasnikov discovered a fascinating aspect of the warp drive, in which an observer on a spaceship cannot create nor control on demand an Alcubierre bubble, with $v > c$, around the ship [177]. It is easy to understand this, as an observer at the origin (with $t = 0$), cannot alter events outside of his future light cone, $|r| \leq t$, with $r = (x^2 + y^2 + z^2)^{1/2}$. Applied to the warp drive, points on the outside front edge of the bubble are always spacelike separated from the centre of the bubble.

The analysis is simplified in the proper reference frame of an observer at the centre of the bubble. Using the transformation $z' = z - z_0(t)$, the metric is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + [dz' + (1-f)vdt]^2. \quad (180)$$

Consider a photon emitted along the $+Oz$ axis (with $ds^2 = dx = dy = 0$):

$$\frac{dz'}{dt} = 1 - (1-f)v. \quad (181)$$

If the spaceship is at rest at the center of the bubble, then initially the photon has $dz/dt = v + 1$ or $dz'/dt = 1$ (because $f = 1$ in the interior of the bubble). However, at some point $z' = z'_c$, with $f = 1 - 1/v$, we have $dz'/dt = 0$ [30]. Once photons reach z'_c , they remain at rest relative to the bubble and are simply carried along with it. Photons emitted in the forward direction by the spaceship never reach the outside edge of the bubble wall, which therefore lies outside the forward light cone of the spaceship. The bubble thus cannot be created (or controlled) by any action of the spaceship crew. This behaviour is reminiscent of an *event horizon*. This does not mean that Alcubierre bubbles, if it were possible to create them, could not be used as a means of superluminal travel. It only means that the actions required to change the metric and create the bubble must be taken beforehand by some observer whose forward light cone contains the entire trajectory of the bubble.

3. Reminiscence of an Event Horizon

The appearance of an event horizon becomes evident in the 2-dimensional model of the Alcubierre space-time [181, 182, 183]. The axis of symmetry coincides with the line element of the spaceship. The metric, Eq. (133), reduces to

$$ds^2 = -(1 - v^2 f^2) dt^2 - 2v f dz dt + dz^2. \quad (182)$$

For simplicity, we consider the velocity of the bubble constant, $v(t) = v_b$, and we have $r = [(z - v_b t)^2]^{1/2}$. If $z > v_b t$, we consider the transformation $r = (z - v_b t)$. Note that the metric components of Eq. (182) only depend on r , which may be adopted as a coordinate.

Using the transformation, $dz = dr + v_b dt$, the metric, Eq. (182) is given by

$$ds^2 = -A(r) \left[dt - \frac{v_b(1 - f(r))}{A(r)} dr \right]^2 + \frac{dr^2}{A(r)}. \quad (183)$$

The function $A(r)$, denoted by the Hiscock function, is given by

$$A(r) = 1 - v_b^2 [1 - f(r)]^2. \quad (184)$$

Its possible to represent the metric, Eq. (183), in a diagonal form, using a new time coordinate

$$d\tau = dt - \frac{v_b [1 - f(r)]}{A(r)} dr, \quad (185)$$

with which Eq. (183) reduces to

$$ds^2 = -A(r) d\tau^2 + \frac{dr^2}{A(r)}. \quad (186)$$

This form of the metric is manifestly static. The τ coordinate has an immediate interpretation in terms of an observer on board of a spaceship: τ is the proper time of the observer, because $A(r) \rightarrow 1$ in the limit $r \rightarrow 0$. We verify that the coordinate system is valid for any value of r , if $v_b < 1$. If $v_b > 1$, we have a coordinate singularity and an event horizon at the point r_0 in which $f(r_0) = 1 - 1/v_b$ and $A(r_0) = 0$.

G. Superluminal subway: The Krasnikov tube

It was pointed out in Section III F, that Krasnikov discovered an interesting aspect of the warp drive, in which an observer on a spaceship cannot create nor control on demand an Alcubierre bubble, i.e., points on the outside front edge of the bubble are always spacelike separated from the centre of the bubble. However, causality considerations do not prevent the crew of a spaceship from arranging, by their own actions, to complete a *round trip* from the Earth to a distant star and back in an arbitrarily short time, as measured by clocks on the Earth, by altering the metric along the path of their outbound trip. Thus, Krasnikov introduced a metric with an interesting property that although the time for a one-way trip to a distant destination cannot be shortened, the time for a round trip, as measured by clocks at the starting point (e.g. Earth), can be made arbitrarily short, as will be demonstrated below.

1. The 2-dimensional Krasnikov solution

The 2-dimensional metric is given by

$$\begin{aligned} ds^2 &= -(dt - dx)(dt + k(t, x)dx) \\ &= -dt^2 + [1 - k(x, t)] dx dt + k(x, t) dx^2. \end{aligned} \quad (187)$$

The form function $k(x, t)$ is defined by

$$k(t, x) = 1 - (2 - \delta)\theta_\varepsilon(t - x) [\theta_\varepsilon(x) - \theta_\varepsilon(x + \varepsilon - D)], \quad (188)$$

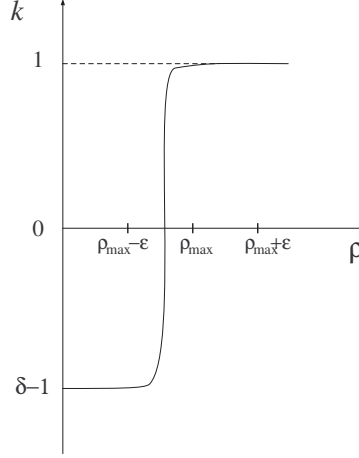


FIG. 7: Graph of the Krasnikov form function k vs ρ at constant x and t , considering only the region of $\epsilon < x < D - \epsilon$.

where δ and ϵ are arbitrarily small positive parameters. θ_ϵ denotes a smooth monotone function

$$\theta_\epsilon(\xi) = \begin{cases} 1, & \text{if } \xi > \epsilon, \\ 0, & \text{if } \xi < 0, \end{cases}$$

which is depicted in Figure 7.

There are three distinct regions in the Krasnikov two-dimensional spacetime, which we shall summarize in the following manner.

