Kerr Geometry

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This paper presents a structural analysis of the space-time around a rotating black hole in vacuum with no electrical charge, namely of the Kerr Metric, followed by certain physical calculations around it as the final report for the PHYS492-Senior Project II course under the supervision of Prof. Ahmet Züfer Eriş.

I. INTRODUCTION

Although General Relativity as a theory of gravitation, and Schwarzschild solution which provides a solution for a non-rotating chargeless black hole in vacuum has been around since 1915, and 1916 respectively; notion of a black hole, as an actual astrophysical object has just been around for less than half a century.

When the Schwarzschild first proposed his solution, it was just thought as the metric around any non-rotating massive object in vacuum; and having any object smaller than 2μ was rather unfathomable to both Einstein, and perhaps Schwarzschild himself, along with nearly all the astrophysicists at the time. Therefore any physical calculation that can be done beyond that point using Schwarzschild's solution was considered as nothing more than a mathematical fantasy.

This way of approaching to the solutions of the Einstein's field equations, to perhaps misfortune of Einstein, did not change up until a couple of years after his death. Yet, when Roy Kerr, a not-so-famous mathematician back then, finally solved the Einstein's field equations in a coordinate system that he proposed himself, for a rotating massive object in vacuum; an exact solution of the Einstein's field equations were not reached for a rotating massive object for nearly half a century, although it was tried very hard by many renowned physicists such as Achilles Papapetrou. And again around those times the idea of a black hole was in its baby steps. Thus, luckily, Kerr's solution born into a scientific environment, in which it could be physically analyzed to most of its potential.

Therefore, in this paper, we revisit some physical consequences displayed by the Kerr metric, namely the metric around a rotating black hole with no electrical charge, in vacuum.

II. KERR METRIC

In general, Kerr solution assumes a stationary, axially symmetric metric, and making the solution is rather a straightforward procedure, such that one first calculates the Christoffel symbols for that general metric;

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2$$

Then using these Christoffel symbols, the Ricci Tensor components $R_{\mu\nu}$ are calculated, and since again we are interested in vacuum solution, Einstein's field equations to solve reduce to;

$$R_{\mu\nu} = 0$$

Yet, one part that I although did not lie; did certainly mislead the reader is that I stated that the solution was straightforward. This statement might cause one to think that it is easy to follow the procedure as it is given, while it is anything but easy, and actually near impossible to solve in the stated form of the given metric due to resulting extremely complicated partial differential equations to solve. Hence, a method which is easier and actually the way this solution was first achieved is to transform this metric to Kerr-Schild form, and then follow the given procedure; which is not explicitly done here.

Yet again, it should also be noted that, the form of the metric as given in the beginning (Boyer-Lindquist coordinates) is as powerful in discussing the physics part of the problem, as it is easy to make a solution in Kerr-Schild form.

Therefore, given the Kerr Metric in Boyer-Lindquist coordinates[1][6];

$$\begin{aligned} ds^2 = {} & \frac{\Delta - a^2 \sin^2\theta}{\rho^2} c^2 dt^2 + \frac{4\mu ar \sin^2\theta}{\rho^2} cdtd\phi \\ & - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \frac{\Sigma^2 \sin^2\theta}{\rho^2} d\phi^2 \end{aligned} \tag{1}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2\mu r + a^2$ and $\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$.

Or, restructured in a tidier form;

$$\begin{aligned} ds^2 = {} & \frac{\rho^2\Delta}{\Sigma^2} c^2 dt^2 - \frac{\Sigma^2 \sin^2\theta}{\rho^2} (d\phi - \omega dt)^2 \\ & - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 \end{aligned} \tag{2}$$

where the extra physical parameter $\omega = 2\mu cra/\Sigma^2$ is introduced, we can start the discussion on the Kerr Geometry.

III. STRUCTURE OF THE SPACETIME AROUND THE KERR BLACK HOLES

A. Intrinsic and Coordinate Singularities

From the eq(1) we can observe that a singularity occurs in the metric when either $\rho = 0$ or $\Delta = 0$. Yet, although it will not be explicitly done here, calculating the Ricci Scalar shows that only $\rho = 0$ is an intrinsic singularity, and $\Delta = 0$ is a coordinate singularity. Hence remembering;

$$\rho^2 = r^2 + a^2 \cos^2 \theta$$

it can be seen that $\rho = 0$ when, $r = 0$ and $\theta = \frac{\pi}{2}$. Then noting that in Boyer-Lindquist coordinates $r = 0$ describes a disc of radius a in the equatorial plane, it can be said that the intrinsic singularity, perhaps surprisingly, occurs at a *ring* of radius a in the equatorial plane, since $\theta = 0$; instead of a single point at the center as in the Schwarzschild Geometry.