The outer region. The outer region is given by the following set

$$\{x < 0\} \cup \{x > D\} \cup \{x > t\}. \quad (189)$$

The two time-independent θ_ϵ -functions between the brackets in Eq. (188) vanish for $x < 0$ and cancel for $x > D$, ensuring $k = 1$ for all t except between $x = 0$ and $x = D$. When this behavior is combined with the effect of the factor $\theta_\epsilon(t - x)$, one sees that the metric (187) is flat, $k = 1$, and reduces to the Minkowski spacetime everywhere for $t < 0$ and at all times outside the range $0 < x < D$. Future light cones are generated by the vectors:

$$\begin{cases} r_O = \partial_t + \partial_x, \\ l_O = \partial_t - \partial_x. \end{cases}$$

The inner region. The inner region is given by the following set

$$\{x < t - \epsilon\} \cap \{\epsilon < x < D - \epsilon\}, \quad (190)$$

so that the first two θ_ϵ -functions in Eq. (188) both equal 1, while $\theta_\epsilon(x + \epsilon - D) = 0$, giving $k = \delta - 1$ everywhere within this region. This region is also flat, but the light cones are *more open*, being generated by the following vectors

$$\begin{cases} r_I = \partial_t + \partial_x \\ l_I = -(1 - \delta)\partial_t - \partial_x. \end{cases}$$

The transition region. The transition region is a narrow curved strip in spacetime, with width $\sim \epsilon$. Two spatial boundaries exist between the inner and outer regions. The first lies between $x = 0$ and $x = \epsilon$, for $t > 0$. The second lies between $x = D - \epsilon$ and $x = D$, for $t > D$. It is possible to view this metric as being produced by the crew of a spaceship, departing from point A ($x = 0$), at $t = 0$, travelling along the x -axis to point B ($x = D$) at a speed, for simplicity, infinitesimally close to the speed of light, therefore arriving at B with $t \approx D$.

The metric is modified by changing k from 1 to $\delta - 1$ along the x -axis, in between $x = 0$ and $x = D$, leaving a transition region of width $\sim \epsilon$ at each end for continuity. But, as the boundary of the forward light cone of the spaceship at $t = 0$ is $|x| = t$, it is not possible for the crew to modify the metric at an arbitrary point x before $t = x$. This fact accounts for the factor $\theta_\epsilon(t - x)$ in the metric, ensuring a transition region in time between the inner and outer region, with a duration of $\sim \epsilon$, lying along the worldline of the spaceship, $x \approx t$. The geometry is shown in the (x, t) plane in Figure 8.

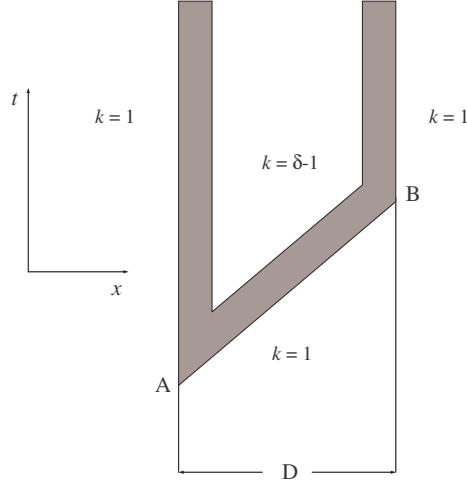


FIG. 8: The Krasnikov spacetime in the (x, t) plane. The vertical lines A and B are the world lines of the stars A and B , respectively. The world line of the spaceship is approximately represented by the line segment AB .

2. Superluminal travel within the Krasnikov tube

The properties of the modified metric with $\delta - 1 \leq k \leq 1$ can be easily seen from the factored form of $ds^2 = 0$. The two branches of the forward light cone in the (t, x) plane are given by $dx/dt = 1$ and $dx/dt = -k$. As k becomes smaller and then negative, the slope of the left-hand branch of the light cone becomes less negative and then changes sign, i. e., the light cone along the negative x -axis “opens out”. See Figure 9.

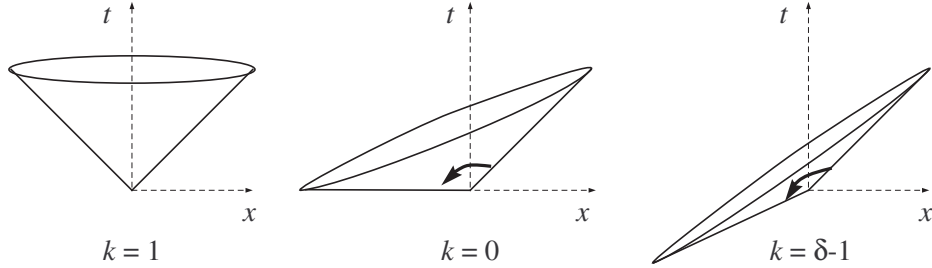


FIG. 9: Forward light cones in the 2-dimensional Krasnikov spacetime for $k = 1$, $k = 0$ and $k = \delta - 1$.

The inner region, with $k = \delta - 1$, is flat because the metric, Eq. (187), may be cast into the Minkowski form, applying the following coordinate transformations

$$dt' = dt + \left(\frac{\delta}{2} - 1\right) dx, \quad dx' = \left(\frac{\delta}{2}\right) dx. \quad (191)$$

The transformation is singular at $\delta = 0$, i.e., $k = -1$. Note that the left branch of the region is given by $dx'/dt' = -1$.

From the above equations, one may easily deduce the following expression

$$\frac{dt}{dt'} = 1 + \left(\frac{2 - \delta}{\delta}\right) \frac{dx'}{dt'}. \quad (192)$$

For an observer moving along the positive x' and x directions, with $dx'/dt' < 1$, we have $dt' > 0$ and consequently $dt > 0$, if $0 < \delta \leq 2$. However, if the observer is moving sufficiently close to the left branch of the light cone, given by $dx'/dt' = -1$, Eq. (192) provides us with $dt/dt' < 0$, for $\delta < 1$. Therefore $dt < 0$, the observer traverses backward in time, as measured by observers in the outer region, with $k = 1$.

The superluminal travel analysis is as follows. Imagine a spaceship leaving star A and arriving at star B , at the instant $t \approx D$. The crew of the spaceship modify the metric, so that $k \approx -1$, for simplicity, along the trajectory.

Now suppose the spaceship returns to star A , travelling with a velocity arbitrarily close to the speed of light, i.e., $\frac{dx'}{dt'} \approx -1$. Therefore, from Eq. (191), one obtains the following relation

$$v_{\text{return}} = \frac{dx}{dt} \approx -\frac{1}{k} = \frac{1}{1-\delta} \approx 1 \quad (193)$$

and $dt < 0$, for $dx < 0$. The return trip from star B to A is done in an interval of $\Delta t_{\text{return}} = -D/v_{\text{return}} = D/(\delta - 1)$. The total interval of time, measured at A , is given by $T_A = D + \Delta t_{\text{return}} = D\delta$. For simplicity, consider ε negligible. Superluminal travel is implicit, because $|\Delta t_{\text{return}}| < D$, if $\delta > 0$, i.e., we have a spatial spacetime interval between A and B . Note that T_A is always positive, but may attain a value arbitrarily close to zero, for an appropriate choice of δ .

Note that for the case $\delta < 1$, it is always possible to choose an allowed value of dx'/dt' for which $dt/dt' = 0$, meaning that the return trip is instantaneous as seen by observers in the external region. This follows easily from Eq. (192), which implies that $dt/dt' = 0$ when dx'/dt' satisfies

$$\frac{dx'}{dt'} = -\frac{\delta}{(2-\delta)}, \quad (194)$$

which lies between 0 and -1 for $0 < \delta < 1$.

3. The 4-dimensional generalization

Soon after the Krasnikov two-dimensional solution, Everett and Roman [30] generalized the analysis to four dimensions, denoting the solution as the *Krasnikov tube*. Consider that the 4-dimensional modification of the metric begins along the path of the spaceship, which is moving along the x -axis, occurring at position x at time $t \approx x$, the time of passage of the spaceship. Also assume that the disturbance in the metric propagates radially outward from the x -axis, so that causality guarantees that at time t the region in which the metric has been modified cannot extend beyond $\rho = t - x$, where $\rho = (y^2 + z^2)^{1/2}$. The modification in the metric should also not extend beyond some maximum radial distance $\rho_{max} \ll D$ from the x -axis. Thus, the metric in the 4-dimensional spacetime, written in cylindrical coordinates, is given by [30]

$$ds^2 = -dt^2 + (1 - k(t, x, \rho))dxdt + k(t, x, \rho)dx^2 + d\rho^2 + \rho^2 d\phi^2, \quad (195)$$

with

$$k(t, x, \rho) = 1 - (2 - \delta)\theta_\varepsilon(\rho_{max} - \rho)\theta_\varepsilon(t - x - \rho)[\theta_\varepsilon(x) - \theta_\varepsilon(x + \varepsilon - D)]. \quad (196)$$

For $t \gg D + \rho_{max}$ one has a tube of radius ρ_{max} centered on the x -axis, within which the metric has been modified. This structure is denoted by the *Krasnikov tube*. In contrast with the Alcubierre spacetime metric, the metric of the Krasnikov tube is static once it has been created.

The stress-energy tensor element T_{tt} given by

$$T_{tt} = \frac{1}{32\pi(1+k)^2} \left[-\frac{4(1+k)}{\rho} \frac{\partial k}{\partial \rho} + 3 \left(\frac{\partial k}{\partial \rho} \right)^2 - 4(1+k) \frac{\partial^2 k}{\partial \rho^2} \right], \quad (197)$$

can be shown to be the energy density measured by a static observer [30], and violates the WEC in a certain range of ρ , i.e., $T_{\mu\nu}U^\mu U^\nu < 0$.

To verify the violation of the WEC, consider the energy density in the middle of the tube and at a time long after it's formation, i.e., $x = D/2$ and $t \gg x + \rho + \varepsilon$, respectively. In this region we have $\theta_\varepsilon(x) = 1$, $\theta_\varepsilon(x + \varepsilon - D) = 0$ and $\theta_\varepsilon(t - x - \rho) = 1$. With this simplification the form function, Eq. (196), reduces to

$$k(t, x, \rho) = 1 - (2 - \delta)\theta_\varepsilon(\rho_{max} - \rho). \quad (198)$$

Consider the following specific form for $\theta_\varepsilon(\xi)$ [30] given by

$$\theta_\varepsilon(\xi) = \frac{1}{2} \left\{ \tanh \left[2 \left(\frac{2\xi}{\varepsilon} - 1 \right) \right] + 1 \right\}, \quad (199)$$

so that the form function of Eq. (198) is provided by

$$k = 1 - \left(1 - \frac{\delta}{2} \right) \left\{ \tanh \left[2 \left(\frac{2\xi}{\varepsilon} - 1 \right) \right] + 1 \right\}. \quad (200)$$

Choosing the following values for the parameters: $\delta = 0.1$, $\varepsilon = 1$ and $\rho_{max} = 100\varepsilon = 100$, the negative character of the energy density is manifest in the immediate inner vicinity of the tube wall, as shown in Figure 10.

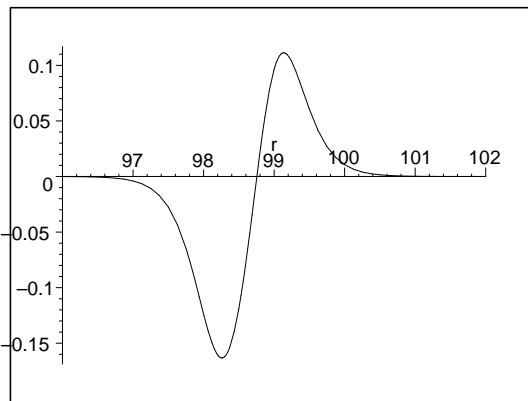


FIG. 10: Graph of the energy density, T_{tt} , as a function of ρ at the middle of the Krasnikov tube, $x = D/2$, and long after it's formation, $t \gg x + \rho + \varepsilon$. We consider the following values for the parameters: $\delta = 0.1$, $\varepsilon = 1$ and $\rho_{max} = 100\varepsilon = 100$.