Then, giving the condition for an event horizon to occur as $g_{rr} = \infty$, where again from eq(1);

$$g_{rr} = -\frac{\rho^2}{\Delta}$$

it is seen that the coordinate singularity occuring at $\Delta = 0$ mentioned above, is also the result for event horizons. Therefore setting;

$$\Delta = r^2 - 2\mu r + a^2 = 0$$

we find that the two surfaces described by;

$$r_+ = \mu + \sqrt{\mu^2 - a^2} \tag{3}$$

$$r_- = \mu - \sqrt{\mu^2 - a^2} \tag{4}$$

are the event horizons for a Kerr black hole, of which geometry is described by the 2D surface line elements;

$$d\sigma^2 = \rho_+^2 d\theta^2 + \left(\frac{2\mu r_+}{\rho_+}\right)^2 \sin^2\theta d\phi^2 \tag{5}$$

$$d\sigma^2 = \rho_-^2 d\theta^2 + \left(\frac{2\mu r_-}{\rho_-}\right)^2 \sin^2\theta d\phi^2 \tag{6}$$

which are when embedded in a 3D Euclidian Space, resemble axisymmetric ellipsoids which are flattened along the rotation axis.

Discussion

From equations (3) and (4) we can observe that a condition for event horizons to exist is;

$$a^2 < \mu^2$$

since real $r_\pm$ values do not exist for $\mu^2 < a^2$

Yet, if the case where $a^2 > \mu^2$ is still considered, then it is found that $\Delta > 0$ throughout the space, which means, as we have inferred above, that event horizons disappear, although ring singularity remains and becomes visible to the outside world, which is called a *naked singularity.* Yet, some astrophysical calculations suggests a limit for the value of a as $a \approx 0.998\mu$, which is within the boundaries of the condition for a real valued event horizon.

B. Stationary Limit Surfaces

For a general axially symmetric metric, the condition for both a surface of infinite redshift and a stationary limit surface is $g_{tt} = 0$. Therefore, again from the eq(1), this condition can be written as;

$$\Delta - a^2 \sin^2\theta = 0$$

Rearranging the equation by inserting the expression for Δ, we rewrite the condition as;

$$r^2 - 2\mu r + a^2\cos^2\theta = 0$$

and see that the solution for both the surface of infinite redshift and stationary limit surface is;

$$r_{S+} = \mu + \sqrt{\mu^2 - a^2\cos^2\theta} \tag{7}$$

$$r_{S-} = \mu - \sqrt{\mu^2 - a^2\cos^2\theta} \tag{8}$$

again of which 2D surface line elements are;

$$\begin{aligned} d\sigma^2 &= \rho_{S\pm}^2 d\theta^2 \\ &\quad + \left[\frac{2\mu r_{S\pm}(2\mu r_{S\pm} + 2a^2\sin^2\theta)}{\rho_{S\pm}^2}\right] \sin^2\theta d\phi^2 \end{aligned} \tag{9}$$

And again, as was the case for eq(5) and eq(6), these line elements given in eq(9), when embedded in 3D Euclidean space, describes not spherically but axially symmetric ellipsoid 2-surfaces, which are oblate around the rotation axis.

Discussion

If we were to take the limits of the expressions given in equations (3-4) and (7-8) as $a \to 0$, which means that the angular momentum approaches to 0, it can be, as expected, seen that the structure of a Schwarzschild Black Hole would be recovered; as the outer surface of redshift reduces $r_{S+} = 2\mu$, and inner surface of redshift reduces to $r_{S-} = 0$. And again it can be observed that these surfaces coincides with the event horizons at the given limit (Schwarzschild Black Holes).

Again returning back to the case with an angular momentum, it can also be observed that, in the equatorial plane, inner stationary limit surface coincides with the ring singularity and again coincides with the inner event horizon at the poles, and remains inside the inner event horizon otherwise (the statement can be checked by replacing the θ in eq(8) by first $\pi/2$, then by 0).

Then if we were to discuss the outer surface of the infinite redshift (outer stationary limit surface), we can again doing the same as above, only this time with eq(7) and eq(3), we again observe that outer stationary limit surface coincides with the outer event horizon at the poles, but this time encloses the outer event horizon entirely otherwise, and this enclosed region between outer stationary limit surface and outer event horizon is then called the *ergoregion* where some interesting physics lies.

C. The Ergoregion

The main property of the ergoregion is that, it is a region where g_{tt} starts to act like a spatial component, such that $g_{tt} < 0$, while other components' signs do not change, which means; being still outside of the horizon, a particle can still escape from this region.

An immediate consequence of g_{tt} being smaller than zero is that a particle cannot stay immobile at a position (r, θ, ϕ), even if it can produce any arbitrary force to move (such as an arbitrarily powerful rocket), since the condition $\vec{u}.\vec{u} = g_{tt}(u^t)^2 = c^2$ cannot be satisfied for a particle with 4-velocity;

$$[u^\mu] = (u^t, 0, 0, 0) \tag{10}$$

if $g_{tt} < 0$.

Yet, it is possible for such a particle to remain at fixed (r, θ), if it rotates around the black hole in the same direction of the Black Hole's rotation, with respect to an infinitely distant observer, which is also a prominent example of dragging of inertial frames.

More mathematically, the 4-velocity of such a particle could be written as;

$$[u^\mu] = u^t(1, 0, 0, \Omega) \tag{11}$$

where, Ω is the angular velocity ($\Omega = d\phi/dt$) of the particle around the black hole, with respect to an infinitely distant observer. The condition for such a configuration is again $\vec{u}.\vec{u} = g_{\mu\nu}u^\mu u^\nu = c^2$, therefore, opening up the expression;

$$\begin{aligned} &g_{tt}(u^t)^2 + 2g_{t\phi}u^t u^\phi + g_{\phi\phi}(u^\phi)^2 \\ &= c^2 = (u^t)^2(g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2) \end{aligned} \tag{12}$$

Then, it can be observed that, the following condition should be satisfied for u^t to be real valued;

$$g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2 > 0 \tag{13}$$

Since g_{tt} is smaller than 0, it then follows that the left hand side of the inequality is a downward opening parabola, which means that the permitted range of angular velocities will be contained between the roots of the function on the left hand side of the inequality, namely $\Omega_- < \Omega < \Omega_+$, where;

$$\begin{aligned} \Omega_\pm &= -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}} \\ &= \omega \pm \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}} \end{aligned} \tag{14}$$

From eq(14), it is easy to see that there are two special cases that stand out; one $g_{tt} = 0$, and other $\omega^2 = g_{tt}/g_{\phi\phi}$.

For the first case, the results are;

$$\begin{aligned} \Omega_- &= 0 \\ \Omega_+ &= 2\omega \end{aligned}$$

and we know that the $g_{tt} = 0$ is the defining property of the stationary limit surfaces. Therefore, the results that we have in this special case is the range of permitted angular momentum values on the outer stationary limit surface (outer surface of the ergoregion), and the $\Omega_- = 0$ is actually the physical definition of a stationary limit surface, such that having 0 angular momentum in the ϕ direction becomes possible on this surface, and inside of this surface, a particle has to rotate in the same direction as the black hole rotates; such that a negative value of Ω only becomes possible for r values larger than r_{S+}.

Passing onto the other special case, the result is simply;

$$\Omega_\pm = \omega$$

As we can check, for the condition for this result to hold, we must have $\Delta = 0$, such that;

$$\begin{aligned} \omega^2 &= \frac{g_{tt}}{g_{\phi\phi}} \\ \left(\frac{2\mu c r a}{\Sigma^2}\right)^2 &= \frac{a^2c^2}{\Sigma^2} \quad \text{for} \quad \Delta = 0 \end{aligned}$$

where again when we insert $\Sigma^2 = (2\mu r)^2$ for $\Delta = 0$, we see that the above equation holds. Therefore, $\Delta = 0$ being the condition for the occurrence of event horizons, we immediately observe that the condition for $\Omega_\pm = \omega$ holds on the outer event horizon, such that continuing from above;

$$\omega^2 = \frac{a^2c^2}{(2\mu r)^2}$$

Hence;

$$\Omega_\pm = \Omega_H = \omega(r_+, \theta) = \frac{ac}{2\mu r_+} \tag{15}$$

Finally, with this calculation, we put a better limit for the maximum permitted angular velocity for a particle at fixed (r, θ) within ergoregion, which is Ω_H.

Discussion

Another interesting consequence of the existence of an ergoregion is that it allows for a process by which energy can be extracted from a spinning black hole; which is called Penrose process.

The process, basically, can be summarized as;

1. A particle enters the ergoregion
2. Particle, at some point, divides/decays into two particles
3. One of those two particles escapes ergoregion and reaches to a stationary infinitely distant observer and other falls into the black hole further
4. At the end of this process the particle escaping the region ends up having more energy than it had when it was entering the ergoregion together with its other part.
5. This process occurs at the expense of the rotational energy of the black hole.

Then, let us continue with a more sound discussion on the Penrose process.