IV. CLOSED TIMELIKE CURVES AND CAUSALITY VIOLATION

As time is incorporated into the proper structure of the fabric of spacetime, it is interesting to note that general relativity is contaminated with non-trivial geometries which generate *closed timelike curves* [2, 26, 28, 91, 134, 188]. A closed timelike curve (CTC) allows time travel, in the sense that an observer which travels on a trajectory in spacetime along this curve, returns to an event which coincides with the departure. The arrow of time leads forward, as measured locally by the observer, but globally he/she may return to an event in the past. This fact apparently violates causality, opening Pandora's box and producing time travel paradoxes [190]. The notion of causality is fundamental in the construction of physical theories, therefore time travel and it's associated paradoxes have to be treated with great caution. A great variety of solutions to the Einstein Field Equations (EFEs) containing CTCs exist, but, two particularly notorious features seem to stand out. Solutions with a tipping over of the light cones due to a rotation about a cylindrically symmetric axis; and solutions that violate the Energy Conditions of general relativity, which are fundamental in the singularity theorems and theorems of classical black hole thermodynamics [15].

A. Stationary and axisymmetric solutions generating CTCs

The tipping over of light cones seem to be a generic feature of some solutions with a rotating cylindrical symmetry. The general metric for a stationary, axisymmetric solution with rotation is given by [2, 204]

$$ds^2 = -F(r) dt^2 + H(r) dr^2 + L(r) d\phi^2 + 2M(r) d\phi dt + H(r) dz^2, \quad (201)$$

The metric components are functions of r alone. It is clear that the determinant, $g = \det(g_{\mu\nu}) = -(FL + M^2)H^2$ is Lorentzian, provided that $(FL + H^2) > 0$.

Due to the periodic nature of the angular coordinate, ϕ , an azimuthal curve with $\gamma = \{t = \text{const}, r = \text{const}, z = \text{const}\}$ is a closed curve of invariant length $s_\gamma^2 \equiv L(r)(2\pi)^2$. If $L(r)$ is negative then the integral curve with (t, r, z) fixed is a CTC. If $L(r) = 0$, then the azimuthal curve is a closed null curve, CNC. Alternatively, consider a null azimuthal curve in the (ϕ, t) plane with (r, z) fixed. It is not necessarily a geodesic, nor will it be a closed curve. The null condition, $ds^2 = 0$, implies

$$0 = -F + 2M\dot{\phi} + L\dot{\phi}^2, \quad (202)$$

with $\dot{\phi} = d\phi/dt$. Solving the quadratic, we have

$$\frac{d\phi}{dt} = \dot{\phi} = \frac{-M \pm \sqrt{M^2 + FL}}{L}. \quad (203)$$

In virtue of the Lorentzian signature constraint, $FL + H^2 > 0$, the roots are real. If $L(r) < 0$ then the light cones are tipped over sufficiently far to permit a trip to the past. By going once around the azimuthal direction, the total

backward time-jump for a null curve is

$$\Delta T = \frac{2\pi|L|}{-M + \sqrt{M^2 - F|L|}}. \quad (204)$$

Roughly, light cones which are tilted over are generic features of spacetimes which contain CTCs.

If $L(r) < 0$ for even a single value of r , then there is a closed causal curve passing through every point of the spacetime. To visualize this consider a null curve beginning at an arbitrary x , reaching r such that $L(r) < 0$. Then follow the null curve that wraps around the azimuth a total of N times. The total backward time-jump is then $N\Delta T$. Finally, follow an ordinary null curve back to the starting point x . So, if $L(r) < 0$ for even a single value of r , the chronology-violation region covers the entire spacetime. See Ref. [2] for more details.

The present Section is far from making an exhaustive search of all the EFE solutions generating CTCs with these features, but the best known spacetimes will be briefly analyzed, namely, the van Stockum spacetime, the Gödel universe, the spinning cosmic strings and the Gott two-string time machine, which is a variation on the theme of the spinning cosmic string.

1. Van Stockum spacetime

The earliest solution to the EFEs containing CTCs, is probably that of the van Stockum spacetime. It is a stationary, cylindrically symmetric solution describing a rapidly rotating infinite cylinder of dust, surrounded by vacuum. The centrifugal forces of the dust are balanced by the gravitational attraction. The metric, assuming the respective symmetries, takes the form of Eq. (201). Consider a frame in which the matter is at rest, and it can be shown that the source is simply positive density dust, implying that all of the energy condition are satisfied [2].

The metric for the interior solution $r < R$, where the surface of the cylinder is located at $r = R$, is

$$ds^2 = -dt^2 + 2\omega r^2 d\phi dt + r^2(1 - \omega^2 r^2)d\phi^2 + \exp(-\omega^2 r^2)(dr^2 + dz^2) \quad (205)$$

where ω is the angular velocity of the cylinder. It is readily verified that CTCs arise if $\omega r > 1$, i.e., for $r > 1/\omega$ the azimuthal curves with (t, r, z) fixed are CTCs. The condition $M^2 + FL = \omega^2 r^4 + r^2(1 - \omega^2 r^2) = r^2 > 0$ is imposed.

The causality violation region could be eliminated by requiring that the boundary of the cylinder to be at $r = R < 1/a$. The interior solution would then be joined to an exterior solution, which would be causally well-behaved. The resulting upper bound to the “velocity” ωR would be 1, although the orbits of the particles creating the field are timelike for all r .

Van Stockum also developed a procedure which generates an exterior solution for all $\omega R > 0$ [213]. It can be shown that the causality violation is avoided for $\omega R \leq 1/2$, but in the region $\omega R > 1/2$, CTCs appear. Consider the exterior metric components for this range, $\omega R > 1/2$:

$$H(r) = \exp(-\omega^2 r^2) (r/R)^{-2\omega^2 r^2}, \quad L(r) = \frac{Rr \sin(3\beta + \gamma)}{2 \sin(2\beta) \cos(\beta)},$$

$$M(r) = \frac{r \sin(\beta + \gamma)}{\sin(2\beta)}, \quad F(r) = \frac{r \sin(\beta - \gamma)}{R \sin(\beta)},$$

with

$$\gamma = \gamma(r) = (4\omega^2 R^2 - 1)^{1/2} \ln(r/R) \quad \text{and} \quad \beta = \beta(r) = \arctan(4\omega^2 R^2 - 1)^{1/2}.$$

As is the interior solution, one may verify that $FL + M^2 = r^2$, so that the metric signature is Lorentzian for $R \leq r < \infty$.