IV. PENROSE PROCESS[6][8]

Imagine a particle A fired from the fixed position of an infinitely distant observer to the Kerr black hole's ergoregion. In the reference frame of the observer at the event of emission $\mathcal{E}$, the energy measured by that observer will be written as;

$$E^{(A)} = p^{(A)}(\mathcal{E}) \cdot \vec{u}_{\text{observer}} = p_t^{(A)}(\mathcal{E}) \tag{16}$$

where $p^{(A)}(\mathcal{E})$ is the particle's 4-momentum at the given event and $\vec{u}_{\text{observer}}$ is the 4-velocity of the observer. And since we have assumed an stationary, infinitely distant observer, the components for the 4-velocity of such an observer will be;

$$[u^\mu_{\text{observer}}] = (1, 0, 0, 0)$$

Now, imagine somewhere in the ergoregion, particle A divides into two different particles, let say, B and C. At the event of division, by using the conservation of momentum, we can write;

$$p^{(A)}(\mathcal{D}) = p^{(B)}(\mathcal{D}) + p^{(C)}(\mathcal{D}) \tag{17}$$

Then, if this division realizes in such a way that, let say, particle-C reaches back to an infinitely distant observer, that observer at the receiving end can write the particle's energy in that receiving event $\mathcal{R}$ as;

$$E^{(C)} = p_t^{(C)}(\mathcal{R}) = p_t^{(C)}(\mathcal{D})$$

where we were able to write the last part of the equation; having the knowledge that covariant time component of the 4-momentum of a particle along the geodesics of the Kerr geometry is conserved, since the metric is stationary. In a similar manner, we can also write for the initial undivided particle;

$$E^{(A)} = p_t^{(A)}(\mathcal{D}) = p_t^{(A)}(\mathcal{R})$$

Hence, we can rewrite the condition for the conservation of momentum (eq(17)), for the time component as;

$$E^{(C)} = E^{(A)} - p_t^{(B)}(\mathcal{D}) \tag{18}$$

where again, throughout the geodesic path pursued by the particle-B, $p_t^{(B)}$ is also conserved. Now, it is useful to note that $p_t^{(B)} = e_t \cdot \vec{p}^{(B)}$, where e_t is the basis vector for t-coordinate, of which squared *length* can be written as;

$$e_t \cdot e_t = g_{tt}$$

Therefore, we can say, if the particle-B were to fall further into black hole, unlike particle-C, it would stay in a region where $g_{tt} < 0$. Which would mean in turn that e_t is spacelike in that region, and then $p_t^{(B)}$ would be spatial momentum component, which can be either positive or negative. Thus, for a division event where it is negative, looking back to eq(18) we see that $E^{(C)} > E^{(A)}$; which means, the divided part of the particle that managed to escape the ergoregion, now has more energy than it had when it was entering the ergoregion undivided, hence the *extraction of energy* from the black hole.

Yet, there is a consequence of such a process for also the black hole; such that the black hole loses some amount of both total mass and angular momentum; as if the falling particle had, interestingly enough, negative mass-energy, which is, as now we know, possible in Kerr geometry. The amount of change in both quantities are then written as;

$$M \to M + \frac{p_t^{(B)}}{c^2} \tag{19}$$

$$J \to J - p_\phi^{(B)} \tag{20}$$

where we can cross-check that the total mass also reduces in the case that $p_t^{(B)}$ is negative. And again it is important to note that p_ϕ is the particle's angular momentum component along the black hole's rotation axis, multiplied by -1.

Now to demonstrate that the angular momentum also decreases due to the falling particle, we first introduce an observer in the ergoregion at a fixed position in (r, θ) coordinates which observes the particle-B as particle passes near it. As we have shown during the calculations for ergoregion, such an observer's 4-velocity is;

$$[u^\mu] = u^t(1, 0, 0, \Omega) \tag{21}$$

Then, the energy of particle-B can be written by this observer as;

$$E^{(B)} = p_\mu^{(B)} u^\mu = u^t \left(p_t^{(B)} + p_\phi^{(B)} \Omega \right)$$

Continuing, since $E^{(B)}$ must be positive, the following condition can be introduced;

$$L < \frac{p_t^{(B)}}{\Omega} \tag{22}$$

where $L = -p_\phi^{(B)}$. Again remembering that $p_t^{(B)}$ was negative, and Ω has to be positive in the reference frame of an observer in the ergoregion, we observe that L is negative also. Hence the angular momentum of a falling particle is *negative*, which in turn decreases the net angular momentum of the black hole.

Finally, considering the calculations done in the previous section, we can actually put a better upper limit for L, such that the eq(22) is expected to hold for any observer at fixed (r, θ) in ergoregion. And since, the angular velocity has the upper bound of $\Omega = \Omega_H$ at the outer horizon $r = r_+$, the condition in eq(22) can be rewritten as;

$$\delta J < \frac{c^2 \delta M}{\Omega_H} \tag{23}$$

where δJ and δM denotes the changes in angular momentum and mass, respectively, which are both negative.

V. GEODESICS IN THE EQUATORIAL PLANE[4][5][6]

For the Kerr geometry, since the metric is not spherically symmetric; it is unnecessarily complicated to work over the entire space. Therefore, we confine the discussion to a much simpler case, which is the geodesics for a constant θ value, namely $\pi/2$, hence the equatorial plane. Then, let us start by first rewriting the Kerr metric (eq(1)) for constant value of $\theta = \pi/2$;

$$\begin{aligned} ds^2 = {} & c^2 \Big(1 - \frac{2\mu}{r}\Big) dt^2 + \frac{4\mu a c}{r} dt d\phi \\ & - \frac{r^2}{\Delta} dr^2 - \Big(r^2 + a^2 + \frac{2\mu a^2}{r}\Big) d\phi^2 \end{aligned} \tag{24}$$

Then, we can use the Euler-Lagrange equation to write the geodesics. Thus, writing the Lagrangian $\mathcal{L}$ as;

$$\begin{aligned} \mathcal{L} = \frac{1}{2}\Big(\frac{ds}{d\tau}\Big)^2 = {} & \frac{1}{2}\Big[c^2 (1 - \frac{2\mu}{r}) \dot{t}^2 + \frac{4\mu a c}{r} \dot{t}\dot{\phi} \\ & - \frac{r^2}{\Delta} \dot{r}^2 - (r^2 + a^2 + \frac{2\mu a^2}{r}) \dot{\phi}^2 \Big] \end{aligned} \tag{25}$$

Then, the Euler-Lagrange equation for t and ϕ coordinates are;

$$\frac{d}{d\tau}\Big(\frac{\partial \mathcal{L}}{\partial \dot{t}}\Big) - \frac{\partial \mathcal{L}}{\partial t} = c^2 \Big(1 - \frac{2\mu}{r}\Big) \ddot{t} + \frac{2\mu a c}{r} \ddot{\phi} = 0 \tag{26}$$

$$\frac{d}{d\tau}\Big(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\Big) - \frac{\partial \mathcal{L}}{\partial \phi} = \frac{2\mu a c}{r} \ddot{t} + \Big(r^2 + a^2 + \frac{2\mu a^2}{r}\Big) \ddot{\phi} = 0 \tag{27}$$

and first integrals of these equations being more useful, we can rewrite equations (26) and (27) as;

$$p_t = c^2 \Big(1 - \frac{2\mu}{r}\Big) \dot{t} + \frac{2\mu a c}{r} \dot{\phi} = k c^2 \tag{28}$$

$$p_\phi = \frac{2\mu a c}{r} \dot{t} + \Big(r^2 + a^2 + \frac{2\mu a^2}{r}\Big) \dot{\phi} = -h \tag{29}$$

Where we have introduce the constants k and h. And finally the solutions for the coupled equations of (28) and (29) are;

$$\begin{aligned} \dot{t} &= \frac{1}{\Delta}\Big[\Big(r^2 + a^2 + \frac{2\mu a^2}{r}\Big) k - \frac{2\mu a}{c r} h\Big] \\ \dot{\phi} &= \frac{1}{\Delta}\Big[\frac{2\mu a c}{r} k + \Big(1 - \frac{2\mu}{r}\Big) h\Big] \end{aligned} \tag{30}$$

Yet, for the case of r, it is more convenient to find the first integral through the invariant length of the 4-momentum $\vec{p}$, rather than using Euler-Lagrange, since the resulting equations from it are quite complicated to solve. Thus, the useful form to use is;

$$g^{\mu\nu} p_\mu p_\nu = \epsilon^2$$

where ϵ^2 is c^2 for a massive particle, and 0 for a photon. Then, knowing that the $p_\theta = 0$ due to our choice of the motion in the equatorial plane, this expression opens up as;

$$g^{tt} (p_t)^2 + 2 g^{t\phi} p_t p_\phi + g^{\phi\phi} (p_\phi)^2 + g^{rr} (p_r)^2 = \epsilon^2 \tag{31}$$

where the contravariant metric components $g^{\mu\nu}$ are;

$$\begin{aligned} g^{tt} &= \frac{1}{c^2 \Delta}\Big(r^2 + a^2 + \frac{2\mu a^2}{r}\Big) \\ g^{t\phi} &= \frac{2\mu a}{c r \Delta} \\ g^{rr} &= -\frac{\Delta}{r^2} \\ g^{\phi\phi} &= -\frac{1}{\Delta}\Big(1 - \frac{2\mu}{r}\Big) \end{aligned}$$

for again constant $\theta = \pi/2$. Yet, before placing the values of the contravariant metric components, it is wiser to massage the eq(31) a little bit further by other substitutions. Noting that $p_t = k c^2$ and $p_\phi = -h$; and remembering that $p_r = g_{rr} \dot{r}$, where $g^{rr} = 1/g_{rr}$, we may obtain the following equation;

$$\dot{r}^2 = g^{rr} (\epsilon^2 - g^{tt} c^4 k^2 + 2 g^{t\phi} c^2 k h - g^{\phi\phi} h^2) \tag{32}$$

Now, substituting the contravariant components of the metric tensor into eq(32), we get;

$$\dot{r}^2 = c^2k^2 - \epsilon^2 + \frac{2\epsilon^2\mu}{r} + \frac{a^2(c^2k^2-\epsilon^2)-h^2}{r^2} + \frac{2\mu(h-kca)^2}{r^3} \tag{33}$$

Since, as the name of the section suggests, the discussion was constrained to the equatorial plane, the equation for θ is irrelevant, since it will not produce any independent equation of motion. Therefore, the results as found in eq(30) and eq(33), define the complete set of null and non-null geodesics in the equatorial plane.