The causality violations arise from the sinusoidal factors of the metric components. Thus, causality violation occur in the matter-free space surrounding a rapidly rotating infinite cylinder, as shown in Figure 11. However, it is not clear that the properties of such a cylinder also hold for realistic cylinders.

The van Stockum spacetime is not asymptotically flat. But, the gravitational potential of the cylinder’s Newtonian analog also diverges at radial infinity. Shrinking the cylinder down to a “ring” singularity, one ends up with the Kerr solution, which also has CTCs (The causal structure of the Kerr spacetime has been extensively analyzed by de Felice and collaborators [214, 215, 216, 217, 218]).

In summary, the van Stockum solution contains CTC provided $\omega R > 1/2$. The causality-violating region covers the entire spacetime. Reactions to the van Stockum solution is that it is unphysical, as it applies to an infinitely long cylinder and it is not asymptotically flat.

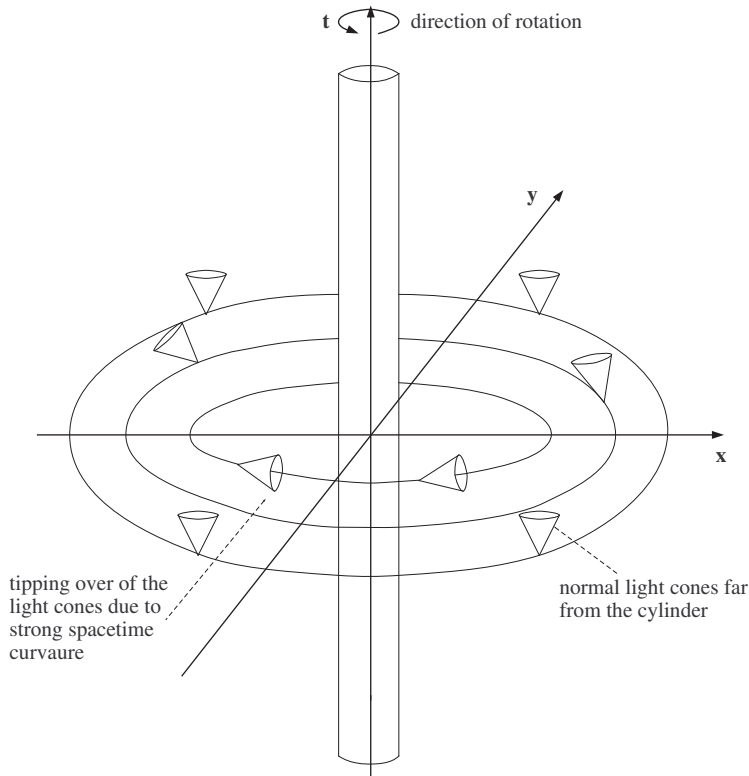


FIG. 11: Van Stockum spacetime showing the tipping over of light cones close to the cylinder, due to the strong curvature of spacetime, which induce closed timelike curves.

2. The Gödel Universe

Kurt Gödel in 1949 discovered an exact solution to the EFEs of a uniformly rotating universe containing dust and a nonzero cosmological constant [219]. It can be shown that the null, weak and dominant energy conditions are satisfied. However, the dominant energy condition is in the imminence of being violated.

Consider a set of alternative coordinates, which explicitly manifest the rotational symmetry of the solution, around the axis $r = 0$, and suppressing the irrelevant z coordinate [15, 219], the metric of the Gödel solution is provided by

$$ds^2 = 2w^{-2}(-dt'^2 + dr^2 - (\sinh^4 r - \sinh^2 r) d\phi^2 + 2(\sqrt{2}) \sinh^2 r d\phi dt). \quad (206)$$

Moving away from the axis, the light cones open out and tilt in the ϕ -direction. The azimuthal curves with $\gamma = \{t = \text{const}, r = \text{const}, z = \text{const}\}$ are CTCs if the condition $r > \ln(1 + \sqrt{2})$ is satisfied [2].

It is interesting to note that in the Gödel spacetime, closed timelike curves are not geodesics. However, Novello and Rebouças [220] discovered a new generalized solution of the Gödel metric, of a shear-free nonexpanding rotating fluid, in which successive concentric causal and noncausal regions exist, with closed timelike curves which are geodesics. A complete study of geodesic motion in Gödel's universe, using the method of the effective potential was further explored by Novello *et al* [221]. Much interest has been aroused in time travel in the Gödel spacetime, from which we may mention the analysis of the geodesical and non-geodesical motions considered by Pfarr [222] and Malament [223, 224].

3. Spinning Cosmic String

Consider an infinitely long straight string that lies and spins around the z -axis. The symmetries are analogous to the van Stockum spacetime, but the asymptotic behavior is different [2, 225]. We restrict the analysis to an infinitely long straight string, with a delta-function source confined to the z -axis. It is characterized by a mass per unit length, μ ; a tension, τ , and an angular momentum per unit length, J . For cosmic strings, the mass per unit length is equal to the tension, $\mu = \tau$.

In cylindrical coordinates the metric takes the following form

$$ds^2 = -[d(t + 4GJ\varphi)]^2 + dr^2 + (1 - 4G\mu)^2 r^2 d\varphi^2 + dz^2, \quad (207)$$

with the following coordinate range

$$-\infty < t < +\infty, \quad 0 < r < \infty, \quad 0 \leq \varphi \leq 2\pi, \quad -\infty < z < +\infty. \quad (208)$$

Consider an azimuthal curve, i.e., an integral curve of φ . Closed timelike curves appear whenever

$$r < \frac{4GJ}{1 - 4G\mu}. \quad (209)$$

These CTCs can be deformed to cover the entire spacetime, consequently, the chronology-violating region covers the entire manifold [2].