Before continuing with specific examples, two things to note about the Kerr geometry are;

1. Equatorial trajectories are depended on if the particle/photon is co-rotating, or counter-rotating.
2. t and ϕ coordinates are both *bad* coordinates near horizons, such that in terms of these coordinates, particles/photons seems to take infinite amount of spiral rotations, and infinite coordinate time to cross the event horizons for an infinitely distant stationary observer (both around r_- and r_+), which is not the case in the reference frame of these moving particles/photons.

A. Equatorial Trajectories of Massive Particles

As mentioned in the general discussion about the equatorial geodesics, the massive particle trajectories are obtained by replacing ϵ^2 with c^2 in eq(33), such that the equation concerning the *energy* becomes;

$$\dot{r}^2 = c^2(k^2-1) + \frac{2c^2\mu}{r} + \frac{a^2(c^2(k^2-1))-h^2}{r^2} + \frac{2\mu(h-kca)^2}{r^3} \tag{34}$$

where, kc^2 and h are respectively the energy and angular momentum per unit mass of the particle, whose trajectory is being defined.

Eq(34), then can be rearranged as;

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r;h,k) = \frac{1}{2}c^2(k^2-1) \tag{35}$$

where the effective potential energy per unit mass is defined as;

$$V_{\text{eff}} = -\frac{\mu c^2}{r} + \frac{h^2 - a^2c^2(k^2-1)}{2r^2} - \frac{\mu(h-kca)^2}{r^3} \tag{36}$$

V_{eff} in the Kerr case also clearly reduces to that of the Schwarzschild case;

$$V_{\text{eff}(S)} = -\frac{\mu c^2}{r} + \frac{h^2}{2r^2} - \frac{\mu h^2}{r^3}$$

as $a \to 0$. Yet, it is important not to be careless about deciphering eq(36) as an effective potential since it also has k (energy of the particle per unit mass) dependence. But still, it is mostly as useful as it is for the Schwarzschild case to consider so.

One last comment on this section is that, a general set of equations of motion will not be derived here, since it is extremely complicated to do so. Hence, the interest will be once more constrained to special cases.

1. *Equatorial motion of massive particles with zero initial angular momentum*

For a particle in free fall with respect to a Kerr black hole, with zero initial angular momentum, h is simply 0. Then, for the sake of brevity, it is also useful to consider the particle to start its motion at rest, from infinity, such that k also becomes 1. Substituting these values, equations (30) and (34) become;

$$\dot{t} = \frac{1}{\Delta}\Big(r^2 + a^2 + \frac{2\mu a^2}{r}\Big) \tag{37}$$

$$\dot{\phi} = \frac{2\mu ac}{r\Delta} \tag{38}$$

$$\dot{r} = \frac{2\mu c^2}{r}\Big(1 + \frac{a^2}{r^2}\Big) \tag{39}$$

As mentioned before, the fact that t and ϕ coordinates are *bad* coordinates around the event horizons ($\Delta = 0$), becomes apparent with the equations (37) and (38), while interestingly, this effect is not observed for the expression of $\dot{r}$.

At this point, writing the equations for trajectories is rather straightforward, making use of the chain rule, such that;

$$\frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} = -\Delta\sqrt{\frac{2\mu c^2}{r}\Big(1+\frac{a^2}{r^2}\Big)} \cdot \Big(r^2 + a^2 + \frac{2\mu a^2}{r}\Big)^{-1} \tag{40}$$

$$\frac{d\phi}{dt} = \frac{\dot{\phi}}{\dot{t}} = \frac{2\mu ac}{r} \cdot \Big(r^2 + a^2 + \frac{2\mu a^2}{r}\Big)^{-1} \tag{41}$$

$$\frac{d\phi}{dr} = \frac{\dot{\phi}}{\dot{r}} = -\frac{2\mu a}{r\Delta} \cdot \Big[\frac{2\mu}{r}\Big(1+\frac{a^2}{r^2}\Big)\Big]^{-1/2} \tag{42}$$

And it is actually possible to observe frame dragging property of the Kerr geometry from the above equations, once they are numerically integrated and a plot of the trajectory of the particle is obtained in the xy-plane.