4. *Gott Cosmic String time machine*

An extremely elegant model of a time-machine was constructed by Gott [226]. The Gott time-machine is an exact solution of the EFE for the general case of two moving straight cosmic strings that do not intersect [226]. This solution produces CTCs even though they do not violate the WEC, have no singularities and event horizons, and are not topologically multiply-connected as the wormhole solution. The appearance of CTCs relies solely on the gravitational lens effect and the relativity of simultaneity.

It is also interesting to verify whether the CTCs in the Gott solution appear at some particular moment, i.e., when the strings approach each other's neighborhood, or if they already pre-exist, i.e., they intersect any spacelike hypersurface. These questions are particularly important in view of Hawking's Chronology Protection Conjecture [140]. This conjecture states that the laws of physics prevent the creation of CTCs. If correct, then the solutions of the EFE which admit CTCs are either unrealistic or are solutions in which the CTCs are pre-existing, so that the time-machine is not created by dynamical processes. Amos Ori proved that in Gott's spacetime, CTCs intersect every $t = \text{const}$ hypersurface [227], so that it is not a counter-example to the Chronology Protection Conjecture.

The global structure of the Gott spacetime was further explored by Cutler [228], and it was shown that the closed timelike curves are confined to a certain region of the spacetime, and that the spacetime contains complete spacelike and achronal hypersurfaces from which the causality violating regions evolve. Grant also examined the global structure of the two-string spacetime and found that away from the strings, the space is identical to a generalized Misner space [229]. The vacuum expectation value of the energy-momentum tensor for a conformally coupled scalar field was then calculated on the respective generalized Misner space, which was found to diverge weakly on the chronology horizon, but diverge strongly on the polarized hypersurfaces. Thus, the back reaction due to the divergent behaviour around the polarized hypersurfaces are expected to radically alter the structure of spacetime, before quantum gravitational effects become important, suggesting that Hawking's chronology protection conjecture holds for spaces with a noncompactly generated chronology horizon. Soon after, Laurence [230] showed that the region containing CTCs in Gott's two-string spacetime is identical to the regions of the generalized Misner space found by Grant, and constructed a family of isometries between both Gott's and Grant's regions. This result was used to argue that the slowly diverging vacuum polarization at the chronology horizon of the Grant space carries over without change to the Gott space. Furthermore, it was shown that the Gott time machine is unphysical in nature, for such an acausal behaviour cannot be realized by physical and timelike sources [231, 232, 233, 234, 235].

B. Solutions violating the Energy Conditions

The traditional manner of solving the EFEs, $G_{\mu\nu} = 8\pi GT_{\mu\nu}$, consists in considering a plausible stress-energy tensor, $T_{\mu\nu}$, and finding the geometrical structure, $G_{\mu\nu}$. But one can run the EFE in the reverse direction by imposing an exotic metric $g_{\mu\nu}$, and eventually finding the matter source for the respective geometry. In this fashion, solutions violating the energy conditions have been obtained. Adopting the reverse philosophy, solutions such as traversable wormholes, the warp drive, the Krasnikov tube and the Ori-Soen spacetime have been obtained. These solutions violate the energy conditions and with simple manipulations generate CTCs.

1. Conversion of traversable wormholes into time machines

Much interest has been aroused in traversable wormholes since the classical article by Morris and Thorne [1]. A wormhole is a hypothetical tunnel which connects different regions in spacetime. These solutions are multiply-connected and probably involve a topology change, which by itself is a problematic issue. One of the most fascinating aspects of wormholes is their apparent ease in generating CTCs [135]. There are several ways to generate a time machine using multiple wormholes [2], but a manipulation of a single wormhole seems to be the simplest way [135, 236]. The basic idea is to create a time shift between both mouths. This is done invoking the time dilation effects in special relativity or in general relativity, i.e., one may consider the analogue of the twin paradox, in which the mouths are moving one with respect to the other, or simply the case in which one of the mouths is placed in a strong gravitational field.

To create a time shift using the twin paradox analogue, consider that the mouths of the wormhole may be moving one with respect to the other in external space, without significant changes of the internal geometry of the handle. For simplicity, consider that one of the mouths A is at rest in an inertial frame, whilst the other mouth B , initially at rest practically close by to A , starts to move out with a high velocity, then returns to its starting point. Due to the Lorentz time contraction, the time interval between these two events, ΔT_B , measured by a clock comoving with B can be made to be significantly shorter than the time interval between the same two events, ΔT_A , as measured by a clock resting at A . Thus, the clock that has moved has been slowed by $\Delta T_A - \Delta T_B$ relative to the standard inertial clock. Note that the tunnel (handle), between A and B remains practically unchanged, so that an observer comparing the time of the clocks through the handle will measure an identical time, as the mouths are at rest with respect to one another. However, by comparing the time of the clocks in external space, he will verify that their time shift is precisely $\Delta T_A - \Delta T_B$, as both mouths are in different reference frames, frames that moved with high velocities with respect to one another. Now, consider an observer starting off from A at an instant T_0 , measured by the clock stationed at A . He makes his way to B in external space and enters the tunnel from B . Consider, for simplicity, that the trip through the wormhole tunnel is instantaneous. He then exits from the wormhole mouth A into external space at the instant $T_0 - (\Delta T_A - \Delta T_B)$ as measured by a clock positioned at A . His arrival at A precedes his departure, and the wormhole has been converted into a time machine. See Figure 12.