2. *Equatorial circular motion of massive particles*

For the motion in a circular orbit, the condition is simply $\dot{r} = 0$, and radial acceleration $\ddot{r}$ should also vanish. Therefore, in terms of the eq(36), for a circular orbit at $r = r_c$, the requirement is;

$$\begin{aligned} V_{\text{eff}}(r_c;h,k) &= \frac{1}{2}c^2(k^2-1) \\ \frac{dV_{\text{eff}}}{dr} &= 0 \quad \text{for} \quad r = r_c \end{aligned} \tag{43}$$

Hence, after some careful but straightforward algebra, we find the results for h and k for circular orbits as;

$$k = \frac{1 - 2\mu u \mp a\sqrt{\mu u^3}}{\sqrt{1 - 3\mu u \mp 2a\sqrt{\mu u^3}}} \tag{44}$$

$$h = \mp \frac{c\sqrt{\mu}(1 + a^2u^2 \pm 2a\sqrt{\mu u^3})}{\sqrt{u(1 - 3\mu u \mp 2a\sqrt{\mu u^3})}} \tag{45}$$

Where $u = 1/r$ and concerning $\pm and \mp$ signs; upper signs denote counter-rotating, and lower signs denote co-rotating circular orbits.

3. *Stability of equatorial circular orbits of massive particles*

In addition to the condition for a circular orbit as given in eq(43), for a borderline stability the following condition should be satisfied;

$$\begin{aligned} \frac{d^2V_{\text{eff}}}{dr^2} &= \frac{d^2V_{\text{eff}}}{du^2}\left(\frac{du}{dr}\right)^2 + \frac{dV_{\text{eff}}}{du}\frac{d^2u}{dr^2} \\ &= u^3\left(u\frac{d^2V_{\text{eff}}}{du^2} + 2\frac{dV_{\text{eff}}}{du}\right) = 0 \end{aligned} \tag{46}$$

Then, again following a careful, yet straightforward calculation; the implicit equation for the innermost stable coordinate radius for a circular orbit becomes

$$r^2 - 6\mu r - 3a^2 \mp 8a\sqrt{\mu r} = 0 \tag{47}$$

where again, the upper sign denotes counter-rotating, while minus sign denotes co-rotating circular orbit.

Finally, substituting two extremes of the Kerr metric;

1. At $a = 0$, we recover the result for the Schwarzschild case that $r = 6\mu$

2. while at $a = \mu$, we find that $r = \mu$ for co-rotating, and $r = 9\mu$ for counter-rotating massive particle.

The general solution is also analytically possible and straightforward to obtain, yet the result is unnecessary to calculate at this point, where a numerical plot provides a better demonstration;

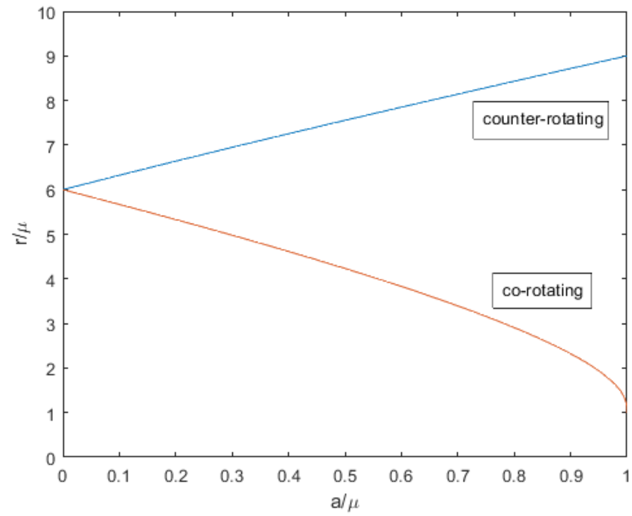

FIG. 1. r/μ for the innermost stable orbit as a function of a/μ

A similar plot of k vs a/μ can be drawn using eq(44), which I will not do here, but using that, efficiency of the accretion disk around a Kerr black hole can also be calculated as;

$$\varepsilon_{\text{acc}} = 1 - k \tag{48}$$

and for the extreme case of $a = \mu$, this value would be;

$$\varepsilon_{\text{acc}} = 1 - \frac{1}{\sqrt{3}} \approx 42\%$$

yet, again more realistically, for a Kerr black hole, this value would be at maximum $\approx 32\%$ with $\frac{a}{\mu} \approx 0.998$.

B. Equatorial Trajectories of Photons

The procedure to follow for this section is nearly identical to the calculations done for the massive particles, yet of course with some differences. Hence, intermediate steps are going to be omitted through most of this section, since an outline of the calculations and results would suffice due to similarities with the previous section.

The main equations to write for the photon trajectories are again simply the eq(30), and eq(33) with $\epsilon^2 = 0$ now. Thus, the eq(33) with the given condition is rewritten as;

$$\dot{r}^2 = c^2k^2 + \frac{k^2c^2a^2 - h^2}{r^2} + \frac{2\mu(h - kca)^2}{r^3} \tag{49}$$

Then, for the calculations, it is helpful to introduce $b = h/(ck)$. In the limit $r \to \infty$, b can be interpreted as a parameter of impact, and since k has a positive sign in this limit, the sign of b is also the same as the sign of $\dot{\phi}$

After these explanations, eq(49) can be restated as;

$$\frac{\dot{r}^2}{h^2} + V_{\text{eff}}(r; b) = \frac{1}{b^2}; \tag{50}$$

where;

$$V_{\text{eff}}(r;b) = \frac{1}{r^2}\left[1 - \left(\frac{a}{b}\right)^2 - \frac{2\mu}{r}\left(1 - \frac{a}{b}\right)^2\right] \tag{51}$$

The most comments that were done for the equatorial trajectories of massive particles are also valid here, such that the results approach to that of Schwarzschild metric, in the limit $a \to 0$, yet it is important to be careful when interpreting V_{eff} as the efficien potential energy, for $a \neq 0$; yet still, it is mostly useful to do so.

1. *Equatorial principle photon geodesics*

Although radial photon geodesics, as expected, do not exist on the equatorial plane of the Kerr geometry due to dragging of inertial frames, it is still possible to have a knowledge about the radial variation of the light-cone structure through investigation of the *principal* null geodesics. With the defining condition that $b = a$, the equations (30) and (33) becomes;

$$\dot{t} = \frac{k}{\Delta}(r^2 + a^2) \tag{52}$$

$$\dot{\phi} = \frac{kca}{\Delta} \tag{53}$$

$$\dot{t} = \pm ck \tag{54}$$

where in eq(54), where $+$ sign denotes outgoing, and $-$ sign denotes incoming photons. Then, let say, for the $\dot{r} = +ck$, we can write;

$$\frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = \frac{(r^2 + a^2)}{c\Delta} \tag{55}$$

$$\frac{d\phi}{dr} = \frac{\dot{\phi}}{\dot{r}} = \frac{a}{\Delta} \tag{56}$$

Then, with the knowledge that $\Delta > 0$ for $r_+ < r < \infty$, it follows that $\frac{dr}{dt} > 0$, hence also verifies that the photon is outgoing. Confining the interest to again $a^2 < \mu^2$, we can outright integrate the equations (55) and (56), and find;

$$\begin{aligned} ct = r &+ \mu\Big(1 + \frac{\mu}{\sqrt{\mu^2 - a^2}}\Big) \ln\Big|\frac{r}{r_+} - 1\Big| \\ &+ \mu\Big(1 - \frac{\mu}{\sqrt{\mu^2 - a^2}}\Big) \ln\Big|\frac{r}{r_-} - 1\Big| + C \end{aligned} \tag{57}$$

$$\phi = \frac{a}{2\sqrt{\mu^2 - a^2}} \ln\Big|\frac{r - r_+}{r - r_-}\Big| + C \tag{58}$$

where C's are simply integration constants and not necessarily equal in two equations. The solution for the incoming photons can then also be calculated following the same procedure.

2. *Equatorial circular motion of photons*

The conditions to write for the equatorial circular motion of photons is the same as it was for the massive particles such that $\dot{r} = 0$ and $\ddot{r}$ also vanishes. Then in terms of the eq(50), for a circular orbit at $r = r_c$, the condition to introduce becomes;

$$\begin{aligned} V_{\text{eff}}(r_c;b) &= \frac{1}{b^2} \\ \frac{dV_{\text{eff}}}{dr} &= 0 \quad \text{for} \quad r = r_c \end{aligned} \tag{59}$$

And after some not-so-long algebra, one might write the result as;

$$r_c = 2\mu\left[1 + \cos\left[\frac{2}{3}\cos^{-1}\Big(\pm\frac{a}{\mu}\Big)\right]\right] \tag{60}$$

$$b = 3\sqrt{\mu r_c} - a \tag{61}$$

where the for $\pm$ $+$ sign denotes retrograde orbit, and $-$ denotes prograde orbit. As always, for $a \to 0$ we get the results for the case in Schwarzschild metric. At this point although making the similar calculations as we have done for the case of massive particles on the stability of equatorial photon orbits is possible and yields some interesting results, we will suffice with stating that the circular photon orbits are unstable in Kerr geometry, as it is the case for the Schwarzschild geometry.

VI. ADMIRATION

After all these calculations were first done about half a century ago, the author of this paper who has written it for an undergraduate senior project, would like to take a moment to appreciate the era he is hopeful to be a physicist in; an era in which an image like FIG.2 is made possible.

FIG. 2. First ever image of a black hole (a Kerr black hole in particular) shadow and the accretion disk around it. Credit:*Event Horizon Telescope Colloboration*

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