For concreteness, following the Morris *et al* analysis [135], consider the metric of the accelerating wormhole given by

$$ds^2 = -(1 + glF(l) \cos \theta)^2 e^{2\Phi(l)} dt^2 + dl^2 + r^2(l) (d\theta^2 + \sin^2 \theta d\phi^2), \quad (210)$$

where the proper radial distance, $dl = (1 - b/r)^{-1/2} dr$, is used. $F(l)$ is a form function that vanishes at the wormhole mouth A , at $l \leq 0$, rising smoothly from 0 to 1, as one moves to mouth B ; $g = g(t)$ is the acceleration of mouth B as measured in its own asymptotic rest frame. Consider that the external metric to the respective wormhole mouths is $ds^2 \cong -dT^2 + dX^2 + dY^2 + dZ^2$. Thus, the transformation from the wormhole mouth coordinates to the external Lorentz coordinates is given by

$$T = t, \quad Z = Z_A + l \cos \theta, \quad X = l \sin \theta \cos \phi, \quad X = l \sin \theta \sin \phi, \quad (211)$$

for mouth A , where Z_A is the time-independent Z location of the wormhole mouth A , and

$$T = T_B + v\gamma l \cos \theta, \quad Z = Z_B + \gamma l \cos \theta, \quad X = l \sin \theta \cos \phi, \quad X = l \sin \theta \sin \phi, \quad (212)$$

for the accelerating wormhole mouth B . The world line of the center of mouth B is given by $Z = Z_B(t)$ and $T = T_B(t)$ with $ds^2 = dT_B^2 - dZ_B^2$; $v(t) \equiv dZ_B/dT_B$ is the velocity of mouth B and $\gamma = (1 - v^2)^{-1/2}$ the respective Lorentz factor; the acceleration appearing in the wormhole metric is given $g(t) = \gamma^2 dv/dt$ [203].

Novikov considered other variants of inducing a time shift through the time dilation effects in special relativity, by using a modified form of the metric (210), and by considering a circular motion of one of the mouths with respect to the other [237]. Another interesting manner to induce a time shift between both mouths is simply to place one of the mouths in a strong external gravitational field, so that times slows down in the respective mouth. The time shift will be given by $T = \int_i^f (\sqrt{g_{tt}(x_A)} - \sqrt{g_{tt}(x_B)}) dt$ [2, 136].

2. The Ori-Soen time machine

A time-machine model was also proposed by Amos Ori and Yoav Soen which significantly ameliorates the conditions of the EFE's solutions which generate CTCs [238, 239, 240, 241]. The Ori-Soen model presents some notable features. It was verified that CTCs evolve from a well-defined initial slice, a partial Cauchy surface, which does not display

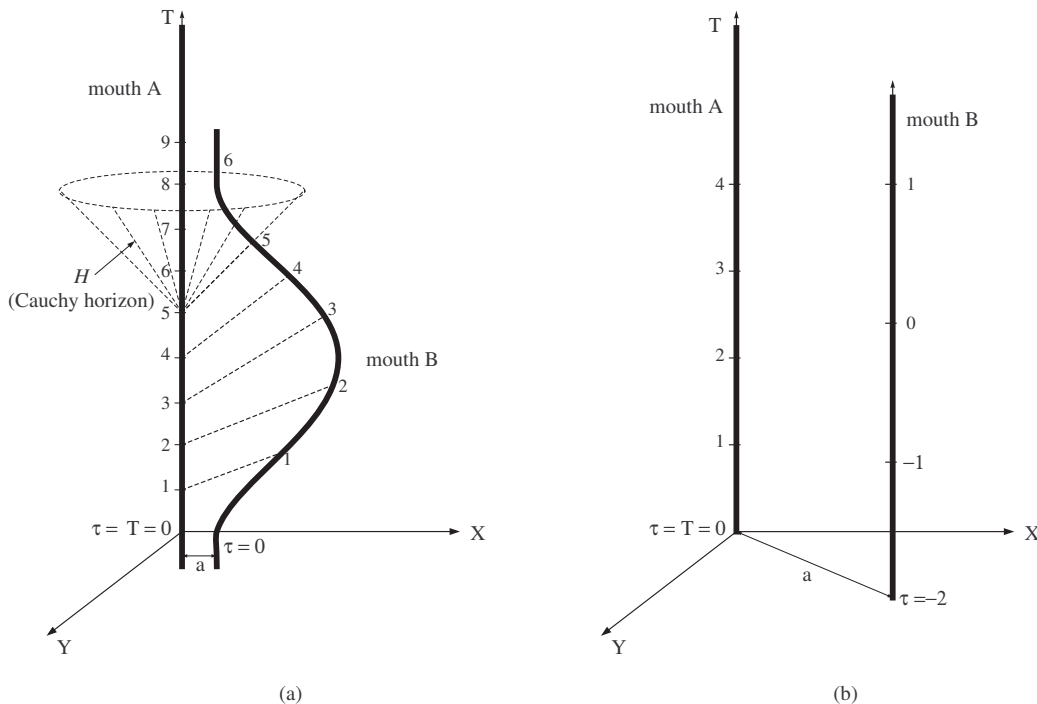


FIG. 12: Depicted are two examples of wormhole spacetimes with closed timelike curves. The wormholes tunnels are arbitrarily short, and its two mouths move along two world tubes depicted as thick lines in the figure. Proper time τ at the wormhole throat is marked off, and note that identical values are the same event as seen through the wormhole handle. In Figure (a), mouth A remains at rest, while mouth B accelerates from A at a high velocity, then returns to its starting point at rest. A time shift is induced between both mouths, due to the time dilation effects of special relativity. The light cone-like hypersurface H shown is a Cauchy horizon. Through every event to the future of H there exist CTCs, and on the other hand there are no CTCs to the past of H . In Figure (b), a time shift between both mouths is induced by placing mouth B in strong gravitational field. See text for details.

causality violation. The partial Cauchy surface and spacetime are asymptotically flat, contrary to the Gott spacetime, and topologically trivial, contrary to the wormhole solutions. The causality violation region is constrained within a bounded region of space, and not in infinity as in the Gott solution. The WEC is satisfied up until and beyond a time slice $t = 1/a$, on which the CTCs appear. Ameliorated versions were recently proposed [242].

3. Warp drive and closed timelike curves

Within the framework of general relativity, it is possible to warp spacetime in a small *bubblelike* region [176], in such a way that the bubble may attain arbitrarily large velocities, $v(t)$. Inspired in the inflationary phase of the early Universe, the enormous speed of separation arises from the expansion of spacetime itself. The model for hyperfast travel is to create a local distortion of spacetime, producing an expansion behind the bubble, and an opposite contraction ahead of it. See Section III B for details.

One may consider a hypothetical spaceship immersed within the bubble, moving along a timelike curve, regardless of the value of $v(t)$. Due to the arbitrary value of the warp bubble velocity, the metric of the warp drive permits superluminal travel, which raises the possibility of the existence of CTCs. Although the solution deduced by Alcubierre by itself does not possess CTCs, Everett demonstrated that these are created by a simple modification of the Alcubierre metric [30], by applying a similar analysis as in tachyons.

4. The Krasnikov tube and closed timelike curves

Krasnikov discovered an interesting feature of the warp drive, in which an observer in the center of the bubble is causally separated from the front edge of the bubble. Therefore he/she cannot control the Alcubierre bubble on

demand. Krasnikov proposed a two-dimensional metric [177], which was later extended to a four-dimensional model [184], as outlined in Section III G. One Krasnikov tube in two dimensions does not generate CTCs. But the situation is quite different in the 4-dimensional generalization. Using two such tubes it is a simple matter, in principle, to generate CTCs. The analysis is similar to that of the warp drive, so that it will be treated in summary.

Imagine a spaceship travelling along the x -axis, departing from a star, S_1 , at $t = 0$, and arriving at a distant star, S_2 , at $t = D$. An observer on board of the spaceship constructs a Krasnikov tube along the trajectory. It is possible for the observer to return to S_1 , travelling along a parallel line to the x -axis, situated at a distance ρ_0 , so that $D \gg \rho_0 \gg 2\rho_{max}$, in the exterior of the first tube. On the return trip, the observer constructs a second tube, analogous to the first, but in the opposite direction, i.e., the metric of the second tube is obtained substituting x and t , for $X = D - x$ and $T = t - D$, respectively in Eq. (195). The fundamental point to note is that in three spatial dimensions it is possible to construct a system of two non-overlapping tube separated by a distance ρ_0 .

After the construction of the system, an observer may initiate a journey, departing from S_1 , at $x = 0$ and $t = 2D$. One is only interested in the appearance of CTCs in principle, therefore the following simplifications are imposed: δ and ε are infinitesimal, and the time to travel between the tubes is negligible. For simplicity, consider the velocity of propagation close to that of light speed. Using the second tube, arriving at S_2 at $x = D$ and $t = D$, then travelling through the first tube, the observer arrives at S_1 at $t = 0$. The spaceship has completed a CTC, arriving at S_1 before it's departure.

V. CONCLUSION

In this Chapter we have considered two particular spacetimes that violate the energy conditions of general relativity, namely, traversable wormholes and “warp drive” spacetimes. It is important to emphasize that these solutions are primarily useful as “gedanken-experiments” and as a theoretician’s probe of the foundations of general relativity. They have also been important to stimulate research in the issues of the energy condition violations, closed timelike curves and the associated causality violations and “effective” superluminal travel.

In the Introduction, we have outlined a review of wormhole physics dating from the “Einstein-Rosen” bridge, the revival of the issue by Wheeler with the introduction of the “geon” concept in the 1960s, the full renaissance of the subject by Thorne and collaborators in the late 1980s, culminating in the monograph by Visser, and detailed the issues that branched therefrom to the present date. In Section II, we have presented a mathematical overview of the Morris-Thorne wormhole, paying close attention to the pointwise and averaged energy condition violations, the concept of the Quantum Inequality and the respective constraints on wormhole geometries, and the introduction of the “Volume Integral Quantifier” which in a certain measure quantifies the amount of energy condition violating matter needed to sustain wormhole spacetimes. We then, treated rotating wormholes and evolving wormholes in a cosmological background, focussing mainly on the energy condition violations, and in particular, on the inflating wormhole and the respective evolution in a flat FRW universe.

In Section III, we have considered the superluminal “warp drive” spacetimes, which also violate the energy conditions and generate closed timelike curves with slight modifications to the spacetime metric. In particular, we have analyzed the Alcubierre and Natário spacetimes, paying close attention to the energy condition violations, reviewing the application of the Quantum Inequality and the superluminal features of these spacetimes, in particular, the appearance of horizons. The discovery that the outer frontal regions of the warp bubble is not in causal contact with a hypothetical observer placed in the center on the bubble, and thus cannot be controlled on demand, inspired the solution known as the Krasnikov spacetime. This spacetime in two and four dimensions were also briefly analyzed.

Using linearized theory, we have also considered a more realistic model of the warp drive spacetime where the warp bubble interacts with a finite mass spaceship. We have tested and quantified the energy conditions to first and second order of the warp bubble velocity. By doing so we have been able to safely ignore the causality problems associated with “superluminal” motion, and so have focussed attention on a previously unremarked feature of the “warp drive” spacetime. If it is possible to realize even a weak-field warp drive in nature, such a spacetime appears to be an example of a “reaction-less drive”. That is, the warp bubble moves by interacting with the geometry of spacetime instead of expending reaction mass, and the spaceship (which in linearized theory can be treated as a finite mass object placed within the warp bubble), is simply carried along with it. We have verified that in this case, the “total amount” of energy condition violating matter (the “net” negative energy of the warp field) must be an appreciable fraction of the positive mass of the spaceship carried along by the warp bubble. This places an extremely stringent condition on the warp drive spacetime, namely, that for all conceivably interesting situations the bubble velocity should be absurdly low, and it therefore appears unlikely that, by using this analysis, the warp drive will ever prove to be technologically useful. Finally, we point out that any attempt at building up a “strong-field” warp drive starting from an approximately Minkowski spacetime will inevitably have to pass through a weak-field regime. Since the weak-field warp drives are already so tightly constrained, the analysis of this work implies additional difficulties for developing

a “strong field” warp drive.

Finally, in Section IV, we have analyzed some solutions to the Einstein field equation that generate closed timelike curves. Far from attempting an exhaustive search of spacetimes that possess closed timelike curves, we have focussed on two particularly notorious properties of these geometries, namely, solutions with a tipping over of the light cones due to a rotation about a cylindrically symmetric axis and solutions that violate the energy conditions of general relativity. Closed timelike curves are troublesome as they apparently violate causality, and it is not clear that even an eventual theory of quantum gravity will provide us with an answer. However, as stated by Kip Thorne, time travel in the form of closed timelike curves, is more than a justification for theoretical speculation, it is a conceptual tool and an epistemological instrument to probe the fundamental foundations of general relativity and to extract some eventual clarifying views.

Acknowledgments

The author was funded by Fundação para a Ciência e a Tecnologia (FCT)–Portugal through the grant SFRH/BPD/26269/2006.

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